Improved Bounds for the Number of ($\leq k$)-Sets, Convex Quadrilaterals, and the Rectilinear Crossing Number of K_n

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Abstract. We use circular sequences to give an improved lower bound on the minimum number of ($\leq k$)-sets in a set of points in general position. We then use this to show that if S is a set of n points in general position, then the number $\square(S)$ of convex quadrilaterals determined by the points in S is at least $0.37553\binom{n}{4} + O(n^3)$. This in turn implies that the rectilinear crossing number $\overline{\text{cr}}(K_n)$ of the complete graph K_n is at least $0.37553\binom{n}{4} + O(n^3)$. These improved bounds refine results recently obtained by Ábrego and Fernández-Merchant, and by Lovász, Vesztergombi, Wagner and Welzl.

1 Introduction

Our aim in this work is to present some selected results and sketches of proofs of our recent work [5] on the use of circular sequences in the problems described in the title. For the reader familiar with the application of circular sequences to these closely related problems, we give in Subsection 1.4 a brief account of what we perceive is the main achievement hereby reported.

It is well-known that the rectilinear crossing number $\overline{\text{cr}}(K_n)$ of the complete graph K_n is closely related to the minimum number $\square(S)$ of convex quadrilaterals in a set S of n points in general position.

Observation 1 *For each positive integer n,*

$$\overline{\text{cr}}(K_n) = \min_{|S|=n} \square(S),$$

with the minimum taken over all point sets S with n elements in general position.

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Working independently, Ábrego and Fernández-Merchant [1], and Lovász, Vesztergombi, Wagner and Welzl [13] recently explored the close connection between $\square(S)$ and the number $\eta_{\leq k}(S)$ of ($\leq k$)-sets of S. The following result is implicitly proved in [1], and the connection with ($\leq k$)-sets was particularly emphasized in [13].

Theorem 1 ([1] and [13]). *Let S be a set of n points in the plane in general position. Then*

$$\square(S) = \sum_{1 \leq k < (n-2)/2} \big(n - 2k - 3\big)\eta_{\leq k+1}(S) + O(n^3),$$

where $\eta_{\leq j}(S)$ denotes the number of ($\leq j$)-sets of S.

We recall that the *rectilinear crossing number* $\overline{\text{cr}}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane in which every edge is drawn as a straight segment. We also recall that if S is a set of points in the plane in general position, then a k-*set* is a subset T of S with $|T| = k$, and such that T can be separated from its complement $T \setminus S$ by a line. An i-set with $1 \leq i \leq k$ is a ($\leq k$)-*set*. As we mentioned above, we use $\eta_{\leq k}(S)$ to denote the number of ($\leq k$)-sets of S.

In this paper we follow the approach, via circular sequences, used by Ábrego and Fernández-Merchant and (independently) by Lovász, Vesztergombi, Wagner and Welzl, to give improved lower bounds for $\eta_{\leq k}(S)$. In view of Observation 1 and Theorem 1, these refined bounds immediately imply improved bounds for $\square(S)$ (for any set S) and for $\overline{\text{cr}}(K_n)$.

1.1 The Relationship Between $\square(S)$ and Circular Sequences

Let S be a set of n points in general position in the plane. In [1] and [13], it is shown that $\square(S)$ is closely related to $\eta_{\leq k}(S)$.

While the important problem of determining, for each k, the maximum number of k-sets remains tantalizingly open (the best current bounds are $O(nk^{1/3})$ and $ne^{\Omega(\log k)}$ (see [8] and [18], respectively), it is known that the maximum number of ($\leq k$)-sets of an n-point set S in the plane is nk (this is attained iff S is in convex position; see [3] and [21]).

In [13] and [21], it is shown that if S is a collection of points in general position, then $\square(S)$ is a linear combination of $\{\eta_{\leq j}(S)\}$. Indeed, Theorem 1 above is a direct consequence of Lemma 9 in [13].

Theorem 1 is exploited in [13] by finding a nontrivial lower bound for $\eta_{\leq k}(S)$ for every $k < n/2$ and every set S of n points in general position (and using an even better bound for k close to $n/2$, which follows from the results in [20]). See Theorems 2 and 4 in [13]. To obtain the bound in their Theorem 2, they follow the approach of circular sequences.

A *circular sequence on n elements Π* is a sequence $(\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ of permutation of the set $\{1, 2, \ldots, n\}$, where π_0 is the identity permutation $(1, 2, \ldots, n)$,

$\pi_{\binom{n}{2}}$ is the reverse permutation $(n, n-1, \ldots, 1)$, and any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions. A transposition that occurs between elements in positions i and $i+1$, or between elements in positions $n-i$ and $n-i+1$ is *i-critical.* A transposition is ($\leq k$)-*critical* if it is critical for some $i \leq k$. We denote the number of ($\leq k$)-critical transpositions in Π by $\chi_{\leq k}(\Pi))$, and use $\mathbf{X}_{\leq k}(n)$ to denote the minimum of $\chi_{\leq k}(\Pi)$ taken over all circular sequences Π on n elements.

Circular sequences can be used to encode any set S of points in general position as follows (see [12]). Let L be a (directed) line that is not orthogonal to any of the lines defined by pairs of points in S. We label the points in S as $p_1, p_2, \ldots, p_n$, according to the order in which their orthogonal projections appear along L. As we rotate L (say counterclockwise), the ordering of the projections changes precisely at the positions where L passes through a position orthogonal to the line defined by some pair of points r, s in S. At the time the projection change occurs, r and s are adjacent in the ordering. and the ordering changes by transposing r and s. By keeping track of all permutations of the projections as L is rotated by 180^o, we obtain a circular sequence Π_S.

The crucial observation is that ($\leq k$)-sets are in one-to-one correspondence with ($\leq k$)-critical transpositions of Π_S.

Observation 2 *Let S be a set of n points in the plane in general position, and let $k < n/2$. Then*

$$\eta_{\leq k}(S) = \chi_{\leq k}(\Pi_S).$$

Combining Theorem 1 and Observation 2 and recalling the definition of $\mathbf{X}_{\leq k}(n)$, one immediately obtains the following statement, obtained independently in [1] and [13].

Theorem 2 ([1] and [13]). *Let S be a set of n points in the plane in general position. Then*

$$\begin{aligned}\Box(S) &= \sum_{1 \leq k < (n-2)/2} \big(n-2k-3\big)\chi_{\leq k+1}(\Pi_S) + O(n^3) \\ &\geq \sum_{1 \leq k < (n-2)/2} \big(n-2k-3\big)\mathbf{X}_{\leq k+1}(n) + O(n^3).\end{aligned}$$

Having reduced the problem of bounding $\Box(S)$ to the problem of bounding $\mathbf{X}_{\leq k}(n)$, Ábrego and Fernández-Merchant [1], and independently Lovász, Vesztergombi, Wagner and Welzl [13], then proceeded to the (combinatorial) problem of deriving good estimates for $\mathbf{X}_{\leq k}(n)$.

1.2 Previous Estimates for $\mathbf{X}_{\leq k}(n)$ and Their Consequences

In [1] and [13], the following was proved:

$$\mathbf{X}_{\leq k}(n) \geq 3\binom{k+1}{2}, \text{ for every positive } n \text{ and every } k < n/2. \tag{1}$$

In [1], this result is applied together with Theorem 2, to obtain the following.

Theorem 3 (Ábrego and Fernández-Merchant [1]). *If S is any set of n points in general position, then*

$$\Box(S) \geq \frac{1}{4}\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n-1}{4}\right\rfloor\left\lfloor\frac{n-2}{4}\right\rfloor\left\lfloor\frac{n-3}{4}\right\rfloor = 0.375\binom{n}{4} + O(n^3). \qquad (2)$$

As a corollary, they obtain $\overline{\mathrm{cr}}(K_n) \geq 0.375\binom{n}{4} + O(n^3)$.

We observe that the bound $\mathbf{X}_{\leq k}(n) \geq 3\binom{k+1}{2}$ is sharp for $k \leq n/3$ (see Example 3 in [13]). Therefore, any improvement on $\Box(S)$ based on the approach of circular sequences must necessarily rely on bounds for $\mathbf{X}_{\leq k}(n)$ that are strictly better than $3\binom{k+1}{2}$ for (some subset of) the interval $n/3 < k < (n-2)/2$. Prior to the present paper, the only such bound reported is the following, which is derived in [13] using a result from [20]:

$$\mathbf{X}_{\leq k}(n) \geq \frac{n^2}{2} - n\sqrt{n^2 - 4k^2} + O(n). \qquad (3)$$

Now (3) is strictly better than (1) for k sufficiently close to $n/2$, namely for $k > k_0(n) := \sqrt{(2\sqrt{13}-5)/9}\,n \approx 0.4956n + O(\sqrt{n})$. Combining (1) (which is also proved in [13] independently of [1]) and (3), and applying Theorem 2, the following was proved in [13].

Theorem 4 (Lovász, Vesztergombi, Wagner and Welzl [13]). *If S is any set of n points in general position, then*

$$\Box(S) > 0.37501\binom{n}{4} + O(n^3).$$

Again, in view of Observation 1 this immediately yields an improved bound for $\overline{\mathrm{cr}}(K_n)$.

Although numerically the improvement (of roughly $1.088 \cdot 10^{-5}$) given in Theorem 4 over 0.375 may seem marginal, conceptually it is most relevant, since it shows that the rectilinear and the ordinary crossing number of K_n (which considers drawings in which the edges are not necessarily straight segments) are different on the asymptotically relevant term n^4. This last observation follows since there are (non-rectilinear) drawings of K_n with exactly $(1/4)\lfloor n/4\rfloor\lfloor(n-1)/4\rfloor\lfloor(n-2)/4\rfloor\lfloor(n-3)/4\rfloor = 0.375\binom{n}{4} + O(n^3)$ crossings. No better (non-rectilinear) drawings of K_n are known, and consequently the (non-rectilinear) crossing number of K_n has been long conjectured to be exactly $(1/4)\lfloor n/4\rfloor\lfloor(n-1)/4\rfloor\lfloor(n-2)/4\rfloor\lfloor(n-3)/4\rfloor$ (see for instance [10]).

1.3 Our Results: Improved Bound for $\mathbf{X}_{\leq k}(n)$ and Its Consequences

The core of this paper is an improved bound on the minimum number $\mathbf{X}_{\leq k}(n)$ of $(\leq k)$-critical transpositions in any circular sequence on n elements. Our bound is given in terms of two functions $F(k,n)$ and $s(k,n)$ defined as follows.

For all positive integers k, n such that $k < n$, let

$$F(k,n) := \left(2 - \frac{1}{s(k,n)}\right) k^2 - \left(\frac{(s(k,n)-1)^2}{s(k,n)}\right) k(n-2k-1)$$

$$+ \left(\frac{s(k,n)^4 - 7s(k,n)^2 + 12s(k,n) - 6}{12s(k,n)}\right)(n-2k-1)^2,$$

where

$$s(k,n) := \left\lfloor \frac{1}{2}\left(1 + \sqrt{\frac{1 + 6\left(\frac{k}{n}\right) - \left(\frac{9}{n}\right)}{1 - 2\left(\frac{k}{n}\right) - \left(\frac{1}{n}\right)}}\right)\right\rfloor .$$

Using this notation, our main result is the following.

Theorem 5 (Main result). *For every positive integer n and every $k < n/2$,*

$$\mathbf{X}_{\leq k}(n) \geq F(k,n) + O(n).$$

This bound is better than the bounds in (1) and (3) for $k > k_1(n) := (1/162)\left(-71 + 71n + \sqrt{19n^2 - 38n + 19}\right) \approx 0.465178n + O(\sqrt{n})$ (see [5]).

The full proof of Theorem 5 is given in [5]. We present a sketch of the general ideas in the proof in Section 2.

By Observation 2, the refined bound for $\mathbf{X}_{\leq k}(n)$ given in Theorem 5 immediately implies improved bounds for $\eta_{\leq k}(S)$, for $k \geq k_1(n)$.

Moreover, in view of Theorem 2, Theorem 5 also gives improved bounds for $\square(S)$, for any set S of n points in general position.

The corresponding calculations (which are somewhat tedious but by no means difficult) are sketched in Section 3, where the following is established.

Proposition 1. *For every positive integer n and every $k < n/2$,*

$$\sum_{1 \leq k < (n-2)/2} (n-2k-3) \cdot \max\left\{3\binom{k+2}{2}, F(k+1,n)\right\} \geq 0.37553\binom{n}{4} + O(n^3).$$

By applying Theorem 5 and Proposition 1 to Theorem 2, we obtain the following.

Corollary 1. *If S is a set of n points in the plane in general position, then*

$$\square(S) \geq 0.37553\binom{n}{4} + O(n^3).$$

In view of Observation 1, we also have the following.

Corollary 2. *For each positive integer n,*

$$\overline{\mathrm{cr}}(K_n) \geq 0.37553\binom{n}{4} + O(n^3).$$

To put this improved lower bound on $\overline{\mathrm{cr}}(K_n)$ into context, first we should point out that the lower bounds on $\overline{\mathrm{cr}}(K_n)$ proved in [1] and [13] represent a remarkable improvement over the previous best general lower bounds. Previous to the successful use of the approach of circular sequences (Edelsbrunner et al. [9] also claimed to have proved that $\mathbf{X}_{\leq k}(n) \geq 3\binom{k+1}{2}$, but their argument seems to have a gap), the best lower bound known was $\overline{\mathrm{cr}}(K_n) \geq 0.3288\binom{n}{4}$ [19].

The improved lower bounds on $\overline{\mathrm{cr}}(K_n)$ reported in [1] and [13] are particularly attractive since they are remarkably close to the best upper bound currently known, namely $\overline{\mathrm{cr}}(K_n) \leq 0.3807\binom{n}{4}$ [2]. This bound was obtained using a computer-generated base case. The best known upper bound derived "by hand" (quoting [13]), namely $\overline{\mathrm{cr}}(K_n) \leq 0.3838\binom{n}{4}$, was obtained by Brodsky, Durocher, and Gethner [6].

We also mention that the exact crossing number of K_n is known for $n \leq 16$. For all $n \leq 9$, the exact value of $\overline{\mathrm{cr}}(K_n)$ can be found for instance in [22]. For $n = 10$ it was determined by Brodsky, Durocher, and Gethner [7], for $n = 11$ and 12 it was calculated by Aichholzer, Aurenhammer, and Krasser [2], and quite recently Aichholzer and Krasser determined it for $n = 13, 14, 15, 16$ (private communication). The most current information on the rectilinear crossing number of K_n for specific values of n is given in Aichholzer's comprehensive web page `http://www.igi.tugraz.at/oaich/triangulations/crossing.html`.

From Corollary 2, the best bounds currently known for $\overline{\mathrm{cr}}(K_n)$ are as follows:

$$0.37553\binom{n}{4} + O(n^3) \leq \overline{\mathrm{cr}}(K_n) \leq 0.3807\binom{n}{4} + O(n^3).$$

1.4 A Brief Discussion on the Main New Results

From our own perspective, the most important contribution of this work is perhaps not the closing of the gap between the lower and upper bounds for $\Box(S)$ and $\overline{\mathrm{cr}}(K_n)$, but the evidence that the technique of circular sequences can be further pushed to yield (substantial, we think) improved results. Indeed, by using exclusively circular sequences we could show that the number of $(\leq k)$-sets is strictly greater than $3\binom{k+1}{2}$ for $k \geq k_1 n \approx 0.465n$, thus closing the gap for roughly 20% of the interval for which this was previously unknown. This success gives us hope that even better results can be obtained by alternative approaches within the technique of circular sequences.

2 Bounding the Number of $(\leq k)$-Critical Transpositions: Sketch of Proof of Theorem 5

Our strategy to prove Theorem 5 is as follows. First we show that the number of $(\leq k)$-critical transpositions in *any* circular sequence Π on n elements is bounded by below by a function that depends on the solution of a maximization problem over a certain family of digraphs. This is done in Section 2.1 (see Proposition 2). Then, in Section 2.2, we find an upper bound for the solution of the maximization problem over this set of digraphs (see Proposition 5).

We will conclude this section with the (by then obvious) observation that Theorem 5 follows from Propositions 2 and 5.

2.1 Bounding the Number of ($\leq k$)-Critical Transpositions in Terms of the Solution of a Digraph Optimization Problem

Our lower bound for the number of ($\leq k$)-critical transpositions in a circular sequence is given in terms of the maximum of an objective function taken over a certain set of digraphs which we now proceed to define. We use $\overrightarrow{uv}$ to denote the directed edge from vertex u to vertex v. The indegree and outdegree of vertex u in the digraph D are denoted $[u]_D^-$ and $[u]_D^+$, respectively.

Definition. Let k, m be integers such that $2 \leq m < k$. A digraph D with vertex set $\{v_1, v_2, \ldots, v_k\}$ is a (k, m)-*digraph* if it satisfies the following conditions:

(i) There is some vertex v_i such that $[v_i]_D^- = 0$.
(ii) For every $i \in \{1, \ldots, k\}$, $[v_i]_D^+ \leq [v_i]_D^- + (m-1)$.
(iii) There is a one-to-one *ordering map* $f_D : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$, such that, for all $i, j \in \{1, 2, \ldots, k\}$, if $\overrightarrow{v_i v_j}$ is in D then $f_D(i) < f_D(j)$.

We let $\mathcal{D}_{k,m}$ denote the set of all (k, m)-digraphs.

The following is one of the core statements of this work. For the sake of brevity, we omit its proof (see [5]).

Proposition 2. *Let Π be any circular sequence on n elements and let $k < n/2$. Define $m := n - 2k$. Then*

$$\chi_{\leq k}(\Pi) \geq 2k^2 + km - \max_{D \in \mathcal{D}_{k,m}} \left\{ 2 \sum_{1 \leq i \leq k} [v_i]_D^- + \sum_{1 \leq i \leq k} \min\left\{ [v_i]_D^- - [v_i]_D^+ + (m-1), m \right\} \right\}.$$

2.2 Bounding the Solution of the Digraph Optimization Problem

The next step is to find a (good) upper bound for the maximization problem in Proposition 2. We achieve this in two steps. First we find a digraph $D_0(k, m)$ in which the maximum is attained, and then we estimate the value of the objective function at $D_0(k, m)$.

Given the nature of the maximization problem in Proposition 2, it is natural to expect that the objective function is maximized in the digraph $D_0(k, m)$ (with vertex set $\{v_1, v_2, \ldots, v_k\}$) in which $[v_i]_{D_0(k,m)}^+$ is maximum possible for each i (subject to the conditions that define $\mathcal{D}_{k,m}$), and in which the $[v_i]_{D_0(k,m)}^+$ directed edges leaving each v_i have endpoints $v_{i+1}, v_{i+2}, \ldots, v_{i+[v_i]_{D_0(k,m)}^+}$ (informally speaking, "there are no gaps"). It can be proved that this is indeed the case, but the proof is long and somewhat technical. For the sake of brevity, we omit the proof of the following statement, and refer the interested reader to [5].

Proposition 3. *The optimal value of the maximization problem in Proposition 2 is attained at the digraph $D_0(k, m)$ with vertex set $\{v_1, v_2, \ldots, v_k\}$ defined as follows:*

(1) $[v_1]^-_{D_0(k,m)} = 0$*;*

(2) $[v_i]^+_{D_0(k,m)} = \min\{[v_i]^-_{D_0(k,m)} + (m-1), k-i\}$*, for every* $i \geq 1$*; and*

(3) For all i, j *such that* $1 \leq i < j \leq k$*, the directed edge* $\overrightarrow{v_i v_j}$ *is in* $D_0(k,m)$ *if and only if* $i+1 \leq j \leq i + [v_i]^+_{D_0(k,m)}$. ∎

For the rest of the section, we denote $D_0(k,m)$ simply by D_0.

In view of this and Proposition 2, our next goal is to estimate a bound for $2\sum_{1\leq i\leq k}[v_i]^-_{D_0(k,m)} + \sum_{1\leq i\leq k}\min\left\{[v_i]^-_{D_0(k,m)} - [v_i]^+_{D_0(k,m)} + (m-1), m\right\}$.

We note that this expression is given in terms of $[v_i]^-_{D_0}$ and $[v_i]^+_{D_0}$. Moreover, in view of the properties of D_0, each $[v_i]^+_{D_0}$ is fully determined by $[v_i]^-_{D_0}$. Thus our first step is to determine (exactly) $[v_i]^-_{D_0}$ for each i. The value of $[v_i]^-_{D_0}$ is given in terms of functions S_m and T_m defined as follows.

For each real number $x \geq 1$, we let $S_m(x)$ denote the (unique) positive integer such that $1 + (S_m(x)-1)S_m(x)(m-1)/2 \leq x < S_m(x)(S_m(x)+1)(m-1)/2$. If $i \geq 1$ is an integer, then we let $T_m(i), U_m(i)$ denote the (unique) integers that satisfy $0 \leq T_m(i) \leq m-2$, $0 \leq U_m(i) \leq S_m(i)-1$, and such that $i = 1 + (S_m(i)-1)S_m(i)(m-1)/2 + S_m(i)T_m(i) + U_m(i)$.

The following statement can be proved by induction on i (see [5]).

Proposition 4. *For each integer* i *such that* $1 \leq i \leq k$*, we have* $[v_i]^-_{D_0} = (S_m(i)-1)(m-1) + T_m(i)$*.* ∎

Once we have the exact value of $[v_i]^-_{D_0}$ for every i, we then proceed to estimate an upper bound for the objective function in Proposition 2, evaluated at D_0. The arguments and calculations needed to prove this bound are not difficult, but somewhat technical and long. We omit the proof of this statement, and refer once again the interested reader to [5]. The upper bound obtained is the right hand side in the inequality in our next statement. Since the objective function is maximized at D_0, we finally conclude the following.

Proposition 5.

$$\max_{D\in\mathcal{D}_{k,m}}\left\{2\sum_{1\leq i\leq k}[v_i]^-_D + \sum_{1\leq i\leq k}\min\left\{[v_i]^-_D - [v_i]^+_D + (m-1), m\right\}\right\} \leq$$
$$\frac{k^2}{S_m(k)} + \frac{(S_m(k)^2 - S_m(k)+1)}{S_m(k)}(m-1)k$$
$$-\left(\frac{S_m(k)^4 - 7S_m(k)^2 + 12S_m(k) - 6}{12S_m(k)}\right)(m-1)^2 + O(k),$$

where

$$S_m(k) = \left\lfloor \frac{1+\sqrt{1+\frac{8(k-1)}{m-1}}}{2} \right\rfloor .$$ ∎

2.3 Proof of Theorem 5

We recall that $m = n - 2k$, and so $s(k,n) = S_m(k)$. Therefore Theorem 5 is an immediate consequence of Propositions 2 and 5 (note that we also used the obvious inequality $km \geq k(m-1)$). ∎

3 Proof of Proposition 1

Our first observation is that, for sufficiently large n, $F(k,n) > 3\binom{k+1}{2}$ for every $k > k_1(n)$ (see Appendix in [5]). We also note that if we define

$$\widetilde{s}(x) := \left\lfloor \frac{1}{2}\left(1 + \sqrt{\frac{1+6x}{1-2x}}\right)\right\rfloor,$$

then it is easy to check that $\widetilde{s}(k/n) = s(k,n)$ (and, moreover, $\widetilde{s}(k/n) = s(k+1,n)$) for all but at most $O(\sqrt{n})$ values of k.

These observations imply that

$$\sum_{k=1}^{(n-2)/2-1} \big(n-2k-3\big)\cdot \max\left\{3\binom{k+2}{2}, F(k+1,n)\right\}$$

$$\geq 3\sum_{k=1}^{\lfloor k_1(n)\rfloor} \big(n-2k-3\big)\binom{k+2}{2} + \sum_{k=\lfloor k_1(n)\rfloor+1}^{(n-2)/2-1} \big(n-2k-3\big)F(k+1,n)$$

$$\geq \frac{3}{2}n^3\cdot\left(\sum_{k=1}^{\lfloor k_1(n)\rfloor}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}\right)^2\right) + n^3\cdot\left(\sum_{k=\lfloor k_1(n)\rfloor+1}^{(n-2)/2-1}\left(1-2\left(\frac{k}{n}\right)\right)\frac{F(k+1,n)}{n^2}\right) + O(n^3)$$

$$\frac{3}{2}n^4\cdot\left(\int_0^{c_1}(1-2x)x^2\,dx\right) + n^4\cdot\left(\int_{c_1}^{1/2}(1-2x)\widetilde{f}(x)\,dx\right) + O(n^3),$$

where $c_1 := 0.465178$ (recall that $k_1(n) \approx 0.465178n + O(\sqrt{n})$), and

$$\widetilde{f}(x) := \left(2 - \frac{1}{\widetilde{s}(x)}\right)x^2 - \left(\frac{(\widetilde{s}(x)-1)^2}{\widetilde{s}(x)}\right)x(1-2x) + \left(\frac{\widetilde{s}(x)^4 - 7\widetilde{s}(x)^2 + 12\widetilde{s}(x) - 6}{12\widetilde{s}(x)}\right)(1-2x)^2.$$

To complete the proof, we note that a numerical evaluation of the integrals in the previous inequality yields

$$\frac{3}{2}\int_0^{c_1}(1-2x)x^2\,dx+\int_{c_1}^{1/2}(1-2x)\widetilde{f}(x)\,dx\approx\frac{0.37553}{24}. \qquad\blacksquare$$

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