Asymptotic Methods for Stability Analysis of Markov Dynamical Systems with Fast Variables

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Summary. We deal with two scale stochastic dynamical systems under Markov perturbations. Our goal is to study asymptotic stability properties of the solutions of such systems. We apply averaging procedures to obtain simpler processes which are then used for the stability analysis of both the slow and the fast components of the original system.

Key words: two-scale system, slow and fast motion, Markov perturbation, stochastic stability, exponential stability, averaging procedure, linear approximation, diffusion approximation, Lyapunov function

Mathematics Subject Classification (2000): 60H10, 93E15, 34F05

1 Introduction: Model and Problems

The averaging principle and diffusion approximation procedures are among the most frequently used asymptotic methods for analysis of nonlinear dynamical systems subjected to random perturbations [1], [5], [6], [9], [13], [15], [16], [18], [19], [20]. It has been recognized that the averaging principle is a powerful tool for analyzing interesting phenomena in the engineering sciences, for example, when studying asymptotically stable multifrequency oscillations, loss of stability due to parametric resonance, etc., see [17] and the references therein. This approach, supplemented recently by probabilistic limit theorems, was used not only in engineering sciences [2] but also applied in social sciences such as economics and medicine [22], [8], [20]. The limit theorems obtained in this area allow us to construct simpler dynamical systems, which are successfully used for approximate analysis of the initial system on finite time intervals and also to describe the asymptotic behavior of the phase coordinates as the time $t \to \infty$, see [1], [10], [11], [13], [20]. It is worth mentioning that mostly in engineering applications only a part of the coordinates have limits as $t \to \infty$, while the rest coordinates undulate and do not have any limit [2]. This creates some difficulties when applying asymptotic methods of nonlinear dynamics and probabilistic limit theorems.

Let us describe the model which we are going to study in this paper. We introduce a "small" positive parameter ε, where $\varepsilon \in (0, \varepsilon_0)$, for some fixed $\varepsilon_0 > 0$. We assume that the system variables, as functions of time, are separated into a fast component (called "radial motion"), and a slow component (called "rotation"). The fast component has "velocity" which is proportional to a negative power of ε, while the slow component has a limit as $\varepsilon \to 0$. We also assume that the dynamical system depends on other fast random variables (that means functions of t/ε) modelled as an ergodic Markov process [13], [15], [19]. Thus we study a system of random differential equations of the following form:

$$\frac{\mathrm{d}x^\varepsilon(t)}{\mathrm{d}t} = F(x^\varepsilon(t), y^\varepsilon(t), \xi^\varepsilon(t), \varepsilon), \tag{1.1}$$

$$\frac{\mathrm{d}y^\varepsilon(t)}{\mathrm{d}t} = \frac{1}{\varepsilon} H(y^\varepsilon(t), \xi^\varepsilon(t), \varepsilon), \ t \geq 0. \tag{1.2}$$

Here $\varepsilon \in (0, \varepsilon_0)$, $F(x, y, z, \varepsilon)$ and $H(y, z, \varepsilon)$ are vector-functions, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{G}$, and $\xi^\varepsilon = (\xi^\varepsilon(t), t \geq 0)$ is a homogeneous right continuous ergodic Markov process on some compact phase space $\mathbb{G}$ with a weak infinitesimal operator Q^ε and an invariant measure μ, which is the same for all ε. If $F(x, y, z, \varepsilon)$ and $H(y, z, \varepsilon)$ are sufficiently smooth functions, then the Cauchy problem for the system (1.1)–(1.2) with initial conditions $x^\varepsilon(s) = x$, $y^\varepsilon(s) = y$ and $\xi^\varepsilon(s) = z$, where $s \geq 0$, has a unique solution $x^\varepsilon(t) = x^\varepsilon(s, t, x, y, z)$, $y^\varepsilon(t) = y^\varepsilon(s, t, x, y, z)$ for any $t \geq s$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{G}$. Let us assume that the trivial solution $x^\varepsilon(t) \equiv 0 \in \mathbb{R}^n$ is an equilibrium point for the slow motion (1.1), that is, $F(0, y, z, \varepsilon) \equiv 0$. One of our goals is to analyze asymptotic stability properties of this equilibrium. For completeness of the presentation we recall some definitions from the classical book [12]. In these definitions ε is fixed and we are interested in the stability of the trivial solution of (1.1) uniformly in $\varepsilon \in (0, \varepsilon_0)$. Examples of systems which are stable in one sense but not in another one can be seen in [12].

We say that equation (1.1), or that its trivial solution, is:

- *locally stable almost surely* (a.s.), if for any $s \geq 0$, $\eta > 0$ and $\beta > 0$, there exists $\delta > 0$ such that the inequality

$$\sup_{y \in \mathbb{R}^m,\ z \in \mathbb{G}} \mathbf{P}\left(\sup_{t \geq s} |x^\varepsilon(s, t, x, y, z)| > \eta\right) < \beta \tag{1.3}$$

  is satisfied for all x in the ball $B_\delta(0) := \{u \in \mathbb{R}^n : |u| < \delta\}$;

- *locally asymptotically stochastically stable*, if it is locally a.s. stable and there exists $\gamma > 0$ such that the trajectories, which do not leave the ball $B_\gamma(0)$, tend to 0 in probability, as $t \to \infty$, that is, for any $c > 0$ and fixed other initial data, we have

$$\lim_{t\to\infty} \mathbf{P}[|x^\varepsilon(s,t,x,y,z)| > c, \{x^\varepsilon(s,t,x,y,z), t \geq s\} \subset B_\gamma(0)] = 0;$$

- *asymptotically stochastically stable*, if it is locally a.s. stable and for any $x \in \mathbb{R}^n$, $s \in \mathbb{R}^+$, and $c > 0$, the following relation holds:

$$\lim_{T\to\infty} \sup_{y\in\mathbb{R}^m,\ z\in\mathbb{G}} \mathbf{P}\left(\sup_{t>T} |x^\varepsilon(s,t,x,y,z)| > c\right) = 0; \tag{1.4}$$

- *exponentially p-stable*, if there are numbers $M > 0$, $\gamma > 0$ such that for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{G}$, $s \geq 0$ and $t > s$ one holds:

$$\mathbf{E}[|x^\varepsilon(s,t,x,y,z)|^p] \leq M|x|^p\, e^{-\gamma(t-s)}. \tag{1.5}$$

The paper is organized as follows. In Section 2 we prove that for linear Markov dynamical systems, the asymptotic stochastic stability of the equilibrium is equivalent to the exponential p-stability for sufficiently small $p > 0$. In Section 3 we show that the exponential p-stability of the linearized Markov system in a neighborhood of its equilibrium state, guarantees the asymptotic (local) stochastic stability of this equilibrium. These results are similar to results in [19] and [20]. However, we have included them here for a better understanding of our approach and for describing a modification of the second Lyapunov method for stochastic stability analysis. Based on the results in Sections 2 and 3, we can analyze the equilibrium stability of the slow motion by rewriting the system (1.1)–(1.2) in the following form:

$$\frac{\mathrm{d}x^\varepsilon(t)}{\mathrm{d}t} = [A_0(y^\varepsilon(t),\xi^\varepsilon(t)) + \varepsilon A_1(y^\varepsilon(t),\xi^\varepsilon(t))]x^\varepsilon(t), \tag{1.6}$$

$$\frac{\mathrm{d}y^\varepsilon(t)}{\mathrm{d}t} = \frac{1}{\varepsilon}h_{-1}(y^\varepsilon(t),\xi^\varepsilon(t)) + h_0(y^\varepsilon(t),\xi^\varepsilon(t)),\ t \geq 0. \tag{1.7}$$

Here $\xi^\varepsilon = (\xi^\varepsilon(t), t \geq 0)$ is a Markov process with infinitesimal operator $Q^\varepsilon = \frac{1}{\varepsilon^2}Q$. The operator Q is supposed to be closed with spectrum $\sigma(Q)$ split into two parts, $\sigma(Q) = \sigma_{-\rho}(Q) \cup \{0\}$, where $\sigma_{-\rho}(Q) \subset \{\mathrm{Re}\lambda \leq -\rho < 0\}$ and zero eigenvalue has multiplicity one. This assumption, see [4], guarantees ergodicity of Markov processes defined by infinitesimal operators $\frac{1}{\varepsilon^2}Q$ and with the same invariant measure μ. To avoid cumbersome formulas an averaging in the Markov phase coordinate $z \in \mathbb{G}$ of any function $f(x,y,z)$ with respect to the invariant measure μ will be denoted by $\bar{f}$, that is, $\bar{f}(x,y) := \int_{\mathbb{G}} f(x,y,z)\mu(\mathrm{d}z)$. In Section 4 we discuss some results for the fast motion assuming that $\bar{h}_{-1}(y) \equiv 0$. In this case, under some assumptions, the stability analysis is based on an averaging procedure for the slow motion (1.6) with a diffusion approximation of the fast motion (1.7):

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = \bar{A}_0(\hat{y}(t))\,\bar{x}(t), \tag{1.8}$$

$$\mathrm{d}\hat{y}(t) = a(\hat{y}(t))\mathrm{d}t + \sigma(\hat{y}(t))\mathrm{d}w(t),\ t \geq 0. \tag{1.9}$$

The coefficients $a(y)$, $\sigma(y)$ are defined by the functions in the right-hand side of (1.7), being respectively the potential of the operator Q and averaging with respect to the invariant measure μ. We prove that the asymptotic stochastic stability of the slow motion (1.1) follows from the exponential p-stability of the random differential equation (1.8).

2 Stochastic Stability of Linear Differential Equations with Markov Coefficients

In this section we deal with the following linear differential equation in $\mathbb{R}^n$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(y(t))\,x(t),\ t \geq 0. \tag{2.1}$$

Here $A(y)$, $y \in \mathbb{R}^m$ is a continuous bounded matrix-valued function and $y(t)$, $t \geq 0$ is a $\mathbb{Y}$–valued stochastically continuous Feller Markov process with weak infinitesimal operator Q and we assume that $\mathbb{Y} \subset \mathbb{R}^m$. The pair $\{x(t), y(t)\}$, $t \geq 0$ forms, see [19], a homogeneous stochastically continuous Markov process whose weak infinitesimal operator, denoted by $\mathcal{L}_0$, is defined as follows:

$$\mathcal{L}_0 v(x,y) = \langle A(y)x, \nabla_x\rangle v(x,y) + Qv(x,y). \tag{2.2}$$

It is clear that there exists a family $\{X(s,t,y),\ 0 \leq s \leq t\}$, of matrix-valued functions defined by the equality $X(s,t,y)x = x(s,t,x,y)$, where $x(s,t,x,y)$, $s \leq t$, denoted simply by $x(t)$, is the solution of the Cauchy problem for (2.1) under the conditions $x(s) = x$ and $y(s) = y$. The matrices $X(s,t,y)$ also satisfy equation (2.1) for all $t > s$ and the initial condition $X(s,s,y) = I$, where I is the unit matrix of order n. This matrix family has the evolution property:

$$X(s,t,y) = X(s,\tau,y(\tau))X(\tau,t,y),\ y \in \mathbb{Y},\ 0 \leq s \leq \tau \leq t. \tag{2.3}$$

The Lyapunov exponent, or p-index, $\lambda^{(p)}$, of equation (2.1) is defined by

$$\lambda^{(p)} = \sup_{x,y} \varlimsup_{t\to\infty} \frac{1}{pt} \ln \mathbf{E}[|X(s,t,y)x|^p]. \tag{2.4}$$

It is not difficult to show that the exponential p-stability of the trivial solution of equation (2.1) is equivalent to the condition $\lambda^{(p)} < 0$. Since for any positive $p_1 < p_2$ we have $(\mathbf{E}[|X(t,s,y)x|^{p_1}])^{1/p_1} \leq (\mathbf{E}[|X(t,s,y)x|^{p_2}])^{1/p_2}$ (Lyapunov inequality), then $p_1 < p_2$ implies that $\lambda^{(p_1)} \leq \lambda^{(p_2)}$, and hence $\lambda^{(p)}$ is a monotone decreasing function as p decreases to 0. In this section we will prove that the asymptotic stochastic stability of (2.1) is equivalent to the following condition: there exists a number $p_0 > 0$, such that $\lambda^{(p)} < 0$ for all $p \in (0, p_0)$.

Lemma 2.1. *If equation* (2.1) *is asymptotically stochastically stable, then it is exponentially p-stable for all sufficiently small positive p.*

Proof. In the definition of a.s. stability we take $\eta = 1$, $\beta = \frac{1}{2}$ and choose $\alpha > 0$ so small that $\sup_{x,y} \mathbf{P}\left(\sup_{t\geq 0} |X(0,t,y)x| > 1\right) < \frac{1}{2}$ for $|x| \leq 2^{-\alpha}$, $y \in \mathbb{Y}$. Since equation (2.1) is linear, then $\sup_{x,y} \mathbf{P}\left(\sup_{t\geq 0} |X(0,t,y)x| > 2^{k\alpha}\right) < \frac{1}{2}$ for $|x| \leq 2^{-\alpha(k-1)}$, $y \in \mathbb{Y}$ and any $k \in \mathbb{N}$. Let us introduce the following notation:

$$g_k := \sup_{|x|\leq 1,\ y\in\mathbb{Y}} \mathbf{P}\left(\sup_{t\geq 0} |\ X(0,t,y)x| \geq 2^{k\alpha}\right).$$

The pair $\{x(t), y(t)\}$, $t \in \mathbb{R}^+$ is a stochastically continuous Markov process. Therefore for any $x \in B_1(0)$ there exits a time $\tau_1(x)$ such that the trajectory $x(0,t,x,y)$ leaves the ball $B_1(0)$. Hence

$$\begin{aligned} g_{k+1} = \sup_{|x|\leq 1,\ y\in\mathbb{Y}} \int_{s=0}^{\infty} \int_{|u|=2^{k\alpha},\ v\in\mathbb{Y}} &\mathbf{P}_{x,y}(\tau_1(x) \in \mathrm{d}s, x(s) \in \mathrm{d}u, y(s) \in \mathrm{d}v) \\ &\times \mathbf{P}\left(\sup_{t\geq 0} |X(0,t,v)u| > 2^{(k+1)\alpha}\right) \\ &\leq \sup_{|x|\leq 2^{k\alpha},\ y\in\mathbb{Y}} \mathbf{P}\left(\sup_{t\geq 0} |X(0,t,y)x| > 2^{(k+1)\alpha}\right) \\ &\times \sup_{|x|\leq 1,\ y\in\mathbb{Y}} \int_{s=0}^{\infty} \int_{|u|=2^{k\alpha},\ v\in\mathbb{Y}} \mathbf{P}_{x,y}(\tau_1(x) \in \mathrm{d}s, x(s) \in \mathrm{d}u, y(s) \in \mathrm{d}v) \\ &\leq \frac{1}{2} \sup_{|x|\leq 1,\ y\in\mathbb{Y}} \mathbf{P}\left(\sup_{t\geq 0} |X(0,t,y)x| \geq 2^{k\alpha}\right) = \frac{1}{2} g_k \end{aligned}$$

and therefore $g_k \leq 2^{-k}$ for any $k \in \mathbb{N}$. Define $\zeta(x,y) := \sup_{t\geq 0} |x(0,t,x,y)|^p$. It is easy to see that for all $p > 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{Y}$ one can write

$$\mathbf{E}[\zeta(x,y)] \leq |x|^p \sup_{|x|\leq 1} \mathbf{E}[\zeta(x,y)] \leq \sum_{k=1}^{\infty} 2^{k\alpha p} \mathbf{P}\left(\sup_{t\geq 0} |x(0,t,x,y)| \geq 2^{(k-1)\alpha}\right).$$

Therefore $\mathbf{E}[\zeta(x,y)] \leq \sum_{k=1}^{\infty} 2^{k\alpha p} 2^{-k} |x|^p := K_1 |x|^p$ for all $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$ and $p \in (0, \alpha^{-1})$. The assumption in Lemma 2.1 implies that the solution $x(0,t,x,y)$, $t \geq 0$ of (2.1) tends to 0 a.s. as $t \to \infty$ uniformly in $y \in \mathbb{Y}$. By the Lebesgue Theorem we conclude that $\lim_{t\to\infty} \sup_{y\in\mathbb{Y}} \mathbf{E}[|x(s, s+t, x, y)|^p] = 0$, for all $x \in \mathbb{R}^n$ and $p \in (0, \alpha^{-1})$. Moreover, it is not difficult to verify that this convergence is uniform in $x \in B_1(0)$ and $s \geq 0$, i.e. $\lim_{t\to\infty} \sup_{x\in B_1(0),\ y\in\mathbb{Y}} \mathbf{E}[|x(s, s+t, x, y)|^p] = 0$. Now we can choose a number T so large that $\sup_{y\in\mathbb{Y}} \mathbf{E}[|x(s, s+t, x, y)|^p] \leq |x|^p e^{-1}$. Further, by using the inequality

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{Y}} \mathbf{P}((k-1)T, x, y, \mathrm{d}u, \mathrm{d}v)\mathbf{E}[|x(0,T,u,v)|^p] \le \frac{1}{e}\mathbf{E}[|x(0,(k-1)T,x,y)|^p],$$

where $\mathbf{P}(t, x, y, \mathrm{d}u, \mathrm{d}v)$ is the transition probability of the homogeneous Markov process $\{x(t), y(t)\}$, $t \ge 0$, one can write

$$\mathbf{E}[|x(0,t,x,y)|^p]{\le}K_1 e^{-[t/T]T}|x|^p,$$

where $[t/T]$ stands for the integer part of the real number t/T. This completes the proof. □

The behavior of the solution of (2.1) for $t \ge u$ with $x(u) = x,\ y(u) = y$, can be studied by using the well-known Dynkin formula:

$$\mathbf{E}^{(u)}_{x,y}[v(x(t), y(t))] = v(x, y) + \int\limits_u^t \mathbf{E}^{(u)}_{x,y}[\mathcal{L}_0 v(x(s), y(s))]\,\mathrm{d}s. \tag{2.5}$$

Sometimes it is necessary to use Lyapunov functions depending also on the time argument t. If $v(t, x, y)$, as a function of x and y, belongs to the domain of the infinitesimal operator $\mathcal{L}_0$ and has continuous t-derivative, we can rewrite formula (2.5) in the form

$$\mathbf{E}^{(u)}_{x,y}[v(t, x(t), y(t))] = v(u, x, y) + \int_u^t \mathbf{E}^{(u)}_{x,y}\left[\left(\frac{\partial}{\partial s} + \mathcal{L}_0\right) v(s, x(s), y(s))\right] \mathrm{d}s. \tag{2.6}$$

Lemma 2.2. *The trivial solution of equation* (2.1) *is exponentially* p*-stable if and only if there exists a Lyapunov function* $v(x, y)$ *and a number* $p > 0$ *such that for some positive constants* c_1, c_2, c_3 *and for all* $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$, *the following two conditions are satisfied:*

$$c_1|x|^p \le v(x, y) \le c_2|x|^p, \quad \mathcal{L}_0 v(x, y) \le -c_3|x|^p. \tag{2.7}$$

Proof. Suppose that there exists such a Lyapunov function. This implies that $\left(\frac{\partial}{\partial s} + \mathcal{L}_0\right)\left(v(x, y)\, e^{c_3 t/c_2}\right) \le 0$, which in combination with formula (2.6) yields $\mathbf{E}_{x,y}[v(x(t), y(t))\, e^{c_3 t/c_2}] \le v(x, y) \le c_2\, |x|^p$ for all $t > 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{Y}$. Hence $\mathbf{E}_{x,y}[|x(t)|^p] \le (c_2/c_1)\, e^{-c_3 t/c_2}\, |x|^p$ and we conclude that equation (2.1) is exponentially p-stable. By using the solutions $x(s, s+t, x, y)$ of (2.1), we can define, for any $T > 0$, the function

$$v(x, y) := \int\limits_0^T \mathbf{E}[|x(s, s+t, x, y)|^p]\,\mathrm{d}t, \tag{2.8}$$

which does not dependent on s because of the homogeneity of the Markov process $y(t)$. Since the matrix $A(y)$ is uniformly bounded, i.e. $\sup_{y\in\mathbb{Y}} \|A(y)\| :=$

$a < \infty$, it is easy to verify that the function $v(x,y)$ satisfies the first condition in (2.7). Let $\mathcal{L}_0$ be the weak infinitesimal operator of the pair $\{x(t), y(t)\}$, $t \geq 0$. If the trivial solution of (2.1) is exponentially p-stable, one can write the relations

$$\begin{aligned}\mathcal{L}_0 v(x,y) &= \lim_{\delta\to 0}\frac{1}{\delta}\left[\int_0^T \mathbf{E}_{x,y}\{\mathbf{E}_{x(\delta),y(\delta)}[|x(t+\delta)|^p]\}\,\mathrm{d}t - \int_0^T \mathbf{E}_{x,y}[|x(s)|^p]\,\mathrm{d}s\right]\\ &= \mathbf{E}_{x,y}[|x(T)|^p] - |x|^p \leq (M\,e^{-\gamma T}-1)|x|^p,\end{aligned}$$

where M and γ are the constants in the definition of the exponential p-stability. Now we take $T = (\ln 2 + \ln M)/\gamma$, and see that the proof is completed. □

Corollary 2.1. *Under the conditions in Lemma 2.2, the trivial solution of equation* (2.1) *is asymptotically stochastically stable.*

Proof. Applying formula (2.6) to the function $\bar{v}(t,x,y) = v(x,y)\,e^{c_3 t/c_2}$ we see that the random process $\theta(t) := v(x(t),y(t))\,e^{c_3 t/c_2}$, $t \geq 0$ is a positive supermartingale. Hence

$$\begin{aligned}&\sup_{y\in\mathbb{Y}}\mathbf{P}\left(\sup_{t\geq 0}|x(0,t,x,y| > \varepsilon\right) \leq \sup_{y\in\mathbb{Y}}\mathbf{P}_{x,y}\left(\sup_{t\geq 0}\{\frac{1}{c_1}v(x(t),y(t))\} > \varepsilon^p\right)\\ &\quad\leq \sup_{y\in\mathbb{Y}}\mathbf{P}_{x,y}\left(\sup_{t\geq 0}\theta(t) > \varepsilon^p c_1\right) \leq (1/\varepsilon^p c_1)\mathbf{E}_{x,y}[\theta(0)] \leq (c_2/\varepsilon^p c_1)\,|x|^p\end{aligned}$$

and the trivial solution of (2.1) is a.s. stochastically stable. Now, to prove the asymptotic stochastic stability, we apply the supermartingale inequality [3]:

$$\begin{aligned}&\sup_{y\in\mathbb{Y}}\mathbf{P}\left(\sup_{t\geq u}|x(u,t,x,y| > c\right) \leq \sup_{y\in\mathbb{Y}}\mathbf{P}^{(u)}_{x,y}\left(\sup_{t\geq u}\{\tfrac{1}{c_1}v(x(t),y(t))\} > c^p\right)\\ &\quad\leq \sup_{y\in\mathbb{Y}}\mathbf{P}^{(u)}_{x,y}\left(\sup_{t\geq u}\{\frac{1}{c_1}\theta(t)\,e^{-c_3 t/c_2}\} > c^p\right) \leq (c_2/c^p c_1)\,|x|^p\,e^{-uc_3/c_2}.\end{aligned}$$

The proof is complete. □

3 Stochastic Stability Based on Linear Approximation

In this section we consider the quasilinear equation

$$\frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = A(y(t))\tilde{x}(t) + g(\tilde{x}(t),y(t)), \quad t \geq 0. \tag{3.1}$$

Here the matrix $A(y)$ and the Markov process $y(t)$, $t \geq 0$ satisfy the conditions given in Section 2. We assume that the function $g(x,y)$ is such that $g(0,y) \equiv 0$, and moreover that $g(x,y)$ obeys bounded continuous x-derivative $D_x g(x,y)$ which is uniformly bounded in the ball $B_r(0)$ for any $r>0$, that is,

$$\sup_{y\in\mathbb{Y},\ x\in B_r(0)} \|D_x g(x,y)\| := g_r < \infty. \tag{3.2}$$

Theorem 3.1. *Suppose that equation* (2.1) *is asymptotically stochastically stable and that* $\lim_{r\to 0} g_r = 0$. *Then equation* (3.1) *is locally asymptotically stochastically stable.*

Proof. Let us mention first that there are many functions $g(x,y)$ satisfying the condition $\lim_{r\to 0} g_r = 0$. A simple example in the one-dimensional case is to take $g(x,y) = h(y)\,x^{\gamma}/(1+x^2)$, where $\gamma = const > 1$ and $h(y)$ is a bounded function.

We consider (2.1) as the linear approximation of equation (3.1). In view of Lemma 2.1 and Lemma 2.2, we can construct the Lyapunov function (2.8) with some small $p>0$. Since the matrix-valued function $D_x\, x(0,t,x,y)$ is the Cauchy matrix of equation (2.1), then the following estimate is valid:

$$\sup_{y\in\mathbb{Y}} \mathbf{E}[\|D_x\, x(s,s+t,x,y)\|^p] \le h_2\, e^{-\gamma t}$$

with some positive constants h, γ and for all $t>0$. Therefore the above Lyapunov function satisfies the conditions (2.7) and by construction for all $x\neq 0$ it has x-derivative which satisfies the inequalities

$$\begin{aligned}&\left|\int_0^T \mathbf{E}[\nabla_x |x(s,s+t,x,y)|^p]\,\mathrm{d}t\right| \\ &\qquad\qquad \le p\,|x|^{p-1}\int_0^T \sup_{y\in\mathbb{Y}} \mathbf{E}[\|D_x\, x(s,s+t,x,y)\|^p]\,\mathrm{d}t \le c_3\,|x|^{p-1}\end{aligned}$$

for some $c_3>0$. Now we estimate the function $\mathcal{L}v(x,y)$, where $\mathcal{L}$ is the weak infinitesimal operator of the pair $\{\tilde{x}(t), y(t)\}$, $t\ge 0$, and we use $\mathcal{L}_0$ as given by (2.2):

$$\begin{aligned}\mathcal{L}v(x,y) &= \mathcal{L}_0 v(x,y) + \langle g(x,y), \nabla_x\rangle v(x,y) \le -\frac{1}{2}\,|x|^p + c_3\,|x|^p\,|g(x,y)| \\ &\le \left(g_r c_3 - \frac{1}{2}\right)|x|^p\end{aligned}$$

for all $x\in B_r(0)$, $r>0$. Hence, in view of the Dynkin formula, we use the estimate

$$\mathbf{E}^{(u)}_{x,y}[v(\tilde{x}(\tau_r(t)), y(\tau_r(t)))] \le v(x,y) + \left(g_r c_3 - \frac{1}{2}\right)\mathbf{E}^{(u)}_{x,y}\left[\int_u^{\tau_r(t)} |\tilde{x}(s)|^p\,\mathrm{d}s\right], \tag{3.3}$$

which is valid for all $y \in \mathbb{Y}$, $x \in B_r(0)$, $r > 0$, $t \geq u \geq 0$. If r is sufficiently small, then the second term in the right-hand side of (3.3) is non-positive. Hence the process $v(\tilde{x}(\tau_r(t)), y(\tau_r(t))$, $t \geq 0$ is a supermartingale, so

$$\mathbf{P}_{x,y}\left(\sup_{t\geq 0} |\tilde{x}(t)| > \varepsilon\right) \leq \mathbf{P}_{x,y}\left(\sup_{t\geq 0} v(\tilde{x}(\tau_r(t)), y(\tau_r(t)) > c_1\varepsilon^p\right) \leq \frac{c_2\delta^p}{c_1\varepsilon^p} \quad (3.4)$$

for all $y \in \mathbb{Y}$, $x \in B_\delta(0)$, $\delta \in (0, \varepsilon)$, $\varepsilon \in (0, r)$ and sufficiently small $r > 0$. The a.s. local stability immediately follows from these estimates. Let us define the function $h_R(r)$ as follows: $h_R(r) = 1$ for $x \in [0, R)$, $h_R(r) = (2R - r)/R$ for $x \in [R, 2R)$, $h_R(r) = 0$ for $x \geq 2R$. Consider the following random differential equation:

$$\frac{\mathrm{d}x_R(t)}{\mathrm{d}t} = A(y(t))\, x_R(t) + h_R(|x_R(t)|)\, g(x_R(t), y(t)), \quad t \geq 0. \quad (3.5)$$

The Cauchy problem for (3.5) with initial condition $x_R(0) = x$ has a unique solution since the function $h_R(|x|)\, g(x, y)$ satisfies the Lipschitz condition with a constant c_{2R}. Hence the pair $\{x_R(t), y(t)\}$, $t \geq 0$ is a Markov process whose weak infinitesimal operator $\mathcal{L}_R$ is defined as follows:

$$\mathcal{L}_R v(x, y) = \mathcal{L}_0 v(x, y) + \langle h_R(|x|)\, g(x, y), \nabla_x \rangle\, v(x, y).$$

Now choosing R so small that $(c_{2R}\, c_3 - \frac{1}{2}) := -c_4 < 0$, one can write the estimate $\mathcal{L}_R\, v(x, y) \leq -c_4|x|^p$. Therefore

$$\mathbf{E}^{(u)}_{x,y}[v(x_R(t), y(t))] \leq v(x, y) - \frac{c_4}{c_1}\int_u^t \mathbf{E}^{(u)}_{x,y}[v(x_R(s), y(s))]\,\mathrm{d}s \quad (3.6)$$

for all $t \geq u \geq 0$. Hence the stochastic process $v(x_R(t), y(t))$, $t \geq 0$ is a positive supermartingale and we have that

$$\mathbf{P}_{x,y}\left(\sup_{t\geq s} v(x_R(t), y(t)) > c_1\varepsilon^p\right) \leq \frac{1}{c_1\varepsilon^p}\, \mathbf{E}_{x,y}[v(x_R(s), y(s))] \quad (3.7)$$

for all $y \in \mathbb{Y}$, $x \in B_R(0)$, $\varepsilon \in (0, R)$ and sufficiently small $R > 0$. We use (3.7) to derive that $\mathbf{E}_{x,y}[v(x_R(t), y(t))] \leq v(x, y)\, e^{-c_4 t/c_1} \leq c_2\, |x|^p\, e^{-c_4 t/c_1}$ and then from (3.6) to conclude that

$$\mathbf{P}_{x,y}\left(\sup_{t\geq s} |x_R(t)| > \varepsilon\right) \leq c_2\, |x|^p \varepsilon^{-p} c_1^{-1} e^{-s c_4/c_1}.$$

Hence all solutions of equation (3.5) starting at $t = 0$ from a position $x(0)$ which is in the ball $B_\varepsilon(0)$ for $\varepsilon \in (0, R)$, and with sufficiently small R, tend to 0 with probability one. For the time before leaving the ball $B_\varepsilon(0)$, the solutions of equations (3.1) and (3.5), with the same initial conditions in the ball $B_\varepsilon(0)$, are coinciding. Hence, all solutions of (3.1), which are in the ball $B_\varepsilon(0)$ for sufficiently small ε, tend to zero with probability one. The proof is complete. □

4 Diffusion Approximation of the Slow Motion and Stability

As mentioned in the Introduction, the operator Q can be considered as the infinitesimal operator of a Markov process $\xi(t),\ t \geq 0$ with the same phase space $\mathbb{G}$. It is assumed that Q is a closed operator such that its spectrum $\sigma(Q)$ is split into two parts, that is, $\sigma(Q) = \sigma_{-\rho}(Q) \cup \{0\}$, where $\sigma_{-\rho}(Q) \subset \{\mathrm{Re}\lambda \leq -\rho < 0\}$ and the zero eigenvalue has multiplicity one. The transition probability $P(t, z, A)$ of this Markov process satisfies the uniform ergodicity condition [4] in the form

$$\sup_{A \in \Sigma_{\mathbb{G}},\ z \in \mathbb{G}} |P(t, z, A) - \mu(A)| \leq e^{-ct},\ \ c = \mathrm{const} > 0,$$

where $\Sigma_{\mathbb{G}}$ is the Borel σ-algebra of subsets of $\mathbb{G}$. This implies that for any $v \in C(\mathbb{G})$, the space of continuous and bounded functions on $\mathbb{G}$, which satisfies the condition

$$\int_{\mathbb{G}} v(z)\mu(\mathrm{d}z) = 0, \tag{4.1}$$

we can define the following continuous function:

$$\Pi v(z) := \int_0^\infty \int_{\mathbb{G}} v(u) P(t, z, \mathrm{d}u)\, \mathrm{d}t, \quad z \in \mathbb{G}.$$

The operator Π, see [4], is said to be the potential of the Markov process. We extend this operator on the whole space $C(\mathbb{G})$ by the equality

$$\Pi v(z) := \int_0^\infty \int_{\mathbb{G}} [v(u) - \bar{v}] P(t, z, du)\, \mathrm{d}t, \text{ where } \bar{v} = \int_{\mathbb{G}} v(y)\mu(\mathrm{d}z). \tag{4.2}$$

We denote its norm by $\|\Pi\| := \sup_{z \in \mathbb{G},\ v \in C(\mathbb{G})} |v(z)|$. Note that, according to [3], the equation $Qf = -v$ has a solution iff v satisfies the orthogonality condition (4.1) and this solution can be taken in the form $f = \Pi v$. It is clear that the Markov process $\xi^\varepsilon(t),\ t \geq 0$ with an infinitesimal operator $Q^\varepsilon = \frac{1}{\varepsilon^2} Q$ can be defined by the formula $\xi^\varepsilon(t) = \xi(t/\varepsilon^2),\ t \geq 0$. In this section we consider the linear equation (1.6) for the slow motion $x^\varepsilon(t),\ t \geq 0$ with a Markov process $\xi^\varepsilon(t) = \xi(t/\varepsilon^2)$ and the fast variable $y^\varepsilon(t),\ t \geq 0$, satisfying equation (1.7). We also suppose that $A(y, z)$, as well as $h_{-1}(y, z)$ and $h_0(y, z)$, are continuous and bounded functions such that their y-derivatives of order up to three are all bounded. The triple $\{x^\varepsilon(t), y^\varepsilon(t), \xi^\varepsilon(t)\},\ t \geq 0$ is a homogeneous Feller Markov process on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{G}$, see [19], and its week infinitesimal operator $\mathcal{L}^{(\varepsilon)}$ is defined for appropriately smooth functions by the equality

$$\begin{aligned} &\mathcal{L}^{(\varepsilon)} v(x, y, z) = \langle A_0(y, z)x, \nabla_x \rangle v(x, y, z) + \varepsilon \langle A_1(y, z)x, \nabla_x \rangle v(x, y, z) \\ &+ \frac{1}{\varepsilon} \langle h_{-1}(y, z), \nabla_y \rangle v(x, y, z) + \langle h_0(y, z), \nabla_y \rangle v(x, y, z) + \frac{1}{\varepsilon^2} Q v(x, y, z). \end{aligned} \tag{4.3}$$

Here ∇_y is the gradient operator in $\mathbb{R}^m$, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^m$ and the operator Q acts on the third argument.

The properties of the pair $\{x^\varepsilon(t), y^\varepsilon(t)\}$, $t \in [0,T]$, for a fixed $T > 0$, considered as a stochastic process in the Skorokhod's space $\mathbb{D}([0,T], \mathbb{R}^n \times \mathbb{R}^m)$, depends essentially on the averaged value $\bar{h}_{-1}(y)$ of the function $h_{-1}(y,z)$ with respect to the invariant measure μ.

We assume that $\bar{h}_{-1}(y) \equiv 0$; the case $\bar{h}_{-1}(y) \neq 0$ needs a separate study. Thus, applying methods and results from [19], under the condition $\bar{h}_{-1}(y) \equiv 0$, one can prove that on any fixed time interval $[0,T]$, as $\varepsilon \to 0$, the pair $\{x^\varepsilon(t), y^\varepsilon(t)\}$, $t \in [0,T]$, converges weekly to a diffusion Markov process $\{\bar{x}(t), \hat{y}(t)\}$, $t \in [0,T]$. Here the Markov process $\hat{y}(t)$, which is said to be the diffusion approximation of $y^\varepsilon(t)$, is given by its infinitesimal operator

$$\hat{\mathcal{L}} v(y) = \langle b(y), \nabla_y \rangle v(y) + \frac{1}{2} \langle \sigma^2(y) \nabla_y, \nabla_y \rangle v(y), \tag{4.4}$$

with $b(y) = \bar{h}_0(y) + \overline{\{D_y \Pi\{h_{-1}\}\}(y,\cdot) h_{-1}(y,\cdot)}$ and the symmetric non-negatively defined matrix $\sigma^2(y)$ given by the formula

$$\sigma^2(y) = \overline{h_{-1}(y,\cdot)\{\Pi h_{-1}\}^T(y,\cdot)} + \overline{\{\Pi h_{-1}\}(y,\cdot)\{h_{-1}(y,\cdot)\}^T}.$$

Moreover, $\bar{x}(t)$, $t \geq 0$ satisfies the random differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{x}(t) = \bar{A}_0(\hat{y}(t))\, \bar{x}(t), \quad t \geq 0, \tag{4.5}$$

with a matrix $\bar{A}_0(\hat{y}(t))$ depending on the above Markov process $\hat{y}$, whose infinitesimal operator is $\hat{\mathcal{L}}$. For further reference it is convenient to define the stochastic process $\hat{y}(t)$, $t \geq 0$ as the solution of an Itô stochastic differential equation. We suppose that this equation is of the form

$$\mathrm{d}\hat{y}(t) = b(\hat{y}(t))\, \mathrm{d}t + \sigma(\hat{y}(t))\, \mathrm{d}w(t), \quad t \geq 0. \tag{4.6}$$

Here the vector $b(y)$ and the matrix $\sigma(y)$ are as given above. The assumptions imposed previously imply that the matrix $\bar{A}_0(y)$, the vector $b(y)$ and the matrix $\sigma(y)$ are three times continuously differentiable and bounded uniformly in $y \in \mathbb{R}^m$ together with their derivatives. We denote by $\bar{x}(s,t,x,y)$, $\hat{y}(s,t,y)$, $t \geq 0$, or simply $\bar{x}(t)$, $\hat{y}(t)$, $t \geq s$, the solution of the system (4.5)–(4.6) with initial conditions $\bar{x}(s) = x$, $\hat{y}(s) = y$. Our goal in this section is to prove that, for sufficiently small ε, the system (4.5)–(4.6) can by successfully used for the exponential p-stability analysis of the slow motion (1.6), which is subjected to the random perturbations $y^\varepsilon(t), \xi^\varepsilon(t)$, $t \geq 0$.

It is easy to see that the pair $\{\bar{x}(t), \hat{y}(t)\}$, $t \geq 0$ is a homogeneous Feller Markov process in the space $\mathbb{R}^n \times \mathbb{R}^m$. The weak infinitesimal operator $\bar{\mathcal{L}}$ of this process is defined for sufficiently smooth functions $v(x,y)$ by the formula

$$\bar{\mathcal{L}} v(x,y) = \langle \bar{A}(y)x, \nabla_x \rangle\, v(x,y) + \langle b(y), \nabla_y \rangle\, v(x,y) + \frac{1}{2} \langle \sigma^2(y) \nabla_y, \nabla_y \rangle\, v(x,y). \tag{4.7}$$

Let us take the function $v(x,y)$ as follows:

$$v(x,y) := \int_0^T \mathbf{E}[|\bar{x}(0,t,x,y)|^p] \,\mathrm{d}t \tag{4.8}$$

with a number $T > 0$ which will be specified later. In order to find useful estimates for this function and its derivatives, we need to estimate the derivatives of the solution of the system (4.5)–(4.6) with respect to the initial conditions $y(0) = y$ and $x(0) = x$. To avoid complicated notations and computations, we consider the process $\hat{y}(t)$ to be 1-dimensional, i.e., $m = 1$. The assumptions on the functions $h_j(y,z)$, $j = -1, 0$ imply that the drift $b(y)$ and the diffusion $\sigma^2(y)$ of the Markov process $\hat{y}$ have at least three continuous uniformly bounded derivatives in y. This property follows from the definition of the potential and the possibility to differentiate in y under the integral sign. By definition, the matrix $\bar{A}(y)$ also has three continuous and uniformly bounded derivatives. Hence, the Markov diffusion process $\{\bar{x}(t), \hat{y}(t)\}$ allows differentiation with respect to the initial data y, where $y = \hat{y}(0)$. We can study these derivatives as solutions of the corresponding equations.

Lemma 4.1. *The solution $\bar{x}(t)$, $t \geq 0$ of equation* (4.5)*, with $\hat{y}(t)$, $t \geq 0$ given by* (4.6)*, admits three continuous y-derivatives for which the following bounds hold for any $r \in \mathbb{N}$:*

$$\sup_{0 \leq t \leq T,\ y \in \mathbb{R}^m} \mathbf{E}_{x,y}[|D_y^j\, \bar{x}(t)|^r] \leq k_r\, |x|^r, \quad j = 1,2,3.$$

Proof. The y-derivative $D_y\bar{x}(t) := D_y\bar{x}(0,t,x,y)$ of the solution of (4.5) satisfies the differential equation

$$\frac{\mathrm{d}D_y\bar{x}(t)}{\mathrm{d}t} = \bar{A}(\hat{y}(t))D_y\bar{x}(t) + D_y\bar{A}^{(1)}(\hat{y}(t))\bar{x}(t), \quad t \geq 0. \tag{4.9}$$

Here and below $\bar{A}^{(j)}(y) = D_y^j \bar{A}(y)$, $j = 1,2,3$. By definition, $D_y\bar{x}(0) = 0$. Now we use the Cauchy integral formula allowing us to write the solution of (4.9), which depends on the parameter y, in the following form:

$$D_y\bar{x}(t) = \int_0^t D_y\hat{y}(s)H^{(1)}(s,t,y)\bar{A}^{(1)}(\hat{y}(s))\bar{x}(s)\,\mathrm{d}s, \tag{4.10}$$

where $H^{(1)}(s,t,y)$ is the Cauchy operator of the corresponding homogeneous equation. Similarly we write the differential equation for the second y-derivative $D_y^2\bar{x}(t)$ of the solution $\bar{x}(t)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D_y^2\bar{x}(t) &= \bar{A}(\hat{y}(t))D_y^2\bar{x}(t) + 2D_y\hat{y}(t)\bar{A}^{(1)}(\hat{y}(t))D_y\bar{x}(t) \\ &\quad + D_y^2\hat{y}(t)\bar{A}^{(1)}(\hat{y}(t))\bar{x}(t) + D_y\hat{y}(t)^2\bar{A}^{(2)}(\hat{y}(t))\bar{x}(t), \quad t \geq 0 \end{aligned} \tag{4.11}$$

with the initial condition $D_y^2\bar{x}(0) = 0$. The equation for the third derivative $D_y^3\bar{x}(t)$ can be written in the same way. All these taken together with the smoothness of the drift and the diffusion imply that the solution of (4.5) admits three y-derivatives and that for any fixed $r \in \mathbb{N}$ there exist constants M_r and γ_r such that

$$\mathbf{E}_y[\|D_y^j\, \hat{y}(t)\| \leq M_r\, e^{\gamma_r t}, \quad j = 1,2,3, \quad t \in [0,T]. \tag{4.12}$$

Let us mention that our assumptions imply also that

$$\sup_{y\in\mathbb{R}^m} \|\bar{A}^{(j)}(y)\| := a_j < \infty, \quad j = 1,2,3. \tag{4.13}$$

It is not difficult to see that the Cauchy operator $H^{(1)}$ in (4.10) is a uniformly bounded continuous matrix-function of t satisfying the following estimate:

$$\|H^{(1)}(s,t,y)\| \leq h_1\, e^{a(t-s)} \tag{4.14}$$

for any $t \in [s, s+T]$. Hence, for some constant $k_{1,r} > 0$, (4.10) implies that for fixed $T > 0$ and for any $r \in \mathbb{N}$, we have

$$\sup_{0\leq t\leq T,\ y\in\mathbb{R}^m} \mathbf{E}_{x,y}[|D_y\bar{x}(t)|^r] \leq k_{1,r}\, |x|^r. \tag{4.15}$$

Using the Cauchy operator $H^{(2)}(s,t,y)$ for equation (4.11) one can obtain a formula similar to (4.10). Further on, we can use (4.12) and (4.13) and derive for $H^{(2)}$ an estimate like (4.14). Thus we conclude finally that

$$\sup_{0\leq t\leq T,\ y\in\mathbb{R}^m} \mathbf{E}_{x,y}[|D_y^2\bar{x}(t)|^r] \leq k_r|x|^r \tag{4.16}$$

with some constant $k_r > 0$ for any $r \in \mathbb{N}$. The third y-derivative of the solution of (4.5) admits a similar estimate. This completes the proof. □

Corollary 4.1. *The Cauchy matrix $X(t)$ of equation (4.5) is three times continuously y-differentiable and for any $T \geq 0$ its derivatives admit the following estimates:*

$$\sup_{0\leq t\leq T,\ y\in\mathbb{R}^m} \mathbf{E}_{x,y}[\|D_y^j X(t)\|] := a_T < \infty, \quad j = 1,2,3 \tag{4.17}$$

Proof. It follows from the fact that the Cauchy matrix $X(t)$ of (4.5) has x-derivatives of its solution and satisfies the same equation under the initial condition $X(0) = I$. □

Lemma 4.2. *The function $v(x,y)$ has three continuous y-derivatives, and there exists a constant $\beta > 0$ such that for any $x \in \mathbb{R}^n$ we have*

$$\|D_y^j v(x,y)\| \leq \beta|x|^p,\ j = 1,2,3.$$

Proof. By definition we can write

$$\nabla_y |x(t)|^p = p\,\langle x(t), D_y x(t)\rangle\,|x(t)|^{p-2}. \tag{4.18}$$

Hence, for any $x \neq 0$ and $p > 0$, we have

$$|\nabla_z\,|x(t)|^p\,| \leq p\,|x(t)|^{p-1}\|D_z x(t)\| \leq p\,e^{a(p-1)t}\,|x|^{p-1}\|D_z x(t)\|. \tag{4.19}$$

Now, using (4.12), we obtain $\sup_{t,y} \mathbf{E}_{x,y}[\|D_y x(t)\|] \leq k_1|x|$, $0 \leq t \leq T$, $y \in \mathbb{R}^m$. Differentiating in y both sides of (4.18) yields $\|D_y^2|x(t)|^p\| \leq p\,\|D_y x(t)\|^2|x(t)|^{p-2} + p\,|(x(t)|^{p-1}\|D_y^2 x(t)\| + p\,|p-2|\,|x(t)|^{p-1}\|D_y x(t)\|$. Each term of the right-hand side of this inequality admits an estimate of the type (4.19), which is also true for $|x(t)|^{p-1}$. Then we can apply Lemma 2.1 for the expectations $\mathbf{E}_{x,y}[\|D_y x(t)\|^j]$, $j = 1, 2$, and for $\mathbf{E}_{x,y}[\|D_y^2 x(t)\|]$. It remains to differentiate twice the inequality (4.18) with respect to y and apply the same estimates for the terms involved thus completing the proof. □

Lemma 4.3. *The vector $V(x, y) := \nabla_x v(x, y)$ and its three y-derivatives admit the following estimates:*

$$\sup_{y\in\mathbb{R}^m} \|D_y^j V(x, y)\| \leq \rho|x|^{p-j}, \quad j = 0, 1, 2, 3 \tag{4.20}$$

for some $\rho > 0$ and any $x \neq 0$.

Proof. For our reasoning we need the following identity: $|x(t)|^p = |X(t)x|^p = \langle X^T(t)X(t)x, x\rangle^{p/2}$, where $X(t)$ is the fundamental matrix of the linear equation (4.5). Let us prove first that $\sup_{t,y} |\nabla_x\,|x(t)|^p| \leq \rho\,|x|^{p-1}$, $0 \leq t \leq T$, $y \in \mathbb{R}^m$ for some $\rho > 0$. Differentiating the above identity for $|x(t)|^p$ in x yields

$$\nabla_x\,|x(t)|^p = p\,\langle X^T(t)X(t)x, x\rangle^{p/2-1}\,X^T(t)X(t)x. \tag{4.21}$$

Hence $|\nabla_x\,|x(t)|^p| \leq p\,|X(t)x|^{p-1}\|X(t)\|$. Since the fundamental matrix of (4.5) is uniformly bounded on any fixed interval $[0, T]$, then the estimate (4.20) is established for $j = 1$. Next is to differentiate (4.21) in y:

$$\begin{aligned} D_y\,\nabla_x\,|x(t)|^p &= p(p-2)|x(t)|^{p/2-2}\langle x(t), D_y x(t)\rangle \\ &\quad \times [X^T(t)x(t) + p|x(t)|^{p-1}(D_y X^T(t)x(t) + X^T(t)D_y x(t))]. \end{aligned} \tag{4.22}$$

The final step is to use the estimate $\|X(t)\| \leq e^{at}$, as well as the estimates for the expectations of the derivatives $D_y x(t)$ and $D_y X(t)$ thus obtaining (4.20). The proof is completed. □

Lemma 4.4. *The matrix $W(x, y) = D_x\nabla_x v(x, y)$ and its two derivatives in y admit the following estimates:*

$$\sup_{y\in\mathbb{R}^m} \|D_y^j W(x, y)\| \leq \delta\,|x|^{p-2}, \quad j = 1, 2 \tag{4.23}$$

for some $\delta > 0$ and all $x \neq 0$.

Proof. The matrix of the second derivatives of $|x(t)|^p$ is as follows:

$$D_x \nabla_x |x(t)|^p = p\,(p-1)\langle X^T(t)X(t)x, x\rangle^{p/2-2}\, X^T(t)x(t)x^T(t)X(t) \\ + p\,\langle X^T(t)X(t)x, x\rangle^{p/2-1}\, X^T(t)X(t). \quad (4.24)$$

We estimate each term in the right-hand side of (3.24) by using the fact that $\|X(t)\| \le e^{at}$ thus arriving at (4.23) for $j = 1$. Similarly, by differentiating once more, we establish (4.23) also for $j = 2$. The proof is completed. □

Theorem 4.1. *Consider the processes* $\bar{x}(t)$ *and* $\hat{y}(t)$ *defined by equations* (4.5) *and* (4.6)*, respectively, and suppose that all the assumptions related to them are satisfied. Suppose now that equation* (4.5) *for* $\bar{x}(t)$*, with* $\hat{y}(t)$*, from* (4.6)*, is exponentially p-stable. Then there is a number* $\varepsilon_0 > 0$ *such that equation* (1.6)*, with coefficients depending on* $y^\varepsilon(t), t \ge 0$*, is exponentially p-stable for all* $\varepsilon \in (0, \varepsilon_0)$.

Proof. It is based on the second Lyapunov method. We use the Lyapunov function of the form $v_\varepsilon(x, y, z) = v(x, y) + \varepsilon\, v_1(x, y, z) + \varepsilon^2\, v_2(x, y, z)$, where $v(x, y)$ is defined by (4.8). Let the functions $v_1(x, y, z)$ and $v_2(x, y, z)$ be the solutions of the following two equations:

$$Q\, v_1(x, y, z) = -\langle A_0(y, z)x, \nabla_y\rangle\, v(x, y), \quad (4.25)$$

$$Q\, v_2(x, y, z) = -\Big\{ \langle [A(y, z) - \bar{A}(y)]x, \nabla_x\rangle\, v(x, y) + \langle h_{-1}(y, z), \nabla_y\rangle\, v_1(x, y, z) \\ - \int_{\mathbb{G}} \langle h_{-1}(y, z), \nabla_y\rangle\, v_1(x, y, z)\mu(\mathrm{d}z) + \langle h_0(y, z) - \bar{h}_0(y), \nabla_y\rangle\, v(y, z) \Big\}. \quad (4.26)$$

The right-hand side of each of these equations, after integration in y with respect of the measure $\mu(\mathrm{d}y)$, is equal to 0. This implies that there exist solutions of both equations. By the definition of a potential, we have $v_1(x, y, z) = \langle \varPi h_{-1}(y, z), \nabla_y v(x, y)\rangle$. The estimates of this function and its derivatives with respect to x and of y can be obtained from appropriate estimates for the scalar product $\langle h_{-1}(y, z), \nabla_y v(x, y)\rangle$ multiplied by $\|\varPi\|$. This follows from the possibility to differentiate the solution of (4.5) and the definition of the potential operator $\varPi$. Hence, there exists a constant $R_1 > 0$, such that the following inequalities are satisfied:

$$|v_1(x, y, z)| \le R_1\, |x|^p, \quad |\nabla_x\, v_1(x, y, z)| \le R_1\, |x|^{p-1}, \quad |\nabla_y\, v_1(x, y, z)| \le R_1\, |x|^p,$$
$$\|D_x \nabla_x\, v_1(x, y, z)\| \le R_1\, |x|^{p-2}, \qquad \|D_y \nabla_x\, v_1(x, y, z)\| \le R_1\, |x|^{p-1},$$
$$\|D_y \nabla_y\, v_1(x, y, z)\| \le R_1\, |x|^p, \qquad \|D_y D_x \nabla_y\, v_1(x, y, z)\| \le R_1\, |x|^{p-1},$$
$$\|D_y D_x \nabla_y\, v_1(x, y, z)\| \le R_1\, |x|^{p-1}, \qquad \|D_x^2 \nabla_y\, v_1(x, y, z)\| \le R_1\, |x|^{p-2}.$$

The same estimates hold also for the function $v_2(x, y, z)$. Hence, using the results in Section 3, we conclude that

$$\|\nabla_y v_2(x,y,z)\| \le R_2\,|x|^p, \qquad \|\nabla_x v_2(x,y,z)\| \le R_2\,|x|^{p-1}$$

for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{G}$ and some $R_2 > 0$.

Let us denote by $\mathcal{A}^{(\varepsilon)}$ the weak infinitesimal operator of the Markov process $\{x^\varepsilon, y^\varepsilon, \xi^\varepsilon\}$ defined by (1.6)–(1.7), with a Markov process ξ^ε. We apply this operator to the function $v^\varepsilon(x,y,z) = v(x,y) + \varepsilon\, v_1(x,y,z) + \varepsilon^2\, v_2(x,y,z)$. By definition

$$\mathcal{A}^{(\varepsilon)} v^\varepsilon(x,y,z) = \langle A_0(y,z)x, \nabla_x\rangle\, v^\varepsilon(x,y,z) + \mathcal{L}^{(\varepsilon)} v^\varepsilon(x,y,z),$$

where $\mathcal{L}^{(\varepsilon)}$ is defined by the formula

$$\mathcal{L}^{(\varepsilon)} = \frac{1}{\varepsilon}\langle h_{-1}(y,z), \nabla_y\rangle + \langle f_0(y,z), \nabla_y\rangle + \frac{1}{\varepsilon^2}\, Q.$$

Hence:

$$\begin{aligned}
\mathcal{A}^{(\varepsilon)} v^\varepsilon(x,y,z) = {}& \frac{1}{\varepsilon}\{Q\, v_1(x,y,z) + \langle h_{-1}(y,z), \nabla_y\rangle v(x,y)\} \\
& + \{\langle A_0(y,z)x, \nabla_x\rangle v(x,y) + \langle h_{-1}(y,z), \nabla_y\rangle v_1(x,y,z) \\
& + \langle h_0(y,z), \nabla_y\rangle v(x,y) + Q v_2(x,y,z)\} \\
& + \varepsilon\{\langle h_{-1}(y,z), \nabla_y\rangle v_2(x,y,z) + \langle A_0(y,z)x, \nabla_x\rangle v_1(x,y,z) \\
& + \langle h_0(y,z), \nabla_y\rangle v_1(x,y,z))\} \\
& + \varepsilon^2\{\langle A_0(y,z)x, \nabla_x\rangle v_2(x,y,z) + \langle h_0(y,z), \nabla_y\rangle v_2(x,y,z)\}.
\end{aligned} \tag{4.27}$$

The expression in the first brackets in the right-hand side of this formula is equal to 0. It follows from (4.25) that the item in the second brackets, by construction, is equal to $\bar{\mathcal{L}} v(x,y)$. Hence, due to our assumption about the exponential p-stability of the averaged system, $\bar{\mathcal{L}} v(x,y)$ does not exceed the quantity $-c_3\,|x|^p$ with some constant $c_3 > 0$. The last items in (4.27) can be estimated by $r|x|^p$ for some $r > 0$. Hence $\mathcal{A}^{(\varepsilon)} v^\varepsilon(x,y,z) \le (-c_3 + \varepsilon r + \varepsilon^2 r)|x|^p$. In addition, $|v_1(x,y,z)| \le \rho|x|^p$, $|v_2(x,y,z)| \le \rho|x|^p$ for some $\rho > 0$. Finally, one can write the inequalities

$$(c_1 - \varepsilon\rho - \varepsilon^2\rho)\,|x|^p \le v^\varepsilon(x,y,z) \le (c_2 + \varepsilon\rho + \varepsilon^2\rho)\,|x|^p$$

for some $c_2 \ge c_1 > 0$. The exponential p-stability of equation (1.6) follows now from these estimates and the estimates for the function v_1 and its derivatives, which have been written above. The theorem is proved. □

We are now in a position to continue the analysis of the system (1.1)–(1.2) with functions $F(x,y,z)$ and $H(y,z)$ in the right-hand sides not depending explicitly on ε. The goal is to show the local asymptotic stochastic stability property for equation (1.1). We introduce first the notation $A_0(y,z) := D_x F(x,y,z)|_{x=0}$ and let $\bar{A}_0(y)$ and $\bar{H}_0(y)$ be the μ–averaged functions, respectively of $A_0(y,z)$ and $H(y,z)$, namely:

$$\bar{A}_0(y) = \int_{\mathbb{G}} A_0(y,z)\,\mu(\mathrm{d}z) \quad \text{and} \quad \bar{H}(y) = \int_{\mathbb{G}} H(y,z)\,\mu(\mathrm{d}z).$$

Corollary 4.2. *Let us suppose that: (i) $F(x,y,z)$ is continuous and bounded; (ii) $F(x,y,z)$ has two uniformly continuous and bounded x–derivatives uniformly in (y,z); (iii) $H(y,z)$ is continuous and bounded with $\bar{H}(y) \equiv 0$. Suppose, finally, that equation (4.5), based on the above $\bar{A}_0(y)$ with $\hat{y}(t)$ satisfying (4.6), is asymptotically stochastically stable. Then equation (1.1) is locally asymptotically stochastically stable for all sufficiently small ε.*

Proof. Together with (1.1) we consider the equation

$$\frac{\mathrm{d}\tilde{x}^\varepsilon(t)}{\mathrm{d}t} = A_0(y^\varepsilon(t), \xi^\varepsilon(t))\,\tilde{x}^\varepsilon(t),\ t \geq 0,$$

where $y^\varepsilon(t)$ satisfies (1.2) and $\xi^\varepsilon(t)$ is the Markov process as defined in the Introduction. The asymptotic stochastic stability of equation (4.5), with $\hat{y}(t)$ from (4.6), combined with the results in Section 1 imply that (4.5) is exponentially p-stable for some $p > 0$. Now, applying Theorem 3.1 we conclude that $\tilde{x}^\varepsilon(t)$ is asymptotically stochastically stable for all sufficiently small ε. Since $F(0,y,z) \equiv 0$, we use the obvious equality

$$F(x,y,z) = (D_x F(0,y,z))x + \left[\int_0^1 [D_x F(tx,y,z) - D_x F(x,y,z)|_{x=0}]\mathrm{d}t\right] x$$

to rewrite the right-hand side of equation (1.1) in the following form:

$$F(x,y,z) = A_0(y,z)x + g(x,y,z).$$

The expressions for $A_0(y,z)$ and $g(y,z)$ are clear. We use the function $g(x,y,z)$ to find first its μ–averaged value $\bar{g}(x,y)$, then the x–derivative $D_x\,\bar{g}(x,y)$ and by (3.2) determine the upper bound, say $\bar{g}_r$, which depends on the radius r of the ball $B_r(0)$. It is not difficult to show that the pair $\{y^\varepsilon(t), \xi^\varepsilon(t)\}$ is a Markov process with values in the space $\mathbb{Y} \times \mathbb{G}$. Hence, we need to refer to Theorem 3.1 and to the assumptions about the function $F(x,y,z)$ which guarantee that the relation $\lim_{r\to 0} \bar{g}_r = 0$ is satisfied and then apply Theorem 2.1 in which stability analysis is based on the linear approximation. The proof is completed. □

Acknowledgments

This study was partly supported by UK-EPSRC Grant No. GR/R71719/01.

We dedicate this paper to Professor Albert Shiryaev on the occasion of his 70th birthday.

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