Pursuit Strategies for Autonomous Agents

J.A. Marshall, Z. Lin, M.E. Broucke, and B.A. Francis

Edward S. Rogers Sr. Department of Electrical and Computer Engineering
University of Toronto, 10 King's College Road, Toronto, ON Canada M5S 3G4
E-mail: {marshall,linz,broucke,francis}@control.toronto.edu

In this paper, we examine the problem of how to achieve certain geometric formations among a group of identical mobile autonomous agents. Our particular view is that the subject of *cooperative control* involves: multiple autonomous agents, such as wheeled robot vehicles; a locally shared information structure; a control strategy that is somehow distributed among the agents; and a global task or desired goal for the group.

Our specific approach is motivated by the historical development of *cyclic pursuit* problems in mathematics. We begin by examining a system of identical linear agents in the plane, a setup that was first proposed circa 1732. The pursuit strategy is particularly simple in that our n agents are ordered such that agent i pursues agent $i+1$ modulo n. In the case of linear agents, we characterize the achievable formations and discuss whether collisions might occur between agents during reconfiguration. Next, we extend this notion to a system of wheeled vehicles, for which we use the unicycle as an example.

The intention here is to illustrate how these ideas might work through descriptive arguments and simulation examples, without employing a rigorous theorem-proof format. For a more mathematically precise view, the reader is referred to related works by the authors [13, 14, 15, 16].

1 Introduction

In 1987, Reynolds [19] developed his *distributed behavioural model*, which may be the most widely recognized artificial example of distributed and apparently self-organizing group behaviour. Reynolds' *boids* (after bird-oids) fly as a coherent flock, each employing only a local control strategy. In fact, it seems that much of the multiple agent robotics research has concentrated on the development of similar reactive or *behaviour-based* techniques. Not unlike Reynolds' boids, this behaviour-based approach is often to mimic biological systems, where *emergent* behaviours result from the local interaction of individual agents that appear to act autonomously.

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From an engineering standpoint, the problem of how to prescribe desired *global* behaviours for a system of mobile autonomous agents through the application of only simple and *local* interactions is of real and practical interest. However, the global outcomes of the above described behaviour-based systems are often difficult to predict analytically [17]. A recent trend is to study "nearest-neighbour" type strategies, whereby agents act based on local information about/from other agents in their neighbourhood. For example, Wang [23, 24] proposed a strategy where agents are instructed to move based on the motions of their nearest neighbours. Certain formation stability properties were then analyzed for the case when one agent is provided a reference trajectory and designated group leader. Early work by Sugihara and Suzuki [21] investigated a set of heuristic algorithms for the generation of geometric patterns in the plane (e.g., lines, circles, or polygons). In [5], a similar setup was presented, but collision avoidance and synchronous group motion (e.g., a matrix formation performing a right turn) were also discussed.

Some have argued that rigorous mathematical analysis of even the most simple interactions can be an almost impractical task. However, this has in fact been the goal of some more recent work [22]. Justh and Krishnaprasad [11, 10] have proposed unicycle steering laws for generating both rectilinear and circular formations in the plane. Their approach uses alignment and separation terms to determine the formation, which is based on the pose of all other vehicles in the group. Jadbabaie et al. [9] proved convergence results for a setup similar to that of Reynolds, guaranteeing that all agents eventually move in an identical fashion, under certain connectivity assumptions, despite the distributed nature of their coordination law.

As already mentioned, these types of patterns seem to appear in nature. One interesting example is Bruckstein's mathematical model [3] explaining the evolution of ant trails. Some have studied aggregate behaviour in *swarms* of organisms (e.g., birds, fish, mammals, and bacteria), where operational models are analyzed for the purpose of potential engineering application (e.g., see [7, 8] and references therein).

1.1 The History of Pursuit

As has already been mentioned, in this paper we discuss the possibilities for employing the notion of *cyclic pursuit* as a distributed coordination strategy for multiple agent systems. We take our inspiration from the so-called "bugs" problem, most prominent in the mathematics literature. The bugs problem refers to what is also variously known as the dogs, mice, ants, or beetles problem, and originally stems from the mathematics of *pursuit curves*, first studied by French scientist Pierre Bouguer (c. 1732). In 1877, Edouard Lucas asked, what trajectories would be generated if three dogs, initially placed at the vertices of an equilateral triangle, were to run one-after-the-other? Three years later, Henri Brocard determined that each dog's pursuit curve would be that of a logarithmic spiral and that the dogs would meet at what is now known as the *Brocard point* of a triangle. Bernhart [2] reports that Gordon Peterson extended this problem to n ordered bugs that start at the vertices of a regular n-polygon, describing his results for the square using four "cannibalistic spiders." If each bug pursues the next modulo n (i.e., *cyclic* pursuit) at fixed speed, the bugs will trace out logarithmic spirals and eventually meet at the polygon's centre. It was in 1969 that Watton and Kydon [25] provided their own solution to this regular n-bugs problem, also noting that the constant-speed assumption is not necessary. Interestingly, the bugs problem has been used for artistic design. For example, plotting the line-of-sight for each bug at regular intervals while tracing out the pursuit curves (see Fig. 1) generates intriguing geometric patterns [18].

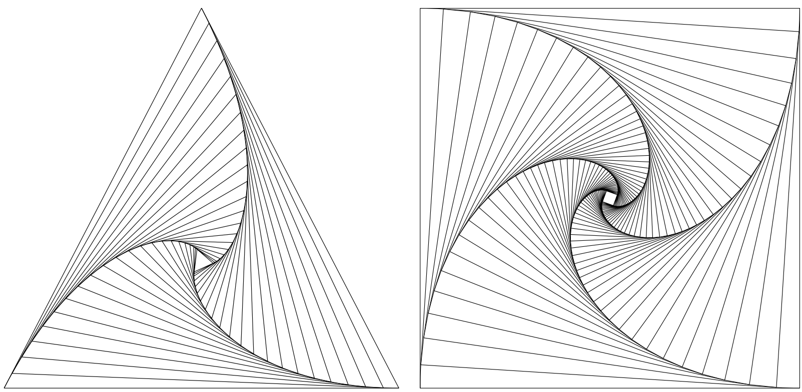

Fig. 1. Pursuit patterns for the regular 3- and 4-bugs problem [18].

Now, suppose our n bugs do not start at the vertices of a regular n-polygon. Klamkin and Newman [12] showed in 1971 that, for three bugs, so long as the bugs are not initially arranged so that they are collinear, they will meet at a common point and this meeting will be mutual. For n bugs, this problem was later examined by Behroozi and Gagnon [1], who proved that "a bug cannot capture a bug which is not capturing another bug [i.e., *mutual* capture], except by head-on collision." They used their result to show that, specifically for the 4-bugs problem, the terminal capture is indeed mutual. Quite recently, Richardson [20] resolved this issue for the general n-bugs problem, showing that "it is possible for bugs to capture their prey without all bugs simultaneously doing so, even for non-collinear initial positions." However, he proved that, if the initial conditions are chosen at random, then the probability of a non-mutual capture is zero.

Other variations on the traditional cyclic pursuit problem have also been investigated. For example, Bruckstein et al. [4] studied both continuous (ants) and discrete (crickets and frogs) pursuit problems, as well as both constant and varying speed scenarios.

1.2 Agents in Pursuit

Suppose we now imagine that each "bug" is instead an autonomous agent in the plane. In what follows, we generalize the n-bugs problem to autonomous agents and discuss its properties as a possible coordination framework for multiple agent systems. Thus, our primary motivation is to follow historical development and study the achievable formations for agents in cyclic pursuit. On the other hand, from a practical viewpoint, cyclic pursuit may become a feasible strategy for multiple agent systems since it is distributed (i.e., decentralized and there is no leader) and relatively simple in that each agent is required to sense information from only one other agent.

Rather than provide a rigorous mathematical treatment of the topic, which has been done to a certain degree in [13, 14, 15, 16], this contribution attempts to motivate the cited research through discussion and example simulations. The intention is to ensure the accessibility of this material to readers from diverse backgrounds (e.g., engineers and biologists alike).

2 Linear Agents in Cyclic Pursuit

We begin by revisiting the classical n-bugs problem, but formalized just slightly such that each bug is now an autonomous agent in the plane. Consider n ordered points in the plane $z_1, z_2, \ldots, z_n$, where each point z_i represents the position of a freely mobile agent, $i = 1, 2, \ldots, n$. Suppose each agent i pursues the next one $i + 1$ modulo n (i.e., $n + 1 = 1$). Henceforth, all agent indices should be evaluated modulo n. Thus, our chosen model is

$$\dot{z}_i = z_{i+1} - z_i, \text{ for } i = 1, 2, \ldots, n,$$

where $\dot{z}_i$ denotes the velocity of agent i. Therefore, the cyclic pursuit strategy is particularly simple. The velocity of each agent i, at each instant, is simply proportional to the distance to its prey, $i + 1$. Moreover, each agent is always heading directly towards its prey. What trajectories are generated?

One strategy for analyzing the collective behaviour of these n interconnected agents is to assemble their individual models into a single *aggregate* model, and then attempt to analyze the behavioural properties of this new n-dimensional model. By doing so, one can verify that for all possible initial agent locations, the centroid (i.e., the average position of the agents) of the points $z_1, z_2, \ldots, z_n$ is actually stationary over time and that every z_i, with $i = 1, 2, \ldots, n$ converges to this centroid [4, 13].

Fig. 2 demonstrates this behaviour for six mobile autonomous agents. In this case, the six agents were ordered and numbered in the clockwise direction. As an additional example, Fig. 3 shows the simulated trajectories for ten autonomous agents in the plane. In this case, the ten agents were numbered and ordered randomly. From this, one might note that, in general, the trajectories of individual agents may overlap.

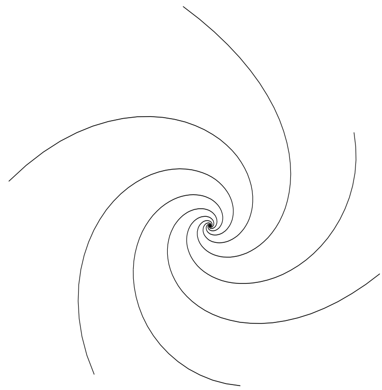

Fig. 2. The trajectories of six ordered agents converging to their centroid.

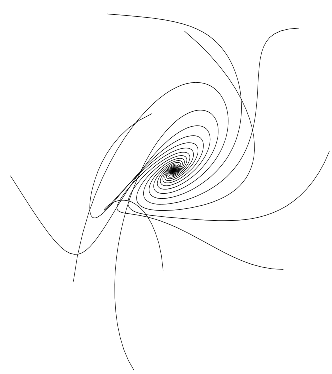

Fig. 3. The trajectories of ten randomly placed agents converging to their centroid.

Convergence to a common location is an instance of an *agreement problem*: the agents, which are not assumed initially to share a common reference frame, eventually come to agree on a common point, which can thereafter serve as an origin. Besides being of interest in its own right, if convergence to a point is achievable, then other formations are achievable by a simple modification, as we now show. Consider a strategy whereby each agent pursues a displacement of the next agent. In other words, the new model is

$$\dot{z}_i = (z_{i+1} + c_i) - z_i, \text{ for } i = 1, 2, \ldots, n.$$

Suppose that the centroid of the fixed points $c_1, c_2, \ldots, c_n$ is the origin. Then, by using what we know from above, it can be shown that the centroid of the points $z_1, z_2, \ldots, z_n$ remains stationary and that every z_i, with $i = 1, 2, \ldots, n$, converges to this point displaced by d_i, where $d_i = d_{i+1} + c_i$ [13].

Fig. 4 shows a simulation of this modified control strategy for six autonomous agents. The initial locations of the six agents were randomly generated and the chosen c_i's were

$$(-5, 5\sqrt{3}),\ (-5, 5\sqrt{3}),\ (10, 0),\ (10, 0),\ (-5, -5\sqrt{3}),\ (-5, -5\sqrt{3})\,.$$

In Fig. 5, the agents converge to a line formation, employing the same control strategy, but where the chosen c_i's were instead

$$(10, 0),\ (10, 0),\ (10, 0),\ (10, 0),\ (10, 0),\ (-50, 0)\,.$$

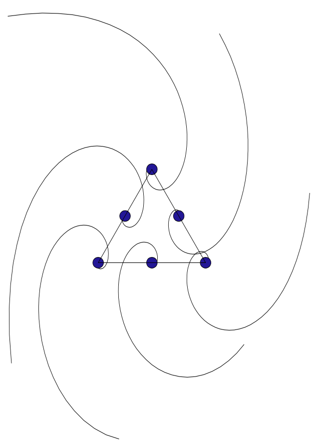

Fig. 4. Six autonomous agents achieving an equilateral triangle formation.

2.1 Collisions among Agents

We now turn our attention to the issue of collisions. A *collision* occurs if $z_i(t) = z_j(t)$ for some time $t \geq 0$ and $i \neq j$. Of course this setup is very ideal-

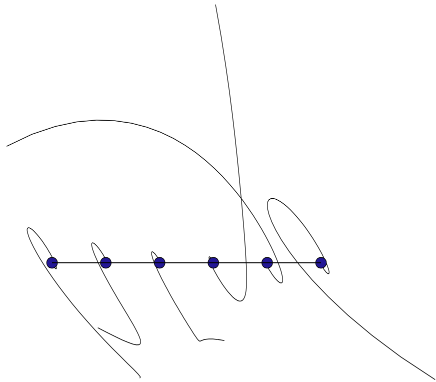

Fig. 5. Six autonomous agents achieving a line formation.

ized, since the agents are modelled as points and collisions must therefore be very rare events indeed. Our discussion is actually more general and concerns how the arrangement of agents evolves.

Let's start by defining some terminology. Consider n distinct points $z_1, z_2, \ldots, z_n$, not all collinear; see Fig. 6. Let z_0 denote their centroid and let r_i be the distance between z_i and the centroid. Let $\overrightarrow{z_i z_j}$ denote the directed line segment from a point z_i to another point z_j. Suppose α_i is the counterclockwise angle from line $\overrightarrow{z_0 z_i}$ to line $\overrightarrow{z_0 z_{i+1}}$. Then, we say that the n points are arranged in a *counterclockwise star formation* if $r_i > 0$, $\alpha_i > 0$, for all $i = 1, 2, \ldots, n$, and $\sum_i^n \alpha_i = 2\pi$. They are said to be arranged in a *clockwise star formation* if $r_i > 0$, $\alpha_i < 0$, for all $i = 1, 2, \ldots, n$, and $\sum_i^n \alpha_i = -2\pi$.

With this in mind, one can prove that if $n > 2$ distinct agents begin in a star formation, then these agents must, in fact, remain in a star formation [13]. In particular, they can *never* collide, which is a desirable result for groups of multiple autonomous agents. The simulation results of Fig. 2 illustrate the case when six agents start in a clockwise star formation.

3 Unicycles in Cyclic Pursuit

Suppose we now extend the above linear cyclic pursuit scenario to one in which each agent is a wheeled vehicle; for simplicity, we choose a unicycle. In this case, just as most automobiles cannot "turn on a dime," depending on the prescribed control signals, each unicycle i will require some finite time to steer itself towards its preassigned prey, $i + 1$. Once again we ask, what trajectories can be generated?

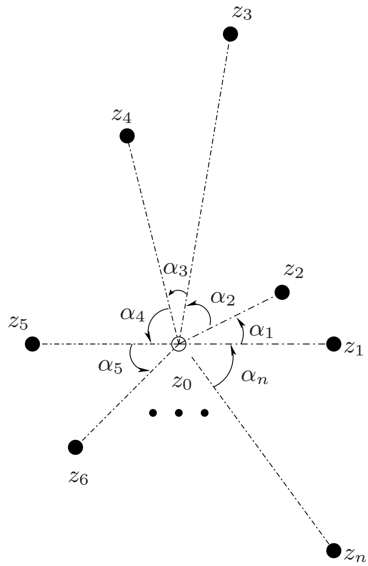

Fig. 6. A counterclockwise star formation.

In this section, let r_i denote the distance between vehicle i and $i+1$, and let α_i be the difference between the i-th vehicle's heading and the heading that would take it directly towards its prey, $i+1$ (see Fig. 7). In analogy with the linear pursuit scheme of the previous section, and as we proposed in [15], an intuitive pursuit law for our system of unicycles is to assign vehicle i's forward speed (call it v_i) in proportion to the distance error r_i, while assigning its angular speed (call it ω_i) in proportion to the heading error α_i. For now, let's fix each vehicle's forward speed, as in [14, 16], and study the possible steady-state formations for multiple agent systems of this sort.

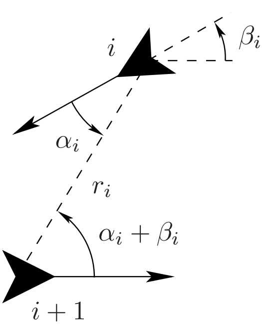

Fig. 7. New coordinates, with vehicle i in pursuit of $i+1$.

3.1 Constant Forward Speed

As suggested, we investigate the case when $v_i = s$ and $\omega_i = k\alpha_i$, where $k, s > 0$ are constants. Preliminary computer simulations suggest the possibility of achieving circular pursuit trajectories in the plane. Fig. 8 shows simulation results for a system of $n = 5$ vehicles, initially positioned at random, under our chosen control law with $k = 3$. Note that the vehicles converge to equally spaced motion around a circle of fixed radius.

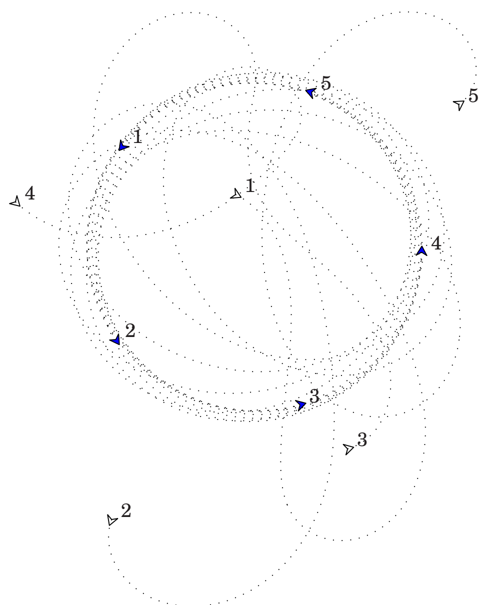

Fig. 8. Five vehicles subject to our chosen control law, with $k = 3$.

It can be shown [14, 16] that every possible steady-state formation for our system of n unicycles can be described as a *generalized regular polygon*, denoted $\{n/d\}$ (after [6]), where n is the number of vehicles and $0 < d < n$ is the polygon *density*. When $d = 1$, $\{n/1\}$ is an *ordinary* regular polygon. However, when $d > 1$ is coprime to n, $\{n/d\}$ is a *star* polygon since its sides intersect at certain extraneous points, which are not included among the vertices [6, pp. 93–94]. If n and d have a common factor $m > 1$, then $\{n/d\}$ has $\tilde{n} = n/m$ distinct vertices and $\tilde{n}$ edges traversed m times.

Fig. 9 illustrates some example possibilities for $\{n/d\}$ when $n = 9$. In the first instance, $\{9/1\}$ is an ordinary polygon. In the second, $\{9/2\}$ is a star polygon since 9 and 2 are coprime. In the third, the edges of $\{9/3\}$ traverse a $\{3/1\}$ polygon 3 times, because $m = 3$ is a common factor of both 9 and 3.

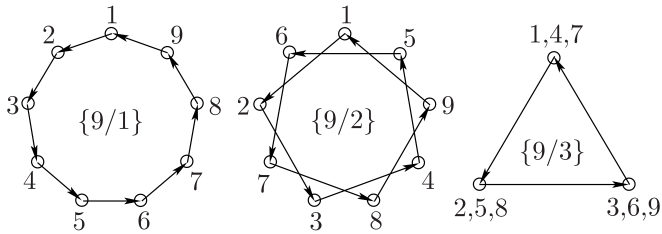

Fig. 9. Example generalized regular polygons $\{9/d\}$, $d \in \{1, 2, 3\}$.

Moreover, it can be proven that the size of the equilibrium polygon (i.e., the distance between vehicles) is dependent on the ratio $s : k$. Thus, for some fixed speed s, the gain k can be used to prescribe a given diameter of pursuit.

The question that remains is, which polygons are asymptotically stable and which are not? Surprisingly, the answer is not entirely intuitive and requires some interesting mathematical tools; the interested reader may refer to [14, 16], where we present details and the results of a full local stability analysis. Table 1 lists all possible equilibrium polygons and gives their stability.

Table 1. Possible equilibrium polygons, with stable polygons shown $\{n/d\}^*$ [14, 16].

$d = 1$	2	3	4	5	6	
$\{2/1\}^*$	$\{3/2\}$	$\{4/3\}$	$\{5/4\}$	$\{6/5\}$	$\{7/6\}$	$\cdots$
$\{3/1\}^*$	$\{4/2\}^*$	$\{5/3\}$	$\{6/4\}$	$\{7/5\}$	$\{8/6\}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\{7/1\}^*$	$\{8/2\}^*$	$\{9/3\}$	$\{10/4\}$	$\{11/5\}$	$\{12/6\}$	$\cdots$
$\{8/1\}^*$	$\{9/2\}^*$	$\{10/3\}^*$	$\{11/4\}$	$\{12/5\}$	$\{13/6\}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\{17/1\}^*$	$\{18/2\}^*$	$\{19/3\}^*$	$\{20/4\}$	$\{21/5\}$	$\{22/6\}$	$\cdots$
$\{18/1\}^*$	$\{19/2\}^*$	$\{20/3\}^*$	$\{21/4\}^*$	$\{22/5\}$	$\{23/6\}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\{49/1\}^*$	$\{50/2\}^*$	$\{51/3\}^*$	$\{52/4\}^*$	$\{53/5\}$	$\{54/6\}$	$\cdots$
$\{50/1\}^*$	$\{51/2\}^*$	$\{52/3\}^*$	$\{53/4\}^*$	$\{54/5\}^*$	$\{55/6\}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Fig. 10 and Fig. 11 provide simulation results for $n = 7$ vehicles, where in each case the forward speed $s = 1$ and gain $k = 4$. However, due to differing initial conditions, the vehicles of Fig. 10 form a $\{7/1\}$ polygon at equilibrium, whereas the vehicles of Fig. 11 converge to a $\{7/2\}$ equilibrium formation.

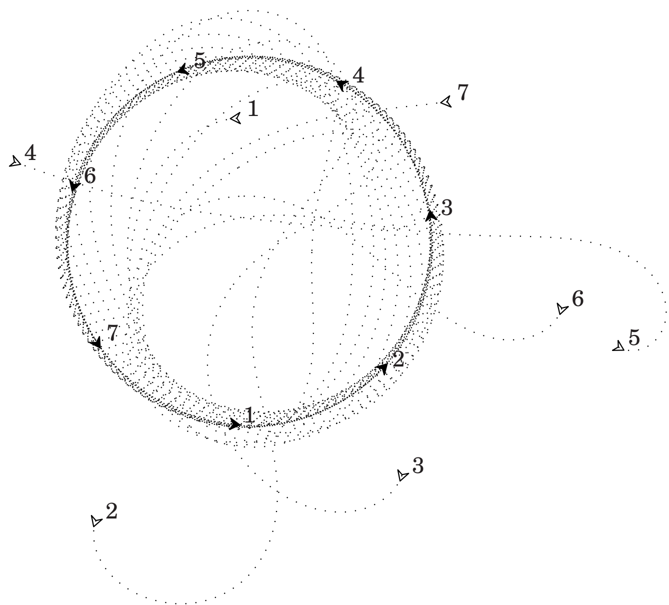

Fig. 10. Fixed forward speed pursuit generating a {7/1} formation.

3.2 Proportional Forward Speed

As was previously suggested, consider now an alternate possibility, where $v_i = k_r r_i$ and $\omega_i = k_\alpha \alpha_i$ with $k_r, k_\alpha > 0$ constant gains. In this case, preliminary simulations show that the system's behaviour is dependent on our choices for k_r and k_α. In fact, it can be verified that only the ratio k_r/k_α has significance [15, Theorem 1]. Therefore, without loss of generality, we may fix k_α and study the system's behaviour as k_r is varied.

Fig. 12, Fig. 13, and Fig. 14 show simulation results for a system of $n = 5$ vehicles, initially positioned at random, where $k_\alpha = 1$ is fixed and k_r is different in each case. In Fig. 12, $k_r = k^* := \frac{\pi}{10} \csc\left(\frac{\pi}{5}\right)$ and the vehicles converge to evenly spaced motion around a circle, similar to our previous results for unicycles with constant forward speed. However, in Fig. 13 with $k_r < k^*$ the vehicles converge to a common point. Finally, in Fig. 14 with $k_r > k^*$ the vehicles diverge, and they appear to do so equally spaced. Preliminary details regarding the results of this section may be found in [15].

4 Conclusion

Our particular approach to the problem of how to achieve certain geometric formations among a group of identical mobile autonomous agents has been inspired by the advancement of cyclic pursuit problems, which have appeared

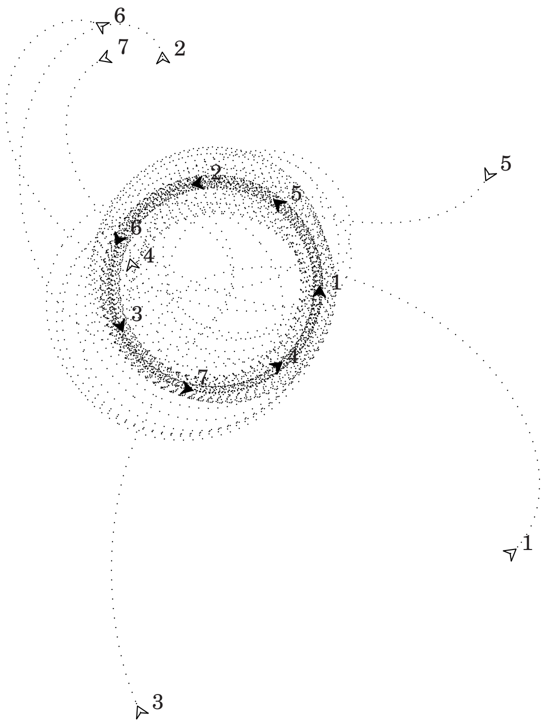

Fig. 11. Fixed forward speed pursuit generating a {7/2} formation.

throughout the last century, in the mathematical literature. Nevertheless, following historical developments, we have attempted to further these ideas for possible use in the coordination of multiple autonomous agents.

In order to reflect the multidisciplinary nature of cooperative control, we have presented ideas and results in a way that is accessible to researchers from varied backgrounds. For those interested in a more thorough mathematical perspective, we have provided the appropriate references.

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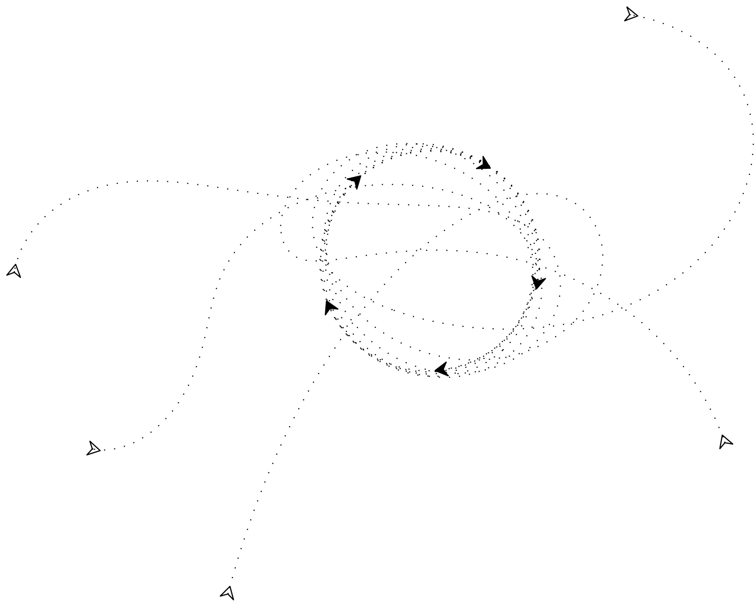

Fig. 12. Five vehicles, $k_\alpha = 1$, $k_r = k^*$.

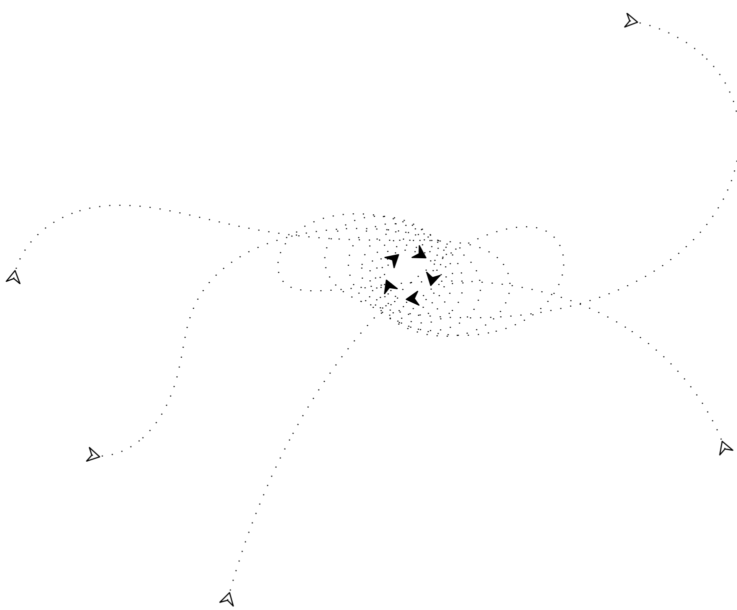

Fig. 13. Five vehicles, $k_\alpha = 1$, $k_r < k^*$.

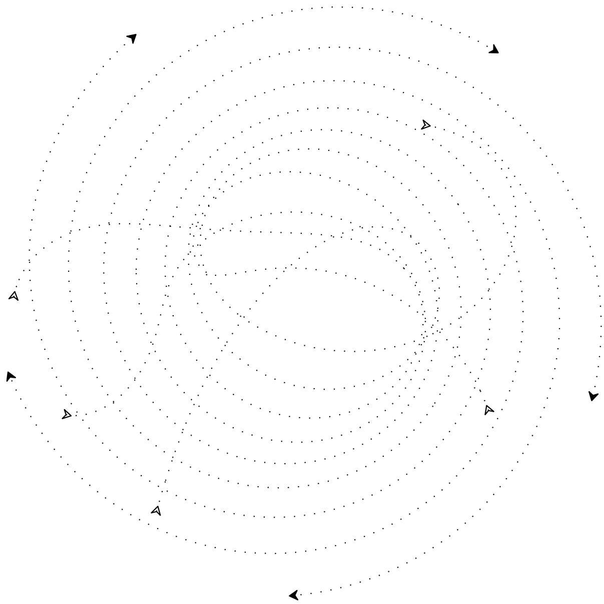

Fig. 14. Five vehicles, $k_\alpha = 1$, $k_r > k^*$.

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