Normal Natural Deduction Proofs (in Non-classical Logics)*

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Abstract. We provide a theoretical framework that allows the direct search for natural deduction proofs in some non-classical logics, namely, intuitionistic sentential and predicate logic, but also in the modal logic S4. The framework uses so-called intercalation calculi to build up broad search spaces from which normal proofs can be extracted, if a proof exists at all. This claim is supported by completeness proofs establishing in a purely semantic way normal form theorems for the above logics. Logical restrictions on the search spaces are briefly discussed in the last section together with some heuristics for structuring a more efficient search. Our paper is a companion piece to [15], where classical logic was treated.

1 Proof Search for ISL

Proofs and Types, the lively and informative book by Girard, Lafont and Taylor [10], expresses a peculiar tension between the presentation of proofs in sequent and natural deduction calculi. Nd calculi are claimed to be limited to intuitionistic logic (p. 8), and yet we are to think of natural deductions as the "true proof objects" (p. 39). Sequent calculi give the "prettiest illustration of the symmetries of logic" and present "numerous analogies with natural deduction, without being limited to the intuitionistic case" (p. 28). However, they have a serious shortcoming from an algorithmic point of view: the lack of a Curry-Howard isomorphism prevents their use "as a typed λ-calculus" (p. 28).

As far as automated theorem proving (via PROLOG or tableau methods) is concerned, the authors of [10] argue that the sequent calculus provides the underlying ideas: "What makes everything work is the sequent calculus with its deep symmetries, and not particular tricks." (p. 28) And yet, as far as proofs are concerned, the system of sequents is not viewed as primitive: the sequent

* This paper is dedicated to Jörg Siekmann, pragmatic visionary.

** Most of this paper was written in May 1998 while the second author was a visiting researcher in the Deparment of Philosophy at Carnegie Mellon University.

D. Hutter, W. Stephan (Eds.): Mechanizing Mathematical Reasoning, LNAI 2605, pp. 169–191, 2005.

calculus “sometimes inconveniently complicates situations” (p. 41), as the “rules of the calculus are in fact more or less complex combinations of rules of natural deduction” (p. 39). The sequent calculus is viewed as “only a system which enables us to work on these objects [i.e., the true proof objects]” (p. 39).

There is a technically convenient and heuristically motivated framework that allows the direct search for normal nd-proofs: the intercalation calculus. This calculus was introduced in 1987 for classical sentential logic in the context of the Carnegie Mellon Proof Tutor project; for a first description see [17]. An appropriate extension to classical predicate logic was given in [14], and a much improved (and corrected) version of this material is contained in [15]. The latter paper exploits also a natural Skolem-Herbrand extension (joined with a generalized unification procedure) in order to transfer strategic considerations for proof search from sentential to predicate logic. The heuristically motivating idea for the intercalation calculus is straightforward: given assumptions $A_1, \ldots, A_n$ and a goal G to be derived from the assumptions, one tries to “close the gap” between $A_1, \ldots, A_n$ and G by systematically using elimination rules “from above” and inverted introduction rules “from below”. We say that formulas are *intercalated* between assumptions and conclusion. The search space of all possible direct ways of closing the gap is generated in this way; it allows us either to extract a normal nd-proof or to obtain a countermodel in case the inference from $A_1, \ldots, A_n$ to G is invalid.

This, obviously, provides a semantic argument of a normal form theorem for nd-proofs (with a family resemblance to the dual considerations for the cut-free sequent calculus). The intercalation framework makes it very natural to consider restricted classes of normal nd-proofs and to investigate the effect of particular strategies on the form of the resulting nd-proofs. (These matters have been pursued by John Byrnes in his dissertation [3].) What is being exploited here are not the left-right symmetries of the rules for sequents, but rather the deep logical structure of branches in normal derivations. For classical logic that is made possible by a suitable formulation of the negation rules; the claim that nd calculi are limited to intuitionistic logic is, so it seems to us, quite incorrect. In any event, our paper is to demonstrate that the basic ideas of proof search in classical logic can be used for the treatment of non-classical logics, paradigmatically, for intuitionistic sentential and predicate logic (in Sect. 2 and 3) and for the modal system $\mathsf{S4}$ (in Sect. 4). We remark that a uniform approach to proof search in non-classical logics has been pursued in many ways: see for example [2], where the authors use labelled Natural Deduction systems for various non-classical logics which admit a Kripke-style semantics; or [1], where the authors exploit a type-theoretical Logical Framework to encode nd-systems for various modal logics. The beginnings of our work go back to 1991, when Cittadini investigated intuitionistic sentential logic from this perspective in his M.S. thesis under Sieg’s direction (see [4]); Cittadini also treated $\mathsf{S4}$ in his paper [5]. A version of the first four sections of this paper appeared in 1998 as [16].

We begin with intuitionistic sentential logic (ISL). As for notation, we follow the conventions of [15], p. 71. The language for ISL has sentential variables, logical connectives ∧, ∨, →, and the logical constant ⊥ for absurdity. Negation $\neg\varphi$ is defined as usual by $\varphi \to \bot$. Nd-rules for ISL are the *proper* elimination (E-) and introduction (I-) rules for ∧, ∨ and → given in [12]:

$$\frac{\varphi_1 \wedge \varphi_2}{\varphi_i}\; i = 1 \text{ or } 2 \qquad\qquad \frac{\varphi_1 \quad \varphi_2}{\varphi_1 \wedge \varphi_2}$$

$$\frac{\varphi_1 \vee \varphi_2 \quad \begin{matrix}[\varphi_1]\\ \vdots \\ \psi\end{matrix} \quad \begin{matrix}[\varphi_2]\\ \vdots \\ \psi\end{matrix}}{\psi} \qquad\qquad \frac{\varphi_i}{\varphi_1 \vee \varphi_2}\; i = 1 \text{ or } 2$$

$$\frac{\varphi_1 \quad \varphi_1 \to \varphi_2}{\varphi_2} \qquad\qquad \frac{\begin{matrix}[\varphi_1]\\ \vdots \\ \varphi_2\end{matrix}}{\varphi_1 \to \varphi_2}$$

plus the following "ex falso quodlibet" rule $\bot_{\mathbf{q}}$, where φ is taken to be different from ⊥:

$$\frac{\bot}{\varphi}$$

In [15], for classical logic, negation ¬ is a primitive connective, and two rules $\bot_{\mathbf{c}}$ and $\bot_{\mathbf{i}}$ are given for it:

$$\frac{\begin{matrix}[\neg\psi]\\ \vdots \\ \varphi\end{matrix} \quad \begin{matrix}[\neg\psi]\\ \vdots \\ \neg\varphi\end{matrix}}{\psi}$$

and

$$\frac{\begin{matrix}[\psi]\\ \vdots \\ \varphi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots \\ \neg\varphi\end{matrix}}{\neg\psi}$$

These rules are considered as both E- and I-rules, but not as proper E- or I-rules. The concepts of *p-normal* and *normal* nd-proof are defined as follows: a proof is called p-normal, when no segment of formula occurrences in the proof is such that the first formula in the segment is the conclusion of a proper I-rule or $\bot_{\mathbf{c}}$ and the last formula the major premise of a proper E-rule; it is called normal, if it is p-normal and satisfies the adjacency condition, i.e., the major premise of a ⊥-rule is not inferred by a ⊥-rule. For the intuitionistic calculus $\bot_{\mathbf{i}}$ is a derived rule, while $\bot_{\mathbf{c}}$ is of course not; the distinction between p-normal and normal does not apply and, consequently, a proof is called normal if it does not contain a segment whose first formula is the conclusion of an I-rule and whose last formula is the major premise of a proper E-rule.

Paths, taken in the sense of [12], through normal proofs have this special property: they contain a uniquely determined *E-part* and *I-part*, consisting of segments that are major premises of proper E- and I-rules, respectively; these two parts are separated by the *minimum segment* that is a premise of an I-rule. The formulas occurring in the segments of the E-part (I-part) are strictly positive subformulas of the formula occurring in the path's first (last) segment; the formula of the minimum segment is a strictly positive subformula of the formula in the first or last segment. This implies the crucial subformula property of normal proofs: every formula occurring in a normal proof is (the negation of) a subformula either of the goal or of one of the assumptions. The parenthetical addition in the last sentence is needed only for the classical calculus.

Now we give the rules for the intuitionistic sentential ic-calculus IIC_0. Those corresponding to proper E-rules and inverted proper I-rules are just as in the classical case (cf. [15], p. 71; for the $\downarrow$-rules, we also include the local side conditions of p. 74.) We recall that the rules are formulated as Post-productions, and the symbol $\Longrightarrow$ has to be understood informally as follows: to answer the question on the left side of $\Longrightarrow$ affirmatively, it suffices to answer the question(s) on the right side of $\Longrightarrow$ affirmatively:

$\wedge_i\downarrow$: $\alpha;\beta?G, \varphi_1\wedge\varphi_2\in\alpha\beta, \varphi_i\notin\alpha\beta \Longrightarrow \alpha;\beta,\varphi_i?G$ for $i=1$ or 2
$\vee\downarrow$: $\alpha;\beta?G, \varphi_1\vee\varphi_2\in\alpha\beta, \varphi_1\notin\alpha\beta, \varphi_2\notin\alpha\beta \Longrightarrow \alpha,\varphi_1;\beta?G$ AND $\alpha,\varphi_2;\beta?G$
$\rightarrow\downarrow$: $\alpha;\beta?G, \varphi_1\rightarrow\varphi_2\in\alpha\beta, \varphi_2\notin\alpha\beta, \varphi_1\neq G \Longrightarrow \alpha;\beta?\varphi_1$ AND $\alpha;\beta,\varphi_2?G$
$\wedge\uparrow$: $\alpha;\beta?\varphi_1\wedge\varphi_2 \Longrightarrow \alpha;\beta?\varphi_1$ AND $\alpha;\beta?\varphi_2$
$\vee_i\uparrow$: $\alpha;\beta?\varphi_1\vee\varphi_2 \Longrightarrow \alpha;\beta?\varphi_i$ for $i=1$ or 2
$\rightarrow\uparrow$: $\alpha;\beta?\varphi_1\rightarrow\varphi_2 \Longrightarrow \alpha,\varphi_1;\beta?\varphi_2$

Moreover, we have the following rule corresponding to "ex falso quodlibet" (with $\perp$ different from G):

$\perp_{\mathbf{q}}$: $\alpha;\beta?G \Longrightarrow \alpha;\beta?\perp$

The search tree (or ic-tree) for ISL is defined just as for classical sentential logic (cf. [15], pp. 72–75) by using all available rules plus $\perp_{\mathbf{q}}$, and is clearly always finite. The assignment of $\mathbf{Y}$ and $\mathbf{N}$ to the nodes of the tree is also straightforward, as is the definition of ic-derivation. From an ic-derivation one can construct uniquely an nd-proof, and that proof is normal. The proof of this fact is the same as for classical logic given in [15], pp. 76–77; the only novel case is that of the rule $\perp_{\mathbf{q}}$, which is trivial thanks to the corresponding rule of natural deduction. So for ISL we easily get the *Proof Extraction Theorem*: for any α^* and G^*, if the ic-tree Σ for $\alpha^*?G^*$ evaluates to $\mathbf{Y}$, then a normal nd-proof of G^* from assumptions in α^* can be found.

In case the ic-tree for $\alpha^*?G^*$ evaluates to $\mathbf{N}$, we want to use the tree itself to define a semantic counterexample to the inference from α^* to G^*. Novel considerations have to come in now, because a semantic counterexample here means a Kripke model $\mathcal{M}=\langle W,R,\Vdash\rangle$ and a world u in W, such that $u\Vdash\varphi$ for all $\varphi\in\alpha^*$, and $u\nVdash G^*$. A *Kripke model for* ISL is a triple $\mathcal{M}=\langle W,R,\Vdash\rangle$, where W is a non-empty set, R is a reflexive and transitive relation on W, and $\Vdash$ is a relation between elements of W and formulas such that, for any $u\in W$:

1. for any sentential variable p, if $u \Vdash p$ and uRv, then $v \Vdash p$;
2. $u \nVdash \bot$;
3. $u \Vdash \varphi_1 \wedge \varphi_2$ iff $u \Vdash \varphi_1$ and $u \Vdash \varphi_2$;
4. $u \Vdash \varphi_1 \vee \varphi_2$ iff $u \Vdash \varphi_1$ or $u \Vdash \varphi_2$;
5. $u \Vdash \varphi_1 \rightarrow \varphi_2$ iff for all v such that uRv, if $v \Vdash \varphi_1$, then $v \Vdash \varphi_2$.

Remark 1. A Kripke model is completely determined by W, R, and the behavior of $\Vdash$ on sentential variables.

Given the natural deduction calculus and Kripke semantics for ISL, the *completeness theorem* is standardly formulated as follows: either there is an intuitionistic nd-proof of G from α, or there exist a Kripke model $\mathcal{M} = \langle W, R, \Vdash \rangle$ and a $u \in W$ such that $u \Vdash \varphi$ for all $\varphi \in \alpha$, and $u \nVdash G$. By using our counterexample construction, we will prove a sharpened version where "intuitionistic nd-proof" is replaced by "normal intuitionistic nd-proof". That allows us then to prove a *normal form theorem* by purely semantic means – the topic of the next section.

2 Normal Form Theorem

Assume that the ic-tree Σ for $\pi_0 = \alpha^*?G^*$ evaluates to $\mathbf{N}$, and let $\preceq$ be the natural order relation on Σ. The first step in the construction of a countermodel consists in choosing a subtree P of Σ, and selecting both a set W of question nodes and sets of formulas from P. The construction proceeds in stages. We put π_0 into W and construct inductively a subtree P_0 of Σ, with π_0 as root, along with two sets of formulas T_0 and F_0. Then we select applications of the rule $\rightarrow\uparrow$ in P_0 and put the question nodes $\pi_1, \ldots, \pi_k$ thus reached into W. Now, we repeat the first stage starting from these nodes, i.e., we construct subtrees P_j and sets of formulas T_j and F_j $(1 \leq j \leq k)$ just as we did for π_0; then we repeat the second stage and so on, as long as possible. The construction has to terminate, since Σ is finite: the subtree P is the union of all the P_j's (actually, we construct the P_j's and P just as sets of nodes; but we can treat them as subtrees, by considering them ordered by the appropriate restrictions of $\preceq$).

We have to be careful in this process to choose an appropriate ordering of the rules. This makes our construction more intricate than that for classical logic: the fact that an application of $\rightarrow\downarrow$ may result in losing track of the goal forces us, in the first stage, to deal with these situations only after all other applicable rules have been tried. Moreover, because of the truth definition for conditionals in a Kripke model, we have to be careful in the second stage when choosing the nodes to which we apply $\rightarrow\uparrow$: we choose only those nodes that have been reached after all rules have been tried, except possibly for $\rightarrow\downarrow$ whose applications lead to the aforementioned situations). The sets T_j and F_j, for $\pi_j \in W$ have good closure properties, and these properties can be used to define a Kripke model $\mathcal{M}$ on W; $\mathcal{M}$ turns out to be a counterexample to the inference from α^* to G^*.

Now, put π_0 into W and construct sets $P_0(n)$ of question nodes all evaluating to $\mathbf{N}$ by induction on the level n of the nodes. P_0 shall be the union of the $P_0(n)$'s. For the base case, let $P_0(0) = \{\pi_0\}$. Assume that $P_0(n)$ has been defined, with all nodes of $P_0(n)$ evaluating to $\mathbf{N}$. Let $P_0(n) = \{\pi_{n,1}, \ldots, \pi_{n,l}\}$. For $1 \leq i \leq l$, we define $P_0(n+1)_i$ in the following way:

Case 1: $\wedge_1 \downarrow$ applies to $\pi_{n,i}$ of the form $\alpha;\beta?G$ with at least one formula of the form $\varphi_1 \wedge \varphi_2$ in $\alpha\beta$, $\varphi_1 \notin \alpha\beta$. Pick the first such formula in the sequence. Above the rule node is a branch leading to $\alpha;\beta,\varphi_1?G$ which evaluates to **N**. Let $P_0(n+1)_i = \{\alpha;\beta,\varphi_1?G\}$.

Case 2: $\wedge_2 \downarrow$ applies to $\pi_{n,i}$. The situation is as in case 1 with $\alpha;\beta,\varphi_2?G$ in place of $\alpha;\beta,\varphi_1?G$.

Case 3: $\vee \downarrow$ applies to $\pi_{n,i}$ of the form $\alpha;\beta?G$ with at least one formula of the form $\varphi_1 \vee \varphi_2$ in $\alpha\beta$, $\varphi_1 \notin \alpha\beta$, $\varphi_2 \notin \alpha\beta$. Pick the first such formula in the sequence. Above the rule node is a conjunctive branching leading to $\alpha,\varphi_1;\beta?G$ and $\alpha,\varphi_2;\beta?G$. At least one of these nodes evaluates to **N**. If $\alpha,\varphi_1;\beta?G$ evaluates to **N**, $P_0(n+1)_i = \{\alpha,\varphi_1;\beta?G\}$; otherwise, $P_0(n+1)_i = \{\alpha,\varphi_2;\beta?G\}$.

Case 4: $\rightarrow\downarrow$ applies to $\pi_{n,i}$ of the form $\alpha;\beta?G$ (with at least one formula $\varphi_1 \rightarrow \varphi_2$ in $\alpha\beta$, where $\varphi_2 \notin \alpha\beta$, $\varphi_1 \neq G$) and leads to $\alpha;\beta,\varphi_2?G$ and $\alpha;\beta?\varphi_1$, *such that the former evaluates to* **N** (the second possibility, with $\alpha;\beta,\varphi_2?G$ evaluating to **Y** and $\alpha;\beta?\varphi_1$ to **N**, will be treated in case 7). Pick the first such formula in the sequence $\alpha\beta$, and let $P_0(n+1)_i = \{\alpha;\beta,\varphi_2?G\}$.

Case 5: $\wedge \uparrow$ applies to $\pi_{n,i}$ of the form $\alpha;\beta?\varphi_1 \wedge \varphi_2$. Above the rule node is a conjunctive branching leading to $\alpha;\beta?\varphi_1$ and $\alpha;\beta?\varphi_2$. At least one of these nodes evaluates to **N**. If $\alpha;\beta?\varphi_1$ evaluates to **N**, $P_0(n+1)_i = \{\alpha;\beta?\varphi_1\}$; otherwise, $P_0(n+1)_i = \{\alpha;\beta?\varphi_2\}$.

Case 6: $\vee \uparrow$ applies to $\pi_{n,i}$ of the form $\alpha;\beta?\varphi_1 \vee \varphi_2$. Above the rule node is a disjunctive branching leading to $\alpha;\beta?\varphi_1$ and $\alpha;\beta?\varphi_2$. Both of these nodes evaluate to **N**. Let $P_0(n+1)_i = \{\alpha;\beta?\varphi_1, \alpha;\beta?\varphi_2\}$.

Case 7: the previous cases do not apply to $\pi_{n,i} = \alpha;\beta?G$. Let $\varphi_1 \rightarrow \psi_1, \ldots,$-$\varphi_r \rightarrow \psi_r$ be the list of all conditionals in $\alpha\beta$, with $\psi_h \notin \alpha\beta$, $\varphi_h \neq G$ (note that r may be 0, in which case the list is empty and we simply put $P_0(n+1)_i = \emptyset$). For $1 \leq h \leq r$, above the rule node is a conjunctive branching leading to $\alpha;\beta,\psi_h?G$ and $\alpha;\beta?\varphi_h$. The latter has to evaluate to **N**, since otherwise case 4 would have applied. Let $P_0(n+1)_i = \{\alpha;\beta?\varphi_1, \ldots, \alpha;\beta?\varphi_r\}$.

To complete the inductive step for $n+1$, define $P_0(n+1) = \bigcup_{1\leq i\leq l} P_0(n+1)_i$. Since Σ is finite, the construction terminates, and there is a natural number m such that for any $n \geq m$ we have $P_0(n) = \emptyset$; with μ being the least such number, we define $P_0 = \bigcup_{0\leq n\leq\mu} P_0(n)$. Let T_0 be the set of all formulas occurring on the left side of the question mark in some node of P_0, and F_0 be the set of all formulas occurring on the right side of the question mark in some node of P_0.

For the second stage of our construction, we select those nodes $\pi = \alpha;\beta?G$ of P_0 to which case 7 applied and where G has the form $\varphi \rightarrow \psi$. Then, above the rule node is a branch leading to $\alpha,\varphi;\beta?\psi$ evaluating to **N**. Let $\pi_1, \ldots, \pi_k$ be the nodes thus reached, and put them into W. The process can be repeated starting from these nodes. The whole construction has to terminate, since Σ is finite. In the end, we let P be the union of all the P_j's. Note that the ordering of the cases in the inductive step is irrelevant, except for case 7, which must be the last one (the reason for this will become clear in the proofs of Lemma 3). Note also that the rule $\bot_{\mathbf{q}}$ is not used in the construction of P: P is used to define sets with good closure properties for which $\bot_{\mathbf{q}}$ is not needed; $\bot_{\mathbf{q}}$ is needed to prove the key property of the Kripke model M formulated in Lemma 4. But we prove first that the sets T_j and F_j, for nodes π_j in $W \subseteq P$, have good closure properties.

Lemma 2. *For any $\pi_j \in W$, the following claims hold:*
(a) $\varphi_1 \wedge \varphi_2 \in T_j$ implies $\varphi_1 \in T_j$ and $\varphi_2 \in T_j$
(b) $\varphi_1 \vee \varphi_2 \in T_j$ implies $\varphi_1 \in T_j$ or $\varphi_2 \in T_j$
(c) $\varphi_1 \rightarrow \varphi_2 \in T_j$ implies $\varphi_1 \in F_j$ or $\varphi_2 \in T_j$
(d) $\varphi_1 \wedge \varphi_2 \in F_j$ implies $\varphi_1 \in F_j$ or $\varphi_2 \in F_j$
(e) $\varphi_1 \vee \varphi_2 \in F_j$ implies $\varphi_1 \in F_j$ and $\varphi_2 \in F_j$
(f) $\varphi_1 \rightarrow \varphi_2 \in F_j$ implies that there exists a $\pi_h \in W$ such that $\pi_j \preceq \pi_h$, $\varphi_1 \in T_h$ and $\varphi_2 \in F_h$.

Proof. For (a)-(e), the key element is that conditionals on the left side of the question mark, all conjunctions and all disjunctions are always dealt with during the construction of P_j. Consider for example (a): if there is a node $\alpha;\beta?G$ in P_j with $\varphi_1 \wedge \varphi_2 \in \alpha\beta$, then this formula is dealt with in cases 1 and 2, hence $\varphi_1 \in T_j$ and $\varphi_2 \in T_j$. Similarly, (b) follows from case 3, (c) from cases 4 and 7, (d) from case 5, (e) from case 6. For (f), if a node $\alpha;\beta?\varphi_1 \rightarrow \varphi_2$ is in P_j, there is a node $\pi_h \in W$ that has been reached by an application of $\rightarrow\uparrow$, such that $\pi_j \preceq \pi_h$, $\varphi_1 \in T_h$ and $\varphi_2 \in F_h$. □

The following lemma shows other important features of the sets T_j and F_j, namely, the T_j's and F_j's do not have common sentential variables, and the T_j's are cumulative.

Lemma 3. *(i) No sentential variable belongs to $T_j \cap F_j$;*[1]
(ii) if $\pi_j, \pi_h \in W$ and $\pi_j \preceq \pi_h$, then $T_j \subseteq T_h$.

Proof. (i) Assume $p \in T_j \cap F_j$, for a sentential variable p. This means that in P_j there are nodes $\rho = \alpha;\beta?G$ with $p \in \alpha\beta$ and $\rho' = \alpha';\beta'?p$. We distinguish three cases.

Case 1: If $\rho \preceq \rho'$, then $p \in \alpha'\beta'$, since no rule of IIC_0 takes away a formula on the left side of the question mark. Thus ρ' evaluates to $\mathbf{Y}$, contrary to the fact that all nodes in P evaluate to $\mathbf{N}$.

Case 2: If $\rho' \prec \rho$, then G must be different from p, since otherwise ρ would evaluate to $\mathbf{Y}$. This means, the formula on the right side of the question mark has been modified in the construction, and since p is a sentential variable this may have happened only through case 7 with an application of $\rightarrow\downarrow$. Let ρ'' be the node to which case 7 applied; clearly, $\rho' \preceq \rho'' \preceq \rho$. Now, the cases that add formulas to the left side of the question mark have been dealt with before case 7, hence they cannot apply above ρ'', and the set of formulas on the left side of the

[1] Actually, one can prove that $T_j \cap F_j = \emptyset$.

question mark remains unchanged in P_j above ρ''. This means that $\rho'' = \alpha;\beta?p$, and since $p \in \alpha\beta$, it evaluates to $\mathbf{Y}$, a contradiction.

Case 3: Assume that ρ and ρ' are on different branches, and let ρ'' be the node at which the highest branching below ρ and ρ' occurred; so either case 6 or case 7 applied to ρ''. But these cases do not change the sequence of formulas on the left side of the question mark, hence any formula that occurs on the left side of the question mark on a branch occurs also on the left side of the question mark on the other branch. Thus, $p \in \alpha'\beta'$, and so ρ' evaluates to $\mathbf{Y}$, a contradiction[2].

(ii) Let $\rho = \alpha;\beta?G$ be a node in P_j, and $\varphi \in \alpha\beta$; we show that $\varphi \in T_h$. If $\rho \preceq \pi_h$, this is immediate since no rule of IIC_0 takes away a formula on the left side of the question mark. If ρ and π_h are on different branches, let $\rho' = \alpha';\beta'?G'$ be the node at which the highest branching occurred. If it occurred through case 6 or 7 in the construction of P_j, then we see as in (i) that any formula that occurs on the left side of the question mark on one branch occurs also on the left side of the question mark on the other branch; hence we conclude $\varphi \in T_h$. Assume then that the branching occurred with an application of $\rightarrow\uparrow$ after the construction of P_j. This means that case 7 applied to ρ'. But then we see, as in (i), that cases 1–4 cannot apply above ρ', and therefore the set of formulas on the left side of the question mark remains unchanged in P_j above ρ'. So $\varphi \in \alpha'\beta'$, and since $\rho' \preceq \pi_h$ we get $\varphi \in T_h$. □

Finally, we come to the definition of the Kripke countermodel: let $\mathcal{M} = \langle W, \preceq, \Vdash \rangle$, where $\preceq$ is restricted to W, and for any $\pi_j \in W$ and any sentential variable p, $\pi_j \Vdash p$ iff $p \in T_j$. This is enough to define a Kripke model, by Remark 1; condition 1 of the definition of Kripke model holds, because of Lemma 3(ii). The following lemma gives the key property of $\mathcal{M}$.

Lemma 4. *For any $\pi_j \in W$ and any formula φ, the following claims hold:*
(1) $\varphi \in T_j$ implies $\pi_j \Vdash \varphi$
(2) $\varphi \in F_j$ implies $\pi_j \nVdash \varphi$.

Proof. By induction on the complexity of φ; we treat the case of atomic formulas and conditionals; the remaining cases of conjunctions and disjunctions are routine.

Assume φ is a sentential variable p. Then (1) follows from the definition of $\Vdash$, and (2) is a consequence of Lemma 3(i).

Assume $\varphi = \bot$. Then (2) follows from the definition of a Kripke model. For (1), suppose there is a node $\rho = \alpha;\beta?G$ in P_j such that $\bot \in \alpha\beta$. Then, when we apply $\bot_{\mathbf{q}}$ to ρ (in the full ic-tree Σ), it leads to a node $\alpha;\beta?\bot$ which evaluates to $\mathbf{Y}$. Hence ρ evaluates to $\mathbf{Y}$, too, contradicting the fact that all nodes in P evaluate to $\mathbf{N}$. So $\bot \notin T_j$, from which (1) follows.

[2] This last case might have been proved in a simpler way by using the fact that the cases in the construction of P_j that add formulas to the left side of the question mark, i.e. cases 1–4, precede also case 6; but we do not want this to be a decisive feature of our countermodel construction, since in the extension to predicate logic we shall not have the same situation.

Assume $\varphi = \varphi_1 \to \varphi_2$. Then, for (1), suppose $\varphi_1 \to \varphi_2 \in T_j$. By Lemma 3(ii), for any $\pi_h \in W$ with $\pi_j \preceq \pi_h$, $\varphi_1 \to \varphi_2 \in T_h$. By Lemma 2(c), this implies $\varphi_1 \in F_h$ or $\varphi_2 \in T_h$. So by induction hypothesis we have $\pi_h \not\Vdash \varphi_1$ or $\pi_h \Vdash \varphi_2$, and since this holds for any $\pi_h \in W$ with $\pi_j \preceq \pi_h$, by definition of $\Vdash$ we obtain $\pi_j \Vdash \varphi_1 \to \varphi_2$. For (2), assume $\varphi_1 \to \varphi_2 \in F_j$. By Lemma 2(f), this implies that there exists a $\pi_h \in W$ with $\pi_j \preceq \pi_h$ such that $\varphi_1 \in T_h$ and $\varphi_2 \in F_h$. So by induction hypothesis we have $\pi_h \Vdash \varphi_1$ and $\pi_h \not\Vdash \varphi_2$, and since $\pi_j \preceq \pi_h$, by definition of $\Vdash$ we obtain $\pi_j \not\Vdash \varphi_1 \to \varphi_2$. □

By applying Lemma 4 to the root node of Σ, we obtain the *Counterexample Extraction Theorem* immediately: if the ic-tree for $\alpha?G$ evaluates to **N**, then it is possible to define from it a counterexample to the inference from α to G, that is, a Kripke model that verifies all the formulas of α and refutes G.

Putting the Proof Extraction Theorem and the Counterexample Extraction Theorem together, we obtain the *Completeness Theorem* for IIC_0 and the sharpened form discussed at the end of Sect. 1 for the nd-calculus.

Theorem 5. *Either the ic-tree for $\alpha?G$ contains an ic-derivation of $\alpha?G$ (from which a normal nd-proof of G from α can be constructed) or it allows the definition of a counterexample to the intuitionistic inference from α to G.*

Soundness and completeness of the ic-calculus provide us with a purely semantic proof of the *Normal Form Theorem* for intuitionistic sentential natural deduction[3]:

Theorem 6. *For every nd-proof there is a normal nd-proof with the same assumptions and conclusion.*

Our completeness proof parallels the one for semantic tableaux given by Fitting in [8]. In that proof, signed formulas are used: i.e., formulas preceded by T (resp. F). These correspond in IIC_0 to formulas on the left (resp. right) side of the question mark. Roughly, the argument in Fitting's proof goes as follows. First one extends the notion of model to signed formulas (more precisely, to sets of signed formulas with suitable closure properties, so-called Hintikka collections). Now assume that the formula φ is not provable in the tableaux system, that is, the set $\{F\varphi\}$ is consistent. Exploiting this hypothesis, construct a Hintikka collection that contains $F\varphi$, and obtain from it a model for this signed formula, i.e. a countermodel for φ. Our proof does not start with a single formula, but with a question $\alpha^*?G^*$. (The approaches are equivalent. Fitting's proof can easily be adapted to start with a set of signed formulas $T\varphi_1, \ldots, T\varphi_n, FG$, with the φ_i's corresponding to the formulas in our α^*.) We assume that $\alpha^*?G^*$ is not "provable" in IIC_0, i.e. the ic-tree for it evaluates to **N**, and use this hypothesis to construct a model for the formulas in α^* which does not verify G^*, i.e. a countermodel for the inference from α^* to G^*. The condition with which a leaf node evaluates to **Y** in IIC_0 corresponds to the condition that makes a set of signed formulas closed; moreover, having an ic-tree evaluating to **Y** corresponds to having a closed tableau.

[3] Because of the finiteness of ic-trees in IIC_0, the ic-calculus also provides a decision procedure for ISL.

3 Extension to Predicate Logic

In this section we extend the metamathematical considerations for ISL to intuitionistic predicate logic (IPL), as was done for classical logic in Sect. 4 of [15]. We use the following nd-rules for the quantifiers (where writing φt assumes that t is free for x in φx or that some bound variables in φx have been renamed):

$$\frac{(\forall x)\varphi x}{\varphi t}\ \forall\text{E} \qquad\qquad \frac{\varphi y}{(\forall x)\varphi x}\ \forall\text{I}$$

$$\frac{(\exists x)\varphi x \quad \begin{matrix}[\varphi y]\\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists\text{E} \qquad\qquad \frac{\varphi t}{(\exists x)\varphi x}\ \exists\text{I}$$

The usual restrictions apply to ∀I (y does not have a free occurrence in any assumption on which the derivation of φy depends) and to ∃E (y must not have free occurrences in ψ or $(\exists x)\varphi x$ nor in any assumption – other than φy – on which the proof of the upper occurrence of ψ depends).

The ic-calculus for IPL, IIC_1, has all the rules of IIC_0 plus the following ones for the quantifiers, where $\mathcal{T}(\gamma)$ denotes the finite set of terms occurring in the formulas of γ:

$\forall\downarrow$: $\alpha;\beta?G$, $(\forall x)\varphi x \in \alpha\beta$, $t \in \mathcal{T}(\alpha\beta, G)$, $\varphi t \notin \alpha\beta \Longrightarrow \alpha;\beta,\varphi t?G$
$\exists\downarrow$: $\alpha;\beta?G$, $(\exists x)\varphi x \in \alpha\beta$, there is no t such that $\varphi t \in \alpha\beta$, y is new for α, $(\exists x)\varphi x, G \Longrightarrow \alpha,\varphi y;\beta?G$
$\forall\uparrow$: $\alpha;\beta?(\forall x)\varphi x$, y is new for α, $(\forall x)\phi x \Longrightarrow \alpha;\beta?\varphi y$
$\exists\uparrow$: $\alpha;\beta?(\exists x)\varphi x$, $t \in \mathcal{T}(\alpha\beta,(\exists x)\varphi x) \Longrightarrow \alpha;\beta?\varphi t$

In the rules $\exists\downarrow$ and $\forall\uparrow$ the new variable y is chosen in a canonical way (say, the first available one in a fixed ordering of the variables).

Ic-trees are defined as in the classical case. Since in general they are not finite, for the evaluation of nodes we use **Y** and **N** as before, but also the value **U** to evaluate partial ic-trees (see [15], pp. 86–87). If the ic-tree Σ for $\alpha^*?G^*$ evaluates to **Y** it is possible, just as for IIC_0, to extract from Σ a normal nd-proof (i.e. a proof extraction theorem holds). In case Σ evaluates to **N** or **U**, we want to construct from Σ a semantic counterexample to the inference from α^* to G^*. To this end, let us recall Kripke semantics for IPL (see e.g. [6], [9]).

Let D be a nonempty set, and $\mathcal{L}(D)$ be the first-order language with constant symbols for elements in D. A *Kripke model for intuitionistic predicate logic over* D is a quadruple $\mathcal{M} = \langle W, R, \delta, \Vdash \rangle$, where W is a non-empty set, R is a reflexive and transitive relation on W, δ is a function from W to nonempty subsets of D satisfying the *monotonicity condition* (i.e. uRv implies $\delta(u) \subseteq \delta(v)$) and $\Vdash$ is a relation between elements of W and sentences of $\mathcal{L}(D)$ such that $\langle W, R, \Vdash \rangle$ is a Kripke model for ISL, and for any $u \in W$ and any quantified sentence of $\mathcal{L}(D)$:

1. $u \Vdash (\exists x)\varphi(x)$ iff $u \Vdash \varphi(c)$ for some $c \in \delta(u)$;
2. $u \Vdash (\forall x)\varphi(x)$ iff for every v such that uRv, $v \Vdash \varphi(c)$ for every $c \in \delta(v)$.

Remark 7. A Kripke model for intuitionistic predicate logic over D is completely determined by W, R, δ and the behavior of $\Vdash$ on atomic sentences of $\mathcal{L}(D)$.

Now let us treat the counterexample extraction. As in classical logic, the case which requires novel considerations with respect to sentential logic is the extraction of a counterexample from an infinite ic-tree Σ evaluating to **U**. We treat this case by extending the technique used in Sect. 2 for ISL. The construction of P_0 goes as that for ISL in cases 1–6. Then we have the following:

Case 7: $\forall\downarrow$ applies to $\pi_{n,i}$ of the form $\alpha;\beta?G$ with at least one formula of the form $(\forall x)\varphi x$ in $\alpha\beta$ and a term $t \in \mathcal{T}(\alpha\beta, G)$ such that $\varphi t \notin \alpha\beta$. Pick the first such formula in the sequence $\alpha\beta$, and the first such term in $\mathcal{T}(\alpha\beta, G)$ (in some fixed ordering of $\mathcal{T}(\alpha\beta, G)$). Above the rule node is a branch leading to a question node $\alpha;\beta,\varphi t?G$ which evaluates to **U**. Let $P_0(n+1)_i = \{\alpha;\beta,\varphi t?G\}$.

Case 8: the previous cases do not apply to $\pi_{n,i} = \alpha;\beta?G$. Let $\varphi_1 \to \psi_1, \ldots, \varphi_r \to \psi_r$ be the list of all conditionals in $\alpha\beta$, with $\psi_h \notin \alpha\beta$, $\varphi_h \neq G$. For $1 \leq h \leq r$, above the rule node is a conjunctive branching leading to nodes $\alpha;\beta,\psi_h?G$ and $\alpha;\beta?\varphi_h$. The latter has to evaluate to **U**, since otherwise case 4 would have applied. Let $(\exists x_1)\vartheta_1 x_1, \ldots, (\exists x_s)\vartheta_s x_s$ be the list of all existentials in $\alpha\beta$, and y_h (for $1 \leq h \leq s$) be the first variable (in the fixed ordering) which is new for $\alpha, (\exists x_h)\vartheta_h x_h, G$. For $1 \leq h \leq s$, above the rule node is a branch leading to a question node $\alpha, \vartheta_h y_h;\beta?G$ which evaluates to **U**. Finally, if $G = (\exists x)\chi x$, i.e. $\exists\uparrow$ applies to $\pi_{n,i}$, above the rule node is a disjunctive branching leading to nodes of the form $\alpha;\beta?\chi t$ (one for each $t \in \mathcal{T}(\alpha\beta, (\exists x)\chi x)$), all evaluating to **U**; in this case, let $X_0(n{+}1)_i = \{\alpha;\beta?\chi t \mid t \in \mathcal{T}(\alpha\beta, (\exists x)\chi x)$, otherwise, let $X_0(n{+}1)_i = \emptyset$. Now let $P_0(n+1)_i = \{\alpha;\beta?\varphi_1, \ldots, \alpha;\beta?\varphi_r, \alpha, \vartheta_1 y_1;\beta?G, \ldots, \alpha, \vartheta_s y_s;\beta?G\} \cup X_0(n+1)_i$ (note that r or s may be 0, in which case the corresponding list is empty; if they are both 0, we simply put $P_0(n+1)_i = X_0(n+1)_i$).

The main reason for having such an intricate case 8 is the rule $\exists\downarrow$. In fact, this rule introduces new variables, so it might cause the construction to go on indefinitely. But we want to treat the special applications of $\to\downarrow$ as the last case; hence we have to reach this at some finite stage. Therefore we are forced to treat them at the same time as the applications of $\exists\downarrow$. The introduction of new variables due to $\exists\downarrow$ also forces us to apply $\exists\uparrow$ (if it is the case) at this stage if we want to get the appropriate closure property.

This construction is of course infinite, but we can still define P_0 as the union of all the $P_0(n)$'s, for $0 \leq n < \omega$, and T_0 and F_0 as in the sentential case, and proceed with the second stage of the construction. We select those nodes $\pi = \alpha;\beta?G$ of P_0 to which case 8 has applied and such that $\to\uparrow$ or $\forall\uparrow$ applies to π, i.e. G has the form $\varphi \to \psi$ or $(\forall x)\vartheta x$. Then, above the rule node is a branch leading to a question node $\alpha, \varphi;\beta?\psi$, which evaluates to **U**, or a branch leading to a question node $\alpha;\beta?\vartheta y$, with y new for $\alpha, (\forall x)\vartheta x$. Let $\pi_1, \ldots, \pi_k, \ldots$ be the nodes thus reached, and put them into W. Now the process can be repeated starting from these nodes, with the definition of P_j, T_j and F_j, and so on. In the end, we let P be the union of all the P_j's[4].

[4] Here, clearly, the index j has to range over countable ordinals and not just natural numbers.

We obtain the following extension of Lemma 2.

Lemma 8. *The closure properties (a)–(f) of Lemma 2 hold for the predicate case. Furthermore, we have the following:*

(g) $(\exists x)\varphi x \in T_j$ *implies* $\varphi t \in T_j$*, for some* t *occurring in* P_j*;*

(h) $(\exists x)\varphi x \in F_j$ *implies* $\varphi t \in F_j$*, for every* t *occurring in* P_j*;*

(i) $(\forall x)\varphi x \in T_j$ *implies that for every* $\pi_h \in W$ *such that* $\pi_j \preceq \pi_h$ *and every* t *occurring in* P_j*,* $\varphi t \in T_h$*;*

(j) $(\forall x)\varphi x \in F_j$ *implies that there exist a* $\pi_h \in W$ *such that* $\pi_j \preceq \pi_h$ *and a* t *occurring in* P_h *such that* $\varphi t \in F_h$*.*

Proof. (a)–(f) are as in the sentential case. (g) follows from case 8. For (h), case 8 gives us $\varphi t \in F_j$ for every t occurring in P_j up to the stage to which the case has applied; but $\exists\downarrow$ and $\exists\uparrow$ are treated at the same time, and the applications of $\exists\downarrow$ do not change the shape of the goal, so the new terms t' that occur in P_j are treated in a successive case 8, and for all of them we get again $\varphi t' \in F_j$. (i) follows from case 7, and the fact that no rule of IIC_1 takes away a formula on the left side of the question mark. For (j), if $(\forall x)\varphi x \in F_j$, then case 8 gives us a new node $\pi_h \in W$ and a term y occurring in P_h such that $\varphi y \in F_h$. □

Moreover, we can prove the following analogue of Lemma 3. The proof requires only slight modifications, hence we omit it.

Lemma 9. *(i) No atomic formula belongs to* $T_j \cap F_j$*;*[5]

(ii) if $\pi_j, \pi_h \in W$ *and* $\pi_j \preceq \pi_h$*, then* $T_j \subseteq T_h$*.*

Finally, for the definition of our countermodel, let D be the set of terms occurring in P, and for any node π_j, $\delta(\pi_j)$ shall be the set of terms occurring in P_j. The monotonicity condition holds, because of the canonical choice of new variables for $\exists\downarrow$ in the construction of each P_j. Let $\mathcal{M} = \langle W, \preceq, \delta, \Vdash \rangle$, where $\preceq$ is restricted to W, and for any $\pi_j \in W$ and any atomic formula φ of $\mathcal{L}(D)$, $\pi_j \Vdash \varphi$ iff $\varphi \in T_j$ (this is enough to define a Kripke model, by Remark 7).

We then obtain the following analogue of Lemma 4. The proof is essentially identical (induction on the complexity of φ, using Lemmas 8 and 9 instead of 2 and 3).

Lemma 10. *For any* $\pi_j \in W$ *and any formula* φ *of* $\mathcal{L}(D)$*, the following hold:*

(1) $\varphi \in T_j$ *implies* $\pi_j \Vdash \varphi$

(2) $\varphi \in F_j$ *implies* $\pi_j \nVdash \varphi$*.*

As in the sentential case, we obtain the *Counterexample Extraction Theorem* by applying the last lemma to the root node of Σ, and again from this we get the *Completeness Theorem* and the *Normal Form Theorem.*

4 The Modal Logic S4

We now apply the ideas underlying intercalation calculi to modal logic by giving an appropriate ic-calculus for the modal system S4. The language contains now

[5] Again, one can prove that $T_j \cap F_j = \emptyset$.

sentential variables, logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, and the modal operator $\Box$. The modal operator $\Diamond$ is defined: $\Diamond\varphi = \neg\Box\neg\varphi$ (this is to follow Prawitz's notation [12], and to save work in the proof of the Counterexample Extraction Theorem; it is not difficult to give a version where $\Diamond$ is primitive, too).

Following [12], we use a nd-system for S4 which has all the rules of classical sentential logic plus the following ones for $\Box$:

$$\frac{\Box\varphi}{\varphi}\ \Box\mathrm{E} \qquad\qquad \frac{\varphi}{\Box\varphi}\ \Box\mathrm{I}$$

The I-rule for $\Box$ has to satisfy certain restrictions. Prawitz gives three versions of such restrictions and shows that the resulting systems are actually equivalent. In the first version, the rule can be applied only if all the open assumptions on which φ depends in the deduction are of the form $\Box\psi$ (in Prawitz's terms, they are *modal formulas*). To define the second version, the notion of *essentially modal formula* is introduced inductively as follows:

1. all modal formulas are essentially modal;
2. if φ_1 and φ_2 are essentially modal, then so are $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$.

The restrictions on $\Box$I are liberalized for the second version: essentially modal formulas are allowed as φ's open assumptions. Finally, in the third version, the restrictions are further liberalized: $\Box$I can be applied to φ when, for each open assumption ψ on which φ depends there is an essentially modal formula ϑ such that

(i) ϑ is ψ or φ, or ϑ occurs on the path from ψ to φ,

and

(ii) all assumptions on which ϑ depends are also assumptions on which φ depends.

The reason for this liberalization is that the first two versions, as Prawitz shows, do not allow a normal form theorem. We shall discuss this issue in connection with the Proof Extraction Theorem; but first, we introduce the S4 ic-calculus.

Here we have all the rules for the classical calculus $\mathrm{IC}_0(\mathcal{F})$ (cf. [15], p. 71), i.e. the $\downarrow$- and $\uparrow$-rules of IIC_0, in particular, the following $\bot$-rules:

$\bot_{\mathbf{c}}(\mathcal{F})$: $\alpha;\beta?G, \varphi \in \mathcal{F}(\alpha, \neg G) \Longrightarrow \alpha, \neg G;\beta?\varphi$ AND $\alpha, \neg G;\beta?\neg\varphi$
$\bot_{\mathbf{i}}(\mathcal{F})$: $\alpha;\beta?\neg G, \varphi \in \mathcal{F}(\alpha, G) \Longrightarrow \alpha, G;\beta?\varphi$ AND $\alpha, G;\beta?\neg\varphi$

where $\mathcal{F}(\gamma)$ is the set of all unnegated proper subformulas of formulas in γ and the unnegated part of all negations which are subformulas of formulas in γ, and the following rules for $\Box$:

$\Box\downarrow$: $\alpha;\beta?G, \Box\varphi \in \alpha\beta \Longrightarrow \alpha;\beta,\varphi?G$
$\Box\uparrow$: $\alpha;\beta?\Box\varphi \Longrightarrow (\alpha\beta)_\nu?\varphi$

where $(\alpha\beta)_\nu$ is the sequence of the modal formulas occurring in $\alpha\beta$. The choice of these rules is inspired by the semantic tableaux version of S4 given by Fitting in [9]. Note that in the rule $\Box\uparrow$ we have $(\alpha\beta)_\nu$ and not $\alpha_\nu;\beta_\nu$ on the left side of the question mark. In fact, when we write $\alpha;\beta?G$ we mean that the formulas of β have been obtained from those of α via $\downarrow$-rules, and we cannot claim in general that formulas of β_ν can be obtained from formulas of α_ν via $\downarrow$-rules. Thus, we take the sequence of modal formulas in $\alpha\beta$ as assumptions of a new proof of φ (clearly, if in this new proof $\downarrow$-rules are used, we find nodes of the form $\alpha';\beta'?G'$ once again).

The definition of the S4 *ic-tree* is straightforward, as are the assignment of **Y** and **N** to the nodes of the tree and the definition of S4 *ic-derivation*. S4 ic-trees are clearly finite.

Now, if we refer to Prawitz's third version of the S4 nd-system, we can easily obtain a *proof extraction theorem*: the argument, as we shall see, proceeds by induction on the height of the ic-derivation, just as in the classical and the intuitionistic case (again, see [15], pp. 76–77). But before doing that, we want to describe why such an argument would not work if we refer to one of the first two versions: this may give a better insight into the liberalization on the constraints for $\Box$I which defines the third version.

The problem is that, in the induction step, some of the ic rules introduce new open assumptions, and these may be neither modal nor essentially modal. Thus, if there is an application of $\Box$I in the nd-proof we obtain by induction hypothesis, the restrictions on $\Box$I in the considered version may be violated. In order to obtain the result, then, the restrictions must be liberalized in such a way that they do not refer only to the shape of the open assumptions, but also to that of the formulas obtained from the open assumptions via $\downarrow$-rules. As the $\downarrow$-rules correspond to the E-rules, we must allow the application of $\Box$I to a formula φ when there are (essentially) modal formulas obtained via E-rules in any path from φ to an open assumption: exactly what Prawitz does with his third version! This, by the way, is not so surprising, since our central concern is to provide a semantic proof of a normal form theorem for S4 natural deduction, and such a theorem does not hold for the first two versions.

We have the following *Proof Extraction Theorem*:

Theorem 11. *For any α and G, if the* S4 *ic-tree for $\alpha?G$ evaluates to* **Y***, then a p-normal nd-proof (in the third version of the* S4 *nd-system) of G from assumptions in α can be found.*

Proof. (sketch): By induction on the height of the ic-derivation. The treatment of classical rules is identical to that for classical logic ([15], pp. 76–77). The ic-rules for $\Box$ are handled with the corresponding nd-rules. In the case of $\Box\uparrow$ the restrictions on $\Box$I are satisfied, thanks to the restriction to $(\alpha\beta)_\nu$ of the set of formulas $\alpha\beta$ on the left side of the question mark. Moreover, the restrictions on $\Box$I are preserved through all induction steps. Indeed, in each case we have by induction hypothesis, for all possible application of $\Box$I to a formula φ and all open assumptions ψ on which φ depends, a formula ϑ satisfying the conditions of the definition: and this ϑ is still present whichever rule we apply, even new open assumptions are introduced. Moreover, the nd-proofs extracted from ic-derivations are clearly p-normal, again exploiting the fact that $\downarrow$-rules are only applied from above and $\uparrow$-rules only from below. $\Box$

Now we give an example to show how the $\downarrow$-rules combine with $\Box\uparrow$ so that the $\mathsf{S4}$ nd-proofs extracted from $\mathsf{S4}$ ic-derivations satisfy the restrictions imposed on $\Box$I (i.e., they are indeed $\mathsf{S4}$ nd-proofs). In this example, an application of $\rightarrow\downarrow$ puts in β a formula of the form $\Box\varphi$, which is not present in α (the original set of hypotheses), but is necessary to make the top node evaluate to $\mathbf{Y}$ after the application of $\Box\uparrow$. Consider the following $\mathsf{S4}$ ic-derivation for $p, p \rightarrow \Box q?\Box\Box q$:

$$\begin{array}{c} \mathbf{Y} \\ | \\ \Box q?\Box q \\ | \\ \Box\uparrow \\ | \\ p, p \rightarrow \Box q; \Box q?\Box\Box q \\ | \\ \rightarrow\downarrow \\ | \\ p, p \rightarrow \Box q?\Box\Box q \end{array}$$

From it we can extract the following $\mathsf{S4}$ nd-proof, where $\Box q$ is the ϑ required for the application of $\Box$I:

$$\dfrac{\dfrac{p \quad p \rightarrow \Box q}{\Box q}}{\Box\Box q}$$

Now we turn to completeness; we start by recalling Kripke semantics for $\mathsf{S4}$. A *Kripke model for* $\mathsf{S4}$ is a triple $\mathcal{M} = \langle W, R, \Vdash \rangle$, where W is a non-empty set, R is a reflexive and transitive relation on W, and $\Vdash$ is a relation between elements of W and formulas such that, for any $u \in W$:

1. $u \Vdash \varphi_1 \wedge \varphi_2$ iff $u \Vdash \varphi_1$ and $u \Vdash \varphi_2$;
2. $u \Vdash \varphi_1 \vee \varphi_2$ iff $u \Vdash \varphi_1$ or $u \Vdash \varphi_2$;
3. $u \Vdash \varphi_1 \rightarrow \varphi_2$ iff $u \Vdash \varphi_1$ implies $u \Vdash \varphi_2$;
4. $u \Vdash \neg\varphi$ iff $u \nVdash \varphi$;
5. $u \Vdash \Box\varphi$ iff for all v such that uRv, $v \Vdash \varphi$.

Clearly, Remark 1 holds also for Kripke models of $\mathsf{S4}$. As in the classical and intuitionistic cases, we want to prove a *Counterexample Extraction Theorem*: that means, in this case, that an $\mathsf{S4}$ ic-tree Σ for $\alpha?G$ which evaluates to $\mathbf{N}$ can be used to define a Kripke model for $\mathsf{S4}$ $\mathcal{M} = \langle W, R, \Vdash \rangle$ and a $u \in W$ such that $u \Vdash \varphi$ for all $\varphi \in \alpha$, and $u \nVdash G$.

Before proceeding with the detailed proofs we sketch the argument for the counterexample extraction from Σ. The proof combines the technique used for the classical case (i.e. the construction of a canonical branch) with that for the intuitionistic case (i.e. for the construction of a Kripke model). The construction proceeds in stages. At the first stage, we select *a single branch* P_0 of Σ, all of whose nodes evaluate to $\mathbf{N}$ (this is done by using the $\perp$-rules systematically, as in the classical case), and put the root node of Σ in a set W. The following stage applies $\Box\uparrow$ to all nodes of P_0 to which this rule is applicable, except for the root (hence these nodes will become branching points in our subtree). Then we put the nodes thus reached in W, and start the construction again from them. In this way we obtain sub-branches $P_1,\ldots,P_k$, and then the process continues. Of course, the construction has to terminate, since the whole tree is finite. The union of all the P_j's will be a subtree of Σ, with W as a subset. Then each node $\pi_j \in W$ will be the root of some P_j. Moreover, it will be possible to prove appropriate closure properties of the P_j's, and then to define a Kripke model on W with the required property of being a counterexample for $\alpha?G$.

Now, assume the ic-tree Σ for $\alpha?G$ evaluates to $\mathbf{N}$. We begin with the construction of P_0. Define φ^- and φ^+ as in the classical case (namely, $\varphi^- = \psi$ if $\varphi = \neg\psi$ and $\varphi^- = \neg\varphi$ otherwise; $\varphi^+ = \psi$ if $\varphi = \neg\neg\psi$ and $\varphi^+ = \varphi$ otherwise), and enumerate $\mathcal{F}(\alpha, G^-)$ by $\langle H_i\rangle_{i\in I}$, where $I = \{i \mid 1 \le i \le n\}$. Put the node $\alpha?G$ in W.

The sequence of nodes $P_0^*(0),\ldots$ is defined as follows. First, let $\alpha_0 = \alpha$, $\lambda_0 = 0$, $G_0 = H_0 = G$. Then, λ_{m+1} is defined according to the following cases:

Case 1: there is a j such that $\lambda_m < j \le n$ and H_j is not of the form $\Box\varphi$ and $H_j \notin \alpha_m$ and $\neg H_j \notin \alpha_m$. Then λ_{m+1} is the least such j.

Case 2: the previous case does not apply, but there is a j such that $\lambda_m < j \le n$ and H_j is of the form $\Box\varphi$ and $H_j \notin \alpha_m$ and $\neg H_j \notin \alpha_m$ and $\alpha_m?\neg H_j$ evaluates to $\mathbf{N}$. Then λ_{m+1} is the least such j.

Case 3: the previous cases do not apply, but there is a j such that $\lambda_m < j \le n$ and H_j is of the form $\Box\varphi$ and $H_j \notin \alpha_m$ and $\neg H_j \notin \alpha_m$ and $\alpha_m?\neg H_j$ evaluates to $\mathbf{Y}$. Then λ_{m+1} is the least such j.

Case 4: the previous cases do not apply. Then let $\lambda_{m+1} = 0$.

Then, let $G_m = \neg H_{\lambda_m}$ if $\alpha_m?\neg H_{\lambda_m}$ evaluates to $\mathbf{N}$ and $G_m = H_{\lambda_m}$ otherwise, $\alpha_{m+1} = \alpha_m, G_m^-$, $P_0^*(2m) = \alpha_m?G_m$, $P_0^*(2m+1) = \bot_{\mathbf{i}}, H_{\lambda_{m+1}}$ if G_m is a negation, $P_0^*(2m+1) = \bot_{\mathbf{c}}, H_{\lambda_{m+1}}$ otherwise.

Let μ be the smallest m with $\lambda_{m+1} = 0$, and define P_0 to be P_0^* restricted to $\{m \mid m \le 2\mu\}$. Now, consider the nodes of the form $\alpha'?\Box\varphi$ in P_0 (excluding the root). These nodes appear in P_0 only because of case 3, hence only after all the formulas of $\mathcal{F}(\alpha, G^-)$ not of the form $\Box\varphi$ have been used (this fact will be crucial for proving the closure properties of the sets P_j). To each of these nodes, the rule $\Box\uparrow$ is applicable, leading to a node of the form $\alpha'_\nu?\varphi$ (which evaluates to $\mathbf{N}$). Call $\{\pi_1,\ldots,\pi_k\}$ the nodes thus obtained. Put each π_j in W, and start from it the construction of a branch P_j, choosing at each stage the following node according to the cases for P_0. Then, repeat the process. Finally, let P be the union of all the P_j's.

Now, let $W = \{\pi_0,\ldots,\pi_r\}$, each π_j being the root of P_j. For $0 \le j \le r$, let $\bar{\pi}_j = \alpha_{\mu_j}?G_{\mu_j}$ be the top node of P_j, and define $A_j = \{\varphi \mid \varphi \in \alpha_{\mu_j}, G^-_{\mu_j}\}$. The following lemma describes the important syntactic closure properties of the sets A_j.

Lemma 12. *For $0 \leq j \leq r$, the following claims hold:*

(i) if $\varphi \in A_j$, then $\varphi^- \notin A_j$;
(ii) if φ is a subformula of an element in A_j, then $\varphi^+ \in A_j$ or $\varphi^- \in A_j$;
(iii) if $\neg\neg\varphi \in A_j$, then $\varphi \in A_j$;
(iv) if $\varphi_1 \wedge \varphi_2 \in A_j$, then $\varphi_1^+ \in A_j$ and $\varphi_2^+ \in A_j$;
if $\neg(\varphi_1 \wedge \varphi_2) \in A_j$, then $\varphi_1^- \in A_j$ or $\varphi_2^- \in A_j$;
(v) if $\varphi_1 \vee \varphi_2 \in A_j$, then $\varphi_1^+ \in A_j$ or $\varphi_2^+ \in A_j$;
if $\neg(\varphi_1 \vee \varphi_2) \in A_j$, then $\varphi_1^- \in A_j$ and $\varphi_2^- \in A_j$;
(vi) if $\varphi_1 \rightarrow \varphi_2 \in A_j$, then $\varphi_1^- \in A_j$ or $\varphi_2^+ \in A_j$;
if $\neg(\varphi_1 \rightarrow \varphi_2) \in A_j$, then $\varphi_1^+ \in A_j$ and $\varphi_2^- \in A_j$;
(vii) if $\Box\varphi \in A_j$, then for each i with $\pi_j \preceq \pi_i$ it holds that $\varphi^+ \in A_i$;
if $\neg\Box\varphi \in A_j$, then there is an i with $\pi_j \preceq \pi_i$ and $\varphi^- \in A_i$.

Proof. (i)-(vi) are proved exactly as in the Closure Lemma for the classical case ([15], pp. 80–82; indeed, the rules of IC_0 are all available here).

To prove the first part of (vii), observe that if $\Box\varphi \in A_j$, then $\Box\varphi$ must appear on the left side of the question mark below any node of P_j to which $\Box\uparrow$ is applied, that is, before any new node is put in W. This is because of the ordering of the cases: indeed, having $\Box\varphi$ on the left side means that this formula has been dealt with in case 2, and new nodes are put in W only when case 3 has applied. Moreover, formulas of the form $\Box\varphi$ are never taken away from the left side of the question mark. From these two facts it follows immediately that for each i such that $\pi_j \preceq \pi_i$ it holds $\Box\varphi \in A_i$. But now it is not possible that $\varphi^- \in A_i$: in fact, an application of $\Box\downarrow$ would make $\bar{\pi_j}$ evaluate to **Y**, while all the nodes in P_j evaluate to **N**. Therefore, by (ii), $\varphi^+ \in A_i$.

For the second part of (vii), observe that if $\neg\Box\varphi \in A_j$, then $\Box\varphi$ has been dealt with in case 3. Thus a new node $\pi_i = \alpha'?\varphi$ has been put in W, and since the rule applied to π_i is either $\bot_\mathbf{i}$ or $\bot_\mathbf{c}$, φ^- appears on the left side of the question mark in P_i, whence $\varphi^- \in A_i$. □

Now consider $\mathcal{M} = \langle W, \preceq, \Vdash \rangle$, where for any sentential variable p and any $\pi_j \in W$ we set $\pi_j \Vdash p$ iff $p \in A_j$.

Lemma 13. *For any $\pi_j \in W$ and any formula φ, if $\varphi \in A_j$, then $\pi_j \Vdash \varphi$.*

Proof. Straightforward induction on the complexity of φ, using the closure properties of Lemma 12. □

By applying Lemma 13 to the root node π_0 of the tree, we conclude that $\pi_0 \Vdash \varphi$, for each $\varphi \in \alpha, G^-$, that is, π_0 verifies all formulas in α and falsifies G. Thus we have a semantic counterexample for the inference from α to G, and so the *Counterexample Extraction Theorem* is proved.

Again, this implies the *Completeness Theorem* for the S4 ic-calculus and, since the nd-proofs obtained from ic-derivations are p-normal, the *p-Normal Form Theorem* for the version of the S4 nd-system considered. Moreover, the finiteness of S4 ic-trees yields another proof of the *Finite Model Property* for this logic (namely, if a formula is not provable in S4, then it is falsified in some *finite* Kripke model for S4), and a decision procedure for it.

To obtain a *normal* form theorem, restrictions on the set F of contradictory pairs available for the $\perp$-rules can be introduced, as for the classical case (see [15], pp. 83–84). With these restrictions, nd-proofs obtained from ic-derivations are normal, and we are still able to prove the Counterexample Extraction Theorem: however, now not only the $\perp$-rules and the $\Box$-rules are involved in the counterexample construction, but possibly all the other rules. We can obtain a sharpened completeness theorem and a normal form theorem (for the third version of Prawitz's system!), in the following form:

Theorem 14. *Either the* S4 *ic-tree for* $\alpha?G$ *contains an* S4 *ic-derivation of* $\alpha?G$ *(and hence allows to construct a normal* S4 *nd-proof of* G *from* α*) or it allows the definition of a counterexample to the inference from* α *to* G*.*

Theorem 15. *For every* S4 *nd-proof there is a normal* S4 *nd-proof with the same assumptions and conclusion.*

5 Heuristics for Search

In this last section we discuss very briefly strategic issues for proof search in intuitionistic sentential as well as predicate logic. Logical restricions on the search space and appropriate heuristics are needed in order to obtain an efficient procedure. As a first step in our discussion we review the coarse structure of proof search in classical predicate logic.

The search for an answer, i.e., an ic-derivation, to the question $\alpha;\beta?G$ involves three distinct components: (i) use of $\downarrow$-rules, (ii) use of $\uparrow$-rules, (iii) use of $\perp$-rules (with a limited set of contradictory pairs of formulas). It is step (i) that is central and that is taken in a *goal-directed way*. If the question:

(*) Is G a strictly positive subformula of a formula in $\alpha\beta$?

has an affirmative answer, this step provides sequences of $\downarrow$-rule applications that *extract* strictly positive occurrences of G in elements of $\alpha\beta$. The connecting formula sequences consist of the major premisses of the $\downarrow$-rules and require in general answers to new questions, namely, those raised in the minor premisses of the rule applications.

The Skolem-Herbrand expansion was introduced in [15], in order to obtain an appropriate generalization of this *extraction strategy*. It will be described below. Here we just emphasize that the goal-directedness of applications of the $\downarrow$-rules (including the quantifier rules) is obtained by generalizing the question (*) to

(**) Is G unifiable with a strictly positive canonical subformula of a formula in $\alpha\beta$?

A subformula is considered to be *canonical*, if quantifiers are instantiated by terms that match the $\downarrow$-quantifier rules of the Skolem-Herbrand expansion, i.e., those terms would be used by the extracting $\downarrow$-rules. Having indicated the point of the Skolem-Herbrand expansion, let us describe it in reasoned detail.

We assume that the language for the intercalation calculus has just the set $X = \{x, x_0, x_1, \ldots\}$ as its set of variables. Then the language of the Skolem-Herbrand expansion has in addition a set $Y = \{y, y_0, y_1, \ldots\}$ of bound variables, a set $Z = \{z, z_0, z_1, \ldots\}$ of parameters, and a set $F = \{f, f_0, f_1, \ldots\}$ of function symbols. (X, Y and Z are all disjoint, and F contains infinitely many function symbols for each arity n, n a natural number.) If γ is a sequence of formulas, by $FV(\gamma)$ we mean (a sequence of all) the parameters from the set Z which occur as terms in the elements of γ. The calculus is obtained by replacing the quantifier rules with the following ones:

$\forall\downarrow$: $\alpha;\beta?G$, $(\forall x)\varphi x \in \alpha\beta \Longrightarrow \alpha;\beta,\varphi z?G$ for some new z
$\exists\downarrow$: $\alpha;\beta?G$, $(\exists x)\varphi x \in \alpha\beta$, $\bar{z} = FV(\alpha,(\exists x)\varphi x, G) \Longrightarrow \alpha,\varphi f(\bar{z});\beta?G$ for some new f
$\forall\uparrow$: $\alpha;\beta?(\forall x)\varphi x$, $\bar{z} = FV(\alpha,(\forall x)\varphi x) \Longrightarrow \alpha;\beta?\varphi f(\bar{z})$ for some new f
$\exists\uparrow$: $\alpha;\beta?(\exists x)\varphi x \Longrightarrow \alpha;\beta?\varphi z$ for some new z

Correctness for the Skolem-Herbrand expansion in the classical case is proved in [15], Sect. 6, using an appropriate notion of unification: a derivation in the intercalation calculus for $\alpha?G$ exists if and only if a derivation in the Skolem-Herbrand expansion for $\alpha?G$ exists. Thus, the expansion can be considered "as a convenient technical tool for automated proof search" ([15], p. 95). A technical tool that is, as we pointed out, of critical importance for generalizing the (sentential) extraction strategy, i.e., the goal-directed use of elimination rules.

Although we do not go into the details, our claim is that the same approach can be pursued for the intuitionistic case. This may look surprising, in view of the considerations of Shankar in [13]. In fact, Shankar claims that "the impermutability of certain pairs of inferences in LJ makes it incorrect to directly use Herbrandization for proof search" (here LJ stands for intuitionistic predicate sequent calculus, and "Herbrandization" stands, roughly, for what we have called "Skolem-Herbrand expansion"). Shankar's remark is quite correct for the calculus he proposes. So let us see, why it does not apply to the intuitionistic intercalation calculus and its Skolem-Herbrand expansion.

Shankar notes that, when using "Herbrandization" for LJ, unwanted unifications may arise, and result in "proving" statements that are not intuitionistically valid. (See his example on pp. 527–528, i.e., the formula $(\forall x)(\varphi x \vee \psi) \rightarrow \psi \vee (\forall x)\varphi x$ – which is classically, but not intuitionistically provable.) Of course, if one attempts to translate such flawed proofs into the sequent calculus, one does not obtain valid sequent proofs (typically, the restrictions on the quantifier rules are violated). In classical logic one can manage to get valid proofs by permuting the applications of certain rules; but these permutations may be disallowed in the intuitionistic case.

The Skolem-Herbrand expansion of the intercalation calculus has an important different feature, however: the introduction of new function symbols in the rules $\exists\downarrow$ and $\forall\uparrow$. The effect of these new function symbols is, indeed, to exclude unwanted unifications. It is easy to check, for example, that the flawed proof of the formula $(\forall x)(\varphi x \vee \psi) \rightarrow \psi \vee (\forall x)\varphi x$ described in [13] cannot arise in the Skolem-Herbrand expansion of the intuitionistic intercalation calculus.

Complications that arise for proof search in intuitionistic logic are also discussed in Wallen's book [20]. In Chap. 1, §4, Wallen points to three kinds of "redundancies" in the search space of the intuitionistic sequent calculus, namely to the *non-permutability* of some inferences (we discussed that already), to *notational redundancy* (the same piece of information can occur repeatedly in the search space), and to *irrelevance* (some of the information may be useless for finding a proof). The last two problems clearly affect also the proof search via the ic-calculus, as the full ic-tree may contain the same piece of information on many of its branches, and "irrelevant" premisses, for example, can become a problem for the efficiency of the search algorithm. An algorithm can deal with notational redundancy, at least partially, by storing information in order to avoid answering the same question more than once: that can be done both for positive answers ("a proof has been found") and negative answers ("all proof attempts failed"). The problem of irrelevance can be addressed by exploiting strong syntactic connections between assumptions and goal: that is more intricate in intuionistic than in classical logic, and is reflected already at the sentential level. One may recall that, in terms of computational complexity, the set of classically provable sentential formulas is in Co-NP, while the set of intuitionistically provable sentential formulas is PSPACE-complete [18]. Let us illustrate some of these complications.

As a first example, we consider a remark of Dyckhoff [7]. In examining the intuitionistic sentential sequent calculus, Dyckhoff notes that the rule for conditional introduction on the left gives rise to the problem of detecting loops, and writes: "We could, following standard practice, use a stack to detect looping – but the looping tests are expensive, and complicate the task of extending the technique to the first-order case" ([7], p. 796). Such a remark applies to the intercalation calculus even more strongly, a looping is the condition for closing a branch of an intercalation tree with **N**: the detection of loopings cannot be avoided. This means that we really have to find good search heuristics, if we want to improve the efficiency of the algorithm.

So let us turn to the case of classical sentential logic and the strategic considerations underlying the implementation of an algorithm based on the ic-calculus (in the Carnegie Mellon Proof Tutor)[6]. When faced with a question $\alpha;\beta?G$, this algorithm can form three different kinds of strategies: extraction strategies (if G is a positive subformula of some formula in $\alpha\beta$), inversion strategies (if G is not atomic), indirect strategies. Then the strategies are ranked: the first two, when available, are preferred to the third one, with the exception of the cases when the goal is an atom, a negation or a disjunction (this is heuristically motivated, by the fact that in many common problems the indirect rules must indeed be used to prove an atom, negation or disjunction). So we ask, how these considerations have to be modified for the intuitionistic case (i.e. where the differences with the classical case actually lie). The extraction and inversion strategies can be formed here as well, and the treatment is exactly the same as in the classical case, except for atoms and disjunctions (recall that we consider $\neg$ a defined connective). As the indirect strategy is not available, the necessary changes concern essentially $\bot$ and $\vee$.

[6] Complementary considerations underly the algorithm MAMBA in Tennant's book [19], pp. 136–140.

For $\bot$ the situation is quite simple, since the only rule we have for it is $\bot_{\mathbf{q}}$. This rule has nothing to do with the shape of the goal (provided it is not $\bot$), and is needed to make sure that anything can be proved from an inconsistent set of assumptions. It is clear that a strategy based on this rule should be tried only as a last resort, that is, if all other strategies have failed. This could be the case, for instance, if the goal G is an atom and is not a positive subformula of one of the assumptions: in fact, in this case we can use neither the extraction nor the inversion strategy, and thus our only hope to get G is finding an inconsistency in the assumptions.

The issue about disjunction is more complex. Yet, in some particular situations, the features of intuitionistic logic can help. This is the case when the goal G is a disjunction $\varphi \vee \psi$, and all assumptions are *Harrop formulas* (i.e. formulas which do not contain disjunctions as strictly positive subformulas). In fact, it follows from a theorem of Harrop [11] that in such a case G is an intuitionistic consequence of the assumptions if and only if at least one among φ and ψ is. This means that, in such a situation, one has to pursue the inversion strategy (even if the goal is a disjunction!) before trying something else.

What if there are non-Harrop formulas among the assumptions? In that case, the presence of disjunctions among the assumptions (or the possibility of extracting them) allows the use of the rule $\vee\downarrow$. Now this rule can play, in some sense, the part of the indirect rules in the classical case: it may make sense to try it before other strategies, if the goal is a disjunction. This is due to the fact that many common problems actually need to use of $\vee\downarrow$ to prove a disjunction (think of the the commutative law for $\vee$, i.e. $\varphi \vee \psi; ?\psi \vee \varphi$). Yet one has to be careful: while it seems not to be too expensive to use the indirect strategy in the classical case (one just has to examine a few contradictory pairs), here the amount of computation may become difficult to deal with, especially if several disjunctions are present. The situation is complicated further by the fact that, in certain cases, the $\vee\downarrow$-strategy would be preferrable even if the goal is a not a disjunction (think of $(\varphi \wedge \psi) \vee (\varphi \wedge \psi); ?\varphi \wedge \psi$).

These examples suggest that we have to include, in our heuristics, looking for connections between the goal and disjunction(s) we have in the assumptions (say, check whether one of the two is a positive subformula of the other, whether they have sentential variables or even a disjunct in common). This, though, is not enough. In fact, there are cases in which $\vee\downarrow$ is needed even if the shape of the goal has nothing to do with that of the disjunction in the assumptions: for instance, $\varphi \vee \psi, \varphi \to \chi, \psi \to \chi; ?\chi$. Similarly, one may consider questions like $\varphi_1 \vee \psi_1, \varphi_2 \vee \psi_2, \varphi_1 \to \chi, \psi_1 \to \chi; ?\chi$: here, clearly, the first disjunction is helpful for proving the goal, while the second is not. These examples show the $\vee\downarrow$-strategy and the extraction strategy have to be somehow combined: while forming extraction strategies for the goal, one should also check whether the open questions met have some relationship with a disjunction that occurs as a positive subformula of one of the assumptions, in order to use $\vee\downarrow$ at the appropriate time.

Summing up this sketch of heuristic considerations, we can conclude that forming strategies in the intuitionistic (sentential) case is almost straightforward: one can form the extraction strategies (when the goal is a positive subformula of one of the assumptions), the inversion strategies (if the goal is not an atom) and the $\vee \downarrow$-strategies (if there are non-Harrop formulas among the assumptions), leaving the $\perp_{\mathbf{q}}$-strategies as a last resort. But complications arise for a good ranking of the strategies, when non-Harrop assumptions occur. (These complications provide a heuristic explanation of the higher computational complexity of the set of intuitionistically provable sentential formulas.) Finally, the Skolem-Herbrand expansion is our tool to extend strategies (particularly, the extraction strategies) to the case of predicate logic.

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