Bisimulation for General Stochastic Hybrid Systems*

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Abstract. In this paper we define a bisimulation concept for some very general models for stochastic hybrid systems (general stochastic hybrid systems). The definition of bisimulation builds on the ideas of Edalat and of Larsen and Skou and of Joyal, Nielsen and Winskel. The main result is that this bisimulation for GSHS is indeed an equivalence relation. The secondary result is that this bisimulation relation for the stochastic hybrid system models used in this paper implies the same kind of bisimulation for their continuous parts and respectively for their jumping structures.

Keywords: stochastic hybrid systems, Markov processes, simulation morphism, zigzag morphism, bisimulation, category theory.

1 Introduction

Significant progress in verification of probabilistic systems has been done mostly for discrete distributions or Markov chains. Continuous stochastic processes are incomparable more difficult to verify. It is notorious that theorem proving of stochastic properties (with the probability one) can be carried out on the unit circle only. Model checking and reachability analysis are strongly conditioned by abstraction techniques. When the state space is not only infinite but also continuous, abstraction techniques must be very strong. Hybrid systems add an extra level of complexity because of the hybrid nature of the state space (discrete and continuous states coexist) and stochastic hybrid systems push further this

* This work has been supported by the European Commission under HYBRIDGE project, IST-2001-32460.

M. Morari and L. Thiele (Eds.): HSCC 2005, LNCS 3414, pp. 198–214, 2005.

complexity by adding non-determinism and uncertainty. Therefore, it is imperious necessary to have an abstraction theory for stochastic processes that can be used for verification and analysis of stochastic hybrid systems.

Reachability analysis and model checking are much easier when a concept of bisimulation is available. The state space can be drastically abstracted in some cases. In this paper, we focus on defining bisimulation relations for stochastic hybrid systems, as a first step towards creating a framework for verification.

Besides of different bisimulation concepts in the concurrency theory, the notion of bisimulation is present

- in the 'deterministic world': continuous and dynamical systems [21] or hybrid systems [15];
- or in the 'probabilistic world': probabilistic discrete systems [18], labelled Markov processes [5], piecewise deterministic Markov processes [22].

In this paper we define a bisimulation concept for some very general models for stochastic hybrid systems (general stochastic hybrid systems, abbreviated GSHS, introduced in [12, 9]). The definition of bisimulation builds on the ideas of Edalat [5, 14] and of Larsen and Skou [18] and of Joyal, Nielsen and Winskel [17]. The main result is that this bisimulation for GSHS, which extends the Edalat definition for labelled Markov processes, is indeed an equivalence relation. This turns out to be a rather hard mathematical result, which employs the whole stochastic analysis apparatus associated to a GSHS (viewed as a strong Markov process defined on Borel space).

Being defined in a category theory context, this stochastic bisimulation, as a notion of system equivalence, enjoys some fundamental mathematical properties. Moreover, we prove that this is a natural notion of bisimulation for GSHS because the bisimilarity of two GSHS implies the bisimilarity of their diffusion components and respectively of their jumping parts.

The rest of the paper is organized as follows. Next section gives a quick tour on stochastic bisimulation. Moreover, it presents the main difficulties, which we have to overcome when we have to define a concept of bisimulation for very general Markov processes. As well, it is stressed that the key point in the construction of bisimulation is the definition of morphism. Section 3 gives a short presentation of GSHS. In section 4 we present different kind of morphism, which might be associated to GSHS. In section 5 we define the concepts of simulation morphism, zigzag morphism and stochastic bisimulation for GSHS. Also, we prove that this bisimulation is an equivalence relation. Section 6 points out the specific features of the bisimulation for GSHS. The paper ends with some conclusions and further work.

2 A Quick Tour in Stochastic Bisimulation

The classical paper of Joyal, Nielsen and Winskel [17] presents a general categorical view of what bisimulation is for deterministic systems. This paper works with a general category of models $\mathbf{M}$, whose objects are the systems in question, and the arrows are the simulation morphisms. More, it is distinguished a subcategory of the **M** called the *path category* **P** of path objects (with morphims expressing how they can be extended). The meaning of a simulation morphism $\psi : X^2 \to X^1$ between two objects X^2, X^1 of **M** is that any path p of X^2 is matched by the path $\psi \circ p$ in X^1. The abstract notion of bisimulation is formulated in terms of certain special morphisms called **P**-*open maps* (which are a stronger version of the simulation morphisms). Two objects X^2 and X^1 are called **P**-bisimilar if and only if there exists an object X together with a span of **P**-*open maps* between them: $\psi^1 : X \to X^1$ and $\psi^2 : X \to X^2$.

For the probabilistic case it is not easy to generalize this bisimulation. The probabilistic bisimulation (for probabilistic systems) in the case of a discrete state space has been developed by Larsen and Skou in [18].

For the continuous case (for Markov processes) this definition can not be adapted straightforward. The main problem is how to define the simulation morphisms and the open maps. In this case, we say that a Markov process M^1 *simulates* another Markov M^2 if there exist a surjective continuous morphism $\psi : X^2 \to X^1$ between their state spaces such that each transition probability on X^2 'is matched' by a transition probability on X^1. The meaning of this 'matching' is that for each measurable set $A \subset X^1$ and for each $u \in X^2$ we have

$$p_t^2(u, \psi^{-1}(A)) \leq p_t^1(\psi(u), A), \forall t \geq 0. \tag{1}$$

where (p_t^2) and (p_t^1) are the transition functions corresponding to M^2, respectively to M^1. A such morphism ψ is called a *simulation morphism.*
The open maps are replaced by the so-called *zigzag morphisms*, which are simulation morphism for which the condition (1) holds with equality.

Practically, a simulation condition as (1) is hard to be checked because the time t runs in a 'continuous' set. Then, it is necessary to require supplementary assumptions about the transition probabilities of the processes we are talking about. This kind of simulation morphisms and zigzag morphisms have been defined for some particular Markov processes: for labelled Markov processes [5] and for stationary Markov processes with discrete time (defined on Polish or analytic spaces) [14]. In these papers, the authors consider the categories of above Markov processes as objects and the zigzag morphisms as morphisms. Then the bisimulation notion for these processes is given in a 'classical' way. Two labelled Markov processes, for example, are probabilistically bisimilar if there exists a *span of zigzag morphisms* between them. In this context, we can point out another reason why only some special kind of Markov processes are considered, as follows. This bisimulation relation is always reflexive and symmetric. But, the transitivity of a such relation (the bisimulation must be an equivalence relation) is usually implied by the existence of *semi-pullbacks* in the Markov process category considered [17, 14]. That means, in the respective category, for any pair of morphisms $\varphi^1 : M^1 \to M$ and $\varphi^2 : M^2 \to M$ (M^1, M^2, M are objects in that category) there exist an object M^0 and morphisms $\pi^i : M^0 \to M^i$ ($i = 1, 2$) such that $\varphi^1 \circ \pi^1 = \varphi^2 \circ \pi^2$ as in the following diagram.

$$\begin{array}{ccccc} & & M^0 & & \\ & \overset{\pi^1}{\swarrow} & & \overset{\pi^2}{\searrow} & \\ M^1 & & & & M^2 \\ & \underset{\varphi^1}{\searrow} & & \underset{\varphi^2}{\swarrow} & \\ & & M & & \end{array}$$

The construction of the semi-pullback in the above categories of Markov processes is strongly based on the stationarity property of the Markov processes considered [5, 14]. In this case the transition probabilities do not depend on time! Then the construction mechanism of the semi-pullback in a such categories of Markov processes is reduced to the construction of the semi-pullback in the category of transition probability functions and surjective transition probability preserving Borel maps (as morphisms in the respective category) (see [14] for the detailed construction).

In this paper, we develop a novel concept of *stochastic bisimulation* for general stochastic hybrid systems. This concept of bisimulation might be formulated, as well, for strong Markov processes defined on Borel spaces. Instead of restricting ourself to some specific categories of Markov processes, we chose to change the definitions of simulation morphisms and the zigzag morphisms. The novelty consists of the way to define these morphisms. Specifically, we replace the condition (1) by a 'global condition' which illustrate that the executions of the simulated process can be matched by the execution of the simulator process. Since, these process are not stationary, we require for these morphisms to 'preserve' the kernel operators (or, dual the infinitesimal generators) of the processes considered. Since the expressions of the generators are known [12], these kind of conditions can be easily checked. Then the bisimulation relation is naturally given via zigzag morphism spans between GSHS. Dually, this bisimulation can be defined using morphisms between the excessive function cones associated to the Markov processes. Moreover, the category of strong Markov processes defined on Borel spaces with these zigzag morphisms as morphisms has semi-pullback, then the bisimulation relation is an equivalence relation (the category of GSHS as objects and with same zigzag morphisms as morphisms is a full subcategory in the above category).

The probabilistic bisimulation (for labelled Markov processes) defined in [5] can be derived from our concept of bisimulation, based on the whole theory that relates the infinitesimal generators and the transition probabilities.

3 Stochastic Hybrid Systems

In this section we give a short presentation of the general model for stochastic hybrid systems, introduced in [12], which is used in the following sections. It is notably that in [4], a quite general model of stochastic hybrid systems that can be related to GSHS as a particular case, has been implemented in Charon [1]).

Definition 1. *A General Stochastic Hybrid System (GSHS) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a countable set of discrete variables;*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces;*
- $\mathcal{X} : Q \to \mathbb{R}^{d(.)}$ *maps each* $q \in Q$ *into an open subset* X^q *of* $\mathbb{R}^{d(q)}$*;*
- $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(.)}$ *is a vector field;*
- $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)\times m}$ *is a* $X^{(\cdot)}$*-valued matrix,* $m \in \mathbb{N}$*;*
- $Init : \mathcal{B}(X) \to [0, 1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ *is a transition measure.*

We call the set $X(Q, d, \mathcal{X}) = \bigcup_{i\in Q}\{i\} \times X^i$ the hybrid state space of the GSHS and $x = (i, x^i) \in X(Q, d, \mathcal{X})$ the hybrid state. The closure of the hybrid state space will be $\overline{X} = X \cup \partial X$, where $\partial X = \bigcup_{i\in Q}\{i\} \times \partial X^i$. It is known that X can be endowed with a metric ρ whose restriction to any component X^i is equivalent to the usual component metric [13]. Then $(X, \mathcal{B}(X))$ is a Borel space (homeomorphic to a Borel subset of a complete separable metric space), where $\mathcal{B}(X)$ is the Borel σ-algebra of X.

We built a GSHS as a *Markov string H* [10] obtained by the concatenation of some diffusion processes (x^i_t), $i \in Q$ together with a jumping mechanism given by a family of stopping times (S^i). Let ω_i be a diffusion trajectory, which starts in $(i, x^i) \in X$. Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x^i_t). Define the function

$$F(t, \omega_i) = I_{(t<t_*(\omega_i))} \exp(-\int_0^t \lambda(i, x^i_s(\omega_i)))ds. \tag{2}$$

This function will be the survivor function for the stopping time S^i associated to the diffusions (x^i_t).

Definition 2 (GSHS Execution). *A stochastic process* $x_t = (q(t), x(t))$ *is called a GSHS execution if there exists a sequence of stopping times* $T_0 = 0 < T_1 < T_2 \leq \ldots$ *such that for each* $k \in \mathbb{N}$*,*

- $x_0 = (q_0, x^{q_0}_0)$ *is a* $Q \times X$*-valued random variable extracted according to the probability measure* $Init$*;*
- *For* $t \in [T_k, T_{k+1})$*,* $q_t = q_{T_k}$ *is constant and* $x(t)$ *is a solution of the SDE:*

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \tag{3}$$

where W_t *is a the* m*-dimensional standard Wiener;*

- $T_{k+1} = T_k + S^{i_k}$ *where* S^{i_k} *is chosen according with the survivor function (2).*
- *The probability distribution of* $x(T_{k+1})$ *is governed by the law* $R\big((q_{T_k}, x(T^-_{k+1})), \cdot\big)$.

It is known, from [9], that any GSHS, H, under standard assumptions (about the diffusion coefficients, non-Zeno executions, transition measure, etc see [9] for a detailed presentation) is a strong Markov process [19] and it has the càdlàg property (i.e. for all $\omega \in \Omega$ the trajectories $t \mapsto x_t(\omega)$ are right continuous on $[0, \infty)$ with left limits on $(0, \infty)$). Here, $(\Omega, \mathcal{F}, P)$ is the underlying probability space associated to H as a Markov process. The model H can be thought of as a family of random variables $(x_t)_{t \geq 0}$. For any $x \in X$, the measure P_x (Wiener probability) is the law of $(x_t)_{t \geq 0}$ under the initial condition $x_0 = x$.

Let (P_t) denote the operator semigroup associated to H which maps $\mathcal{B}^b(X)$ (the set of all bounded measurable functions $f : X \to \mathbb{R}$) into itself given by

$$P_t f(x) = E_x f(x_t), \tag{4}$$

where E_x is the expectation w.r.t. P_x. As well, we define the *resolvent operators* associated to the semigroup (4) by $V^\alpha f := \int_0^\infty e^{-\alpha t} P_t f dt$, $\alpha \geq 0$ for all positive $\mathcal{B}$-measurable functions f . We write V for V^0 and we call it the *kernel operator*. Then a function f is *excessive* (w.r.t. the semigroup (P_t) or the resolvent (V^α)) if it is measurable, non-negative and $P_t f \leq f$ for all $t \geq 0$ and $P_t f \nearrow f$ as $t \searrow 0$. Let denote by $\mathcal{E}_H$ the set of all excessive functions associated to H. The strong Markov property can be characterized in terms of excessive functions [19].

For a GSHS, H, as a Markov process, the expression of the infinitesimal generator L is given in [12]. For $f \in \mathcal{D}(L)$ (the domain of generator) Lf is given by

$$Lf(x) = L_{cont} f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x)) R(x, dy) \tag{5}$$

where:

$$L_{cont} f(x) = \mathcal{L}_b f(x) + \frac{1}{2} Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x)). \tag{6}$$

For a strong Markov process defined on a Borel space (which is the case for GSHS), the opus of the kernel operator is the inverse operator of the infinitesimal generator of the process [19].

A stochastic differential equation generates a much richer structure than just a family of stochastic processes, each solving the stochastic differential equation for a given value. In fact, it gives a flow of random diffeomorphism, i.e. it generates a random dynamical system (RDS) [2]. Therefore, the construction of a GSHS as a Markov string (see [10]) of diffusions does not only generate a Markov process, but it also generates an RDS (which is a 'string' of the RDS components). The theory of random dynamical systems is relatively new and we refer to [2], as the first systematic presentation of this theory. We present only the necessary definitions that we need in this paper.

Let $\theta_t : \Omega \to \Omega$ for all $t \in [0, \infty)$. $(\Omega, \mathcal{F}, P, \theta_t)$ (abbreviated θ) is called a *metric dynamical system*, if: 1.The map $\theta : \Omega \times [0, \infty) \to \Omega$, $(\omega, t) \mapsto \theta_t(\omega)$ is measurable from $(\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}([0, \infty))$ to $(\Omega, \mathcal{F})$; 2. θ satisfies the flow properties: (i) $\theta_0 = id_\Omega$ and (ii) $\theta(t + s) = \theta_t \circ \theta_s$ $\forall s, t \in [0, \infty)$; 3. θ is measure preserving, i.e. $\theta_t P = P$ $\forall t \in [0, \infty)$ (where $fP := P \circ f^{-1}$). The metric dynamical system is necessary to model the random perturbations of an RDS.

A measurable *random dynamical system* on the measurable space $(X, \mathcal{B})$ over the metric dynamical system θ with time $[0, \infty)$ is a map $\varphi : [0, \infty) \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ with the following properties: 1. φ is $\mathcal{B}([0, \infty)) \otimes \mathcal{F} \otimes \mathcal{B}/\mathcal{B}$

- measurable; 2. If $\varphi(t,\omega) = \varphi(t,\omega,\cdot)$ then φ forms a perfect cocycle over θ, i.e. φ has the properties: (i) $\varphi(0,\omega) = id_X$ and (ii) $\varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)$ $\forall \omega \in \Omega \ \ \forall s,t \in [0,\infty)$.

The RDS associated to a GSHS arises from its construction as a Markov string: the shift operator (θ_t) of the corresponding Markov string is exactly the metric dynamical system for the RDS and for each $x \in X$, $\omega \in \Omega$, $t \geq 0$ the value of the RDS cocycle $\varphi(t,\omega,x)$ is exactly $x_t(\omega)$ with x as the starting point (or $\varphi(t,\omega,x)$ is the execution of GSHS with x as the starting point). In other words, the cocycle φ is a replacement of the flow from the determinist case.

In the next section we will define some concepts of morphism for stochastic hybrid systems. The definitions will employ notions specific to the Markov process theory as: kernel operator, excessive functions, etc. The three faces of a stochastic hybrid system - Markov process, random dynamical system or dynamical system - will give more intuitions about the notion of morphism which will be proposed next. Some connections with theory of dynamical systems might be available.

4 Morphisms Associated to GSHS

In this section we define a concept of morphism between GSHS intimately connected with the morphisms between the associated cones of excessive functions.

Let H a GSHS defined as in section 3. We assume that H as Markov process is transient (i.e. there exists a strict positive Borel measurable function q such that Vq is a bounded function). We define a preorder relation $\prec_H$ on X as

$$x \prec_H y \Longleftrightarrow Vf(y) \leq Vf(x), \forall f \in \mathcal{B}^b(X), f \geq 0.$$

$\prec_H$ is an order on the trajectories of H. That means: $x \prec_H y$ if and only if there exist some time $t \in [0,+\infty)$ and $\omega \in \Omega$ such that $y = \varphi(t,\omega,x)$. For each fixed ω, the trajectory $[\varphi(t,\omega,\cdot)]_{t\geq 0}$ is totally ordered w.r.t. $\prec_H$. If H degenerates in a dynamical system then the relation $\prec_H$ is an order relation because H is supposed to be transient. We will call $\prec_H$ the *trajectory (pre)order* of H.

One can define on X the *fine topology*, denoted by τ_H^f, which consists of the sets $G \subseteq X$ with the following property: $\forall x \in G$, $\forall \omega \in \Omega$ $\exists t_0 \in (0,\zeta(\omega))$ such that $\varphi(t,\omega,x) \in G, \forall t \in (0,t_0)$ (each trajectory starting from x remains for a while in G)[1]. The fine topology is the coarsest topology on X, which makes continuous all excessive functions. The fine topology τ_H^f is separated and is finer than the initial topology.

In the first step, we define the morphisms between the cones of excessive functions. Let H^1, H^2 be two GSHS with state spaces $X^{(1)}$, respectively $X^{(2)}$. Let

[1] Note that the fine topology can be defined in terms of hitting times for a Markov process.

$\mathcal{E}_{H^1}, \mathcal{E}_{H^2}$ the associated cones of excessive functions. An $\mathcal{E}$-*morphism* (between these two cones) can be defined as an application

$$\Psi : \mathcal{E}_{H^1} \rightarrow \mathcal{E}_{H^2} \tag{7}$$

such that the following properties hold: (i) $\Psi(f+g) = \Psi(f) + \Psi(g), \forall f, g \in \mathcal{E}_{H^1}$; (ii) $f \leq g \Rightarrow \Psi(f) \leq \Psi(g)$; $f_k \nearrow f \Rightarrow \Psi(f_k) \nearrow \Psi(f)$; (iv) $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g), \forall f, g \in \mathcal{E}_{H^1}$; (v) $\Psi(1) = 1$. An $\mathcal{E}$-morphism Ψ is called *finite* if $f < +\infty \Rightarrow \Psi(f) < +\infty$.

Proposition 1. *If $\psi : X^{(2)} \rightarrow X^{(1)}$ is measurable, monotone (i.e. $u \prec_{H^2} v \Rightarrow \psi(u) \prec_{H^1} \psi(v)$) and finely continuous then $\Psi : \mathcal{E}_{H^1} \rightarrow \mathcal{E}_{H^2}$ given by*

$$\Psi(f) = f \circ \psi \tag{8}$$

for all $f \in \mathcal{E}_{H^1}$, is a finite $\mathcal{E}$-morphism.

In some papers [20], an application ψ as in the Prop. 1 is called *H-map*. Intuitively, in the formula (8) the H-map ψ can be thought of as a *variable change*, i.e. for all $f \in \mathcal{E}_{H^1}$

$$\Psi(f)(u) = f(\psi(u)), \forall u \in X^{(2)}. \tag{9}$$

Remark 1. (i) The map Ψ defined by (8) can be extended as a map between the two cones of measurable positive functions defined on $X^{(1)}$, respectively $X^{(2)}$, loosing the property of finely continuity. Prop.1 shows how a function between the state spaces of H^1, H^2 can provide an $\mathcal{E}$-morphism.

(ii) Conversely, if Ψ is an $\mathcal{E}$-morphism as in (7) then there exists a unique measurable monotone and finely continuous application $\overline{\psi}$ from $X^{(2)}$ to an extension of $X^{(1)}$ such that: $\Psi(f) = f \circ \overline{\psi}, \forall f \in \mathcal{E}_{H^1}$. To obtain this result one can use results from [20].

In the next section the notion of stochastic bisimulation will be defined based on the concept of H-map. For this purpose the following results will guide us in building the notions of simulation morphism and zigzag morphism.
A surjective H-map $\psi : X^{(2)} \rightarrow X^{(1)}$ induces an equivalence relation $\sim_\psi$ on $X^{(2)}$

$$u \sim_\psi v \Leftrightarrow \psi(u) = \psi(v). \tag{10}$$

In this way, to each $x \in X^{(1)}$ we can associate an equivalence class $\widehat{u}$ w.r.t. $\sim_\psi$ such that $\widehat{u} = \psi^{-1}(x)$. Then, using (9), each function g belonging to the range of Ψ can be extended to $X^{(2)}/_{\sim\psi}$, i.e. $g(\widehat{u}) = f(x)$ provided that $\widehat{u} = \psi^{-1}(x)$ and $g = \Psi(f)$.

Proposition 2. *If $\psi : X^{(2)} \rightarrow X^{(1)}$ is a surjective and finely open H-map such that each excessive function $g \in \mathcal{E}_{H^2}$ has the property*

$$u \sim_\psi v \Rightarrow g(u) = g(v) \tag{11}$$

then the $\mathcal{E}$-morphism $\Psi : \mathcal{E}_{H^1} \rightarrow \mathcal{E}_{H^2}$ given by formula (8) is surjective.

Proof. For each $g \in \mathcal{E}_{H^2}$ we have to define $f \in \mathcal{E}_{H^1}$ such that $\Psi(f) = g$. Let $f : X^{(1)} \to [0, \infty)$ defined by $f(x) = g(u)$ for each $x \in X^{(1)}$, where $u \in X^{(2)}$ is such that $\psi(u) = x$ (there exists a such u since ψ is surjective). The function f is well defined because of (11). Then f can be written as $f = g \circ \psi^{-1}$ and for any open set $D \subset [0, \infty)$ we have $f^{-1}(D) = \psi(g^{-1}(D))$. Since ψ is a finely open map we obtain that $f^{-1}(D)$ is finely open in $X^{(1)}$. Then $f \in \mathcal{E}_{H^1}$. □

Remark 2. It is easy to check that if in the Prop. 1 both ψ and Ψ are surjective then Ψ must be bijective. Therefore the two excessive function cones can be identified and the two processes are equivalent.

5 Stochastic Bisimulation

In this section we develop a novel concept of bisimulation for GSHS. This concept is inspired by the bisimulation concept for labelled Markov processes [5] or stationary Markov processes with discrete time [14]. Because, our models *are not stationary* Markov processes, we can not use the Edalat's bisimulation.

To define the notion of bisimulation for GSHS, we need to give the definition of *simulation morphism and zigzag morphisms* between GSHS. The main difference from the similar notions from [5] is that we replace the conditions about the transition probabilities (which, in the non-stationary case, should depend on time) with *global conditions* written in terms of kernel operators or excessive functions associated to the GSHS. Similarly, these morphisms can be defined for strong Markov processes with càdlàg property defined on Borel spaces.

Definition 3. *A simulation morphism between two GSHS, H^1 and H^2 (the process H^1 simulates the process H^2), is a H-map (i.e. measurable, monotone, finely continuous application) $\psi : X^{(2)} \to X^{(1)}$ such that*

$$V^2(f \circ \psi) \leq V^1 f \circ \psi, \forall f \in \mathcal{B}^b(X^{(1)}),\ f \geq 0, \tag{12}$$

where V^1 (resp. V^2) is the kernel operator associated to H^1 (resp. H^2).

The definition 3 illustrates, in terms of kernel operators, that the simulating process can make all the transitions of the simulated process with greater probability than in the process being simulated. More intuitively, a simulation morphism ψ is not only monotone, but it also refines the "distances" on the trajectories since the trajectory order relations are defined by means of the kernel operators. On the other hand, the finely continuity of ψ illustrates the fact that to a trajectory of H^1 corresponds a class of trajectories of H^2.

Remark 3. Replacing the simulation condition (12) with an weaker one, using the $\mathcal{E}$-morphism Ψ generated by ψ with formula (8), one can define a simulation morphism as follows

$$V^2 \circ \Psi \leq \Psi \circ V^1. \tag{13}$$

Definition 4. *A surjective simulation morphism* ψ *between two GSHS,* H^1 *and* H^2 *is called zigzag morphism if the formula (12) holds with equality, i.e.*

$$V^2(f \circ \psi) = V^1 f \circ \psi, \forall f \in \mathcal{B}^b(X^{(1)}),\ f \geq 0. \tag{14}$$

Remark 4. For a zigzag morphims the monotony is already implied by the zigzag condition (14) (easy consequence of the way to define the order relations on the spaces $X^{(2)}$ and $X^{(1)}$).

Using the $\mathcal{E}$-morphism Ψ generated by ψ, the condition (14) becomes

$$V^2 \circ \Psi = \Psi \circ V^1 \tag{15}$$

i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_{H^1} & \stackrel{\Psi}{\rightarrow} & \mathcal{E}_{H^2} \\ V^1 \uparrow & & \uparrow V^2 \\ \mathcal{E}_{H^1} & \stackrel{\Psi}{\rightarrow} & \mathcal{E}_{H^2} \end{array}$$

Then we can define a *zigzag* $\mathcal{E}$*-morphim* Ψ (between two GSHS, H^1 and H^2) as a surjective $\mathcal{E}$-morphism such that the condition (15) yields.

Next, we define the *stochastic bisimulation* for GSHS as the existence of a span of zigzag morphisms.

Definition 5. *Let* H^1 *and* H^2 *be two GSHS.* H^1 *is stochastic bisimilar to* H^2 *(written* $H^1 \sim H^2$*) if there exists a span of zigzag morphisms between them, i.e. there exists a GSHS* H^{12} *and zigzag morphisms* ψ^1 *(where* $\psi^1 : X^{12} \to X^{(1)}$*) and* ψ^2 *(where* $\psi^2 : X^{12} \to X^{(2)}$*) such that*

$$\begin{array}{ccccc} & & H^{12} & & \\ & \stackrel{\psi^1}{\swarrow} & & \stackrel{\psi^2}{\searrow} & \\ H^1 & & & & H^2 \end{array}$$

Notice that if there is a zigzag morphism between two systems, they are bisimilar since the identity is a zigzag morphism.

Remark 5. The notions of simulation morphism, zigzag morphism and stochastic bisimulation can be formulated in a similar way for strong Markov processes defined on Polish spaces (a Polish space is a homeomorphic image of complete separable metric space) or analytic spaces (an analytic space is the continuous image of a Polish space into another Polish space and is equipped with the subspace topology of the latter space). In this paper, since the GSHS state space is a Borel space, we consider only Markov processes defined on Borel spaces.

Remark 6. We can define a weak version of the stochastic bisimulation via $\mathcal{E}$-morphisms, i.e. $H^1 \sim H^2$ if there exist a cospan of zigzag $\mathcal{E}$-morphisms Ψ^1 and Ψ^2 between their excessive function cones

$$\begin{array}{ccccc} & & \mathcal{E}_{H^{12}} & & \\ & \overset{\Psi^1}{\nearrow} & & \overset{\Psi^2}{\nwarrow} & \\ \mathcal{E}_{H^1} & & & & \mathcal{E}_{H^2} \end{array}$$

Let us consider the category of the strong Markov processes defined on Borel spaces as the objects and zigzag morphisms as the morphisms. This category contains as a full subcategory the category of GSHS as the objects and zigzag morphisms as the morphisms.

Proposition 3. *The category of the strong Markov processes on Borel spaces as the objects and zigzag morphisms as the morphisms has semi-pullbacks.*

Proof. Let M^1, M^2, M be strong Markov processes defined on the Borel spaces $X^{(1)}, X^{(2)}, X$, respectively. Suppose that there exist two zigzag morphisms

$$\psi^1 : X^{(1)} \to X,\ \psi^2 : X^{(2)} \to X. \tag{16}$$

We have to prove that there exist another object M^0 (a strong Markov process defined on a Borel space $X^{(0)}$) and two zigzag morphisms $\pi^1 : X^{(0)} \to X^{(1)}$ and $\pi^2 : X^{(0)} \to X^{(2)}$ such that the following diagram commutes

$$\begin{array}{ccccc} & & X^{(0)} & & \\ & \overset{\pi^1}{\swarrow} & & \overset{\pi^2}{\searrow} & \\ X^{(1)} & & & & X^{(2)} \\ & \underset{\psi^1}{\searrow} & & \underset{\psi^2}{\swarrow} & \\ & & X & & \end{array}$$

Let $X^{(0)} = \{(x^1, x^2) | \psi^1(x^1) = \psi^2(x^2)\}$ equipped with the subspace topology of the product topology on $X^{(1)} \times X^{(2)}$. Note that $X^{(0)}$ is nonempty since ψ^1 and ψ^2 are supposed surjective. As well, $X^{(0)}$ is a measurable set of $X^{(1)} \times X^{(2)}$ (equipped with its Borel σ-algebra). We take M^0 as the part of the product of the Markov processes M^1, M^2 restricted to $X^{(0)}$, the process product is "killed" outside of $X^{(0)}$ [13] For the relationships which exist between the kernel operators of the processes M^1, M^2 and the kernel operator of their product, see [11] and the references therein. Then π^1 and π^2 can be taken as the projection maps and the equality $\psi^1 \circ \pi^1 = \psi^2 \circ \pi^2$ trivially holds.

On the other hand, if we define the stochastic bisimulation defined via zigzag $\mathcal{E}$-morphisms, then the pullback existence for the category of Markov processes (with morphisms given by zigzag $\mathcal{E}$-morphisms) is equivalent with the *pushout existence* in the category of their excessive function cones (with the morphisms given by zigzag $\mathcal{E}$-morphisms). Let us take the following span of morphims between the excessive function cones

$$\begin{array}{ccccc} & & \mathcal{E}_{M} & & \\ & \overset{\Psi^1}{\swarrow} & & \overset{\Psi^2}{\searrow} & \\ \mathcal{E}_{M^1} & & & & \mathcal{E}_{M^2} \end{array}$$

Naturally, we consider $\mathcal{E}$ as the tensor product $\mathcal{E}_{M^1} \otimes \mathcal{E}_{M^1}$ of the cones $\mathcal{E}_{M^1}, \mathcal{E}_{M^1}$ (which correspond to the product of operator semigroups or to Markov process product defined on $X^{(1)} \times X^{(2)}$). Then the 'inclusions' $\mathcal{E}_{M^1} \overset{\Gamma^1}{\hookrightarrow} \mathcal{E}$, $\Gamma^1(f^1) = \Psi^1(f) \otimes \Psi^2(f)$ if $f^1 = \Psi^1(f)$ and $\mathcal{E}_{M^2} \overset{\Gamma^2}{\hookrightarrow} \mathcal{E}$, $\Gamma^2(f^2) = \Psi^1(f) \otimes \Psi^2(f)$ if $f^2 = \Psi^2(f)$ (essentially, Ψ^1 and Ψ^2 are surjective) gives the desired pushout construction, i.e. the following diagram commutes

$$\begin{array}{ccc} & \mathcal{E}_M & \\ {}^{\Psi^1}\swarrow & & \searrow{}^{\Psi^2} \\ \mathcal{E}_{M^1} & & \mathcal{E}_{M^2} \\ \searrow_{\Gamma^1} & & \swarrow_{\Gamma^2} \\ & \mathcal{E} & \end{array}$$

Proposition 4. *The stochastic bisimulation defined by Def. 5 on GSHS (or strong Markov processes on Borel spaces) is an equivalence relation.*

6 Specific Features of Bisimulation for GSHS

A zigzag morphism $\psi : X^{(2)} \to X^{(1)}$ between two GSHS, H^1 and H^2, induces a relation $\mathcal{R} \subset X^{(2)} \times X^{(1)}$ as follows: $u\mathcal{R}x \Leftrightarrow \psi(u) = x$. Then the equivalence relation $\sim_\psi$on $X^{(2)}$ can be thought of as the equivalence relation induced by $\mathcal{R}$ in sense of [22], i.e. $u \sim_\psi v$ iff there exists $x \in X^{(1)}$ such that $u\mathcal{R}x$ and $v\mathcal{R}x$ (which is exact the meaning of (10)). The equivalence relation induced by $\mathcal{R}$ on $X^{(2)}$ is the trivial one (x can be equivalent only with itself).

The space $X^{(2)}/_{\sim_\psi}$ can be endowed with the σ-algebra $\mathcal{B}^*(X^{(2)})$, which is the "saturation" of the Borel σ-algebra of $X^{(2)}$ w.r.t. $\sim_\psi$ (i.e. the collection of all Borel sets of $X^{(2)}$ in which any equivalence class of $X^{(2)}$ is either totally contained or totally not contained). A function on $g : X^{(2)} \to \mathbb{R}$, which is measurable w.r.t. $\mathcal{B}^*(X^{(2)})$ will be called *saturated measurable function*. It is clear that a function measurable g is saturated measurable iff (11) holds. Each function $f : X^{(1)} \to \mathbb{R}$ measurable w.r.t. $\mathcal{B}(X^{(1)})$ can be identified with a saturated measurable function g such that $g = f \circ \psi$.

The morphism ψ can be viewed as a bijective mapping $\psi : X^{(2)}/_{\sim_\psi} \to X^{(1)}$. It is clear that ψ is a measurable application. To identify the two above measurable spaces ψ^{-1} must be measurable. The main idea, which results from this reasoning, is that the measurable space $(X^{(1)}, \mathcal{B}(X^{(1)}))$ can be embedded in the measurable space $(X^{(2)}, \mathcal{B}(X^{(2)}))$ and the measurable function on $X^{(1)}$ can be identified with the saturated measurable functions on $X^{(2)}$.

Based on the theory of semigroups of Markov processes, one can obtain from the zigzag condition (14): for almost all $t \geq 0$ (i.e. except with a zero Lebesgue measure set of times) the following equalities (versions of (1)) hold

$$p_t^2(u, \psi^{-1}(A)) = p_t^1(x, A), \forall x \in X^{(1)}, \forall u \in \widehat{u} = \psi^{-1}(x), \forall A \in \mathcal{B}(X^{(1)}) \quad (17)$$
$$P_t^2(f \circ \psi)(u) = P_t^1 f(x), \forall x \in X^{(1)}, \forall u \in \widehat{u} = \psi^{-1}(x), \forall f \in \mathcal{B}^b(X^{(1)})$$

Note that $\psi^{-1}(A) \in \mathcal{B}^*(X^{(2)})$. Therefore the transition probabilities of H^1 simulates 'equivalence classes' of transition probabilities of H^2.

Remark 7. The connection between the kernel operator and the infinitesimal generator of the strong process Markov process allows us transform the conditions (15) and (14) as follows

$$L^{(2)} \circ \Psi = \Psi \circ L^{(1)}$$
$$L^{(2)}(f \circ \psi) = L^{(1)} f \circ \psi, \forall f \in \mathcal{D}(L^{(1)}) \tag{18}$$

where $L^{(1)}$ (resp. $L^{(2)}$) is the infinitesimal generator of H^1 (resp. H^2). The equality (18) holds provided that for each $f \in \mathcal{D}(L^{(1)})$ (the domain of L^1) the function $f \circ \psi$ belongs to $\mathcal{D}(L^{(2)})$ (the domain of $L^{(2)}$).

Since for GSHS the expression of the infinitesimal generator is known, to check if the formula (18) is true for two given GSHS is only a computation exercice.
Recall that a GSHS has been constructed as a Markov string, i.e. a sequence of diffusion processes with a jumping structure. Then the cone of excessive functions associated to a GSHS can be characterized as a 'sum' of the excessive function cones associated to the diffusion components. This characterization 'explains' the following result.

Proposition 5. *A zigzag morphism ψ between two GSHS H^1 and H^2 defined as in Def. 4 preserves the continuous parts of the two models.*

Proof. Suppose that the two GSHS state spaces are $X^{(1)} = \bigcup_{i \in Q^1} \{i\} \times X^{i(1)}$ and $X^{(2)} = \bigcup_{q \in Q^2} \{q\} \times X^{q(2)}$. We can suppose without loosing the generality that each two modes have empty intersection and therefore $X^{(1)} = \bigcup_{i \in Q^1} X^{i(1)}$ and $X^{(2)} = \bigcup_{q \in Q^2} X^{q(2)}$. The function ψ maps $X^{(2)}$ into $X^{(1)}$. From the construction of H^1, as Markov string, we have $V^1 f = \sum_{i \in Q^1} V^{i1} f^i, \forall f \in \mathcal{B}^b(X^{(1)})$,where, for each $i \in Q^1$, V^{i1} is the kernel operators of the component diffusion of H^1 which operates on $X^{i(1)}$ and $f^i = f|_{X^{i(1)}} \in \mathcal{B}^b(X^{i(1)})$. A similar expression can be written for V^2 (i.e. $V^2 g = \sum_{q \in Q^2} V^{q2} g^q$, $g \in \mathcal{B}^b(X^{(2)})$).

Let f be an arbitrary positive bounded measurable function on $X^{(1)}$. Then for each $i \in Q^1$ consider f^i as before. Let $Y^{i(2)} = \psi^{-1}(X^{i(1)})$ (note that $Y^{i(2)}$ is an open set) and ψ^i be the restriction of ψ, which maps $Y^{i(2)}$ into $X^{i(1)}$. Denote $g^i = f^i \circ \psi^i \in \mathcal{B}^b(Y^{i(2)})$ and $g^{iq} = g^i|_{Y^{i(2)} \cap X^{q(2)}}$. The zigzag condition (14) becomes $W^{i2}(f^i \circ \psi^i) = V^{i1} f^i \circ \psi^i$, where W^{i2} is the 'restriction' of V^2 to $Y^{i(2)}$, i.e. $W^{i2} g^i = \sum_{q \in Q^2} V^{q2} g^{iq}$ (more intuitively, W^{i2} is the sum of kernels associated to the component diffusions of H^2, which operate on $Y^{i(2)}$). Then, for all $x \in X^{i(1)}$ we have

$$W^{i2} g^i(u) = V^{i1} f^i(x), \tag{19}$$

provided that $\psi^i(u) = x$. Because V^{i1} corresponds to a diffusion process, it must be the case that in the left hand side of (19) the 'jumping part' to not longer exist (at least for the saturated measurable functions). Then the kernel W^{i2} corresponds to a continuous process (which might be a diffusion or a switching diffusion process). □

Any zigzag morphism ψ can be extended by (finely) continuity to the boundary of the state spaces. Or, we can suppose from the beginning that the zigzag morphims operate on the closures of the state spaces. We have to assume that the zigzag morphims 'keep' the boundary points, or, in other words, $\psi : \partial X^{(2)} \to \partial X^{(1)}$ is surjective.

Remark 8. The finely continuity of a zigzag morphism between two GSHS is important only when we use the connection with the associated excessive function cones. Otherwise, we can replace this continuity with the continuity w.r.t. to the initial topologies of the state spaces.

Proposition 6. *A zigzag morphism ψ between two GSHS H^1 and H^2 defined as in Def. 4 preserves the jumping structure of the two models.*

Proof. For each $x \in X^{(1)}$ there exist, by surjectivity of ψ, some elements $u \in X^{(2)}$ such that $\psi(u) = x$. Then, for each $f \in \mathcal{D}(L^{(1)})$, a simple computation of the right hand side of (18) gives

$$L^{(1)} f(x) = L^{(1)}_{cont} f(x) + \lambda^1(x) \int_{\overline{X}^{(1)}} (f(y) - f(x)) R^1(x, dy) \tag{20}$$

and after, the left hand side of (18) is

$$L^{(2)}(f \circ \psi)(u) = L^{(2)}_{cont}(f \circ \psi)(u) + \lambda^2(u) \int_{\overline{X}^{(2)}} [(f \circ \psi)(v) - (f \circ \psi)(u)] R^2(u, dv). \tag{21}$$

From the Prop. 5 we have the equality of the continuous parts of (20) and (21). Then the jumping parts (20) and (21) must coincide. Then

$$\lambda^1(x) \int_{\overline{X}^{(1)}} (f(y) - f(x)) R^1(x, dy) = \lambda^2(u) \int_{\overline{X}^{(2)}} [(f \circ \psi)(v) - (f \circ \psi)(u)] R^2(u, dv).$$

The construction of GSHS H^1 and H^2, as Markov strings, shows that the transition measures R^1 and R^2 play the role of the transition probabilities when the processes jump from one diffusion to another (see Def.2). Then they satisfy (17), i.e. $R^2(u, \psi^{-1}(A)) = R^1(x, A), \forall A \in \mathcal{B}(X^{(1)})$. It easily follows that $\lambda^1(x) = \lambda^2(u), \forall u \in \widehat{u} = \psi^{-1}(x)$. □

Therefore, the stochastic bisimulation between two GSHS reduces to the bisimulations between their continuous components and between their jump structures. In this way our concept of bisimulation can be related with the bisimulation for piecewise deterministic Markov processes (which are particular class of GSHS) defined in terms of an equivalence relation between the deterministic flows and the probabilistic jumps [22].

7 Conclusions

In this paper we develop a notion of stochastic bisimulation for a category of general models for stochastic hybrid systems (which are Markov processes) or, more generally, for the category of strong Markov processes defined on Borel spaces. The morphisms in this category are the zigzag morphims. A zigzag morphism between two Markov processes is a surjective (finely) continuous measurable functions between their state spaces which 'commutes' with the kernel operators of the processes considered. The fundamental technical contribution is the proof that this stochastic bisimulation is indeed an equivalence relation.

The secondary result of the paper is that this bisimulation relation for GSHS (the stochastic hybrid system models we are dealing in this paper) implies the same kind of bisimulation for their continuous parts and respectively for their jumping structures.

This work is intended to be a foundation for applying formal methods to stochastic hybrid systems. The category of GSHS we have introduced can be used to employ various methodologies from formal methods that admit a categorical support, like viewpoints and formal testing [6].

8 Further Work

From stochastic analysis viewpoint, most of the models of stochastic hybrid systems are strong Markov processes. Then, many tools available for the Markov process studying can be used to characterize their main features. On the other hand, some of them can be included in the class of random dynamical systems (stochastic extensions of the dynamical systems). Therefore the whole ergodic theory or stability results available for random dynamical systems might be applied to them. As well, stability results of random dynamical systems [3] can be lifted to these models of stochastic hybrid systems. Moreover, because in the deterministic case there are characterizations of the Lyapunov functions in terms of excessive function [16], it might be possible to investigate similar connections in the stochastic case.

From the verification and analysis of stochastic hybrid systems perspective, a concept of stochastic bisimulation can facilitate the way towards a model checking of stochastic hybrid systems.

The work presented in this paper and the above discussion allow us to point out some possible research directions in the stochastic hybrid system framework:

- Use the stochastic bisimulation to get manageable sized system abstractions;
- Use the stochastic bisimulation to investigate the reachability problem;
- Make a comparative study of the different approaches on reachability analysis for stochastic hybrid systems: 1. the approach based on the hitting times and hitting probabilities for a target set [7]; 2. the approach based on the so-called Dirichlet forms and excessive functions [8]; 3. the approach based on Lyapunov function (for the switching diffusion processes) [23].

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