Approximation Algorithms for Mixed Fractional Packing and Covering Problems

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We study general mixed fractional packing and covering problems (MPC_ϵ) of the following form: Given a vector $f : B \rightarrow \mathbb{R}_+^M$ of M nonnegative continuous convex functions and a vector $g : B \rightarrow \mathbb{R}_+^M$ of M nonnegative continuous concave functions, two M - dimensional nonnegative vectors a, b, a nonempty convex compact set B and a relative tolerance $\epsilon \in (0,1)$, find an approximately feasible vector $x \in B$ such that $f(x) \leq (1+\epsilon)a$ and $g(x) \geq (1-\epsilon)b$ or find a proof that no vector is feasible (that satisfies $x \in B$, $f(x) \leq a$ and $g(x) \geq b$).

The fractional packing problem with convex constraints, i.e. to find $x \in B$ such that $f(x) \leq (1+\epsilon)a$, is solved in [4, 5, 8] by the Lagrangian decomposition method in $O(M(\epsilon^{-2} + \ln M))$ iterations where each iteration requires a call to an approximate block solver $ABS(p,t)$ of the form: find $\hat{x} \in B$ such that $p^T f(\hat{x}) \leq (1+t)\Lambda(p)$ where $\Lambda(p) = \min_{x \in B} p^T f(x)$. Furthermore, Grigoriadis et al. [6] proposed also an approximation algorithm for the fractional covering problem with concave constraints, i.e. to find $x \in B$ such that $g(x) \geq (1-\epsilon)b$, within $O(M(\epsilon^{-2} + \ln M))$ iterations where each iteration requires here a call to an approximate block solver $ABS(q,t)$ of the form: find $\hat{x} \in B$ such that $q^T g(\hat{x}) \geq (1-t)\Lambda(q)$ where $\Lambda(q) = \max_{x \in B} q^T g(x)$. Both algorithms solve also the corresponding min-max and max-min optimization variants within the same number of iterations. Furthermore, the algorithms can be generalized to the case where the block solver has arbitrary approximation ratio [7, 8, 9].

Further interesting algorithms for the fractional packing and fractional covering problem with linear constraints were developed by Plotkin et al. [13] and Young [15]. These algorithms have a running time that depends linearly on the width - an unbounded function of the input instance. Several relatively complicated techniques were proposed to reduce this dependence. Garg and Könemann [3] described a nice algorithm for the fractional packing problem with linear constraints that needs only $O(M\epsilon^{-2} \ln M)$ iterations.

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For the mixed packing and covering problem (with linear constraints), Plotkin et al. [13] proposed also approximation algorithms where the running time depends on the width. Young [16] described an approximation algorithm for a special mixed packing and covering problem with linear constraints and special convex set $B = \mathbb{R}_+^N$. The algorithm has a running time of $O(M^2\epsilon^{-2}\ln M)$. Recently, Fleischer [1] gave an approximation scheme for the optimization variant (minimizing $c^T x$ such that $Cx \geq b$, $Px \leq a$ and $x \geq 0$ where a, b, and c are nonnegative integer vectors and P and C are nonnegative integer matrices).

Young [16] posed the following interesting open problem: find an efficient width-independent Lagrangian-relaxation algorithm for the mixed packing and covering problem (with linear constraints): find $x \in B$ such that $Px \leq (1+\epsilon)a$, $Cx \geq (1-\epsilon)b$, where P, C are nonnegative matrices, a, b are nonnegative vectors and B is a polytope that can be queried by an optimization oracle (given a vector c, return $x \in B$ minimizing $c^T x$) or some other suitable oracle.

New results: We found an approximation algorithm for the general mixed problem with M convex and M concave functions f_m, g_m that uses an suitable oracle of the form: find $\hat{x} \in B$ such that $p^T f(\hat{x}) \leq \sum_{m=1}^M p_m$ and $q^T g(\hat{x}) \geq \sum_{m=1}^M q_m$ [10]. The algorithm uses $O(M\epsilon^{-2}\ln(M\epsilon^{-1}))$ iterations or coordination steps, where in each iteration an oracle of the form above is called. Recently we found an improved width-independent Lagrangian-relaxation algorithm for the general mixed problem [11]. The algorithm uses a variant of the Lagrangian or price directive decomposition method. This is an iterative strategy that solves (MPC_ϵ) by computing a sequence of triples (p, q, x) as follows. A coordinator uses the current vector $x \in B$ to compute two price vectors $p = p(x) \in \mathbb{R}_+^M$ and $q = q(x) \in \mathbb{R}_+^M$ with $\sum_{m=1}^M p_m + q_m = 1$. Then the coordinator calls here a feasibility oracle to compute a solution $\hat{x} \in B$ of the block problem $BP(p, q, t)$

$$\text{find } \hat{x} \in B \text{ s.t. } p^T f(\hat{x})/(1+t) \leq q^T g(\hat{x})(1+t) + 2\bar{p} - 1,$$

(where $t = \Theta(\epsilon)$ and $\bar{p} = \sum_{m=1}^M p_m$) and makes a move from x to $(1-\tau)x + \tau\hat{x}$ with an appropriate step length $\tau \in (0, 1)$. Such a iteration is called a coordination step. In case $\bar{p}$ is close to $1/2$, we use a slightly different block problem $BP'(p, q, t)$ of the form:

$$\text{find } \hat{x} \in B \text{ s.t. } p^T f(\hat{x})/(1+8t) \leq q^T g(\hat{x})(1+8t) + (2\bar{p} - 1 - t).$$

Our main result is the following: There is an approximation algorithm that for any given accuracy $\epsilon \in (0, 1)$ solves the general mixed fractional packing and covering problem (MPC_ϵ) within

$$N = O(M(\epsilon^{-2}\ln\epsilon^{-1} + \ln M))$$

iterations or coordination steps, where each of which requires a call to the block problem $BP(p, q, t)$ or $BP'(p, q, t)$.

Independently, Khandekar and Garg [2] proposed an approximation algorithm for the general mixed problem that uses $O(M\epsilon^{-2}\ln M)$ iterations or coordination steps.

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