A $\frac{5}{4}$-Approximation Algorithm for Biconnecting a Graph with a Given Hamiltonian Path*

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Abstract. Finding a minimum size 2-vertex connected spanning subgraph of a k-vertex connected graph $G = (V, E)$ with n vertices and m edges is known to be NP-hard and APX-hard, as well as approximable in $O(n^2 m)$ time within a factor of 4/3. Interestingly, the problem remains NP-hard even if a Hamiltonian path of G is given as part of the input. For this input-enriched version of the problem, we provide in this paper a linear time and space algorithm which approximates the optimal solution by a factor of no more than $\min\left\{\frac{5}{4}, \frac{2k-1}{2(k-1)}\right\}$.

1 Introduction

The problem of finding a minimum size 2-vertex connected (simply *biconnected*, in the following) spanning subgraph (*MBSS problem*) of a biconnected, undirected graph $G = (V, E)$, with n vertices and m edges, is one of the classical problems in computer science and combinatorial optimization [9]. It is known to be NP-hard, since its decision version contains as a special case the *Hamiltonian cycle* problem (i.e., the problem of deciding whether a graph G contains a simple cycle that includes all the vertices), which is well-known to be NP-complete [5].

Due to its relevance and to the great number of applications it finds in different fields, several approximation algorithms for solving this problem have been devised in the past few years. Khuller and Vishkin [10] introduced the notions of *carving* of a graph to establish an approximation factor of no more than 5/3. Their algorithm has been firstly improved by Garg *et al.* [6], who obtained an approximation ratio of 3/2. After, this ratio was improved to 4/3 by Vempala and Vetta [13]. Concerning inapproximability results, the problem is known to be APX-hard [11].

The weighted version of the problem has been deeply investigated as well. In this case, the problem admits a $\left(2 + \frac{1}{n}\right)$-approximation algorithm [10], while if G satisfies the triangle inequality, then it is approximable within 3/2 [4]. Moreover, for any integer $d = o(\log n)$, if G is complete and Euclidean in $\mathbb{R}^d$

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(i.e., G is embedded in the Euclidean d-dimensional space and the edge weights correspond to the Euclidean d-dimensional distance between the corresponding endvertices), then the problem admits a PTAS [2]. Concerning inapproximability results, the problem is not approximable (unless $\mathsf{P}{=}\mathsf{NP}$) within $\frac{68569}{68564} - \varepsilon$, for any constant $\varepsilon > 0$ [1].

As far as the edge version of the problem is concerned, i.e., that of finding a minimum size 2-edge connected spanning subgraph of a 2-edge-connected, undirected graph, the best known approximation ratio is 5/4 [7]. In the same paper, the authors claimed that their algorithm can be extended, by preserving the approximation ratio, to the vertex version of the problem, but unfortunately this seems not to be the case [8]. As a consequence, there is currently a gap between the approximability of the vertex and the edge version of the problem, i.e., 4/3 versus 5/4.

A question which naturally arises is that of studying whether the approximation guarantee can be improved once the input of the problem is enriched. In particular, Papadimitriou and Steiglitz [12] proved that the problem of determining whether a graph contains a Hamiltonian cycle remains NP-complete even if a Hamiltonian path is given as part of the input. It follows that the problem of determining whether a graph admits a biconnected spanning subgraph of size $k \geq n$, once a Hamiltonian path is given as part of the input, is NP-complete as well. In this paper we consider the optimization version of this latter problem (*MBSSHP problem* for short).

To the best of our knowledge, for the MBSSHP problem the same approximation factor as for the MBSS problem holds, also when the edge-version of the problem is considered. Hence, also in this case, there is a gap between the approximability of the vertex and the edge version of the problem, i.e., 4/3 versus 5/4.

In this paper, we get to the target of closing this gap. Indeed, we show that the MBSSHP problem can be approximated in linear time and space with a performance guarantee of 5/4. Moreover, we show that if G is k-vertex connected, $k > 3$, then our algorithm can be enhanced to return a $\frac{2k-1}{2(k-1)}$-approximated solution. Our approach deviates significantly from that proposed for the MBSS problem by Vempala and Vetta [13], since their algorithm cannot guarantee an approximation factor better than 4/3 when adapted to the MBSSHP problem.

From an application point of view, our algorithm has a practical impact on *chain communication networks*, where we have a set of vertices $v_1, v_2, \ldots, v_n$ which mutually exchange messages through a chain of links $(v_i, v_{i+1}), i = 1, \ldots, n-1$. Suppose now we have a set of potential additional links (v_i, v_j), $1 \leq i < j+1 \leq n$, such that the graph resulting from the chain enriched of the additional edges is biconnected. Then, one might be interested in making the communication between vertices immune to single vertex failures, by using a minimum number of links. In this case, our algorithm computes an approximated solution in linear time and space with a performance guarantee of 5/4.

The paper is organized as follows: in Section 2 we introduce basic definitions and notations used in the paper. In Section 3 we present two simple algorithms for the MBSSHP problem. In Section 4, after analyzing the known lower bounds for the MBSS problem, we first present a new lower bound, then we refine the algorithms of Section 3.

2 Basic Definitions

Let $G = (V, E)$ be a simple undirected graph (i.e., without loops and parallel edges), where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Let $n \geq 3$ and m be the number of vertices and the number of edges, respectively. For all $v \in V$, $\delta_G(v)$ denotes the *degree* of v in G, i.e., the number of vertices adjacent to v with respect to the edge set of G. A graph $H = (V(H), E(H))$ is called a *subgraph* of G if $V(H) \subseteq V$ and $E(H) \subseteq E$. If $V(H) = V$ then H is called a *spanning subgraph* of G. For any $U \subseteq V$, the graph $G_U = (U, E_U)$ where $E_U = \{(v, v') \in E \mid v, v' \in U\}$ is said to be *induced* from U while the graph $G' = (U \cup \{x_U\}, E_U \cup E')$, where $x_U \notin V$ and $E' = \{(u, x_U) \mid (u, v) \in E \text{ with } u \in U \wedge v \in V \setminus U\}$, is said to be obtained from G by *shrinking* $V \setminus U$ into one vertex x_U.

A *simple path* Π (or a *path* for short) in G is a subgraph with $V(\Pi) = \{v_1, \ldots, v_k \mid v_i \neq v_j \text{ for } i \neq j\}$ and $E(\Pi) = \{e_i = (v_i, v_{i+1}) \mid 1 \leq i < k\}$, also denoted as $\Pi(v_1, v_k)$ or $\langle v_1, v_2, \ldots, v_k \rangle$. Path Π is said to go from v_1 to v_k, called the *endvertices* of Π, passing through the *internal* vertices $v_2, v_3, \ldots, v_{k-1}$. A *cycle* is a path whose endvertices coincide. G is said to be *Hamiltonian* if it has a spanning cycle. A spanning path $\Pi_G = \langle v_1, v_2, \ldots, v_n \rangle$ of G is called a *Hamiltonian path*. Edges in $E(\Pi_G)$ are called *path edges*, while edges in $F = E \setminus E(\Pi_G)$ are called *cycle edges*. By $E(e_i)$ we denote the set of all cycle edges forming a *fundamental cycle* with e_i, i.e., a cycle containing only one cycle edge. If $f \in E(e_i)$ then we say f *covers* e_i. Thus, $f = (v_j, v_h)$, with $j < h$, covers e_i iff $j \leq i < h$. For any cycle edge $f = (v_i, v_j)$, with $i < j$, we call v_i (resp., v_j) the *left* (resp., *right*) endvertex of f. We denote by $\mathcal{G}^n$ the class of graphs of n vertices having a Hamiltonian path.

A graph G is *connected* if, for any $u, v \in V$, there exists a path in G going from u to v. The *connected components* of a graph G are the maximal (w.r.t. vertex insertion) connected subgraphs of G. A graph G is *k-vertex connected* (or simply *k-connected*) if $n \geq k + 1$ and for any $V' \subsetneq V$ of $k - 1$ vertices, the graph induced by $V \setminus V'$ is connected. When $k = 2$, G is said to be *biconnected*. The maximum integer k such that G is k-connected is said to be the *connectivity number* of G and it is denoted by $\kappa(G)$.

A subset C of V, with $|C| = k$ is a *k-separator* (or simply *separator*) if $G_{V \setminus C}$ is not connected, that is, there exists a two partition V_ℓ, V_r of $V \setminus C$ such that G has no edges with one endvertex in V_ℓ and the other in V_r. We say that V_ℓ, C, V_r is a *k-separation* of G. A separator C is said to be *minimal* if no proper subset of C is a separator. Observe that $\kappa(C) \geq \kappa(G)$, for every separator C. Let $e_i \in E(\Pi_G)$. By $\kappa_\ell(e_i)$ (resp., $\kappa_r(e_i)$) we denote the cardinality of a minimal (unique) separation V_ℓ, C, V_r such that $C \cup V_\ell = \{v_1, v_2, \ldots, v_i\}$ (resp., $C \cup V_r = \{v_{i+1}, v_{i+2}, \ldots, v_n\}$). Then, $\kappa(e_i) = \min\{\kappa_\ell(e_i), \kappa_r(e_i)\}$.

In the rest of the paper, $\text{Opt}(G)$ will denote the size of a *minimum biconnected spanning subgraph* (MBSS) of a biconnected graph G. Clearly, $\text{Opt}(G) \geq n$.

3 Basic Algorithm

In this section we describe a very simple algorithm for finding a biconnected spanning subgraph of an undirected graph G, with the hypothesis that we have a Hamiltonian path Π_G of G and $\kappa(G) > 1$. We show that the approximation ratio depends on $\kappa(G)$. Henceforth, unless stated otherwise, $\Pi_G = \langle v_1, v_2, \ldots, v_n \rangle$. Moreover, by f_i we denote a cycle edge of $E(e_i)$ with right endvertex of maximum index.

The basic idea of the algorithm is simple. Starting from a spanning subgraph H of G with no edges, the algorithm processes each vertex in order from v_1 to v_n. At each step it augments $E(H)$ by adding edges. The invariant property maintained by the algorithm is the following: if it is currently exploring Π_G from v_i to v_j, $i < j$, then the subgraph H' of H induced from $\{v_1, v_2, \ldots, v_i\}$ is already biconnected. The set of edges to be added is determined by the function *Expand*. Thus, the more powerful the function *Expand*, the lower the size of the computed biconnected spanning subgraph. We propose a first simple version of the function *Expand*.

Algorithm $BSS(G, \Pi_G = \langle v_1, v_2, \ldots v_n \rangle)$;
Input: A biconnected graph G and a Hamiltonian path Π_G of G;
Output: A biconnected spanning subgraph H of G.
begin
  $H = (V, \emptyset)$;
  $\Pi_R = \Pi_G(v_2, v_n)$;
  while $\Pi_R \neq \langle v_n \rangle$ **do**
    $(F', E_\Pi) = Expand(G, \Pi_R)$; $\%F' \subseteq F = E \setminus E(\Pi_G), E_\Pi \subseteq E(\Pi_G)$
    $i = \max\{j \mid (v_h, v_j) \in F', h < j\}$;
    $E(H) = E(H) \cup F' \cup E_\Pi$;
    $\Pi_R = \Pi_G(v_i, v_n)$;
  end while
  return H;
end.

Function $Expand(G, \Pi_R = \langle v_{i+1}, v_{i+2}, \ldots, v_n \rangle)$;
Input: A graph G and a path Π_R of G;
Output: A set of cycle edges $F' \subseteq E \setminus E(\Pi_G)$ and a set of path edges $E_\Pi \subseteq E(\Pi_R)$.
begin
  Let $E_\Pi \subseteq E(\Pi_R)$ be the set of path edges covered by f_i;
  return $(\{f_i\}, E_\Pi)$;
end.

Theorem 1. *The algorithm BSS computes a $\frac{\kappa(G)}{\kappa(G)-1}$-approximated solution in $\mathcal{O}(m)$ time and space.*

Proof. At a given iteration, let $\Pi_R = \Pi_G(v_{i+1}, v_n)$. Since $\kappa(e_i) \geq \kappa(G)$, we have that $f_i = (v, v_{i+j+1})$, with $j \geq \min\{n-i-1, \kappa(G)-1\}$. So we add at most one cycle edge every $\kappa(G)-1$ edges of Π_R plus one extra cycle edge if Π_R has a length less than $\kappa(G)-1$. Since Π_G is a subgraph of H, we achieve the following approximation ratio:

$$\frac{|E(H)|}{\textsc{Opt}(G)} \leq \frac{n-1+\left\lceil \frac{n-1}{\kappa(G)-1} \right\rceil}{n} \leq 1 + \frac{\left\lceil \frac{n}{\kappa(G)-1} \right\rceil - 1}{n} \leq 1 + \frac{\frac{n}{\kappa(G)-1}}{n} = \frac{\kappa(G)}{\kappa(G)-1}.$$

Moreover, at each iteration we only have to explore the cycle edges belonging to $E(e_i)$. As we take the one whose right endvertex has maximum index and since Π_R updates to $\Pi(v_{i+j+1}, v_n)$, then the algorithm does not need to explore these edges any more. Hence, the time and space complexity is $\mathcal{O}(m)$. □

The function *Expand*(G, Π_R) defined above uses little information implied by $\kappa(G)$. To improve the performance of the algorithm we introduce the relation of *semi-adjacency* between cycle edges. So we say that two cycle edges $f = (v_i, v_{j+1})$, $f' = (v_j, v_k)$, with $i < j < k-1$, are *semi-adjacent.* Path edge e_j is said to be the *middle* of f and f'. We can now prove the following:

Lemma 1. *For every e_i, either f_i or a pair semi-adjacent edges f, f' (one of which belongs to $E(e_i)$) cover* $\min\{n-i-1, 2(\kappa(G)-1)\}$ *edges of* $\Pi_G(v_{i+1}, v_n)$.

Proof. Let v_{j+1} be the right endvertex of f_i. If f_i covers $\min\{n-i-1, 2(\kappa(G)-1)\}$ edges of $\Pi_G(v_{i+1}, v_n)$ then the claim is true. So we can assume that f_i covers $k < \min\{n-i-1, 2(\kappa(G)-1)\}$ edges of $\Pi_G(v_{i+1}, v_n)$. We say that v_h, with $i < h \leq j$ is a *potential semi-adjacent* vertex if v_{h+1} is an endvertex of some edges in $E(e_i)$. Indeed, if some edge in $E(e_j)$ has v_h as endvertex, then this edge is semi-adjacent to some edge in $E(e_i)$. Since $\kappa(e_i) \geq \kappa(G)$, then there are $k_1 \geq \kappa(G)-1$ vertices of $\Pi_G(v_{i+2}, v_{j+1})$ that are endvertices of some edge in $E(e_i)$. Hence, k_1 vertices of $\Pi_G(v_{i+1}, v_j)$ are potential semi-adjacent vertices, while the remaining

$$k_2 = k - k_1 \leq k - \kappa(G) + 1$$

are not. Since $\kappa(e_j) \geq \kappa(G)$, then at least $\kappa(G)-1$ vertices of $\Pi_G(v_{i+1}, v_j)$ are endvertices of some edges in $E(e_j)$. As $k_2 \leq k - \kappa(G) + 1 < \kappa(G) - 1$ this means that there exist edges in $E(e_j)$ semi-adjacent to some edges in $E(e_i)$. Among such edges, choose any one (say f) whose right endvertex has maximum index. We claim that f covers at least $\min\{n-j-1, 2(\kappa(G)-1)-k\}$ edges of $\Pi_G(v_{j+1}, v_n)$. To prove this, suppose it is not true, that is f covers $h < \min\{n-j-1, 2(\kappa(G)-1)-k\} \leq 2(\kappa(G)-1)-k$ edges of $\Pi_G(v_{j+1}, v_n)$. In this case, if we remove the first $h+1$ vertices of $\Pi_G(v_{j+1}, v_n)$ and all non potential semi-adjacent vertices in $\Pi_G(v_{i+1}, v_j)$, we break the graph into two connected components. But the number of vertices removed is

$$k_2 + h + 1 \leq k - \kappa(G) + 1 + h + 1 < k - \kappa(G) + 1 + 2\kappa(G) - 2 - k + 1 = \kappa(G)$$

and so G is not $\kappa(G)$-connected. We have obtained a contradiction. □

Using Lemma 1 we can implement a new powerful function *Expand* such that:

Theorem 2. *The algorithm BSS computes a* $\frac{2\kappa(G)-1}{2(\kappa(G)-1)}$*-approximated solution in* $\mathcal{O}(m)$ *time and space.*

Proof. At a given iteration, let v_{j+1} in $\Pi_R = \Pi_G(v_{i+1}, v_n)$ be the right endvertex of f_i. If f_i covers $\min\{n-i-1, 2(\kappa(G)-1)\}$ edges of Π_R, set $E_C = \{f_i\}$ and $E_\Pi = \{e_{i+1}, e_{i+2}, \ldots, e_j\}$, otherwise from Lemma 1 there are two semi-adjacent edges f, f', one of which belongs to $E(e_i)$, covering $\min\{n-i-1, 2(\kappa(G)-1)\}$ edges of Π_R. In this case E_Π it the set of path edges of Π_R covered by f, f' minus the middle one, while $E_C = \{f, f'\}$. So we add at most one cycle edge every $2(\kappa(G)-1)$ edges of Π_R plus one extra cycle edge if Π_R has a length less than $2(\kappa(G)-1)$. Thus, we achieve the following approximation ratio:

$$\frac{|E(H)|}{\mathrm{OPT}(G)} \leq \frac{n-1+\left\lceil \frac{n-1}{2(\kappa(G)-1)} \right\rceil}{n} \leq 1 + \frac{\frac{n}{2(\kappa(G)-1)}}{n} = \frac{2\kappa(G)-1}{2(\kappa(G)-1)}.$$

From Lemma 1 it also follows that at each iteration we have to explore the edges belonging to $E(e_i)$ and the edges belonging to $E(e_j)$. Since the next iterations the algorithm will not explore edges in $E(e_i)$ any more, then the time and space complexity is $\mathcal{O}(m)$. $\square$

The second function *Expand* we have just defined, uses $\kappa(G)$ as a lower bound for $\kappa(e_i), 1 \leq i \leq n-1$. Looking at the proof of Lemma 1 one may convince that

Remark 1. The practical approximation ratio we obtain is given by the minimum value $\kappa(e_i)$ of all sets $E(e_i)$ the algorithm considers.

The following lemma shows that the approximation ratio of Theorem 2 is tight when compared to the trivial lower bound n.

Lemma 2. $\forall k \geq 2$*, there exists a* k*-connected graph* $G \in \mathcal{G}^n$ *for which the ratio between* $\mathrm{OPT}(G)$ *and* n *is equal to* $\frac{2k-1}{2(k-1)}$.

Proof. The proof is constructive. Let $n = 2j(k-1)+4k-1$, where j is a positive integer. We first build a bipartite graph $G' \in \mathcal{G}^n$. We number the vertices of G' from 1 to n. Let $V_C = \{u^i = v_{2k+2i(k-1)} | i = 0, \ldots, j\}$. Let $V_0 = \{v_1, v_2, \ldots, u^0\}$ and $V_{j+1} = \{u^j, \ldots, n\}$, while

$$V_i = \{u^{i-1}, \ldots, u^i\}, \text{ with } i = 1, \ldots, j.$$

Notice that $V_i \cap V_{i+1} = \{u^i\}$. For every $i = 0, 1, \ldots, j$, let V_i^e (resp., V_i^o) be the set of even (resp., odd) vertices of V_i. The set of edges of G' is defined as follows:

$$E(G') = \bigcup_{i=0}^{j+1} \left(V_i^e \times V_i^o\right).$$

Notice that $\langle v_1, v_2, \ldots, v_n \rangle$ is a Hamiltonian path of G'. Clearly $\kappa(G') = 1$, since every vertex in V_C is a *cut-vertex*, i.e., a vertex whose removal disconnects G'. From G' we build a new graph $G = (V, E)$ with $\kappa(G) = k$. Let $V = V(G')$ and $E = E(G') \cup E'$, where

$$E' = \{(u, v) | u \in V_i^e, v \in V_{i+1}^e, i = 0, 1, \ldots, j\}.$$

It is easy to see that G is k-connected. Let us consider the topological structure of a MBSS $\mathcal{H}$ of G. As every vertex $u^i \in V_C$ in G' is a cut-vertex, it follows that $E(\mathcal{H})$ must contain a cycle edge in E' *covering* u^i (i.e., an edge whose addition to G' makes u_i not to be a cut-vertex). By construction, the cycle edges covering u^i do not cover u^j, with $j \neq i$. So $\mathcal{H}$ has at least

$$|V_C| = j + 1 = \frac{n - 4k + 1}{2(k-1)} + 1 = \frac{n - 2k - 1}{2(k-1)}$$

edges of E'. Moreover, since there is no edge between pairs of odd vertices, this implies that $\mathcal{H}$ must have $2\left\lceil \frac{n}{2} \right\rceil = n + 1$ edges of $E(G')$, where the equality follows from the fact that n is odd. So the approximation ratio is

$$\lim_{n \to +\infty} \frac{n + 1 + |V_C|}{n} = 1 + \lim_{n \to +\infty} \frac{1 + \frac{n-2k-1}{2(k-1)}}{n} = 1 + \lim_{n \to +\infty} \frac{n - 3}{2n(k-1)} = \frac{2k-1}{2(k-1)}.$$

□

4 Improving the Algorithm

4.1 Considerations About Well-Known Lower Bounds

As seen in the previous section, the approximation ratio we can achieve is strictly related to the connectivity value of G. Since $\forall \kappa(G) \geq 3$, $\frac{2\kappa(G)-1}{2(\kappa(G)-1)} \leq \frac{5}{4}$, our aim now is to improve the approximation ratio for graphs G with $\kappa(G) = 2$. In [13], the authors give an improvement for the lower bound of $\textsc{Opt}(G)$ based on the following definitions:

Definition 1. *A vertex v is a* beta-vertex *if there exist two vertices u_1, u_2 such that the graph induced from $V \setminus \{u_1, u_2\}$ has at least three connected components, one of which only contains v.*

Definition 2. *Two vertices v_1, v_2 are a* beta-pair *if there exist two vertices u_1, u_2 such that the graph induced from $V \setminus \{u_1, u_2\}$ has at least three connected components, one of which only contains v_1, v_2.*

Definition 3. *A graph G is* beta-free *if it has no* beta-structures, *i.e., neither beta-vertices nor beta-pairs.*

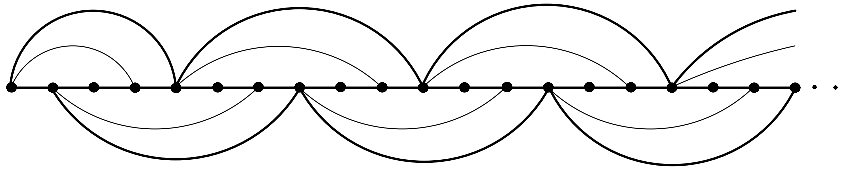

Fig. 1. A tight example for the lower bound given in [13] and in [7]. The edges of a MBSS are represented in bold

In [13], the authors only consider instances for which G is beta-free. They first show that the case of beta-pairs can be reduced to that of beta-vertices. Then, they consider the case of a beta-vertex v of G. Assume u_1, u_2 are the two vertices adjacent to v. Let $G' = G \setminus \{v\}$. Since the two edges incident to v are forced in any MBSS of G, an α-approximated solution for G can be obtained from an α-approximated solution for G' by adding the edges incident to v. In this case, it is easy to build a Hamiltonian path $\Pi_{G'}$ of G' from a Hamiltonian path Π_G of G: simply remove the beta-vertex v and add edge (u_1, u_2). If v is an endvertex of Π_G, then the graph induced from $V \setminus \{u_1, u_2\}$ has another beta-vertex u different from v. In this case remove u (as it cannot be an endvertex of Π_G) and add edge (u_1, u_2).

Now we describe a linear time algorithm that uses Π_G to remove all beta-vertices and all beta-pairs from G. For every vertex v_i, $1 < i \leq n-3$, if $\delta_G(v_{i+1}) = 2$ and f_{i-1} does not cover e_{i+2}, then v_{i+1} is a beta-vertex. Otherwise, if $v_{i+3} \neq v_n$, v_{i+1} is at most adjacent to v_i, v_{i+2}, v_{i+3}, while v_{i+2} is at most adjacent to v_{i+1}, v_{i+3}, v_i and f_{i-1} does not cover e_{i+3}, then v_{i+1}, v_{i+2} is a beta-pair.

In [13] it is shown that one can find a 4/3-approximated solution in $\mathcal{O}(n^2 m)$ time and linear space. The lower bound used there to estimate $\textsc{Opt}(G)$ (the same lower bound was used in [7]) is given by the number of beta-structures removed from G plus the size of a minimum spanning subgraph H with $\delta_H(v) \geq 2, \forall v \in V(H)$, of the graph computed from the beta-free reduction of G. However, whenever $G \in \mathcal{G}^n$ is a beta-free graph, this lower bound is at most $n+1$. Indeed, a subgraph H of G made up by a Hamiltonian path Π_G, plus two extra edges $(v_1, u), (u', v_n) \in E$, is a spanning subgraph of G such that $\delta_H(v) \geq 2, \forall v \in V$. In Figure 1 we show a beta-free graph $G \in \mathcal{G}^n$ whose MBSS has size asymptotically equal to $\frac{4}{3}n$.

4.2 Our Lower Bound

As seen before, the core of the MBSS problem is not just the achievement of a better algorithm, but also the careful estimate of the size of an optimal solution. The purpose of this section is to present a new lower bound.

Let $\alpha(G)$ denote the *independence number* of G, i.e., the size of a largest set of vertices U (called *maximum independent set*) of G that induces an *empty graph*, i.e., a graph with no edges. The following two lemmas are well-known.

Lemma 3. *[3] Every graph G with $n \geq 3$ and $\kappa(G) \geq \alpha(G)$ is Hamiltonian.* □

Lemma 4. *[6] For a biconnected graph G,* $\textsc{Opt}(G) \geq \max\{n, 2\alpha(G)\}$. $\square$

The *join* $G = G_1 + G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, is the *graph union* $G_1 \cup G_2$ (i.e., the graph $(V_1 \cup V_2, E_1 \cup E_2)$) together with all the edges joining V_1 and V_2.

Let $\mathcal{K}(|U|, \mathcal{K}_k)$ denote the *join* of the empty graph of $|U|$ vertices and the *complete graph* $\mathcal{K}_k$ (i.e., the graph having an edge between every pair of its k vertices). We can now prove the following:

Lemma 5. *For every integer $k \geq 2$, $\mathcal{K}(k+1, \mathcal{K}_k)$ is a maximum-size k-connected non-Hamiltonian graph with $2k+1$ vertices. Moreover, any other k-connected non-Hamiltonian graph G with $2k+1$ vertices is isomorphic to a subgraph of $\mathcal{K}(k+1, \mathcal{K}_k)$.*

Proof. From Lemma 3, a non-Hamiltonian k-connected graph must have at least $2k+1$ vertices. Since $\alpha(\mathcal{K}(k+1, \mathcal{K}_k)) = k+1$, from Lemma 4 it follows that $\mathcal{K}(k+1, \mathcal{K}_k))$ is not Hamiltonian (notice that, in every biconnected spanning subgraph of $\mathcal{K}(k+1, \mathcal{K}_k)$, at least 2 vertices of $V(\mathcal{K}(k+1, \mathcal{K}_k)) \setminus U$ must have degree 3). The insertion of an edge in $\mathcal{K}(k+1, \mathcal{K}_k)$ causes the independence number to decrease to k, and so from Lemma 3 the graph becomes Hamiltonian. Every non-Hamiltonian k-connected graph G with $2k+1$ vertices must have $\alpha(G) = k+1$, and so there exists a subgraph of $\mathcal{K}(k+1, \mathcal{K}_k)$ isomorphic to G. The claim follows. $\square$

Let $G = (V, E)$ be a biconnected graph. For every k-partition $V_1, V_2, \ldots, V_k$ of V, there exist two edges with one endvertex in V_i and the other in $V \setminus V_i$, $i = 1, \ldots, k$. Looking at the proof of Lemma 5, we deduce the following:

Lemma 6. *Let C be a k-separator and let $V_1, V_2, \ldots V_{k+1}$ be a $(k+1)$-partition of $V \setminus C$. If by shrinking each V_i into a node x_i we obtain a graph isomorphic to a subgraph of $\mathcal{K}(k+1, \mathcal{K}_k)$ (with $x_1, x_2, \ldots, x_{k+1}$ mapped to U), then G is not Hamiltonian, i.e.,* $\textsc{Opt}(G) \geq n+1$. *Moreover, every biconnected spanning subgraph of G has two vertices of C with degree greater than 2.*

Now it becomes trivial to prove the following:

Corollary 1 (Lower Bound). *Let G be a biconnected graph and let $C_1, C_2, \ldots, C_p$ be disjoint separators. If $C_j, j = 1, \ldots, p$, satisfies conditions of Lemma 6, then* $\textsc{Opt}(G) \geq n+p$. $\square$

Before ending this subsection we introduce a new topological structure that let us allow to design a better algorithm w.r.t. the one described in Theorem 2.

Definition 4. *A path edge $e_{i+3} = (v_{i+3}, v_{i+4})$ in Π_G,* generates a left hook *(see Figure 2) if the following conditions hold:*

(i) $\delta_G(v_{i+2}) = 2$;
(ii) $E(e_i) \cap E(e_{i+1}) = \{(v_j, v_{i+3}), (v_j, v_{i+4})\}$, *for a unique $j \leq i$ (v_j is the* tip *of the hook);*

(iii) *$E(e_{i+4})$ contains at least one edge with v_{i+1} as left endvertex, plus (possibly) edges with v_{i+4} as left endvertex; these are the only admissible covering edges for e_{i+4};*

(iv) *there exists a vertex $v_t \neq v_j$, with $t \leq i$.*

Definition 5. *A path edge (v_i, v_{i+1}) in $\Pi_G = \langle v_1, v_2, \ldots, v_n \rangle$* generates a right hook *if (u^{n-i}, u^{n-i+1}) generates a left hook in $\Pi_G^{-1} = \langle u^1 \equiv v_n, u^2 \equiv v_{n-1}, \ldots, u^{n-i+1} \equiv v_i, \ldots, u^n \equiv v_1 \rangle$.*

Definition 6. *Let $G \in \mathcal{G}^n$ and let Π_G a Hamiltonian path of G. G* has a hook *if there exists a path edges that generates either a left or a right hook.*

Looking at the definition of left hook, it is easy to devise an $\mathcal{O}(m)$ time algorithm that finds all the hooks of a graph G. In the next subsection we will prove that, if G has k hooks, then $\text{OPT}(G) \geq n+k$. Moreover, we will prove that we can remove all hooks from G by creating a new graph and a Hamiltonian path for it, and without altering the size of MBSS.

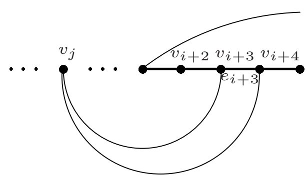

Fig. 2. A left hook generated by e_{i+3}

4.3 Graph Decomposition

Let G be a biconnected graph and let V_ℓ, C, V_r be a 2-separation. Henceforth, G_ℓ (resp., G_r) will denote the graph obtained from G by shrinking V_r (resp., V_ℓ) into one vertex x_ℓ (resp., x_r). Notice that G_ℓ, G_r are biconnected. Moreover, by $G_{\ell,r} = (V', E')$, where $V' = V_\ell \cup C \cup V_r$ and $E' = \{(u,v) \in E(G_\ell) \cup E(G_r) \mid u, v \in V'\}$, we denote the graph *built from G_ℓ and G_r w.r.t. x_ℓ, C, x_r*. Observe that $G = G_{\ell,r}$.

Lemma 7. *If H is a MBSS of G then H_ℓ, H_r are MBSS of G_ℓ, G_r, respectively. If H_ℓ, H_r are respectively MBSS of G_ℓ, G_r, then $H_{\ell,r}$ is a MBSS of G.*

Proof. Suppose H is a MBSS of G, but, w.l.o.g., H_ℓ is not a MBSS of G_ℓ. Let $\mathcal{H}_\ell^*$ be a biconnected spanning subgraph of G_ℓ and $|E(\mathcal{H}_\ell^*)| < |E(H_\ell)|$. It is easy to show that the graph H' built from $\mathcal{H}_\ell^*$ and H_r w.r.t. x_ℓ, C, x_r is biconnected. As x_ℓ, x_r have degree 2 in G_ℓ, G_r, respectively, it follows that

$$|E(H')| = |E(\mathcal{H}_\ell^*)| - 2 + |E(H_r)| - 2 < |E(H_\ell)| - 2 + |E(H_r)| - 2 = |E(H)|$$

and so H cannot be a MBSS of G, thus obtaining a contradiction.

Suppose now that H_ℓ, H_r are respectively MBSS of G_ℓ, G_r, but $H_{\ell,r}$ is not a MBSS of $G_{\ell,r}$. Let $\mathcal{H}$ be a biconnected spanning subgraph of G with $|E(\mathcal{H})| < |E(H)|$. Then, we have

$$|E(\mathcal{H})| = |E(\mathcal{H}_\ell)| - 2 + |E(\mathcal{H}_r)| - 2 < |E(H_\ell)| - 2 + |E(H_r)| - 2 = |E(H_{\ell,r})|,$$

from which we deduce

$$|E(\mathcal{H}_\ell)| + |E(\mathcal{H}_r)| < |E(H_\ell)| + |E(H_r)|.$$

Since all graphs used in the equation above are biconnected, we can claim that either H_ℓ or H_r is not a MBSS for the respective graph, thus obtaining a contradiction. □

Let Π_G be a Hamiltonian path of a biconnected graph G. We say that G is *decomposable* if there exists a path edge e_i with $\kappa(e_i) = 2$ and the associated 2-separation V_ℓ, C, V_r is such that $|V_\ell|, |V_r| \neq 1$. The pair G_ℓ, G_r is a *decomposition* of G. A non decomposable graph is called *prime*. A *prime decomposition* $G_1, G_2, \ldots, G_k$ of a decomposable graph G is the repeated decomposition of non-prime graphs of a decomposition of G, until G_j is prime, $j = 1, 2, \ldots, k$.

Remark 2. How much does it cost to decompose G w.r.t. the 2-separation V_ℓ, C, V_r? Let $\delta_G(v) = \delta_\ell(v) + \delta_r(v)$, where $v \in C$ and $\delta_\ell(v)$ (resp., $\delta_r(v)$) is the number of edges incident to v and to a vertex in V_ℓ (resp., V_r). Then decomposing G into G_ℓ, G_r costs $\mathcal{O}\left(\max_{v \in C}\{\min\{\delta_\ell(v), \delta_r(v)\}\}\right)$ time. Moreover $\delta_{G_\ell}(v) = \delta_\ell(v) + 1$ and $\delta_{G_r}(v) = \delta_r(v) + 1$, $\forall v \in C$.

Remark 3. If $G \in \mathcal{G}^n$ is prime, then $\kappa(e_i) \geq 3$, $4 \leq i \leq n-4$.

We can now prove the following lemma:

Lemma 8. *Let $G \in \mathcal{G}^n$ be a prime graph and let Π_G be a Hamiltonian path of G. If G has a left hook, then* $\text{OPT}(G) \geq n+1$.

Proof. First note that a prime graph cannot have more than one left hook. Indeed, if e_{i+3} generates a left hook, then $\{v_{i+1}, v_{i+4}\}$ is a 2-separator. Let v be the tip of the hook. It is not hard to see that every MBSS of G is such that either of v_{i+3}, v_{i+4} or v has degree at least 3, while v_{i+1} must always have degree at least 3. The claim follows. □

The previous lemma naturally extends to right hooks. Looking at its proof, it is not hard to see that if G has a left hook generated by e_{i+3} with tip in v, then we can remove edge (v, v_{i+4}) without altering the size of a MBSS. Note that this process makes v_{i+2} become a beta-vertex.

Remark 4. From now on, we will consider biconnected beta-free graphs having no hooks.

Looking at Remarks 1 and 3, there is an advantage if the instance of our problem is a prime graph in $\mathcal{G}^n$. We can try to work with prime graphs if Lemma 7 applies in a nice way. Let us assume that there exists a path edge e_i such that $\kappa(e_i) = 2$. W.l.o.g. assume that $\kappa_\ell(e_i) = 2$, and let $V_\ell, C = \{v_j, v_i\}, V_r$ be the associated 2-separation such that $v_1 \in V_\ell$. Let G_ℓ (resp., G_r) be the graph obtained by shrinking the set V_r (resp., V_ℓ) into one vertex x_ℓ (resp., x_r). The pair G_ℓ, G_r is a decomposition of G. It is easy to see that $\langle v_1, v_2, \ldots, v_j, \ldots, v_i, x_\ell \rangle$ is a Hamiltonian path of G_ℓ, while $\langle v_j, x_r, v_i, v_{i+1}, v_{i+2}, \ldots, v_n \rangle$ is a Hamiltonian path of G_r. Moreover, note that x_r has degree 2 in G_r. Thus, by removing x_r from G_r and by adding (v_j, v_i) to G_r we obtain a new graph G'_r and $\langle v_j, v_i, v_{i+1}, v_{i+2}, \ldots, v_n \rangle$ is a Hamiltonian path of G'_r. It is easy to see that if $\mathcal{H}'_r$ is a MBSS of G'_r chosen among all biconnected spanning subgraphs of G'_r having edge (v_j, v_i), then Lemma 7 still holds. Indeed, the graph obtained from $\mathcal{H}'_r$ minus edge (v_j, v_i), plus vertex x_r and edges $(v_j, x_r), (x_r, v_i)$, is a MBSS of G_r. Moreover, notice that:

Remark 5. If H_ℓ, H'_r are biconnected spanning subgraphs of G_ℓ, G'_r, respectively, then the graph H built from H_ℓ and H'_r w.r.t. x_ℓ, C, x_r, is such that $|E(H)| = |E(H_\ell)| + |E(H'_r)| - 3$.

The pair G_ℓ, G'_r is said to be a *simplified decomposition* of G. A *simplified prime decomposition* $G_1, G_2, \ldots, G_k$ of a decomposable graph G is the repeated simplified decomposition of non-prime graphs of a simplified decomposition of G, until G_j is prime, $j = 1, \ldots, k$.

4.4 The Final Algorithm

In this section we improve the algorithms described in Section 3. Before showing the final algorithm, we first describe a linear time and space algorithm for decomposing a graph G into a collection of prime graphs. We assume that a Hamiltonian path Π_G of G is given in input. The algorithm begins by making a copy of G (say G') and by assuming that G' is the initial partial decomposition of G. Then, it decomposes the computed partial decomposition of G until each graph of the decomposition is prime. The algorithm explores path edges in order from e_1 to e_n. At a given iteration, let e_i be the path edge the algorithm must examine. Let v_h be the right endvertex of f_i and let v_{j+1} be the second right endvertex (different from h) of some edge in $E(e_i) \cup \{e_i\}$ (say f'_i) having maximum index. If $v_{j+1} = v_{i+1}$, then $\kappa_r(e_i) = 2$ and $C = \{v_{i+1}, v_h\}$ is a 2-separator. If a graph of the computed partial decomposition of G is decomposable w.r.t. C, then decompose it and skip to $e_j = e_{i+1}$. Otherwise, $\kappa_r(e_t) \geq 3, t = i, \ldots, j-1$, and the algorithm can directly skip to e_j. Moreover, when examining e_j, it suffices to explore only cycle edges in $E(e_j)$ having the left endvertex with index greater than i. However, remember that f_i could be either f_j or f'_j. The same algorithm can be easily adapted to compute $\kappa_\ell(e_i)$. About the time and space complexity, notice that each cycle edge is explored a constant number of times. Moreover, as the number of vertices of a prime decomposition of G is $\mathcal{O}(n)$, then from Remark 2 it follows that the time and space complexity is $\mathcal{O}(m)$.

Lemma 9. *A biconnected beta-free graph $G \in \mathcal{G}^k$, with $k \leq 6$, is Hamiltonian.*

Proof. Since G is biconnected, from Lemma 3 it follows that graphs in $\mathcal{G}^3, \mathcal{G}^4$ are Hamiltonian. Let us assume that $G \in \mathcal{G}^6$ is not Hamiltonian and let $\mathcal{H}$ be a MBSS of G. In this case there are 2 vertices u_1, u_2 of $\mathcal{H}$ having degree 3. The graph obtained from $\mathcal{H}$ after the removal of u_1, u_2 has at least 3 connected components, two of these made by a single vertex v and v', respectively. Both v, v' cannot be adjacent to other vertices different from u_1, u_2 in G, otherwise G is Hamiltonian and $\mathcal{H}$ is not an optimal solution. So v is a beta-vertex in a beta-free graph. We have obtained a contradiction. The same technique can be used to prove that a beta-free graph $G \in \mathcal{G}^5$ is Hamiltonian. $\square$

Now we can implement a new more powerful function *Expand* such that

Lemma 10. *For a prime graph $G \in \mathcal{G}^n$ and a given Hamiltonian path Π_G of G, if $\text{OPT}(G) \geq n + k$, then the algorithm BSS computes in $\mathcal{O}(m)$ time and space a solution H such that $|E(H)| \leq n + k + \lceil \frac{n-6}{4} \rceil < \frac{5}{4}\text{OPT}(G)$.*

Proof. If $n \leq 6$ the claim follows from Lemma 9. Notice that as G is prime and has no beta-structures, it is not hard to see that we can cover the first (resp., last) 4 edges of Π_G by adding only one extra edge. So, looking at Remark 3 it is easy to prove the claim when $n \leq 9$. Hence, we can assume that $n \geq 10$. Moreover suppose we have added all edges of Π_G to $E(H)$. So we will remove all useless path edges and add cycle edges. We prove that we need to add one extra edge for the first (resp., last) 5 edges of Π_G, and one extra edge every 4 edges of $\Pi_G(v_6, v_n)$ (in an amortized sense). We prove this for the first path edges of Π_G, since for the last 5 path edges of Π_G the problem is symmetric. Let $\lambda = n$ be an initial lower bound for $|E(\mathcal{H})|$.

We first prove that we add only one extra edge for the first 5 path edges of Π_G. We must test sequentially the following (mutually exclusive) conditions.

(i) If f_1 covers 5 edges of $\Pi_G(v_1, v_n)$, then take it. Otherwise, if there are 2 semi-adjacent edges (one of which in $E(e_1)$) covering 5 edges of $\Pi_G(v_1, v_n)$, then take them and remove the middle one.

(ii) As G is prime, then $f_1 = (v_1, v_5)$ is the only cycle edge covering $E(e_1)$, and so it must be added to H. If there is an edge in $E(e_4)$ covering the first edges of $\Pi_G(v_5, v_n)$, then take it. Otherwise, if there are 2 semi-adjacent edges (one of which in $E(e_4)$) covering the first 5 edges of $\Pi_G(v_5, v_n)$, then take them and remove the middle one.

(iii) Add $f_1 = (v_1, v_5)$. In this case $E(e_5) \subseteq \{(v_2, v_j), (v_3, v_i) \mid 6 \leq i, j \leq 9\}$. Note that e_2 cannot generate a right hook (see Remark 4). Now the proof breaks into mutually exclusive cases that must be tested sequentially:

   (a) $(v_2, v_6) \in E(e_4)$ (resp., $(v_3, v_6) \in E(e_4)$). Add (v_2, v_6) (resp., (v_3, v_6)) plus one cycle edge in $E(e_5)$ with endvertex v_3 (resp., v_2), and remove e_2, e_5 (see Figure 3 (a)). In this case we add one extra edge for at least 6 path edges.

   (b) $(v_2, v_j), (v_3, v_{j+1}) \in E(e_4)$ (resp., $(v_3, v_j), (v_2, v_{j+1}) \in E(e_4)$). Add both edges and remove e_2, e_j (see Figure 3 (b)). In this case we add one extra edge for at least 7 path edges.

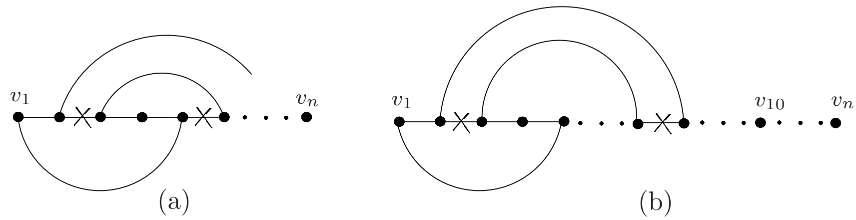

Fig. 3. Cases (a) and (b)

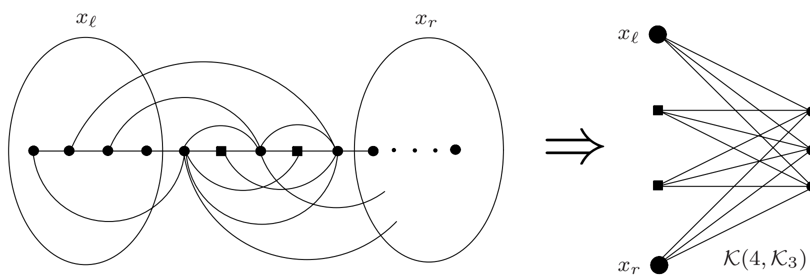

Fig. 4. Case (c)

(c) Otherwise, it must be $E(e_5) \subseteq \{(v_2, v_7), (v_2, v_9), (v_3, v_7), (v_3, v_9)\}$ and $E(e_9) \subseteq \{(v_5, v_j), (v_7, v_i), (v_9, v_k) \mid 10 \leq j, i, k\}$. Let G' be the graph obtained by shrinking the set $\{v_1, v_2, v_3, v_4\}$ (resp., $\{v_{10}, v_{11}, \ldots, v_n\}$) into one vertex x_ℓ (resp., x_r). Since $C = \{v_5, v_7, v_9\}$ is a separator and G is prime, it is not hard to see that G' is isomorphic to a subgraph of $\mathcal{K}(4, \mathcal{K}_3)$ (see Figure 4). In this case, add either $\{(v_2, v_7), (v_3, v_9)\}$ or $\{(v_3, v_7), (v_2, v_9)\}$, and remove e_2. Since from Corollary 1 we can increase λ by 1, it follows that in this case we add one extra edge for 9 path edges.

Since $\kappa(e_i) \geq 3$ for $4 \leq i \leq n-4$, from Lemma 1 we have that we are able to add one cycle edge every 4 path edges. Let $\lambda = n + k \leq \textsc{Opt}(G)$ be our final lower bound. The computed solution H has size

$$|E(H)| \leq n - 1 + \left(1 + k + \left\lceil \frac{n-1-5}{4} \right\rceil\right) = n + k + \left\lceil \frac{n-6}{4} \right\rceil < \frac{5}{4}(n+k).$$

Comparing the size of H with $\textsc{Opt}(G)$ the approximation ratio follows. About the time and space complexity, as each edge is explored a constant number of times (see also Theorem 2), then the time and space complexity is $\mathcal{O}(m)$. $\square$

We can finally prove the following:

Theorem 3. *The algorithm BSS returns a* $\min\left\{\frac{5}{4}, \frac{2\kappa(G)-1}{2(\kappa(G)-1)}\right\}$*-approximated solution for the MBSSHP-problem in* $\mathcal{O}(m)$ *time and space.*

Proof. If $\kappa(G) \geq 3$, the claim follows from Theorem 2. So, assume $\kappa(G) = 2$. Let $G_1, G_2, \ldots, G_i$ be the simplified prime decomposition of G and let Π_{G_j} be the given Hamiltonian path of G_j, $j = 1, 2, \ldots, i$. We first find a biconnected spanning subgraph H_j for each instance G_j, Π_{G_j}. Notice that we can build a biconnected spanning subgraph H of G from $H_1, H_2, \ldots, H_i$ in $\mathcal{O}(m)$ time and space. Let $p_j + 1$, with $j = 1, i$, be the length of path Π_{G_j}, and let $p_j + 1 + k_j$ be the lower bound for the size of a MBSS of G_j computed as in Lemma 10. Moreover, let $p_j + 2$, with $j = 2, \ldots, i-1$, be the length of path Π_{G_j}, and let $p_j + 2 + k_j$ be the lower bound for the size of a MBSS of G_j computed as in Lemma 10. Then, let $k = \sum_{j=1}^{i} k_j$.

From Lemma 10, from Remark 5 and since $n \geq 1 + \sum_{j=1}^{i} p_j$, we have that

$$|E(H)| = \sum_{j=1,i} \left(p_j + 2 + k_j + \left\lceil \frac{p_j - 4}{4} \right\rceil \right) + \sum_{j=2}^{i-1} \left(p_j + 3 + k_j + \left\lceil \frac{p_j - 3}{4} \right\rceil \right) - 3(i-1)$$

$$\leq k + n + \sum_{j=1}^{i} \left\lceil \frac{p_j - 3}{4} \right\rceil \leq k + n + \frac{1}{4} \sum_{j=1}^{i} p_j \leq \frac{5}{4}(n+k),$$

where the second inequality follows from the fact that, for every integer $1 \leq h \leq m$, being $m = hq + r$, where q and $r < h$ are positive integers, then:

$$\left\lceil \frac{m - (h-1)}{h} \right\rceil = \left\lceil \frac{hq + (r + 1 - h)}{h} \right\rceil \leq q + \left\lceil \frac{r + 1 - h}{h} \right\rceil \leq q \leq \frac{m}{h}.$$

As Lemma 7 and Corollary 1 imply that $\textsc{Opt}(G) \geq n + k$, then the approximation ratio follows. The time and space complexity follows from Lemma 10 and from the fact we can find a prime decomposition of G in linear time and space. □

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