The Multi-Agent Rendezvous Problem. An Extended Summary*

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Summary. This paper is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's *sensing region* is meant a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location. This paper describe two types of strategies for solving the problem. The first consists of agent strategies which are mutually synchronized in the sense that all depend on a common clock. The second consists of strategies which can be implemented independently of each other, without reference to a common clock.

Current interest in cooperative control has led to the development of a number of distributed control algorithms capable of causing large groups of mobile autonomous agents to perform useful tasks [1] – [14]. Of particular interest here are provably correct algorithms which solve what we shall refer to as the "multi-agent rendezvous problem." This problem, which was posed in [1], is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's *sensing region* is meant a closed disk of positive radius r centered at the agent's current position. The

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multi-agent rendezvous problem is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location.

In this paper, as in [1], we consider distributed strategies which guide each agent toward rendezvous by performing a sequence of "stop-and-go" maneuvers. A *stop-and-go maneuver* takes place within a time interval consisting of two consecutive sub-intervals. The first, called a *sensing period*, is an interval of fixed length during which the agent is stationary. The second, called a *maneuvering period*, is an interval of variable length during which the agent moves from its current position to its next 'way-point' and again come to rest. Successive way-points for each agent are chosen to be within r_M units of each other where r_M is a pre-specified positive distance no larger than r. It is assumed that there has been chosen for each agent i, a positive number τ_{M_i}, called a *maneuver time*, which is large enough so that the required maneuver for agent i from any one way-point to the next can be accomplished in at most τ_{M_i} seconds. Since our interest here is exclusively with devising of *high level* strategies which dictate when and where agents are to move, we will use point models for agents and shall not deal with how maneuvers are actually carried out or with how vehicle collisions are to be avoided.

In the sequel we describe two families of stop-and-go strategies. The first, which includes the specific strategies proposed in [1], consists of agent strategies which are mutually synchronized in the sense that all depend on a common clock. The second consists of strategies which can be implemented independently of each other, without reference to a common clock.

In the synchronous case §1, the *kth* maneuvering periods of all n agents begin at the same time $\bar{t}_k$. The *kth* way-point of each agent is a function of the positions of its " registered neighbors" at time $\bar{t}_k$. Agent i's registered neighbors at time $\bar{t}_k$ are all those other agents positioned within its sensing region at time $\bar{t}_k$. This notion of a neighbor induces a *symmetric* relation on the agent group since agent j is a registered neighbor of agent i at time $\bar{t}_k$ just in case agent i is a registered neighbor of agent j at the same time. Because of this it is possible to characterize neighbor relationships at time $\bar{t}_k$ with a simple graph whose vertices represent agents and whose edges represent existing neighbor relationships {§1.1}. Although the neighbor relation is symmetric, it is clearly not transitive. On the other hand if agent i is at the same position as neighbor j at time $\bar{t}_k$, then any registered neighbor of agent j at time $\bar{t}_k$ must certainly must be a registered neighbor of agent i at the same time. It is precisely because of this *weak transitivity* property that one can infer a *global* condition of the entire agent group from a *local* condition of one agent and its neighbors. In particular, if the graph characterizing neighbor relationships at time $\bar{t}_k$ is connected, and any one agent is at the same position as all of its neighbors, then the weak transitivity property guarantees at once that all n agents have rendezvoused at time $\bar{t}_k$.

One way to ensure that a neighbor graph is connected at time $\bar{t}_k$, assuming it is connected when the rendezvousing process begins, is to constrain each agent's way-points to be positioned in such a way so that no agent can lose any of its registered neighbors when it moves from one way-point to the next. This can be accomplished using a clever idea taken from [1]. An immediate consequence is that each agent's set of registered neighbors is non-decreasing and, because of this, ultimately converges to a fixed neighbor set for $\bar{t}_k$ sufficiently large.

A second local constraint is to require the way-point of each agent i at the beginning of its kth maneuvering period to lie in the "local" convex hull $\mathcal{H}_i(k)$ of agent i's own position at time $\bar{t}_k$ and the sensed positions of its registered neighbors at the same time. It is quite easy to prove that doing this causes the global convex hull $\mathcal{H}(k+1)$ of all n agent positions at time $\bar{t}_{k+1}$ to be contained in the corresponding global convex hull $\mathcal{H}(k)$ at time $\bar{t}_k$.

A third constraint is to stipulate that for each i, the only condition under which agent i's kth way-point can be positioned at a corner of $\mathcal{H}_i(k)$, is when $\mathcal{H}_i(k)$ is a single point. The global implication of doing this is that the diameter of $\mathcal{H}(k+1)$ must either be strictly smaller than the diameter of $\mathcal{H}(k)$ or every agent must be at the same position as all of its registered neighbors at time $\bar{t}_k$ – and this is true whether or not the graph characterizing neighbor relationships at time $\bar{t}_k$ is connected.

In §1.3, a more or less standard Lyapunov-style argument is used to prove that if the preceding constraints are adopted by all agents and if the graph characterizing initial neighbor positions is connected, then all n agents will eventually rendezvous at a single point. Not surprisingly, the Lyapunov function used for this purpose is the diameter of the global convex hull. However, although connectivity of the graph characterizing initial neighbor positions is sufficient for rendezvousing, it is not necessary. An example illustrating this is given in §1.2. The example deals with the situation when the initial neighbor graph consists of two connected components, with one "encircling" the other in a suitably defined sense.

The strategy described in §1 cannot be regarded as truly distributed because each agent's decisions must be synchronized to a common clock shared by all other agents in the group. In §2 we redefine the strategies so that a common clock is not required. To do this it is necessary to modify somewhat what is meant by a registered neighbor of agent i at time $\bar{t}_{ik}$, where $\bar{t}_{ik}$ is the time at which agent i's kth maneuvering period begins. Our definition is guided by considerations discussed above for the synchronous case. For example, the new definition is crafted to retain versions of the symmetry and weak transitivity properties of the neighbor relation inherent in the synchronous case. Doing this is challenging, because unlike the synchronous case, the time each agent registers its neighbors and its neighbor's positions is not synchronized with the times its neighbors do the same thing.

Exactly the same way-point update rules considered in the synchronous case are adopted for the asynchronous case. Thus the only functional differences between the two cases are the definitions of registered neighbors and registered neighbor positions. Of course in the asynchronous case, way-point updates are computed asynchronously, whereas in the synchronous case they are not.

Not surprisingly, the analysis of the asynchronous version of the problem is considerably more challenging than is the analysis of the synchronous version. For example, while it is more or less obvious in the synchronous case that agents retain their neighbors as the system evolves, proving that this is also true in the asynchronous case involves a number of steps {§2.1}.

Just as in the synchronous case, it is possible to characterize neighbor relationships with a graph. This is done in §2.1 by first merging together into a single ordered time set the distinct "event times" $\bar{t}_{ik}$, $i \in \{1, 2, \ldots, n\}$, $k \geq 1$ generated by all n agents. The elements of this set are then relabelled as $t_1, t_2, \cdots$ in such a way so that $t_j < t_{j+1}$, $j \in \{1, 2, \ldots\}$. With this notation, agent i's registered neighbors at its *kth* event time $\bar{t}_{ik}$ are its registered neighbors at time $t_{S_i(k)}$ where $S_i(k)$ denotes that value of s for which $t_s = \bar{t}_{ik}$. For each $i \in \{1, 2, \ldots, n\}$, the domain of definition of agent i's registered neighbors is then extended from the set $\{t_{S_i(k)} : k \geq 1\}$ to the set $\{t_s : s \geq S_i(1)\}$ by stipulating that for values of t_s which are between two successive event times of agent i, say between $\bar{t}_{ik}$ and $\bar{t}_{i(k+1)}$, agent i's registered neighbors are the same as its registered neighbors at time $\bar{t}_{ik}$. This means that registered neighbors of each agent are defined at each time $t_s \geq t_{\bar{s}}$ where $\bar{s} \stackrel{\Delta}{=} \max\{S_1(1), S_2(1), \ldots S_n(1)\}$. Because of this, it is possible to describe neighbor relationships with a directed graph with vertex set $\{1, 2, \ldots, n\}$ and directed edge set defined so that (i, j) is a directed edge from vertex i to vertex j just in case agent j is a registered neighbor of agent i at event time t_s. The main result of this paper {Corollary 3} is that if this graph is ever weakly connected[4], then rendezvous of all n agents will eventually occur.

The remainder of the paper is devoted to a proof of this claim. This is done in §3 by first "embedding" the n agent *asynchronous* processes in a suitably defined *synchronous* discrete-time, hybrid dynamical system $\mathbb{S}$ which captures the salient features of the agent system {§3.1}. Not captured, however, are the details of individual agent maneuvers. As a result, $\mathbb{S}$'s next state map $f(\cdot)$

[4] Recall that a directed graph is *weakly connected* if there is an undirected path between each pair of vertices [15].

is set-valued and $\mathbb{S}$ is consequently *nondeterministic*. Fixing the value of $\mathbb{S}$'s state recursively at one of the possible states specified by f results in a well-defined trajectory. For a given initial state x, the set of all such trajectories $\mathcal{T}(x)$ turns out to be countable. To claim that a signal can be generated by $\mathbb{S}$ initialized at x, is to claim that the signal can be generated along at least one of the trajectories in $\mathcal{T}(x)$. In §3.3 it is shown that with proper interpretation of $\mathbb{S}$, all of the way-point sequences generated by all n agents can be simultaneously generated in this manner after a certain finite amount of time. A Lyapunov-style argument is used in §3.4 to prove Theorem 3 which states that trajectories of $\mathbb{S}$ admitting this interpretation must converge if the underlying neighbor graph of the agents is ever weakly connected. The paper's main result mentioned above {Corollary 3} is an immediate consequence of this theorem.

1 Synchronous Case

In the synchronous case, the maneuvering times for all agents are all the same length positive value τ_M. Along any trajectory of the system to be considered, the real time axis can be partitioned into a sequence of time intervals $[0, t_1), [t_1, t_2), \ldots [t_{k-1}, t_k), \ldots$, each of length at least τ_M. Each interval consists of a sensing period followed by a maneuvering period of fixed length τ_M. All agents function in synchronization in the sense that all are at rest during sensing periods and all can maneuver only during maneuvering periods. In particular, all agents actions are synchronized to the time sequence $\bar{t}_1, \bar{t}_2, \ldots \bar{t}_k \ldots$ where $\bar{t}_k$ denotes the real time $t_k - \tau_M$ at which the kth maneuvering period begins. Agent i's *registered neighbors* at the beginning of its kth maneuvering period $[\bar{t}_k, t_k)$, are those agents, except for agent i, which are within agent i's sensing region at time $\bar{t}_k$. Note that this definition is a symmetric relation on the set of all agents; i.e., if agent i is a registered neighbor of agent j at the beginning of maneuvering period k, then agent j is a registered neighbor of agent i at the beginning of the same maneuvering period. As we shall see, special steps will have to be taken to achieve a similar property in the asynchronous case.

A pair of agents which are registered neighbors at the beginning of maneuvering period k, are said to satisfy the *pairwise motion constraint* during the period if the positions to which they move at time t_k are both within a closed disk of diameter r centered at the mean of their registered positions at time $\bar{t}_k$. The definition implies that any two agents which are registered neighbors at the beginning of maneuvering period k will be registered neighbors at the beginning of maneuvering period $k+1$ if they satisfy the pairwise motion constraint during the kth. We are interested in strategies possessing this property and accordingly make the following assumption.

Cooperation Assumption: During each maneuvering period k, each pair of agents which are registered neighbors at the beginning of the period, restrict their motions to satisfy the pairwise motion constraint.

Agent i's *kth way-point* is the point to which agent i is to move to at time t_k. Thus if $x_i(t)$ denotes the position of agent i at time t represented in a world coordinate system, then $x_i(t_k)$ and agent i's *kth* way-point are one and the same. The rule which determines each such way-point is a function depending only on the number and relative positions of agent i's registered neighbors. In particular, if agent i has m_i registered neighbors at time $\bar{t}_k$, positioned relative to agent i at points

$$z_j \stackrel{\Delta}{=} x_{i_j}(\bar{t}_k) - x_i(\bar{t}_k), \ j \in \{1, 2, \ldots, m_i\} \tag{1}$$

then agent i's *kth* way-point is

$$x_i(t_{k-1}) + u_{m_i}(z_1, z_2, \ldots, z_{m_i}) \tag{2}$$

where $u_0 = 0$, $u_m : \mathbb{D}^m \to \mathbb{D}_M, \ m \in \{1, \ldots, n-1\}$, and $\mathbb{D}$ and $\mathbb{D}_M$ are the closed disks of radii r and r_M respectively, centered at the origin in $\mathbb{R}^2$. In other words, if agent i has no registered neighbors at time $\bar{t}_k$, {i.e., $m_i = 0$}, it does not move during the *kth* maneuvering period. On the other hand, if agent i has $m_i > 0$ neighbors at time $\bar{t}_k$ with relative positions $z_1, z_2, \ldots, z_{m_i}$, then agent i moves to the position $x_i(t_{k-1}) + u_{m_i}(z_1, z_2, \ldots, z_{m_i})$ at time t_k. Thus

$$\begin{aligned} x_i(t_k) = x_i(t_{k-1}) + u_{m_i(\bar{t}_k)}(&x_{i_1}(\bar{t}_k) - x_i(\bar{t}_k), \ x_{i_2}(\bar{t}_k) - x_i(\bar{t}_k), \\ &\ldots, \ x_{i_{m_i(\bar{t}_k)}}(\bar{t}_k) - x_i(\bar{t}_k)) \end{aligned} \tag{3}$$

In the sequel we will explain how the u_m are defined. At the very least we will require each to be a continuous function.

1.1 Definition of u_m

We've already defined $u_0 = 0$. To define u_m for $m > 0$ it is necessary to take into account the pairwise motion constraint. Toward this end, for each $z \in \mathbb{D}$, let $\mathcal{C}(z)$ denote the closed disk of diameter r centered at the point $\frac{1}{2}z$. More generally, for each $\{z_1, z_2, \ldots, z_m\} \in \mathbb{D}^m$, let

$$\mathcal{C}(z_1, z_2, \ldots, z_m) = \bigcap_{j=1}^{m} \mathcal{C}(z_j) \tag{4}$$

Note that 0 is in each $\mathcal{C}(z_i)$ and moreover that each such $\mathcal{C}(z_i)$ is closed and strictly convex. Consequently $\mathcal{C}(z_1, z_2, \ldots, z_m)$ is either the singleton $\{0\}$ or a strictly convex, closed set containing 0. We can now define u_m to be any continuous function on $\mathbb{D}^m$ satisfying

$$u_m(z_1, z_2, \ldots, z_m) \in \mathbb{D}_M \cap \mathcal{C}(z_1, z_2, \ldots, z_m) \cap \langle 0, z_1, z_2, \ldots, z_m \rangle, \\ \forall \{z_1, z_2, \ldots, z_m\} \in \mathbb{D}^m \tag{5}$$

where $\langle 0, z_1, z_2, \ldots, z_m \rangle$ is the convex hull of the points $0, z_1, z_2, \ldots, z_m$. The u_m are further required to have the property that

$$u_m(z_1, z_2, \ldots, z_m) \neq \text{a corner}^5 \text{ of } \langle 0, z_1, z_2, \ldots, z_m \rangle \tag{6}$$

unless $z_1 = z_2 = \cdots = z_m = 0$. In other words, u_m is required to be (i) a continuous function on $\mathbb{D}^m$ which maps each $\{z_1, z_2, \ldots, z_m\} \in \mathbb{D}^m$ into $\mathbb{D}_M \cap \mathcal{C}(z_1, z_2, \ldots, z_m) \cap \langle 0, z_1, z_2, \ldots, z_m \rangle$ and (ii) a function with the property that $u_m(z_1, z_2, \ldots, z_m)$ is not a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ unless $z_1 = z_2 = \cdots = z_m = 0$. Examples of functions satisfying these conditions will be given in the sequel.

One way to go about defining specific u_m which are continuous and which satisfy (5) and (6), is by first defining what we shall refer to as a "target point." By a *target point* is meant a continuous function $\tau : \mathbb{D}^m \to \langle 0, z_1, z_2, \ldots, z_m \rangle$ defined in such a way that for each $\{z_1, z_2, \ldots, z_m\} \in \mathbb{D}^m$ for which 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \ldots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has positive length. For should it be possible to define such a τ, one could satisfy (5) and (6) as well as the continuity requirement with a control of the form

$$u_m = g(z_1, z_2, \ldots, z_m)\tau(z_1, z_2, \ldots, z_m)$$

where $g : \mathbb{D}^m \to \mathbb{R}$ is any continuous, positive definite function satisfying

$$g < \max_{(0,1]} \{\mu : \mu\tau \in \mathbb{D}_M \bigcap \mathcal{C}(z_1, z_2, \ldots, z_m)\}$$

Note that $g\tau \in \langle 0, z_1, z_2, \ldots, z_m \rangle$, $\forall g \in [0, 1]$ because $0 \in \langle 0, z_1, z_2, \ldots, z_m \rangle$. The role of g is therefore to scale down the magnitude of τ enough to insure that $g\tau$ is in the constraint set $\mathbb{D}_M \bigcap \mathcal{C}(z_1, z_2, \ldots, z_m)$.

It might be thought that one could choose for τ, the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ or perhaps the average of the z_i and 0, namely

$$\tau \triangleq \frac{1}{m+1} \sum_{i=1}^{m} z_i,$$

Both candidate definitions satisfy the requirement that $\tau(z_1, z_2, \ldots, z_m)$ must be a point in $\langle 0, z_1, z_2, \ldots, z_m \rangle$. Unfortunately, simple examples show that

[5] Recall that a point x in a polytope $\mathbb{P}$ in $\mathbb{R}^m$ is a *corner* if the only points y and z in $\mathbb{P}$ for which x is a convex combination are $y = z = x$.

the centroid definition does not necessarily yield a function which satisfies the continuity requirement while the averaging definition may lead to a function which fails to satisfy the requirement that when 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \dots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \dots, z_m)$ has positive length. For example, the centroid of the convex hull of the points $(0, 0)$, $z_1 = (0, 1)$ and $z_2 = (p, 1)$ is at $(\frac{p}{3}, \frac{2}{3})$ for $p > 0$ and at $(0, \frac{1}{2})$ for $p = 0$ so the centroid is discontinuous at $p = 0$. As a counterexample to the use of coordinate averaging to define a target point, note that the average of the four points located at $(0, 0)$, $z_1 = (-r, 0)$, $z_2 = (\frac{2r}{3}, \frac{r}{2})$, and $z_3 = (\frac{r}{3}, \frac{r}{2})$ is at $(0, \frac{r}{4})$ while the constraint set $\mathcal{C}(z_1, z_2, z_3)$ determined by these points must be contained in the constraint disk $\mathcal{C}(z_1)$. Since the line $\mathcal{L}$ from $(0, 0)$ to $(0, \frac{r}{4})$ is tangent to this disk at the origin, the intersection of $\mathcal{L}$ with $\mathcal{C}(z_1, z_2, z_3)$ is just the point $(0, 0)$ and consequently not a line segment of positive length.

In the sequel we shall approach the problem of defining of τ in a slightly different way. We begin by stating the following proposition which provides a simple condition on $\tau(\cdot)$, which if satisfied, automatically implies satisfaction of the requirement that when 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \dots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \dots, z_m)$ has positive length.

Proposition 1. *Let $z_1, z_2, \dots, z_m$ be a set of $m > 0$ points in $\mathbb{D}$ which are not all 0. If 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$ and z is any non-zero point in $\mathbb{D}$ within r units of each point in $\{z_1, z_2, \dots, z_m\}$, then the segment of the line from 0 to z which lies in $\mathcal{C}(z_1, z_2, \dots, z_m)$, has positive length.*

Proposition 1 suggest the following approach for defining a target point. First, for each $z \in \mathbb{D}$, let $\mathcal{D}(z)$ denote a closed disk of radius r centered at z. More generally for any set of $m > 0$ point $z_1, z_2, \dots, z_m$ in $\mathbb{D}$, write

$$\mathcal{D}(z_1, z_2, \dots, z_m) = \bigcap_{i=1}^{m} \mathcal{D}(z_i)$$

By construction, each point in $\mathcal{D}(z_1, z_2, \dots, z_m)$ is within r units of each point in $\{z_1, z_2, \dots, z_m\}$ and conversely. Thus $0 \in \mathcal{D}(z_1, z_2, \dots, z_m)$ because $z_i \in \mathbb{D}$, $i \in \{1, 2, \dots, m\}$.

Second, note that if $z_1, z_2, \dots, z_m$ is any set of $m > 0$ points in $\mathbb{D}$ which are not all zero and for which 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$, then by Proposition 1 the segment of the line from 0 to any non-zero point $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \dots, z_m)$ which lies in $\mathcal{C}(z_1, z_2, \dots, z_m)$, must have positive length. It follows that any continuous function $\tau : \mathbb{D}^m \to \langle 0, z_1, z_2, \dots, z_m \rangle$ which satisfies

$$\tau(z_1, z_2, \dots z_m) \in \mathbb{D} \bigcap \mathcal{D}(z_1, z_2, \dots, z_m) \bigcap \langle 0, z_1, z_2, \dots, z_m \rangle$$

and which is non-zero whenever 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and $z_1, z_2, \ldots, z_m$ are not all zero, fulfills all the conditions required to be a target point. In the sequel we will show that there are at least two different ways to so define τ.

The centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$

In order for the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ to be a target point, it must depend continuously on the z_i and, in addition, must have the property that it is non-zero for any set of m points in $\mathbb{D}$ which are not all zero and for which 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$. These properties are guaranteed by the following two propositions.

Proposition 2. *Let $z_1, z_2, \ldots, z_m$ be a set of $m > 0$ points in $\mathbb{D}$ which are not all 0. Then the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is in $\langle 0, z_1, z_2, \ldots, z_m \rangle$. If, in addition, 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, then $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ has a non-empty interior and the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ cannot be at 0.*

Proposition 3. *The function which assigns to each set of $m > 0$ points $z_1, z_2, \ldots, z_m$ in $\mathbb{D}$, the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$, is continuous.*

Examination of the proof of Proposition 3, given in the full-lenght version of this paper, reveals that the continuity of the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ depends crucially on the fact that the centroid is at 0 whenever the area of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is zero. This property is not shared by the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and it is for this reason that the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ is not a continuous function of the z_i.

It turns out that Propositions 2 and 3 both hold if the set $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is replaced throughout by the constraint set $\mathbb{D} \cap \mathcal{C}(z_1, z_2, \ldots, z_m)$. This can be shown using essentially the same proofs of the propositions as those given in the appendix. What this means then is that the centroid of $\mathbb{D} \cap \mathcal{C}(z_1, z_2, \ldots, z_m)$ is also a valid target point.

The center of the smallest circle containing $\langle 0, z_1, z_2, \ldots, z_m \rangle$

It is also possible to define τ to be the center of the smallest circle containing $\langle 0, z_1, z_2, \ldots, z_m \rangle$. To understand why this is so, let us note first that for any set of points $z_i \in \mathbb{D}$, $i \in \{1, 2, \ldots, m\}$, the set of points $\mathcal{Q} \triangleq \{0, z_1, \ldots, z_m\}$ is contained in a circle of radius r centered at 0. It follows that the center of this circle is at most r units from every point in $\mathcal{Q}$. This suggests that one might choose for $\tau(z_1, z_2, \ldots, z_m)$ the center $\tau_C(z_1, z_2, \ldots z_m)$ of the smallest circle containing $\mathcal{Q}$ or equivalently $\langle 0, z_1, z_2, \ldots, z_m \rangle$, since $\tau_C(z_1, z_2, \ldots, z_m)$ would have to be within r units of every point in $\mathcal{Q}$. It is known that there is such a smallest circle [16] and that if the z_i are not all zero, $\tau_C(z_1, z_2, \ldots z_m)$ is either the midpoint between two of the points in $\mathcal{Q}$ or a point within the interior of a triangle formed from at least one set of three points in $\mathcal{Q}$ [1]. In either case it is clear that $\tau_C(z_1, z_2, \ldots z_m) \in \langle 0, z_1, z_2, \ldots, z_m \rangle$ and, if the z_i are not all zero and 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, that $\tau_C(z_1, z_2, \ldots z_m)$ is nonzero as well. Furthermore it can be shown that $\tau_C(z_1, z_2, \ldots z_m)$ depends continuously on the z_i [17]. In other words, $\tau_C(z_1, z_2, \ldots z_m)$ satisfies all the conditions required to be a target point. This elegant choice for τ is the one proposed in [1].

1.2 Main Results

Define $t_0 = 0$. Note that because agents don't move during sensing periods, for $k \geq 1$ the position of each agent at time t_{k-1} is the same as its position at time $\bar{t}_k$. Thus (3) can be re-written as

$$x_i(t_k) = x_i(t_{k-1}) + u_{m_i(t_{k-1})}(x_{i_1}(t_{k-1}) - x_i(t_{k-1}),\ x_{i_2}(t_{k-1}) - x_i(t_{k-1}),\\ \ldots,\ x_{i_{m_i(t_{k-1})}}(t_{k-1}) - x_i(t_{k-1})) \tag{7}$$

where $m_i(t_{k-1}) \stackrel{\Delta}{=} m_i(\bar{t}_k)$. Because of this, the system just defined admits the model of a nonlinear discrete-time system with state $x(t_k) =$ column $\{x_1(t_k), x_2(t_k), \ldots x_n(t_k)\}$ evolving on the time set $t_0, t_1, \ldots t_k, \ldots$. Analysis of this system depends on the relationships between neighbors and how they evolve with time. These relationships can be conveniently described by a simple, undirected graph with vertex set $\{1, 2, \ldots, n\}$ which is defined so that (i, j) is one of the graph's edges just in case agents i and j are registered neighbors at the beginning of maneuvering period k. Since these relationships can change from one maneuvering period to the next, so can the graph which describes them. In the sequel we use the symbol $\mathcal{P}$ to denote a suitably defined set, indexing the class of all simple graphs $\mathbb{G}_p$ on n vertices. Let us partially order the set $\{\mathbb{G}_p : p \in \mathcal{P}\}$ by agreeing to say that $\mathbb{G}_p$ is contained in $\mathbb{G}_q$ if the edge set of $\mathbb{G}_p$ is a subset on the edge set of $\mathbb{G}_q$. It is natural then to define the *union* of a collection of such graphs, $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$, to be the simple graph $\mathbb{G}$ with vertex set $\{1, 2, \ldots, n\}$ and edge set equaling the union of the edge sets of all of the graphs in the collection.

Let $\sigma(k)$ denote the index of the graph in $\{\mathbb{G}_p : p \in \mathcal{P}\}$ which describes the relationship between registered neighbors at the beginning of maneuvering period k. Because of the cooperation assumption, we know that each agent keeps all of its registered neighbors as the system evolves. What this means is the sequence of graphs $\mathbb{G}_{\sigma(1)}, \mathbb{G}_{\sigma(2)}, \ldots, \mathbb{G}_{\sigma(k)}, \ldots$ forms the ascending chain

$$\mathbb{G}_{\sigma(1)} \subset \mathbb{G}_{\sigma(2)} \subset \cdots \mathbb{G}_{\sigma(k)} \cdots \tag{8}$$

Because $\{G_p : p \in \mathcal{P}\}$ is a finite set, the chain must converge to the graph

$$\mathbb{G} \stackrel{\Delta}{=} \bigcup_{k=1}^{\infty} \mathbb{G}_{\sigma(k)} \tag{9}$$

in a finite number of steps. Since the sequence of graphs stops changing in a finite number of steps, rendezvousing at a single point can only occur if $\mathbb{G}$ is a complete graph. There is however, no a priori guarantee that along a particular trajectory, $\mathbb{G}$ will turn out to be complete. On the other hand, it is clear that $\mathbb{G}$ will always be at least connected if the initial graph $\mathbb{G}_{\sigma(1)}$ in the ascending chain is. It turns out that connectivity of $\mathbb{G}_{\sigma(1)}$ implies not only that $\mathbb{G}$ is connected but also that the types of distributed control strategies just described actually cause all agents to rendezvous at a single point.

Theorem 1. *Let $u_0 = 0 \in \mathbb{D}_M$ and for each $m \in \{1, 2, \ldots, n-1\}$, let $u_m : \mathbb{D}^m \to \mathbb{D}_M$ be any continuous function satisfying (5) and (6). For each set of initial agent positions $x_1(0), x_2(0), \ldots, x_n(0)$, each agent's position $x_i(t)$ converges to a unique point $p_i \in \mathbb{R}^2$ such that for each $i, j \in \{1, 2, \ldots, n\}$, either $p_i = p_j$ or $||p_i - p_j|| > r$. Moreover, if agents i and j are registered neighbors at any time t, then $p_i = p_j$.*

Theorem 1 states that the strategies under consideration cause all agents positions to converge to points in the plane with the property that each two such points are either equal to each other, or separated by a distance greater than r units. The theorem further states that if two agents are ever registered neighbors of each other, then their positions converge to the same point. We are led to the following corollary.

Corollary 1. *If the graph characterizing registered neighbors at the beginning of period* 1 *is connected, then the positions of all n agents converge to a common point in the plane.*

It is quite straight forward to extend these results to the leader-follower case when the rendezvous point is specified at the outset. This can be accomplished by simply fixing one additional agent {i.e., a virtual agent} at the desired rendezvous point and letting the remaining n agents maneuver just as before. With initial graph connectivity of all $n+1$ agent positions, convergence to the position of the virtual agent is then assured.

A more interesting case occurs when two virtual agents are fixed at distinct points in the plane. In this case it can be shown that with initial connectivity of the $n + 2$ - agent graph, all n agents will eventually move to positions on the line connecting the two virtual agents and will distribute themselves in a predictable manner depending only the number of agents, r and the distance between the two fixed, virtual agents. This behavior will be explored in greater depth in another paper dealing with forming formations using distributed control.

Trapping

While the graph connectivity hypothesis of Corollary 1 is sufficient for rendezvousing, it is not necessary. For example, suppose that the $\mathbb{G}_{\sigma(1)}$ has a connected component $\mathbb{G}_C$ which contains a simple closed cycle whose vertices are $i_1, i_2, \ldots, i_m$. Then in the plane, the geometric form obtained by connecting by a straight line, the initial position of each agent $i_j \in \{i_1, i_2, \ldots, i_m\}$ with its registered neighbors with labels in $\{i_1, i_2, \ldots, i_m\}$, will be a simple, closed, polygon $\mathbb{P}$. It turns out that if the initial positions of all agents whose labels are not in the vertex set of $\mathbb{G}_C$, are within $\mathbb{P}$, then rendezvous will necessarily occur. While this conclusion might appear to be an obvious consequence of the established property that agents $i_j \in \{i_1, i_2, \ldots, i_m\}$ eventually rendezvous at a point, actually proving that this is so is not so straight forward. There are two reasons for this. First there is no guarantee that the polygon $\mathbb{P}(k)$ formed by the positions at time t_k of agents $i_j \in \{i_1, i_2, \ldots, i_m\}$ will remain simple as the system evolves, even if it is initially; thus just what it means for an agent to be "inside" of $\mathbb{P}(k)$ requires a more sophisticated notion of interior than the obvious one for a simple closed curve in the plane and this in turn complicates the analysis. Second, it is quite possible that an agent initially positioned inside of $\mathbb{P}(0)$, will be outside of $\mathbb{P}(k)$ for some $k > 0$. In the full length version of this paper we explain how to overcome both of these difficulties.

The closed curves of interest here are of a specific type determined by finite point sets in $\mathbb{R}^2$. In particular, let us note that any ordered set of $m > 0$ points $\{y_1, y_2, \ldots, y_m\}$ in $\mathbb{R}^2$ uniquely determines a continuous, piecewise-linear, closed curve $c : [0, m] \to \mathbb{R}^2$ defined so that

$$c(t) = (t + 1 - i)y_{i+1} + (i - t)y_i, \qquad i - 1 \leq t \leq i, \qquad i \in \{1, 2, \ldots, m\}$$

where $y_{m+1} = y_1$. An ordered set $\{y_1, y_2, \ldots, y_m\}$ of three or more such points is called a *cycle* if $||y_{i+1} - y_i|| \leq r,\ i \in \{1, 2, \ldots, m\}$; in the sequel we denote such a cycle by $\lceil y_1, y_2, \ldots, y_m \rceil$. A point $z \in \mathbb{R}^2$ is called an *interior point* of $\lceil y_1, y_2, \ldots, y_m \rceil$ if it is an interior point of the closed, piece-wise linear curve c determined by $\{y_1, y_2, \ldots, y_m\}$.

A point $z \in \mathbb{R}^2$ is said to be *linked* to a non-empty set of vectors $\{y_1, y_2, \ldots, y_m\}$ in $\mathbb{R}^2$, if for some $i \in \{1, 2, \ldots, m\}$, $||z - y_i|| \leq r$. More generally, z is *connected* to $\{y_1, y_2, \ldots, y_m\}$ through a set of vectors $\{x_1, x_2, \ldots, x_n\}$ in $\mathbb{R}^2$ if there exists a subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ with $x_{i_k} \in \{y_1, y_2, \ldots, y_m\}$ such that $||z - x_{i_1}|| \leq r$ and $||x_{i_{s-1}} - x_{i_s}|| \leq r$, $i \in \{2, 3, \ldots k\}$. The following corollary to Theorem 1 is our main result on trapping.

Corollary 2. *Suppose that the set of initial positions* $\{x_1(0),\ x_2(0),\ \ldots,$ $x_n(0)\}$ *of the* n *agents contains a cycle* $\lceil x_{i_1}(0), x_{i_2}(0), \ldots, x_{i_m}(0)\rceil$. *Then all agents initially positioned inside the cycle eventually rendezvous at one point with all agents with positions initially connected to the cycle through* $\{x_1(0),\ x_2(0),$
$\ldots,\ x_n(0)\}$.

1.3 Analysis

The aim of this section is to establish the correctness of Theorem 1. Towards this end, let $\{\{x_1(t_k),\ x_2(t_k), \ldots, x_n(t_k)\} : k \geq 1\}$ be a system trajectory determined by (7) and any initial set of agent positions. Let k^* denote the value of k for which the ascending chain shown in (8) converge to the limit graph $\mathbb{G}$ in (9). Thus for $t_k \geq t_{k^*}$, the neighbors of each agent do not change. For each $i \in \{1, 2, \ldots, n\}$, let $\{i_1, i_2, \ldots, i_{m_i}\}$ denote the set of indices labelling the neighbors of agent i. For simplicity, we will only deal with the case when each agent has at least one neighbor. This means that all m_i are positive integers. These assumptions imply that for $k \geq k^*$, the system under consideration will have a state $\{x_1(t_k), x_2(t_k), \ldots, x_n(t_k)\}$ taking values in the space

$$\mathcal{X} = \{\{x_1, x_2, \ldots x_n\} : ||x_j - x_i|| \leq r \quad j \in \{i_1, i_2, \ldots, i_{m_i}\}, \quad i \in \{1, 2, \ldots, n\}\} \tag{10}$$

Error System: To analyze system behavior it is convenient to introduce a suitably defined "error system." For $\{x_1, x_2, \ldots, x_n\} \in \mathcal{X}$, define

$$e_i = x_i - x_n, \quad i \in \{1, 2, \ldots, n\} \tag{11}$$

and note that $e_n = 0$. Let $e \stackrel{\Delta}{=} \{e_1, e_2, \ldots, e_{n-1}\}$. In view of (10) and the fact that $e_j - e_i = x_j - x_i$ for all $i, j \in \{1, 2, \ldots, n\}$, we see that e takes values in the closed space

$$\begin{aligned}\mathcal{E} = \{\{e_1, e_2, \ldots e_{n-1}\} : e_n = 0, ||e_j - e_i|| \leq r \quad j \in \{i_1, i_2, \ldots, i_{m_i}\},\\ i \in \{1, 2, \ldots, n\}\}\end{aligned} \tag{12}$$

Note that

$$x_{i_j}(t_{k-1}) - x_i(t_{k-1}) = e_{i_j}(t_{k-1}) - e_i(t_{k-1}), \ \ j \in \{1, 2, \ldots, m_i\}, \ i \in \{1, 2, \ldots, n\}$$

It follows that the update equation (7) for x_i can be written as

$$x_i(t_k) = x_i(t_{k-1}) + f_i(e(t_{k-1})), \quad k \geq k^* \tag{13}$$

where $f_i : \mathcal{E} \to \mathbb{D}$ is the continuous function

$$\{e_1, e_2, \ldots, e_{n-1}\} \longmapsto u_{m_i}(e_{i_1} - e_i,\ e_{i_2} - e_i,\ \ldots,\ e_{i_{m_i}} - e_i)|_{e_n=0}$$

In view of (13) and the definition of the e_i,

$$e_i(t_k) = e_i(t_{k-1}) + f_i(e(t_{k-1})) - f_n(e(t_{k-1})), \quad k > k^*, \quad i \in \{1, 2, \ldots, n-1\} \tag{14}$$

This enables us to define the *error system*

$$e(t_k) = e(t_{k-1}) + f(e(t_{k-1})), \quad k > k^* \tag{15}$$

where $f(e) = \{f_1(e) - f_n(e),\ f_2(e) - f_n(e),\ \ldots,\ f_{n-1}(e) - f_n(e)\}$.

Proving Convergence in the Style of Lyapunov

In the sequel, we will prove that under certain conditions $e(t_k) \to 0$ as $k \to \infty$. We will do this using the positive definite function $V : \mathcal{E} \to \mathbb{R}$ defined by

$$V(e) = \text{dia}\{e_1, e_2, \ldots, e_{n-1}, 0\} \tag{16}$$

where for any set of vectors $y_1, y_2, \ldots, y_m$ in $\mathbb{R}^2$, $\text{dia}\{y_1, y_2, \ldots, y_m\}$ denotes the diameter[6] of $\langle y_1, y_2, \ \ldots, y_m\rangle$. The following proposition is central to the proof of Theorem 1.

Proposition 4. *The difference function $\Delta : \mathcal{E} \to \mathbb{R}$ defined by*

$$\Delta(e) = V(e + f(e)) - V(e) \tag{17}$$

is negative semi-definite. Moreover if $\mathbb{G}$ is connected, then Δ is negative definite.

Proof of Theorem 1: In general the graph $\mathbb{G}$ to which the ascending chain (8) converges for some finite $k = k^*$ consists of a finite set of connected components. Suppose that $\mathbb{G}_c$ is any one of these. To prove Theorem 1 it is enough to show that the positions of those agents whose indices are the vertices of $\mathbb{G}_c$, converge to a common point. For simplicity we will do this only for the case when $\mathbb{G}_c = \mathbb{G}$, since, except for notation, the proof is essentially the same even if $\mathbb{G}_c \neq \mathbb{G}$.

[6] Recall that the *diameter* of a closed set $\mathcal{S} \subset \mathbb{R}^2$ is the maximum of $||s_1 - s_2||$ over all $s_1, s_2 \in \mathcal{S}$.

By hypothesis $n > 1$. Note that if $e(t_k) = 0$, for some $k = \bar{k}$, then all agents are in the same position at time $t_{\bar{k}}$; moreover any such position will remain fixed for all $t \geq t_{\bar{k}}$ because $f(0) = 0$. Therefore to complete the proof it is enough to show that $e(t_k)$ tends to 0 as $k \to \infty$.

Let $V : \mathcal{E} \to \mathrm{I\!R}$ be defined as in (16). Note that

$$V(e(t_k)) = \mathrm{dia}\{x_1(t_k), x_2(t_k), \ldots, x_n(t_k)\} \tag{18}$$

because the diameter of a convex set in $\mathrm{I\!R}^2$ is invariant under translation of the set. From this and Proposition 4, it follows that the difference function

$$\varDelta(e(t_k)) = V(e(t_k) + f(e(t_k))) - V(e(t_k))$$

is non-positive for $k \geq k^*$. Thus $V(e(t_k))$ is a monotone non-increasing function of k for $k \geq k^*$. Since for $k \geq k^*$, $V(e(t_k))$ is bounded above by $V(e(t_{k^*}))$ and below by 0, there must exist a finite limit

$$V^* \stackrel{\varDelta}{=} \lim_{k \to \infty} V(e(t_k))$$

We claim that $V^* = 0$. To prove this claim, suppose that it is false. Then $V^* > 0$. Let $\mathcal{B}$ denote the set of all points $e \in \mathcal{E}$ such that $V^* \leq V(e) \leq V(e(t_{k^*}))$. Note that $\mathcal{B}$ is closed and bounded because $V(\cdot)$ is continuous and $\mathcal{E}$ is closed. Moreover $0 \notin \mathcal{B}$ because $V(\cdot)$ is positive definite and bounded away from zero on $\mathcal{B}$. By Proposition 4, $\varDelta(\cdot)$ is negative definite. Therefore for all $e \in \mathcal{B}$, $\varDelta(e) < 0$. From this, the compactness of $\mathcal{B}$ and the continuity of $\varDelta(\cdot)$, it follows that

$$\mu \stackrel{\varDelta}{=} \max_{e \in \mathcal{B}} \varDelta(e)$$

is a finite negative number. Since $e(t_k) \in \mathcal{B}$ for $k \geq k^*$, it must therefore be true that

$$V(e(t_{k+1})) - V(e(t_k)) = \varDelta(e(t_k)) \leq \mu, \quad k \geq k^*$$

Thus by summing,

$$V(e(t_k)) \leq V(e(t_{k^*})) + (k - k^*)\mu, \quad k \geq k^*$$

Therefore, for k sufficiently large $V(e(t_k))$ must be negative because $\mu < 0$,. But this is impossible because $V(\cdot)$ is positive definite. Hence V^* cannot be positive. ■

The proof just given is basically a standard Lyapunov argument applied to the system (17). It is worth pointing out here that the continuity of $\varDelta(\cdot)$ is crucial to the proof as is the fact that $\mathcal{E}$ is closed. If $\mathcal{E}$ were not a closed set, the preceding proof would break down because one could not conclude that $\mathcal{B}$ is closed. The closure of $\mathcal{E}$ is a direct consequence of the fact that sensing regions are defined to be closed sets. The continuity of $\varDelta(\cdot)$ is a consequence of the requirement that the $u_m(\cdot)$ be continuous functions. In summary, for the present analysis to go through, it is essential that sensing regions be closed sets and that the $u_m(\cdot)$ be continuous functions. Whether or not these requirements can be relaxed by approaching convergence differently remains to be seen.

2 Asynchronous Case

The strategy described in the previous section cannot be regarded as truly distributed because each agent's decisions must be synchronized to a common clock shared by all other agents in the group. In this section we redefine the strategies so that a common clock is not required. To do this it will be necessary to modify somewhat what is meant by a registered neighbor and by a registered neighbor's position.

In the asynchronous case, for each agent i, the real time axis can be partitioned into a sequence of time intervals $[0, t_{i1})$, $[t_{i1}, t_{i2})$, $\ldots$, $[t_{i(k_i-1)}, t_{ik_i})$, $\ldots$ each of length at most $\tau_D + \tau_{M_i}$ where τ_D is a number greater than τ_{M_i} called a *dwell time*. Each interval $[t_{i(k_i-1)}, t_{ik_i})$ consists of a *sensing period* $[t_{i(k_i-1)}, \bar{t}_{ik_i})$ of fixed length τ_D during which agent i is stationary, followed by a maneuvering period $[\bar{t}_{ik_i}, t_{ik_i})$ of length at most τ_{M_i} during which agent i moves from its current position to its next way-point. Although all agents use the same dwell time, they operate asynchronously in the sense that the time sequences $t_{i1}, t_{i2}, \cdots, \quad i \in \{1, 2, \ldots, n\}$ are uncorrelated. Thus each agent's strategy can be implemented independent of the rest, without the need for a common clock.

Because of the asynchronous nature of the control strategies under consideration, care must be exercised in defining what is meant by a registered neighbor if one is to end up with something similar to the symmetry property of the neighbor relationship defined in the synchronous case. For the asynchronous case, agent i's *registered neighbors* at the beginning of its *kth* maneuvering period $[\bar{t}_{ik}, t_{ik})$ are taken to be those agents which are fixed at one position within agent i's sensing region for at least $\tau_S > 0$ seconds during agent i's *kth* sensing period $\mathcal{T}_i(k) \stackrel{\Delta}{=} [t_{i(k-1)}, \bar{t}_{ik})$. Here τ_S is a positive number called a *sensing time*. For reasons to be made clear below, we shall require τ_S to satisfy

$$\tau_S \leq \frac{1}{2}(\tau_D - \tau_{M_i}) \qquad \forall i \in \{1, 2, \ldots, n\} \tag{19}$$

For any agent j, there may be more than one distinct interval of length at least τ_S within $\mathcal{T}_i(k)$ during which agent j is stationary. Let t^* denote the end time of the last of these. For purposes of calculation, agent i takes the *registered position* of agent j at the beginning of its *kth* maneuvering period, to be the actual position of agent j at *registration time* t^*. To attain a symmetry-like property for the asynchronous case, it is necessary make sure that the *registration interval* $[t^* - \tau_S,\ t^*)$ lies within one of agent j's sensing periods. One way to guarantee that this is so is to require each agent to keep moving during each of its maneuvering periods except possibly for brief periods which are each shorter than τ_S. Another way is equip each agent with a signaling device {such as a light in the case of visual sensing} which is on just in case the agent is in one of its sensing periods. In the sequel we will assume that registration of each agent j during one of agent i's sensing periods always occurs at the end of a registration interval $[t^* - \tau_S,\ t^*)$ which also lies within one of agent j's sensing periods. Note that this and the requirement that agent j is stationary during its sensing periods together imply that agent j's registered position $x_j(t^*)$ is equal to $x_j(\bar{t}_{jk^*})$ where k^* is the sensing/maneuvering interval of agent j during which registration takes place.

2.1 Cooperation Assumption

Prompted by the preceding, let us agree to say that for each $i, j \in \{1, 2, \dots, n\}$, agent j's *qth* sensing period $\mathcal{T}_j(q)$ *overlaps* agent i's *kth* sensing period $\mathcal{T}_i(k)$ if $\mathcal{T}_j(q) \cap \mathcal{T}_i(k)$ is a non-empty interval of length at least τ_S. Let us note that because all sensing periods of all agents are τ_D seconds long, the largest number of sensing periods of any agent j which a given sensing period of agent i can overlap, is two. On the other hand, each sensing period of agent i must overlap at least one sensing period of each agent j. To understand why this is so, note first that the maximal possible amount of time between two successive sensing periods of agent j is τ_{M_j}; but τ_{M_j} is bounded above by $\tau_D - 2\tau_S$ because of (19). Thus the maximal possible amount of time between two successive sensing periods of agent j is no greater than $\tau_D - 2\tau_S$. Given this and the fact that all sensing periods are τ_D seconds long, it follows that each sensing period of agent i must overlap at least one sensing period of each agent j.

For agent j to be a registered neighbor of agent i at the beginning of agent i's *kth* maneuvering period, it is necessary and sufficient that a sensing period $\mathcal{T}_j(q)$ overlapping $\mathcal{T}_i(k)$ exist, and that agent j be "within range of agent i" {i.e., within agent i's sensing region} during the last τ_S seconds of the overlap period $\mathcal{T}_j(q) \cap \mathcal{T}_i(k)$. Since both agents are stationary during their sensing periods, the range requirement can be written as $||x_j(\bar{t}_{jq}) - x_i(\bar{t}_{ik})|| \leq r$. If agent j is a registered neighbor of agent i, this inequality will always hold with $q = k^*$ where k^* is the index of the sensing/maneuvering period within which registration of agent j takes place. In fact, $\mathcal{T}_j(k^*)$ must be the *last* sensing period of agent j during which agent i is in range and which overlaps $\mathcal{T}_i(k)$. Moreover, registration of agent j will always occur at $\bar{t}_{ik}$ or $\bar{t}_{jk^*}$, whichever time comes first; thus agent j's registered position will be at $x_j(\bar{t}_{ik})$ if $\bar{t}_{ik} \leq$

$\bar{t}_{jk^*}$ or at $x_j(\bar{t}_{jk^*})$ if $\bar{t}_{ik} > \bar{t}_{jk^*}$. But if $\bar{t}_{ik} \leq \bar{t}_{jk^*}$, then $x_j(\bar{t}_{ik}) = x_j(\bar{t}_{jk^*})$ because agent j does not move during its sensing periods. In other words, under any conditions the registered position of agent j is always equal to $x_j(\bar{t}_{jk^*})$. The following proposition summarizes these observations.

Proposition 5. *Agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period if and only if for some sensing period $\mathcal{T}_j(q)$ which overlaps $\mathcal{T}_i(k)$,*

$$||x_j(\bar{t}_{jq}) - x_i(\bar{t}_{ik})|| \leq r \tag{20}$$

If agent j is such a registered neighbor, then the index k^* of the sensing/maneuvering period of agent j during which registration takes place is the largest value of q for which $\mathcal{T}_j(q)$ overlaps $\mathcal{T}_i(k)$ and (20) holds. Under these conditions, $x_j(\bar{t}_{jk^*})$ is the registered position of agent j at the beginning of agent i's kth maneuvering period.

There are three possible situations in which agent j will be a registered neighbor of agent i at the beginning of agent i's kth maneuvering period. In the first two situations, shown in Figure 1a and 1b, agent $i's\ kth$ sensing period $\mathcal{T}_i(k)$ overlaps exactly one of agent j's sensing periods. On the other hand in the situation shown in Figure 1c, $\mathcal{T}_i(k)$ overlaps two of agent j's sensing periods. Two overlaps are always possible because $2\tau_S \leq \tau_D$ as can be deduced from (19), and because two such successive sensing periods of agent j can occur with zero maneuvering time separating them. In cases (a) and (b), agent j will be a registered neighbor of agent i and t^* will be as shown, provided agent j is within range of agent i during the corresponding overlap period shown in each case. In situation (c), agent j will also be such a registered neighbor of agent i provided agent j is within range of agent i during at least one of the two overlapping periods shown. If agent j is in range of agent i during the latter overlap period, then t^* will be located at time t_2^* as shown in Figure 1c. On the other hand, if j is not within range of agent i during the latter but is within the former, then $t^* = t_1^*$.

The definition of a registered neighbor determines a relationship between agents similar to the symmetric relation determined by the definition of a registered neighbor in the synchronous case. Suppose that agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period. As before, let t^* be the registration time of agent j during agent i's kth sensing/maneuvering period and write k^* for the index of the sensing/maneuvering period of agent j during which registration takes place. In view of Proposition 5, $\mathcal{T}_j(k^*)$ overlaps $\mathcal{T}_i(k)$ and $||x_j(\bar{t}_{jk^*}) - x_i(\bar{t}_{ik})|| \leq r$. Because of the obvious symmetry of these conditions, Proposition 5 also implies that agent i is a registered neighbor of agent j at the beginning of agent j's k^*th maneuvering interval. Let t_R denote the registration time of agent i during agent j's k^*th sensing/maneuvering period, With reference to Figure 1, let us note that t_R

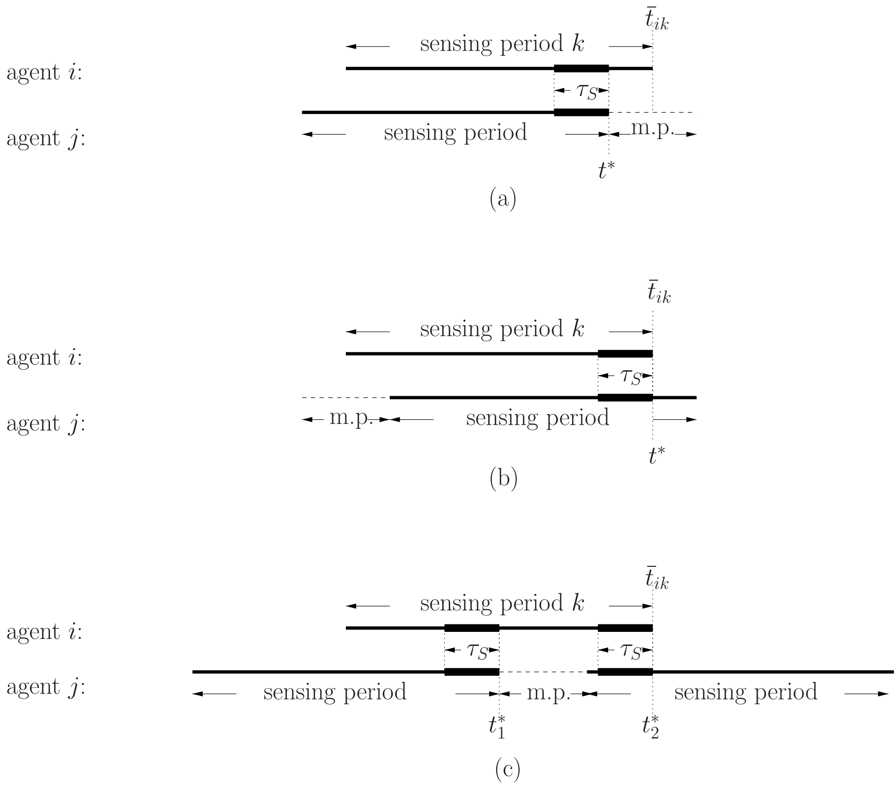

Fig. 1. Sensing Period Overlaps

is at t^* in case (a) and in case (c) when $t^* = t_1^*$. In these cases, agent i's corresponding registered position is equal to $x_i(\bar{t}_{ik})$ because t_R falls within the closure of agent i's kth sensing period, which is a period during which agent i does not move. In case (b) and (c) when $t^* = t_2$, there are two possibilities. Either registration occurs during agent i's kth sensing/maneuvering period or during its $k+1st$. If the former is true, then like case (a), registration is at $\bar{t}_{ik}$ in which case agent i's registered position is again $x_i(\bar{t}_{ik})$. If the latter is true, this would mean that agent j's k^*th sensing period overlaps with agent i's $k+1st$ sensing period and registration occurs during the closure of the overlap; under these conditions agent i's registered position would be at $x_i(\bar{t}_{i(k+1)})$ since agent i does not move on its $k+1st$ sensing period. We summarize.

Proposition 6. *If agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period and k^* is the sensing/maneuvering inter-*

val of agent j during which registration takes place, then agent i is a registered neighbor of agent j at the beginning of agent j's k^*th maneuvering period. In addition, the registered position of agent i at the beginning of agent j's k^*th maneuvering period is $x_i(\bar{t}_{iq})$, where q is that index in $\{k, k+1\}$ labelling the sensing/maneuvering period of agent i during which registration takes place.

The notion of a pairwise motion constraint introduced in the synchronous case can be replaced with the following constraint which is appropriate for the asynchronous case. Agent i is said to satisfy the *motion constraints induced by its neighbors*, if for each $j \in \{1, 2, \ldots, n\}$ for which $j \neq i$ and each $k \in \{1, 2, \ldots\}$ for which agent j is a registered neighbor of agent i at the beginning of maneuvering period k, the position to which agent i moves at the end of the period is within a closed disk of diameter r centered at the mean of agent i's position at the beginning of the period {i.e., at time $\bar{t}_{ik}$} and the registered position of agent j at the beginning of the period. In the synchronous case, satisfaction of the pairwise motion constraint by agent i and neighbor j causes each to retain the other as a neighbor. The following proposition implies that essentially the same thing is true in the asynchronous case when the induced motion constraints are satisfied by agents i and j.

Proposition 7. *Suppose that agents i and j satisfy the motion constraints induced by their registered neighbors. If agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period, and k^* is the sensing/maneuvering period of agent j during which registration of agent j takes place, then agent j is also a registered neighbor of agent i at the beginning of agent i's $k+1$st maneuvering period. Moreover, sensing periods $\mathcal{T}_j(k^*+1)$ and $\mathcal{T}_i(k)$ do not overlap.*

We are interested in strategies which cause agents to retain their registered neighbors. We therefore make the following assumption.

Cooperation Assumption: Each agent i satisfies the motion constraints induced by each of its registered neighbors.

Suppose that the cooperation assumption is satisfied. Proposition 7 states that if agent j is a registered neighbor of agent i during maneuvering interval k then it will also be a registered neighbor of agent i during maneuvering interval $k+1$. In other words, if the cooperation assumption is satisfied, each agent retains all of its prior registered neighbors as the system evolves. Thus if $\mathcal{N}_i(k)$ denotes the sent of labels of agent i's neighbors at the beginning of its *kth* maneuvering period, then $\mathcal{N}_i(k) \subset \mathcal{N}_i(k+1),\ k \geq 1$.

Just like the synchronous case, agent i's *kth way-point* $\bar{x}_i(k)$ is the point to which agent i moves at the end of its *kth* maneuvering period. The rule which determines $\bar{x}_i(k)$ is essentially the same as in the synchronous case, except that now $\bar{x}_i(k)$ depend on agent i's its own position at the beginning of its kth maneuvering period and the relative positions of agent i's registered neighbors at the beginning of the period. Thus if agent i has no registered neighbors at time $\bar{t}_{ik}$, agent i does not move during its kth maneuvering period. On the other hand, if agent i has $m_{ik} > 0$ registered neighbors at time $\bar{t}_{ik}$ with registered positions $z_1, z_2, \ldots, z_{m_{ik}}$ relative to agent i's, then agent i moves to the position $\bar{x}_i(k) = x_i(t_{i(k-1)}) + u_{m_{ik}}(z_1, \ldots, z_{m_{ik}})$ at the end of the period where

$$z_j = x_{i_j}(t_j^*) - x_i(t_{i(k-1)}), \quad j \in \{1, 2, \ldots, m_{ik}\}, \tag{21}$$

t_j^* is the time neighbor i_j is registered within agent i's kth sensing/maneuvering period, and $u_{m_{ik}}(\cdot)$ is exactly as before. Thus at the end of its kth maneuver, agent i's position is given by

$$x_i(t_{ik}) = x_i(t_{i(k-1)}) + u_{m_{ik}}(x_{i_1}(t_1^*) - x_i(t_{i(k-1)}), \ldots,\ x_{i_{m_{ik}}}(t_{m_{ik}}^*) - x_i(t_{i(k-1)})) \tag{22}$$

Of course the set $\{i_1, i_2, \ldots, i_{m_{ik}}\}$, and the registration times $t_1^*, t_2^*, \ldots, t_{m_{ik}}^*$ all depend on i and k.

Main Results

Note that because agents do not move during sensing periods, for each $i \in \{1, 2, \ldots, n\}$ the positions of agent i at times $t_{i(k-1)}$ and t_{ik} are the same as at times $\bar{t}_{ik}$ and $\bar{t}_{i(k+1)}$ respectively. Thus (22) can also be written as

$$x_i(\bar{t}_{i(k+1)}) = x_i(\bar{t}_{ik}) + u_{m_{ik}}(x_{i_1}(t_1^*) - x_i(\bar{t}_{ik}), \ldots,\ x_{i_{m_{ik}}}(t_{m_{ik}}^*) - x_i(\bar{t}_{ik})) \tag{23}$$

The n equations given by (23) for $i \in \{1, 2, \ldots, n\}$, completely describes the evolution of the positions of the n agents under consideration as each maneuvers from way-point to way-point. Just as in the synchronous case, the analysis of these equations depends on the relationships between registered neighbors and how these relationships evolve with time. To characterize these relationships, we first extend the domain of definition of each agent's registered neighbors from its set of maneuvering period start times to a suitably defined set of "event times" common to all n agents. By an *event time* is meant any time $\bar{t}_{ik}$ at which any maneuvering period $[\bar{t}_{ik}, t_{ik})$ of any agent begins. Let $\{\bar{t}_{ik} : i \in \{1, 2, \ldots, n\},\ k \geq 1\}$ denote the set of all distinct event times. Label this set's elements as $t_1, t_2, \cdots$ in such a way so that $t_j < t_{j+1},\ j \in \{1, 2, \ldots\}$. For $i \in \{1, 2, \ldots, n\}$, let $S_i(k)$ denote that value of s for which $t_s = \bar{t}_{ik}$. Thus with this notation, agent i's registered neighbors at its kth event time $t_{S_i(k)}$, are its registered neighbors at time $\bar{t}_{ik}$. For each $i \in \{1, 2, \ldots, n\}$ we extend the domain of definition of agent i's registered neighbors from the set $\{t_{S_i(k)} : k \geq 1\}$ to the set $\{t_s : s \geq S_i(1)\}$ by stipulating that for values of t_s

which are between two successive event times of agent i, say between t_{ik} and $t_{i(k+1)}$, agent i's registered neighbors are the same as its registered neighbors at time t_{ik}.

Let $\mathcal{T} \stackrel{\Delta}{=} \{t_{\bar{s}}, t_{\bar{s}+1}, t_{\bar{s}+2} \dots\}$ denote the set of all event times greater than or equal to $t_{\bar{s}}$ where $\bar{s} \stackrel{\Delta}{=} \max\{S_1(1), S_2(1), \dots S_n(1)\}$. Note that the registered neighbors of each agent are defined at each time in $\mathcal{T}$. For each $s \geq \bar{s}$, it is therefore possible to describe neighbor relationships using a directed[7] graph $\mathbb{G}_s$ with vertex set $\{1, 2, \dots, n\}$ and directed edge set defined so that (i, j) is a directed edge from vertex i to vertex j just in case agent j is a registered neighbor of agent i at event time t_s.

Let us partially order the set of all directed graphs with vertex set $\{1, 2, \dots, n\}$ by agreeing to say that $\mathbb{G}$ is contained in $\bar{\mathbb{G}}$ if the edge set of $\mathbb{G}$ is a subset on the edge set of $\bar{\mathbb{G}}$. It is natural then to define the *union* of a collection of such graphs to be the directed graph with vertex set $\{1, 2, \dots, n\}$, and edge set equaling the union of the edge sets of all of the graphs in the collection. Because of the cooperation assumption and Proposition 7, we know that each agent keeps all of its registered neighbors as the system evolves. What this means is the sequence of graphs $\mathbb{G}_{\bar{s}}, \mathbb{G}_{\bar{s}+1}, \dots, \mathbb{G}_s, \dots$ forms the ascending chain

$$\mathbb{G}_{\bar{s}} \subset \mathbb{G}_{\bar{s}+1} \subset \cdots \mathbb{G}_s \cdots \tag{24}$$

Because the set of directed graphs on vertices $\{1, 2, \dots, n\}$ is a finite set, the chain must converge to the graph

$$\mathbb{G} \stackrel{\Delta}{=} \bigcup_{s=\bar{s}}^{\infty} \mathbb{G}_s \tag{25}$$

in a finite number of steps. More is true. Suppose that agent i has agent j as a registered neighbor at the beginning of one of agent i's maneuvering periods. Then because of Proposition 6, agent i must be a registered neighbor of agent j at the beginning of one of agent j's maneuvering periods. These observations together with the cooperation assumption imply that agents i and j must both eventually become and remain registered neighbors of each other. As a consequence, there must be directed arcs in $\mathbb{G}$ from vertex i to vertex j as well as from vertex j to vertex i. Clearly $\mathbb{G}$ must be a directed graph with the property that for each distinct pair of vertices - say i and j - either there is no directed arc connecting one to the other or there are two directed arcs one from vertex i to vertex j and the other from vertex j to vertex i. Directed graphs with this property are usually regarded as simple graphs whose edges represent such pairs of directed arcs [15]. In the sequel we

[7] It will soon be clear that the aforementioned symmetry of the neighbor relationship will ultimately enable us to characterize neighbor relationships with a simple, undirected graph as in the synchronous case.

shall adopt this viewpoint and refer to $\mathbb{G}$ as a simple graph. Our main result is as follows.

Theorem 2. *Let* $u_0 = 0 \in \mathbb{D}_M$ *and for each* $m \in \{1, 2, \ldots, n-1\}$, *let* $u_m : \mathbb{D}^m \to \mathbb{D}_M$ *be any continuous function satisfying (5) and (6). For each set of initial agent positions* $x_1(0), x_2(0), \ldots, x_n(0)$, *each agent's position* $x_i(t)$ *converges to a unique point* $p_i \in \mathbb{R}^2$ *such that for each* $i, j \in \{1, 2, \ldots, n\}$, *either* $p_i = p_j$ *or* $||p_i - p_j|| > r$. *Moreover, if agent* j *is a registered neighbor of agent* i *at the beginning of one of agent* i*'s maneuvering periods, then* $p_i = p_j$.

An outline of the proof of this theorem is given in §3.

Theorem 2 states that the strategies under consideration cause all agents positions to converge to points in the plane with the property that each pair of such points are either equal to each other, or separated by a distance greater than r units. The theorem further states that if one agent is ever a registered neighbor of another, then both converge to the same point. Thus all n agents position will converge to a single point if any one directed graph in the ascending chain is weakly connected. We are led to the following corollary.

Corollary 3. *If at any event time* $t_s \geq t_{\bar{s}}$, *the directed graph characterizing registered neighbors is weakly connected, then positions of all* n *agents converge to a common point in the plane.*

3 Analysis

The aim of this section is to outline the proof of the correctness of Theorem 2. This requires the analysis of the asymptotic behavior of the *asynchronous* process described by (23) for $i \in \{1, 2, \ldots, n\}$. Despite the apparent complexity of this process, it is possible to capture its salient features for t_s sufficiently large using a suitably defined *synchronous* discrete-time, hybrid dynamical system $\mathbb{S}$. Interestingly, $\mathbb{S}$ turns out to be *non-deterministic* in a sense which will be made clear in the sequel.

3.1 Definition of $\mathbb{S}$

We will define $\mathbb{S}$ to be a synchronous dynamical system representing n "nodes" evolving on the time set $\mathcal{I} = \{1, 2, \ldots, \}$. In order to avoid introducing lots of extra notation, and without sacrificing clarity, we will often use the same symbols in defining $\mathbb{S}$ as already used to describe the agent system. Thus for example,we associate with each node $i \in \{1, 2, \dots, \}$ a strictly monotone increasing function $S_i : \{0, 1, 2, \dots\} \to \{0, 1, 2, \dots, \}$ whose value $S_i(k)$ at $k \in \mathcal{I}$ is node i's *kth event time* and whose value at 0 is $S_i(0) = 0$. We write $\mathcal{S}_i$ for the image of S_i. We shall require the S_i to satisfy the following conditions:

1. For any integer $i \in \{1, 2, \dots, n\}$ and any two successive values $S_i(k),\ S_i(k+1)$ in $\mathcal{S}_i$
$$S_i(k+1) - S_i(k) \le 2n - 1 \tag{26}$$

2. For any integers $i, j \in \{1, 2, \dots, n\}$ and any two successive values $S_i(k)$, $S_i(k+1)$ in $\mathcal{S}_i$ there are at most two successive values $S_j(p),\ S_j(p+1)$ in $\mathcal{S}_j$ such that
$$S_i(k) \le S_j(p) < S_j(p+1) \le S_i(k+1) \tag{27}$$

For each $i \in \{1, 2, \dots, n\}$ we will often make use of the function $k_i : \{0, 1, \dots\} \to \{0, 1, \dots\}$ whose value at s, written $k_i(s)$, is the unique value of k such that $S_i(k) \le s < S_i(k+1)$. It is easy to verify that k_i is a left inverse for S_i; i.e., $k_i(S_i(k)) = k$ for all $k \in \{0, 1, \dots, \}$. We take as given n nonempty subsets $\mathcal{N}_i = \{i_1, i_2, \dots, i_{m_i}\} \subset \{1, 2, \dots, n\}$, with $i \notin \mathcal{N}_i$. The $\mathcal{N}_i$ are all required to have the following *symmetry property:* If $j \in \mathcal{N}_i$ then $i \in \mathcal{N}_j$. Because of the symmetry property we can associate with the $\mathcal{N}_i$ a simple graph $\mathbb{G}$ with vertex set $\{1, 2, \dots, n\}$ and edge set defined in such a way that (i, j) is in the edge set just in case $i \in \mathcal{N}_j$ and $j \in \mathcal{N}_i$. We will take as the state space of $\mathbb{S}$, the space $\mathcal{X}$ of all lists $\{y_1, y_2, \dots y_n, w_1, w_2, \dots w_n, v_{11}, \dots, v_{1m_1}, \dots v_{ij}, \dots, v_{n1}, \dots, v_{nm_n}\}$ satisfying

$$\left.\begin{array}{l} y_i, w_i, v_{ij} \in \mathbb{R}^2, \\[2ex] ||y_i - y_j|| \le r \\[2ex] ||y_i - \frac{1}{2}(w_i + v_{ij})|| \le \frac{1}{2} r \end{array}\right\} j \in \mathcal{N}_i, \quad i \in \{1, 2, \dots, n\} \tag{28}$$

In the sequel we often write y for $\{y_1, y_2, \dots y_n\}$, w for $\{w_1, w_2, \dots w_n\}$ and v_i for $\{v_{ii_1}, v_{ii_2} \dots, v_{ii_{m_i}}\}$, $i \in \{1, 2, \dots, n\}$. We sometimes refer to $\{y_i, w_i, v_i\}$ as the state of node i. We now define $\mathbb{S}$ to be a time-varying system with state $\{y, w, v_1, v_2, \dots, v_n\}$ which evolves on $\mathcal{I}$ according to update equations defined for $i \in \{1, 2, \dots, n\}$ by

$$y_i(s) = y_i(s-1) + u_{m_i}(v_{ii_1}(s) - y_i(s-1), \dots,\ v_{ii_{m_i}}(s) - y_i(s-1)), \tag{29}$$
$$w_i(s) = y_i(s-1), \tag{30}$$
$$v_{ij}(s) \in \mathbf{V}_{ij}(s, y_i(s-1), y_j(s-1), w_j(s-1), v_{ji}(s-1)), \qquad j \in \mathcal{N}_i \tag{31}$$

for $s \in \mathcal{S}_i$, and by

$$y_i(s) = y_i(s-1), \tag{32}$$
$$w_i(s) = w_i(s-1), \tag{33}$$
$$v_{ij}(s) = v_{ij}(s-1), \qquad j \in \mathcal{N}_i \tag{34}$$

for $s \notin \mathcal{S}_i$. The function $\mathbf{V}_{ij} : \mathcal{I} \times \mathrm{I\!R}^2 \times \mathrm{I\!R}^2 \times \mathrm{I\!R}^2 \times \mathrm{I\!R}^2 \rightarrow 2^{\mathrm{I\!R}^2}$ is a set-valued map depending on S_i and S_j and defined by the flow diagram shown in Figure 2. As the diagram shows, the value of $\mathbf{V}_{ij}$ at $\{s, y_i, y_j, w_j, v_{ji}\}$ is either the two element set $\{y_j, w_j\}$ or the singleton $\{y_j\}$, depending on the value of $\{s, y_i, y_j, w_j, v_{ji}\}$.

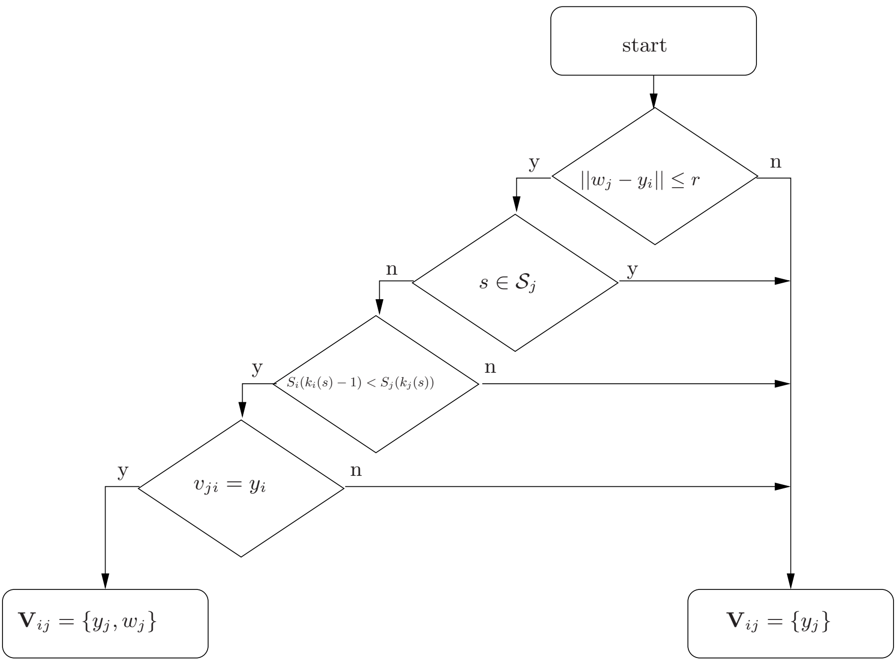

Fig. 2. Flow Diagram for $\mathbf{V}_{ij}(s, y_i, y_j, w_j, v_{ji})$

Note that $\mathbb{S}$ is nondeterministic because there can be ambiguity in the value which $v_{ij}(s)$ takes when $s \in \mathcal{S}_i$. Fixing $v_{ij}(s)$ in (31) at one of its possible values whenever $s \in \mathcal{S}_i$ results in a deterministic discrete dynamical system. The family of all such possible deterministic systems, namely $\mathcal{F}$, is clearly countable. By a *trajectory* of $\mathbb{S}$ is meant a trajectory of at least one of the systems in $\mathcal{F}$. Thus the set of all trajectories of all systems in $\mathcal{F}$ is the set of trajectories of $\mathbb{S}$. To claim that a deterministic signal can be generated by $\mathbb{S}$ is to claim that the signal can be generated along a trajectory of at least one system in $\mathcal{F}$. In the full length version of this paper it is shown that with proper interpretation of the y_i, w_i, and v_{ij}, one such system in $\mathcal{F}$ can simultaneously generate all n agent position vector sequences $x_i(\bar{t}_{ik})$ for all $i \in \{1, 2, \ldots, n\}$ and all sufficiently large $\bar{t}_{ik}$. Before doing this however we need to show that $\mathbb{S}$ is well-defined in the sense that under all conditions, the update equations (29) - (34) map any state in $\mathcal{X}$ into next states which are also in $\mathcal{X}$. The following proposition settles this issue.

Proposition 8. *Let $s \in \mathcal{I}$ be fixed and let $\{y, w, v_1, v_2, \ldots, v_n\}$ be any given state in $\mathcal{X}$. Let $\{\bar{y}, \bar{w}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ be any one of the possible vector lists which results when update equations (29) - (34) are applied to $\{s, y, w, v_1, v_2, \ldots, v_n\}$. Then $\{\bar{y}, \bar{w}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\} \in \mathcal{X}$.*

3.2 Properties of $\mathbb{S}$

In Section 3.1 we introduced $\mathbb{S}$ and claimed in Proposition 8 that $\mathbb{S}$ is a well-defined, synchronous, non-deterministic dynamical system. In this section we outline several important properties of $\mathbb{S}$.

In the sequel we write $\mathcal{H}_i(s)$ for the *ith local convex hull*

$$\mathcal{H}_i(s) = \langle y_i(s-1), y_{i_1}(s-1), \ldots, y_{i_{m_i}}(s-1), w_i(s-1), w_{i_1}(s-1), \ldots, w_{i_{m_i}}(s-1)\rangle$$

where $\{i_1, i_2, \ldots, i_{m_i}\} = \mathcal{N}_i$. We also write $\mathcal{H}(s)$ for the {global} convex hull

$$\mathcal{H}(s) = \langle y_1(s-1), y_2(s-1), \ldots, y_n(s-1), w_1(s-1), w_2(s-1), \ldots, w_n(s-1)\rangle,$$

and $\mathcal{K}(s)$ for the set of corners of $\mathcal{H}(s)$. Clearly

$$\mathcal{H}_i(s) \subset \mathcal{H}(s), \qquad i \in \{1, 2, \ldots, n\} \tag{35}$$

This fact plays a role in the proof of the following lemma which established a fundamental property of $\mathbb{S}$:

Lemma 1.

$$\mathcal{H}(s+1) \subset \mathcal{H}(s), \quad s \in \mathcal{I} \tag{36}$$

Let us agree to say that node i is *stationary* at time $s \in \mathcal{S}_i$ if

$$y_i(s) = w_i(s) = v_{ii_1}(s) = \cdots = v_{ii_{m_i}}(s)$$

The terminology is prompted by the fact that if node i is stationary at s, then $y_i(s) = y_i(s-1)$; this can be seen from (29), (30) and from the definition of u_{m_i}. The definition of u_{m_i} also implies that if $y_i(s)$ is a corner of $\langle y_i(s-1), v_{ii_1}(s), \ldots, v_{ii_{m_i}}(s)\rangle$, then node i must be stationary at s. This is also true if $y_i(s)$ is a corner of $\mathcal{H}(s)$.

Lemma 2. *Fix $i \in \{1, 2, \ldots, n\}$. If $y_i(s) \in \mathcal{K}(s)$ for some $s \in \mathcal{S}_i$, then node i must be stationary at s.*

By an *equilibrium state* of $\mathbb{S}$ we mean a state which does not change under the action of (29) - (34) under any conditions for every value of $s \in \mathcal{I}$. It is easy to see that equilibrium states are precisely those states $\{y, w, v_1, v_2, \ldots, v_n\} \in \mathcal{X}$ for which

$$y_i = w_i = v_{ii_1} = \cdots = v_{ii_{m_i}}, \quad \forall i \in \{1, 2, \ldots, n\}$$

We call the set of such states $\mathcal{E}$, the *equilibrium set* of $\mathbb{S}$. Note that $\mathcal{E}$ is an invariant set under the action of (29) - (34) under any and all possible conditions. Note also that $\mathbb{S}$ is at an equilibrium state at time s just in case each node of $\mathbb{S}$ is stationary at s.

In the sequel we will say node $i \in \{1, 2, \ldots, n\}$ has *locally rendezvoused* at time s if $\mathcal{H}_i(s+1)$ is a single point; i.e., if $y_i(s) = y_{i_1}(s) = \cdots = y_{i_{m_i}}(s) = w_i(s) = w_{i_1}(s) = \cdots = w_{i_{m_i}}(s)$. Note that if a node has locally rendezvoused at s it must be stationary at s. The following proposition provides a criterion for a node of $\mathbb{S}$ to be locally rendezvoused.

Proposition 9. *Let $s_1 < s_2 < s_3 < s_4$ be four successive values of s in $\mathcal{S}_i$. If $y_i(s_4) \in \mathcal{K}(s_1)$, then node i is locally rendezvoused at $s = s_3$.*

It is possible to describe $\mathbb{S}$ concisely by a recursive inclusion of the form

$$\bar{x}(s) \in f(s, \bar{x}(s-1)), \quad s \in \mathcal{I}$$

where $\bar{x}$ is the state $\{y, w, v_1, v_2, \ldots, v_n\}$, $f : \mathcal{I} \times \mathcal{X} \to 2^{\mathcal{X}}$ is the next state map defined by (29) - (34), and $2^{\mathcal{X}}$ is the power set of $\mathcal{X}$.

Lemma 3. *There exists a finite index set $\mathcal{P}$, a finite collection of subsets $\mathcal{P}(s) \subset \mathcal{P}$, $s \in \mathcal{I}$, and a finite set of continuous functions $F_p : \mathcal{X}_p \to \mathcal{X}$ with closed domains such that the following statement is true. For any $s \in \mathcal{I}$ and any pair of states $x \in \mathcal{X}$ and $\bar{x} \in f(s, x)$, there is a $p \in \mathcal{P}(s)$ for which*

$$\bar{x} = F_p(x) \tag{37}$$

The implication of Lemma 3 is that if $\{\bar{x}(s) : s \geq 0\}$ is a trajectory of $\mathbb{S}$, then there are indices $p(s) \in \mathcal{P}(s),\ s \in \mathcal{I}$ such that

$$\bar{x}(s) = F_{p(s)}F_{p(s-1)} \cdots F_{p(\tau+1)}(\bar{x}(\tau)),\ \ s > \tau \geq 0 \tag{38}$$

Here $F_{p(s)}F_{p(s-1)} \cdots F_{p(\tau+1)}$ is a "composed function", where by the composition of functions F_p and F_q we mean the function $F_qF_p : \mathcal{X}_{qp} \rightarrow \mathcal{X}$, whose domain $\mathcal{X}_{qp}$ is the inverse image $\mathcal{X}_q$ under F_p, and whose action on x is $x \longmapsto F_q(F_p(x))$. Composition is an associative operation and because of this, the operation extends unambiguously to finite families of F_p. Note that any such composed function $F = F_{p_1}F_{p_2} \cdots F_{p_k}$ has a closed domain on which it is continuous.

Suppose that $\bar{s} > 0$ is fixed. If follows from the preceding that there are $p(s) \in \mathcal{P}(s)$ such that

$$\bar{x}(s + \bar{s}) = F_{p(s+\bar{s})}F_{p(s+\bar{s}-1)} \cdots F_{p(s+1)}(\bar{x}(s)),\ \ s \in \mathcal{I} \tag{39}$$

It is important to recognize that even though the composed function $F_{p(s+\bar{s})}$ $F_{p(s+\bar{s}-1)} \cdots F_{p(s+1)}$ depends on s, there can be only a finite number of such composed functions. This is because the family of maps $F_p : p \in \mathcal{P}\}$ is a finite set and because the composed functions in question are all compositions of exactly $\bar{s}$ maps in the family. The following proposition summarizes these observations.

Proposition 10. *Let $\bar{s} > 0$ be fixed. There exist a finite index set $\mathcal{Q}$, a finite set of closed subsets $\bar{\mathcal{X}}_q \subset \mathcal{X}$, and a finite set of continuous maps $D_q : \bar{\mathcal{X}}_q \rightarrow \mathcal{X}$, $q \in \mathcal{Q}$ with the following property. For each trajectory $\{x(s) : s \in \mathcal{I}\}$ of $\mathbb{S}$, and each $s \in \mathcal{I}$, there is a $q \in \mathcal{Q}$ such that*

$$x(\bar{s} + s) = D_q(x(s)) \tag{40}$$

3.3 Representing the Agent System

In Section 3.1 we defined $\mathbb{S}$ without any reference to the actual agent system. In this section we will explain the manner in which $\mathbb{S}$ can represent the agent system. We will do this for all values of $s > s^*$ where s^* is the smallest value of $s \geq \bar{s}$ for which the ascending chain shown in (24) has converged to the limit graph $\mathbb{G}$ in (25). Thus for $t_s \geq t_{s^*}$, the registered neighbors of each agent do not change. For $\bar{t}_{ik} \geq t_{s^*}$, the position update equation (23) for agent i can thus be written as

$$x_i(\bar{t}_{i(k+1)}) = x_i(\bar{t}_{ik}) + u_{m_i}(x_{i_1}(t_1^*) - x_i(\bar{t}_{ik}), \ldots,\ x_{i_{m_i}}(t_{m_i}^*) - x_i(\bar{t}_{ik})) \tag{41}$$

For simplicity, we will only deal with the case when each agent has at least one neighbor. This means that all the m_i are positive numbers. For each $i \in \{1, 2, \ldots, n\}$, the subset $\mathcal{N}_i$ introduced in the definition of $\mathbb{S}$ is interpreted as the set of indices $\{i_1, i_2, \ldots, i_{m_i}\}$ labelling agent i's neighbors. The required symmetry property of the $\mathcal{N}_i$ is an immediate consequence of the agent neighbor relationship which we have already discussed. Accordingly, we take the neighbor graph $\mathbb{G}$ associated with $\mathbb{S}$ to be one and the same as the simple neighbor graph defined by (25).

We shall interpret S_i used in defining $\mathbb{S}$ just as we did when we used the same symbol in discussing the actual agent system. Thus $t_{S_i(k)}$ is the *kth* event time of agent i. The following lemma serves to justify constraints (26) and (27) imposed on the S_i.

Lemma 4. *For any integer $i \in \{1, 2, \ldots, n\}$ and any two successive event times $\bar{t}_{ik}$ and $\bar{t}_{i(k+1)}$ of agent i, the number of distinct event times in the set $\{\bar{t}_{jq} : j \in \{1, 2, \ldots, n\},\ j \neq i,\ q \geq 1\}$ which satisfy*

$$\bar{t}_{ik} \leq \bar{t}_{jq} \leq \bar{t}_{i(k+1)} \tag{42}$$

does not exceed $2(n-1)$. Moreover, for any particular integer $j \in \{1, 2, \ldots, n\}$ there are at most two distinct event times in the set $\{\bar{t}_{jq} : q \geq 1\}$ which satisfy (42).

The following result establishes the connection between the asynchronously functioning agent trajectories defined by (41) and the trajectories of $\mathbb{S}$.

Proposition 11. *For each $i \in \{1, 2, \ldots, n\}$ and each $s \geq s^*$, define*

$$y_i(s) \triangleq x_i(\bar{t}_{i(k_i(s)+1)}), \tag{43}$$

$$w_i(s) \triangleq x_i(\bar{t}_{ik_i(s)}), \tag{44}$$

$$v_{ij}(s) \triangleq \left\{ \begin{array}{ll} x_j(\bar{t}_{j(p_{ij}(s)+1)}), & \text{if } \mathcal{T}_i(k_i(s)) \text{ overlaps } \mathcal{T}_j(p_{ij}(s)+1) \\ x_j(\bar{t}_{jp_{ij}(s)}), & \text{otherwise} \end{array} \right\} j \in \mathcal{N}_i \tag{45}$$

where $k_i(s)$ is the unique integer k for which $S_i(k) \leq s < S_i(k+1)$ and $p_{ij}(s) \triangleq k_j(S_i(k_i(s)))$. Under the hypotheses of Theorem 2, $\{\{y(s), w(s), v_1(s), \ldots, v_n(s)\} : s \geq s^*\}$ is a trajectory of $\mathbb{S}$.

3.4 Global Rendezvous

In this section we shall be concerned with those trajectories of $\mathbb{S}$ for which (43) - (45) hold. It is natural to say that the n nodes of $\mathbb{S}$ have {globally} *rendezvoused* at time s if $\mathcal{H}(s+1)$ is a single point; i.e., if $y_1(s) = y_2(s) = \cdots = y_n(s) = w_1(s) = w_2(s) = \cdots = w_n(s)$. In view of the definitions of t_s and the y_i and w_i in (43) – (44), it is clear that the rendezvousing of all n nodes at time s implies the rendezvousing of all n agents at time t_s. It is also clear that the rendezvousing of all n nodes at time s implies that each node has locally rendezvoused at s. This in turn implies that each node is stationary at s. In other words, points at which global rendezvousing occurs are equilibrium states of $\mathbb{S}$. It can be shown that the converse is also true if $\mathbb{G}$ is a connected graph. The following lemma is key.

Lemma 5. *Suppose $\mathbb{G}$ is a connected graph. Suppose in addition that $\{y(s), w(s), v_1(s), v_2(s), \ldots, v_n(s) : s \in \mathcal{I}\}$ is a trajectory of $\mathbb{S}$ along which (43) - (45) hold. If for some $i \in \{1, 2, \ldots, n\}$ and $s \in \mathcal{S}_i$, node i is locally rendezvoused, then the n nodes of $\mathbb{S}$ are globally rendezvoused.*

Establishing the preceding result requires one to be able to conclude that if for some for some $i, j \in \{1, 2, \ldots, n\}$ and some $s \in \mathcal{S}_i$, nodes i and j are in the same "position" in the sense that $y_i(s) = y_j(s)$ and $w_i(s) = w_j(s)$, then $\mathcal{N}_j \subset \mathcal{N}_i$. In words, what this is roughly saying is that if node j is in the same position as node i, then node j's "neighbors" must also be neighbors of node i. This *weak transitivity property* is not necessarily true for $\mathbb{S}$ but it is true if $y(s)$ and $w(s)$ are defined by (43) and (44) respectively. This is a consequence of the following lemma.

Lemma 6. *Let $y(s)$ and $w(s)$ be defined by (43) and (44) respectively. Let $i \in \{1, 2, \ldots, n\}$ and $s \in \mathcal{S}_i$ be fixed. Suppose that for some $j \in \{1, 2, \ldots, n\}$,*

$$||y_i(s) - y_j(s)|| \le r \tag{46}$$

$$||w_i(s) - y_j(s)|| \le r \tag{47}$$

Then $j \in \mathcal{N}_i$

The following proposition shows that if $\mathcal{H}$ does not change for a sufficiently long period of time, then the n nodes have rendezvoused.

Proposition 12. *Suppose $\mathbb{G}$ is a connected graph. Suppose in addition that $\{y(s), w(s), v_1(s), v_2(s), \ldots, v_n(s) : s \in \mathcal{S}\}$ is a trajectory of $\mathbb{S}$ along which (43) - (44) hold. Suppose that s_a and s_b are values in $\mathcal{S}$ for which $s_b - s_a \ge 8n$ and*

$$\text{dia}\{\mathcal{H}(s_a)\} = \text{dia}\{\mathcal{H}(s_b)\} \tag{48}$$

Then the n nodes of $\mathbb{S}$ have rendezvoused at $s = s_b$.

The following theorem is our main convergence result concerning $\mathbb{S}$. The main result of this paper, Theorem 2, is an immediate consequence.

Theorem 3. *Let* $\{\{y(s), w(s), v_1(s), \ldots, v_n(s)\} : s \in \mathcal{S}\}$ *be a trajectory of* $\mathbb{S}$ *along which (43)-(45) hold. If* $\mathbb{G}$ *is a connected graph, then*

$$\lim_{s \to \infty} \mathrm{dia}\langle y_1(s), y_2(s), \ldots, y_n(s), w_1(s), w_2(s), \ldots, w_n(s)\rangle = 0 \qquad (49)$$

Proof of Theorem 3: In the sequel we often write $x(s)$ for $\{y(s), w(s), v_1(s), \ldots, v_n(s)\}$. Let $V : \mathcal{X} \to \mathbb{R}$ denote the diameter function $\{y, w, v_1, v_2, \ldots, v_n\} \longmapsto \mathrm{dia}\langle y_1, y_2, \ldots, y_n, w_1, w_2, \ldots, w_n\rangle$. As a consequence of Lemma 1, $V(x(s))$ is a monotone non-increasing function of s Clearly $V(x(s))$ is bounded below by 0. Moreover $V(x(s))$ is bounded above by $V(x(0))$ because $V(\cdot)$ is continuous and $\mathcal{X}$ is compact. Therefore there must exist a finite limit

$$V^* = \lim_{s \to \infty} V(x(s))$$

We claim that $V^* = 0$. To prove this claim, suppose that is false. Then $V^* > 0$. This means that the trajectory $\{x(s) : s \in \mathcal{S}\}$ cannot contain any points in the set $\mathcal{E} = \{x : V(x) = 0\}$. To proceed, fix $\bar{s} > 8n$ and let $\Delta(x(s))$ denote the difference

$$\Delta(x(s)) = V(x(\bar{s} + s)) - V(x(s)) \qquad (50)$$

Since $V(x(s))$ is monotone non-increasing, $\Delta(x(s)) \leq 0, \quad s \in \mathcal{S}$. In the light of Proposition 12 and the fact that $\mathcal{E}$ has no points in common with $\{x(s) : s \in \mathcal{S}\}$, one can conclude that $\Delta(x(s)) \neq 0, \quad s \in \mathcal{S}$. Therefore

$$\Delta(x(s)) < 0, \quad s \in \mathcal{S} \qquad (51)$$

According to Proposition 10, for each $s \in \mathcal{S}$ there is a continuous function D_q such that $x(s + \bar{s}) = D_q(x(s))$. Let $\mathcal{W}_q$ denote the set of state pairs $(x(s + \bar{s}), x(s))$ along the given trajectory for which this formula holds. It follows that

$$\{(x(s + \bar{s}), x(s)) : s \in \mathcal{S}\} = \bigcup_{q \in \mathcal{Q}} \mathcal{W}_q$$

and that each $\mathcal{W}_q$ is a closed set. For $(\bar{x}, x) \in \mathcal{W}_q$ define $\Delta_q : \mathcal{W}_q \to \mathbb{R}$ so that $(\bar{x}, x) \longmapsto V(D_q(\bar{x})) - V(x)$. Note that Δ_q is a continuous function on $\mathcal{W}_q$ whose value at each point $(\bar{x}, x) \in \mathcal{W}_q$ agrees with $\Delta(x(s))$ for some s. It follows from (51) that

$$\Delta_q(\bar{x}, x) < 0, \ \ (\bar{x}, x) \in \mathcal{W}_q$$

Define

$$\mu_q = \inf_{(\bar{x}, x) \in \mathcal{W}_q} \Delta_q(\bar{x}, x)$$

Since $\mathcal{W}_q$ is compact and Δ_q is negative and continuous on $\mathcal{W}_q$, it must be true that $\mu_q < 0$. Let

$$\mu = \max_{q \in \mathcal{Q}} \mu_i$$

Since $\mathcal{Q}$ is finite, $\mu < 0$. Clearly

$$\Delta_q(\bar{x}, x) \leq \mu \quad (\bar{x}, x) \in \mathcal{W}_q, \quad q \in \mathcal{Q} \tag{52}$$

Note that by construction, for each $s \in \mathcal{S}$ there must be a $q \in \mathcal{Q}$ such that $\Delta(x(s)) = \Delta_q(x(s + \bar{s}), x(s)))$ ¿From this and (52) it follows that

$$\Delta(x(s)) \leq -\mu, \quad s \in \mathcal{S}$$

Note that

$$V(x(s + \bar{s})) - V(x(s)) = \Delta(x(s)) \leq \mu, \quad s \in \mathcal{S}$$

Thus by summing,

$$V(x(s + k\bar{s})) \leq V(x(s))k\mu, \quad k \geq 1$$

Therefore, for k sufficiently large $V(x(s+k\bar{s}))$ must be negative because $\mu < 0$. But this is impossible because $V(\cdot)$ is positive definite. Hence V^* cannot be positive. ■

4 Concluding Remarks

The approach taken in this paper appears to have much in common with the embedding process discussed in Chapter 7 of [18] for analyzing "partially asynchronous iterative algorithms." This suggests that the tools developed in [18] may be helpful in understanding the asynchronous system considered in this paper.

In summary, if one is to avoid detailed maneuvering models in studying the multi-agent rendezvous problem, then one must deal with an event driven system - actually a hybrid system - in which *non-deterministic* state transitions can occur. This somewhat surprising fact seems to be related to our objective of only using high level models. Understanding this phenomenon and how to analyze systems with this property, is a topic for future research.

References

1. H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita. Distributed memoryless point convergence algorithm for mobile robots with limited visibility. *IEEE Transactions on Robotics and Automation*, pages 818–828, oct 1999.

2. L. Zhiyun, M. Brouche, and B. Francis. Local control strategies for groups of mobile autonomous agents. Ece control group report, University of Toronto, 2003.
3. T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, pages 1226–1229, 1995.
4. A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, pages 988–1001, june 2003. also in Proc. 2002 IEEE CDC, pages 2953 - 2958.
5. J. A. Fax and R. M. Murray. Graph laplacians and vehicle formation stabilization. CDS Technical Report 01-007, California Institute of Technology, 2001.
6. T. Eren, P. N. Belhumeur, and A. S. Morse. Closing ranks in vehicle formations based on rigidity. In *Proceedings of the 2002 IEEE Conference on Decision and Control*, pages 2959 – 2964, dec 2002.
7. T. Eren, P. N. Belhumeur, B. D. O. Anderson, and A. S. Morse. A framework for maintaining formations based on rigidity. In *Proceedings of the 2002 IFAC Congress*, pages 2752–2757, 2002.
8. D. E. Chang and J. E. Marsden. Gyroscopic forces and collision avoidance. In *Proc. Symposium on New Trends in Nonlinear Dynamics and Control and their Applications*, oct 2002. to appear.
9. J. Alexander Fax and Richard M. Murray. Graph laplacians and stabilization of vehicle formations. *15th IFAC Congress, Barcelona, Spain*, 2002.
10. N. Leonard and E. Friorelli. Virtual leaders, artificial potentials and coordinated control of groups. *IEEE Conference on Decision and Control, Orlando, FL*, 2001.
11. J. P. Desai, J. P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE Transactions on Robotics and Automation*, 17(6):905–908, 2001.
12. Y. Liu, K.M. Passino, and M. Polycarpou. Stability analysis of one-dimensional asynchronous swarms. In *American Control Conference, Arlington, VA*, pages 716–721, June, 2001.
13. P. Tabuado, G. J. Pappas, and P. Lima. Feasible formations of multi-agent systems, 2001.
14. H. Tanner, A. Jadbabaie, and G. Pappas. Distributed coordination strategies for groups of mobile autonomous agents. Technical report, ESE Department, University of Pennsylvania, December 2002.
15. C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, 2001.
16. H. Rademacher and O. Toeplitz. *The Enjoyment of Mathematics*. Princeton University Press, Princeton, N.J., 1957.
17. Rangarajan K. Sundaram. *A First Course in Optimization Theory*. Cambridge University Press, Cambridge, UK, 1996.
18. D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation*. Prentice Hall, 1989.