On the Existence of Optimal Portfolios for the Utility Maximization Problem in Discrete Time Financial Market Models *

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Summary. We consider an investor whose preferences are described by a concave nondecreasing function $U : (0, \infty) \to \mathbb{R}$ and prove that in an arbitrage-free discrete-time market model there is a strategy attaining the supremum of expected utility at the terminal date provided that this supremum is finite.

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1 Introduction and main result

In this paper we study the existence of optimal portfolios for maximizing expected utility of the terminal wealth. His or her preferences are described by a concave nondecreasing function $U : (0, \infty) \to \mathbb{R}$, trading dates occur at discrete time instants.

Recently, [8, 9] have treated the same problem, concentrating rather on the construction of pricing operators using optimal strategies. In this paper we apply the machinery which was developed in [7] for utility functions $U : \mathbb{R} \to \mathbb{R}$ and establish the existence of optimal strategies under minimal conditions (U is concave nondecreasing, absence of arbitrage, the value function is finite). This general theorem has already been anticipated in Section 3.1 of [3] where the authors proved it for a one-step model and nonnegative price process.

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A usual setting for discrete-time market models is considered: a probability space $(\Omega, \mathcal{F}, P)$; a filtration $(\mathcal{F}_t)_{0\le t\le T}$ such that $\mathcal{F}_0$ contains P-null sets and a d-dimensional adapted process $(S_t)_{0\le t\le T}$ describing the prices of d risky assets in a given economy.

It is implicitly assumed that investors also dispose of a risk-free asset $S_t^0 := 1,\ 0 \le t \le T$; hence trading strategies can be arbitrary d-dimensional predictable processes $(\varphi_t)_{1\le t\le T}$, where φ_t^i denotes the investor's holding in asset i at time t. Predictability means that φ_t is $\mathcal{F}_{t-1}$-measurable, i.e. the portfolio is chosen before new prices S_t are revealed. Let Φ denote the family of all predictable trading strategies.

The value of a portfolio φ starting from initial capital c is given by

$$V_t^{c,\varphi} = c + \sum_{i=1}^{t} \langle \varphi_i, \Delta S_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product in $\mathbb{R}^d$, $\Delta S_i := S_i - S_{i-1}$ and $c > 0$.

Introduce for each $t = 1, ..., T$ a random subset $D_t(\omega)$ of $\mathbb{R}^d$: the affine hull of the support of the (regular) conditional distribution of ΔS_t given $\mathcal{F}_{t-1}$, see Proposition 4.1.

In this paper we impose the following (fairly natural) trading constraint: portfolio value should not become negative. Define for $c > 0$ the set of admissible trading strategies as

$$\mathcal{A}(c) := \{\varphi \in \Phi :\ V_t^{c,\varphi} \ge 0 \text{ a.s.},\ 0 \le t \le T\}. \tag{1.1}$$

In what follows, Ξ_t will denote the set of $\mathcal{F}_t$-measurable d-dimensional random variables. When a date t is fixed, φ_t is called admissible for the initial capital x if $\varphi_t \in \Xi_{t-1}^x$, where

$$\Xi_t^x := \{\xi \in \Xi_t :\ x + \langle \xi, \Delta S_{t+1} \rangle \ge 0 \text{ a.s.}\},\quad x \in [0, \infty).$$

Define for any $\mathcal{F}_t$-measurable nonnegative random variable H

$$\Xi_t(H) := \{\xi \in \Xi_t :\ H + \langle \xi, \Delta S_{t+1} \rangle \ge 0 \text{ a.s.}\},$$

and also

$$\tilde{\Xi}_t := \{\xi \in \Xi_t : |\xi(\omega)| = 1,\ \xi(\omega) \in D_{t+1}(\omega) \text{ a.s.}\}.$$

Assumption 1.1 $U : (0, \infty) \to \mathbb{R}$ *is a concave nondecreasing function.*

We extend U by continuity to zero ($U(0) = U(0+)$ may be $-\infty$) and set $U(x) = -\infty,\ x < 0$. By convention, $U'(x)$ denotes the *left-hand* derivative of U at x; U^+ is the positive part of U.

We are dealing with maximizing the expected utility of the terminal wealth:

$$EU(V_T^{c,\varphi}) \to \max,\quad \varphi \in \mathcal{A}(c). \tag{1.2}$$

So as to have a well-posed problem the following *absence of arbitrage* (NA) property will be imposed:

$$\text{(NA)} \quad \forall c > 0 \ \forall \varphi \in \mathcal{A}(c) \ \ (V_T^{c,\varphi} \geq c \text{ a.s.} \Longrightarrow V_T^{c,\varphi} = c \text{ a.s.}). \tag{1.3}$$

Theorem 1.1. *Let Assumption 1.1 hold and let S satisfy* (1.3). *Suppose that the expectations in the definition below exist (though might take the value $-\infty$)*

$$u(c) := \sup_{\varphi \in \mathcal{A}(c)} EU(V_T^{c,\varphi}), \tag{1.4}$$

and

$$u(c) < \infty \text{ for all } c \in (0, \infty). \tag{1.5}$$

Then for each $c \in (0, \infty)$ there exists a strategy $\varphi^*(c)$ satisfying

$$u(c) = EU(V_T^{c,\varphi^*(c)}),$$

moreover one has $\varphi_t^*(c) \in D_t$ a.s.

We will present the proof of Theorem 1.1 in Sections 2 and 3. A possible extension (Theorem 3.1) to random utility functions is sketched in Remarks 2.2 and 3.1.

Remark 1.1. In fact, it is sufficient to suppose that there exists $c > 0$ such that $u(c) < \infty$. In this case Lemma 2.2 entails that for any strategy φ and any $\lambda \geq 1$ we have the bound

$$U^+(V_T^{\lambda c,\varphi}) \leq 2\lambda[U^+(V_T^{c,\varphi/\lambda}) + U(2)],$$

with the right-hand side having a finite expectation as $u(c) < \infty$. This means that for any $c' > c$ the expectations in the definition (1.4) of $u(c')$ exist. It is easy to see that $u(\cdot)$ is concave, hence if we had $u(c') = \infty$ for some $c' > c$ then

$$u(c/2) = u(\alpha c' + (1-\alpha)c/4) \geq \alpha u(c') + (1-\alpha)u(c/4) = \infty,$$

where $\alpha \in (0,1)$ is a suitable number. But this is impossible, as by monotonicity

$$u(c/2) \leq u(c) < \infty.$$

Remark 1.2. Theorem 1.1 fails to be true in general semimartingale models. As it was shown by counterexamples of [6], in the continuous-time case certain additional properties have to be imposed on U to guarantee the existence of optimal strategies.

We mention a uniqueness result whose proof is omitted as it is identical to that of Theorem 2.8 in [7].

Theorem 1.2. *If U is strictly concave then there is a unique optimal strategy φ^* satisfying*

$$\varphi_t^* \in D_t \text{ a.s.}$$

We will need an alternative characterization of (NA), see the Proposition below. This statement is implicit in Theorem 3 of [4], where it is shown that absence of arbitrage is equivalent to the fact that the origin lies in the relative interior of the convex hull of the support of conditional distribution of ΔS_t given $\mathcal{F}_{t-1}$. We make this more explicit and "quantitative":

Proposition 1.1. *Under (NA) the set $D_t(\omega)$ is a linear subspace of $\mathbb{R}^d$, almost surely. The (NA) condition implies the existence of $\mathcal{F}_t$-measurable random variables $\beta_t, \kappa_t > 0$, $0 \leq t \leq T-1$, such that for any $p \in \tilde{\Xi}_t$*

$$P(\langle p, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) \geq \kappa_t \tag{1.6}$$

almost surely.

Proof. The "standard" absence of arbitrage property is the following

$$\text{(NA')} \quad \forall \varphi \in \Phi \ \ (V_T^{0,\varphi} \geq 0 \text{ a.s.} \Rightarrow V_T^{0,\varphi} = 0 \text{ a.s.})$$

It follows from Theorem 3 of [4] and Proposition 3.3 of [7] that if (NA') holds then D_t is a linear subspace and (1.6) holds. So it suffices to establish that (NA) and (NA') are equivalent. The (NA') condition trivially implies (NA) since if we had a φ violating (NA) we would immediately get

$$V_T^{0,\varphi} = V_T^{c,\varphi} - c \geq 0, \quad P(V_T^{0,\varphi} > 0) > 0,$$

which contradicts (NA'). The other direction is also clear: if there is φ such that (NA') fails then we know from the implication $(b) \Rightarrow (a)$ of Theorem 3 in [4] that there is ψ such that $V_t^{0,\psi} \geq 0$, $0 \leq t \leq T$ and $P(V_T^{0,\psi} > 0) > 0$. For such a strategy

$$V_t^{c,\psi} \geq c \text{ a.s.}, \ 0 \leq t \leq T, \quad P(V_T^{c,\psi} > c) > 0,$$

so $\psi \in \mathcal{A}(c)$ and (NA) is violated.

2 Optimal strategy in the one-step case

Let $V : [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$ be a function such that for almost all ω, $V(\cdot, \omega)$ is a nondecreasing continuous concave function, $V(x, \omega)$ is finite for $x \in (0, \infty)$ and $V(x, \cdot)$ is $\mathcal{F}$-measurable for any fixed x. Let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra containing P-null sets. Let Y be a d-dimensional random variable. Denote by Ξ the family of $\mathcal{H}$-measurable d-dimensional random variables. Put

$$\tilde{\Xi} := \{\xi \in \Xi : \ |\xi(\omega)| = 1, \ \xi(\omega) \in D(\omega) \text{ a.s.}\},$$
$$\Xi^x := \{\xi \in \Xi : \ x + \langle \xi, Y \rangle \geq 0 \text{ a.s.}\}, \ x \in [0, \infty),$$

here D denotes the smallest affine subspace containing the support of the conditional distribution of Y with respect to $\mathcal{H}$ (see Section 4). We suppose that D is actually a linear subspace a.s. and that

$$P(\langle p, Y \rangle < -\delta | \mathcal{H}) \geq \kappa, \text{ for all } p \in \tilde{\Xi}, \tag{2.1}$$

with some $\mathcal{H}$-measurable random variables $\kappa, \delta > 0$.

Introduce also

$$\Xi^H := \{\xi \in \Xi : \ H + \langle \xi, Y \rangle \geq 0 \text{ a.s.}\},$$

for each $\mathcal{H}$-measurable nonnegative random variable H.

This setting will be applied in Section 3 with $\mathcal{H} = \mathcal{F}_{t-1}$, $D = D_t$, and $Y = \Delta S_t$; V will be the supremum of conditional expected utility if trading begins at time t.

Assume that

$$V(1) \geq 0 \text{ a.s.} \tag{2.2}$$

and for all $x \in [0, \infty)$

$$\operatorname{ess.}\sup_{\xi \in \Xi^x} E(V(x + \langle \xi, Y \rangle) | \mathcal{H}) < \infty \quad \text{a.s.} \tag{2.3}$$

We need some preparatory results.

Proposition 2.1. *Let $\xi \in \Xi^x$ be fixed. There exists a version of*

$$y \to E(V(y + \langle \xi, Y \rangle) | \mathcal{H}), \ y \geq x,$$

such that it is a nondecreasing upper semicontinuous concave function (perhaps taking the value $-\infty$), for almost all ω.

Proof. Fix a version of $F(q, \omega) := E(V(q + \langle \xi, Y \rangle) | \mathcal{H})$ for $q \in \mathbb{Q}_+$. The following inequalities hold almost surely for any pairs $q_1 \leq q_2$ of rational numbers:

$$F(q_1) \leq F(q_2), \quad F(\frac{q_1 + q_2}{2}) \geq \frac{F(q_1) + F(q_2)}{2}.$$

Let us fix a P-zero set N such that outside this set the above inequalities hold. Fix $y \in [x, \infty)$ and take rationals $q_n \searrow y$. The monotone convergence theorem yields

$$F(y+) = \lim_n F(q_n) = \lim_n E(V(q_n + \langle \xi, Y \rangle) | \mathcal{H}) =$$
$$E(V(y + \langle \xi, Y \rangle) | \mathcal{H}), \text{ a.s.}$$

showing that the right-continuous pathwise extension of F is as required.

Remark 2.1. If $E(V(x+\langle\xi,Y\rangle)|\mathcal{H})$ is almost surely finite then, by concavity, we get an almost surely continuous version from the above proposition.

Proposition 2.2. *Let $x>0$, $\xi\in\Xi^x$. Let $\hat{\xi}(\omega)$ be the orthogonal projection of $\xi(\omega)$ on the subspace $D(\omega)$. Then $\hat{\xi}\in\Xi^x$. Furthermore,*

$$E(V(x+\langle\hat{\xi},Y\rangle)|\mathcal{H})=E(V(x+\langle\xi,Y\rangle)|\mathcal{H}),$$

almost everywhere.

Proof. To check that

$$x+\langle\hat{\xi},Y\rangle\geq 0 \text{ a.s.} \tag{2.4}$$

we proceed as follows: take a regular version $\mu(dx,\omega)$ of $P(Y\in dx|\mathcal{H})$. Notice that for almost all ω:

$$\text{supp}\,\mu(\cdot,\omega)\subset D(\omega),\qquad \mu(\{y:\ x+\langle\xi(\omega),y\rangle\geq 0\},\omega)=1,$$

so necessarily

$$\mu(\{y:\ x+\langle\hat{\xi}(\omega),y\rangle\geq 0\},\omega)=1,$$

which shows (2.4). For the rest of this technical proof we refer to Proposition 4.6 of [7].

Lemma 2.1. *Let us fix $x_0>0$. There exists a $\mathcal{H}$-measurable random variable $K=K(x_0)>0$ such that for any $x\leq x_0$ and $\xi\in\Xi^x$ satisfying $\xi\in D$ we have $|\xi|\leq K$ almost surely.*

Proof. Indeed, we know from (2.1) that if $|\xi|>x_0/\delta$ then necessarily for any $x\leq x_0$

$$P(x+\langle\xi,Y\rangle<0|\mathcal{H})\geq\kappa>0,$$

which means that $\xi\notin\Xi^x$, hence we may set $K:=x_0/\delta$.

When showing the existence of an optimal strategy we will use a Fatou-lemma argument for which we need the two lemmata below.

Lemma 2.2. *Let $V:(0,\infty)\to\mathbb{R}$ be a concave nondecreasing function such that $V(1)\geq 0$. Then for all $x>0$ and $\lambda\geq 1$*

$$V^+(\lambda x)\leq 2\lambda[V^+(x)+V(2)].$$

Proof. First let us suppose $x\geq 2$. In this case

$$V^+(\lambda x)=V(\lambda x)\leq V(x)+V'(x)(\lambda x-x)\leq$$
$$V(x)+\frac{V(x)-V(1)}{x-1}x(\lambda-1)\leq V(x)+2(\lambda-1)(V(x)-V(1))\leq$$
$$2\lambda V(x),$$

where we used the concavity and the inequalities $x \geq 2$ and $V(x) \geq V(1) \geq 0$. For $x < 2$ by monotonicity

$$V^+(\lambda x) \leq V(2\lambda) \leq 2\lambda V(2).$$

Putting these estimations together, we get, for any $x > 0$, that

$$V^+(\lambda x) \leq 2\lambda \max\{V(2), V^+(x)\} \leq 2\lambda[V^+(x) + V(2)],$$

as desired.

Lemma 2.3. *Fix $x > 0$. There exists a nonnegative random variable L such that for any $\xi \in \Xi^x,\ \xi \in D$*

$$V^+(x + \langle \xi, Y \rangle) \leq L, \qquad E(L|\mathcal{H}) < \infty \ a.s. \tag{2.5}$$

Proof. Take the random set $M(\omega, x)$ of Proposition 4.2 and its linear span $R(\omega, x)$, see Proposition 4.3. It suffices to carry out the majoration separately on the sets

$$A_k := \{\omega : \ \dim R(\omega) = k\} \in \mathcal{H}, \quad 0 \leq k \leq d,$$

i.e. finding L_k such that

$$V^+(x + \langle \xi, Y \rangle) I_{A_k} \leq L_k, \qquad E(L_k|\mathcal{H}) < \infty.$$

The case $k = 0$ being trivial we may and will suppose that dim $R = m \geq 1$ is a fixed constant. Let the $\mathbb{R}^d$-valued random variables $\zeta_j,\ 1 \leq j \leq m$, be such that they form a (random) orthonormal bases of R, almost surely. Define $W := \{-1, +1\}^m$ and introduce the vectors

$$\theta_i := \sum_{j=1}^m i(j)\zeta_j, \quad i \in W.$$

It is clear from Lemma 2.1 that $M(x)$ is contained in the m-dimensional cube with edges $K\theta_i,\ i \in W$, almost surely. As a linear function defined on a polyhedral set attains its maximum on the extreme points, we immediately have for all selectors $\xi \in M(x)$, i.e. for any $\xi \in \Xi^x,\ \xi \in D$

$$x + \langle \xi, Y \rangle \leq \bigvee_{i \in W} (x + K\langle \theta_i, Y \rangle) \ \text{ a.s.}$$

So by monotonicity

$$V(x + \langle \xi, Y \rangle) \leq \bigvee_{i \in W} V(x + K\langle \theta_i, Y \rangle) \ \text{ a.s.}$$

Thus,

$$V^+(x + \langle \xi, Y \rangle) \leq \sum_{i \in W} V^+(x + K\langle \theta_i, Y \rangle) \ \text{ a.s.} \tag{2.6}$$

The relative interior ri M is also a random set by Proposition 4.3. Let ρ be an $\mathcal{H}$-measurable selector of ri M. Then the projection on Ω of each set

$$B_i := \{(\omega, a) \in \Omega \times (0,1] :\ \rho + a(K\theta_i - \rho) \in M(x)\} \in \mathcal{H} \otimes \mathcal{B}((0,1]), \quad i \in W,$$

is of full measure. Hence B_i admit $\mathcal{H}$-measurable selectors ψ_i. Now Lemma 2.2 implies that

$$\begin{aligned} V^+(x + K\langle\theta_i, Y\rangle) = V^+(x + \langle\rho, Y\rangle + \langle K\theta_i - \rho, Y\rangle) \leq \\ 2\frac{1}{\psi_i}[V^+(\psi_i(x + \langle\rho, Y\rangle) + \psi_i\langle K\theta_i - \rho, Y\rangle) + V(2)] \leq \\ \frac{2}{\psi_i}[V^+(x + \langle\rho, Y\rangle + \langle\psi_i(K\theta_i - \rho), Y\rangle) + V(2)], \quad i \in W. \end{aligned} \tag{2.7}$$

where we used Lemma 2.2, monotonicity of V, $\psi_i \leq 1$ and $\rho \in \Xi^x$. Define

$$L := 2\sum_{i\in W}\frac{1}{\psi_i}[V^+(x + \langle\rho, Y\rangle + \langle\psi_i(K\theta_i - \rho), Y\rangle) + V(2)].$$

As ψ_i is chosen in such a manner that

$$\rho + \psi_i(K\theta_i - \rho) \in M(x), \quad i \in W,$$

we have, using (2.3)

$$\begin{aligned} E(L|\mathcal{H}) = 2\sum_{i\in W}\frac{1}{\psi_i}E(V^+(x + \langle\rho, Y\rangle + \langle\psi_i(K\theta_i - \rho), Y\rangle)|\mathcal{H}) + \\ + 2^{m+1}E(V(2)|\mathcal{H}) < \infty. \end{aligned}$$

The bounds (2.6) and (2.7) imply (2.5).

Now a regular version of the essential supremum is shown to exist.

Proposition 2.3. *There is a function* $G : [0,\infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$ *which is a version of*

$$\operatorname{ess.\,sup}_{\xi\in\Xi^x} E(V(x + \langle\xi, Y\rangle)|\mathcal{H})$$

for each fixed $x \in [0,\infty)$*; nondecreasing, concave, continuous on* $[0,\infty)$ *and finite valued for* $x \in (0,\infty)$*, for almost all* ω*.*

Proof. Take a version $G(q,\omega)$ of the essential supremum, for $q \in \mathbb{Q}_+$. As $0 \in \Xi^x$ for all x, $E(V(x + \langle\xi, Y\rangle)|\mathcal{H})$ is almost surely finite-valued for each $x \in (0,\infty)$. Outside a P-null set the monotonicity and convexity relations

$$G(q_1) \leq G(q_2), \text{ if } q_1 \leq q_2, \quad G\left(\frac{1}{2}(q_1 + q_2)\right) \geq \frac{G(q_1) + G(q_2)}{2}, \ q_1, q_2 \in \mathbb{Q}_+,$$

hold, hence on a set of probability one we may extend G by monotonicity to a nondecreasing concave function on $(0, \infty)$ which is finite-valued (and hence continuous).

Take any $x \in (0, \infty)$ and two sequences of rationals $q_n \nearrow x$, $r_n \searrow x$. As for $y \le z$ the relation $\Xi^y \subseteq \Xi^z$ holds, we get that

$$\text{ess.}\sup_{\xi \in \Xi^x} E(V(x + \langle \xi, Y \rangle)|\mathcal{H}) \ge \limsup_n G(q_n) = G(x),$$
$$\text{ess.}\sup_{\xi \in \Xi^x} E(V(x + \langle \xi, Y \rangle)|\mathcal{H}) \le \liminf_n G(r_n) = G(x),$$

showing that $G(x)$ is a version of the essential supremum for each $x \in (0, \infty)$. By construction $G(0)$ is a version of the essential supremum at $x = 0$, so it remains to check the continuity of G at zero, i.e. the equality

$$\lim_{l \to \infty} \text{ess.}\sup_{\xi \in \Xi^{1/l}} E(V(1/l + \langle \xi, Y \rangle)|\mathcal{H}) = \text{ess.}\sup_{\xi \in \Xi^0} E(V(\langle \xi, Y \rangle)|\mathcal{H}). \quad (2.8)$$

The limit exists by monotonicity on a set of probability one and certainly greater than or equal to the right-hand side above. The particular structure of the family whose essential supremum is taken guarantees that for each $l \in \mathbb{N}$ there exists $\eta_l \in \Xi^{1/l}$ such that

$$|\text{ess.}\sup_{\xi \in \Xi^{1/l}} E(V(1/l + \langle \xi, Y \rangle)|\mathcal{H}) - E(V(1/l + \langle \eta_l, Y \rangle)|\mathcal{H})| \le 1/l \quad \text{a.s.}$$

We may supppose $\eta_l \in D$ by Proposition 2.2. Then Lemmata 2.1 and 4.1 imply that a random subsequence η_{l_k} exists such that $\eta_{l_k} \to \tilde{\eta}$ a.s., as $k \to \infty$ and $\tilde{\eta} \in \cap_{x>0} \Xi^x = \Xi^0$. The continuity of V, Lemma 2.3 and the Fatou lemma guarantee that

$$\lim_{k \to \infty} E(V(1/l_k + \langle \eta_{l_k}, Y \rangle)|\mathcal{H}) \le E(V(\langle \tilde{\eta}, Y \rangle)|\mathcal{H}) \le \text{ess.}\sup_{\xi \in \Xi^0} E(V(\langle \xi, Y \rangle)|\mathcal{H}),$$

hence assertion (2.8) follows.

We construct a sequence of strategies converging to the optimal value for all $x \in (0, \infty)$.

Lemma 2.4. *There exist $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{H}$-measurable functions $\xi_n(x, \omega)$ and suitable versions $G_n(x, \omega)$ of*

$$E(V(x + \langle \xi_n(x), Y \rangle)|\mathcal{H}),$$

such that outside a fixed P-null set we have for all $x \in (0, \infty)$

$$\lim_{n \to \infty} G_n(x) = G(x), \quad (2.9)$$

and the limit is attained in a nondecreasing way.

Proof. It suffices to prove this for $x \in [1,2)$; in an analogous way we get sequences ξ_n for all the intervals $[n,n+1)$, $[1/(n+1),1/n)$, $n \in \mathbb{N}$, and then by "pasting together" we finally get an approximation along all the positive axis.

Fix a version $G(\cdot,\omega)$ of the essential supremum given by Proposition 2.3. First notice that, for fixed $x \in (0,\infty)$, the family of functions

$$E(V(x+\langle \xi, Y\rangle)|\mathcal{H}),\ \xi \in \Xi^x, \tag{2.10}$$

is directed upwards, so there is a sequence $\eta_n(x) \in \Xi^x$ such that

$$\lim_{n\to\infty} \uparrow E(V(x+\langle \eta_n(x), Y\rangle)|\mathcal{H}) = \text{ess.}\sup_{\xi\in\Xi^x} E(V(x+\langle \xi, Y\rangle)|\mathcal{H}),$$

almost surely. Let us fix such a sequence for each dyadic rational $q \in [1,2)$. Now set

$$\xi_0(x,\omega) := 0.$$

Let us suppose that $\xi_0,\dots,\xi_{n-1}$ have been defined, as well as $\xi_n(x,\omega)$ for $x \in [1, 1+k/2^n)$ for some $0 \le k \le 2^n - 1$. If $k = 0$ we set $\xi_n(x,\omega) := \kappa_n^0$ for $x \in [1,1+1/2^n)$, where κ_n^0 is chosen such that

$$\begin{aligned} &E(V(1+\langle \kappa_n^0, Y\rangle)|\mathcal{H}) \\ &\qquad \ge E\left(V\left(1+\langle \xi_{n-1}(1), Y\rangle\right)|\mathcal{H}\right) \vee E\left(V\left(1+\langle \eta_n(1), Y\rangle\right)|\mathcal{H}\right). \end{aligned}$$

If $1 \le k \le 2^n - 1$ we set

$$\xi_n(x,\omega) := \kappa_n^k(\omega), \quad x \in \left[1+\frac{k}{2^n}, 1+\frac{k+1}{2^n}\right),$$

where $\kappa_n^k \in \Xi^{1+k/2^n}$ is chosen in such a way that almost everywhere

$$E(V(1+k/2^n+\langle \kappa_n^k, Y\rangle)|\mathcal{H}) \ge u_n^k \vee v_n^k \vee w_n^k. \tag{2.11}$$

Here we use the notations

$$\begin{aligned} u_n^k &:= E\left(V\left(1+\frac{k}{2^n}+\left\langle \xi_n\left(1+\frac{k-1}{2^n}\right), Y\right\rangle\right)\Big|\mathcal{H}\right), \\ v_n^k &:= E\left(V\left(1+\frac{k}{2^n}+\left\langle \eta_n\left(1+\frac{k}{2^n}\right), Y\right\rangle\right)\Big|\mathcal{H}\right), \\ w_n^k &:= E\left(V\left(1+\frac{k}{2^n}+\left\langle \xi_{n-1}\left(1+\frac{k}{2^n}\right), Y\right\rangle\right)\Big|\mathcal{H}\right). \end{aligned}$$

This is possible, as the family (2.10) is directed upwards and $\Xi^y \subseteq \Xi^z$ for $y \le z$. The latter fact implies also that actually $\kappa_n^k \in \Xi^y$ for y from the interval $[1+k/2^n, 1+(k+1)/2^n)$, so $\xi_n(x) \in \Xi^x$ for all $x \in [1,2)$.

Using Propositions 2.1 and 2.3 as well as (2.11) it is easy to see that there is a P-null set N such that outside this set $G(\cdot,\omega)$ is continuous and suitable versions $G_n(\cdot,\omega)$ of

$$E(V(x + \langle \xi_n(x), Y \rangle)|\mathcal{H})(\omega)$$

are nondecreasing and continuous on subintervals of the form $[1 + k/2^n, 1 + (k+1)/2^n)$, $0 \le k \le 2^n - 1$, for all $n \in \mathbb{N}$. By the definitions of $\eta_n(x)$ and $\xi_n(x)$ we see immediately that (outside another P-null set N') for all dyadic rationals $q \in [1, 2)$

$$G(q) = \lim_{n \to \infty} \uparrow G_n(q).$$

Consequently, outside $N \cup N'$ the sequence $G_n(x)$ is nondecreasing in n, for all $x \in [1, 2)$. For any $x \in [1, 2)$ and dyadic rationals $q_1 < x < q_2$,

$$G_n(q_1) \le G_n(x) \le G_n(q_2)$$

outside N, so necessarily

$$G(q_1) \le \liminf_n G_n(x) \le \limsup_n G_n(x) \le G(q_2),$$

outside $N \cup N'$. The function G being continuous at x, we get almost sure convergence to G in all points $x \in [1, 2)$.

The following lemma contains the actual construction of the one-step optimal strategy.

Lemma 2.5. *There exists a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{H}$-measurable function $\tilde{\xi}(x, \omega)$ such that for each $x \in (0, \infty)$*

$$E(V(x + \langle \tilde{\xi}(x), Y \rangle)|\mathcal{H}) = \text{ess.} \sup_{\xi \in \Xi^x} E(V(x + \langle \xi, Y \rangle)|\mathcal{H}).$$

Proof. It suffices to prove this, e.g., when $x \in [1, 2)$, then one can "paste together" the optimal strategy for $x \in (0, \infty)$. We take an approximating sequence ξ_n as provided by Lemma 2.4, then change to the projections $\hat{\xi}_n$ figuring in Proposition 2.2. Using Proposition 2.1 and the structure of the approximating sequence one can see that G_n is a version of

$$E(V(x + \langle \hat{\xi}_n, Y \rangle)|\mathcal{H}),$$

and almost surely

$$E(V(x + \langle \hat{\xi}_n, Y \rangle)|\mathcal{H}) \to G(x), \text{ for all } x \in [1, 2).$$

Then take $x_0 := 2$ and apply Lemma 2.1. It follows that, almost surely,

$$|\hat{\xi}_n(x)| \le K(x_0), \text{ for all } x \in [1, 2).$$

Now use Lemma 4.1 to find a random subsequence $\tilde{\eta}_k := \hat{\xi}_{n_k}$ of $\hat{\xi}_n$ converging to some $\tilde{\xi}$. Apply the Fatou lemma (we shall justify its use in a while):

$$E(V(x + \langle \tilde{\xi}(x), Y \rangle)|\mathcal{H}) \ge \limsup_{k \to \infty} E(V(x + \langle \tilde{\eta}_k(x), Y \rangle)|\mathcal{H}).$$

By the structure of the random subsequence in Proposition 4.1

$$E(V(x + \langle \tilde{\eta}_k(x), Y \rangle)|\mathcal{H}) \geq E(V(x + \langle \xi_{n_k}(x), Y \rangle)|\mathcal{H}),$$

so the construction of the approximating sequence in Lemma 2.4 implies that for all x

$$E(V(x + \langle \tilde{\xi}(x), Y \rangle)|\mathcal{H}) \geq G(x) \text{ a.s.}$$

hence by the definition of G

$$E(V(x + \langle \tilde{\xi}(x), Y \rangle)|\mathcal{H}) = G(x) \text{ a.s.}$$

It remains to check that we were allowed to invoke the Fatou lemma. This follows from Lemma 2.3, the random variable L figuring there is a suitable majorant.

Proposition 2.4. *The $\tilde{\xi}$ constructed in the proof of Lemma 2.5 is such that $\tilde{\xi}(H) \in \Xi^H$ and*

$$G(H) = E(V(H + \langle \tilde{\xi}(H), Y \rangle)|\mathcal{H}) = \text{ess.} \sup_{\xi \in \Xi^H} E(V(H + \langle \xi, Y \rangle)|\mathcal{H}) \text{ a.s.},$$

for any $\mathcal{H}$-measurable $[0, \infty)$-valued random variable H; here G is the function constructed in Proposition 2.3.

Proof. By the piecewise constant structure of the approximating sequence of Lemma 2.4 we have that

$$P(\forall x \ \forall n \quad x + \langle \hat{\xi}_n(x, \omega), Y \rangle \geq 0) = 1.$$

Random subsequences do not change this, so

$$P(\forall x \quad x + \langle \tilde{\xi}(x, \omega), Y \rangle \geq 0) = 1,$$

which implies that $\tilde{\xi}(H) \in \Xi^H$.

For the proof of "$\leq$" in the first equality we refer to Proposition 4.10 of [7]. The left-hand side of the second equality is clearly not greater than the right-hand side, so we need only to show that for fixed $\xi \in \Xi^H$ we have:

$$G(H, \omega) \geq E(V(H + \langle \xi, Y \rangle)|\mathcal{H}) \text{ a.s.} \tag{2.12}$$

For step functions H (2.12) is clearly true. Now for general H take a decreasing step-function approximation H_n of H. Then $\xi \in \Xi^H \subseteq \Xi^{H_n}$ for all n, hence

$$G(H_n) \geq E(V(H_n + \langle \xi, Y \rangle)|\mathcal{H}) \text{ a.s.},$$

the left-hand side converges by path regularity of G, the right-hand side by monotone convergence, so (2.12) is proved.

Remark 2.2. Results of the present section may be extended to a slightly more general setting. We briefly sum up the major modifications.

Let $V : [0,\infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$ be a function such that $V(x,\cdot)$ is $\mathcal{F}$-measurable for each x and for almost all ω the function $V(\cdot,\omega)$ is nondecreasing, concave and upper semicontinuous. Put

$$\Theta(\omega) := 0 \vee \sup\{q \in \mathbb{Q}_+ : \ V(q,\omega) = -\infty\}.$$

Assume that Θ is a bounded random variable and introduce the random variable

$$\theta := \text{ess.}\inf\{X : \ \sigma(X) \subset \mathcal{H}, \ \exists \varphi \in \Xi \text{ s.t. } X + \langle \varphi, Y\rangle \geq \Theta \text{ a.s.}\}.$$

Redefine Ξ^H for each $\mathcal{H}$-measurable $H \geq \theta$ as

$$\Xi^H := \{\xi \in \Xi : \ H + \langle \xi, Y\rangle \geq \Theta \text{ a.s.}\}.$$

Replace (2.3) by

$$\forall x \in [0,\infty) \quad \text{ess.}\sup_{\xi \in \Xi^{\theta+x}} E(V(x + \langle \xi, Y\rangle)|\mathcal{H}) < \infty$$

and (2.2) by

$$V(F) \geq 0, \tag{2.13}$$

where $F > 0$ is some constant. Otherwise let the notations and hypotheses at the beginning of this section be valid.

One needs to construct regular versions of

$$y \to E(V(\theta + y + \langle \xi, Y\rangle)|\mathcal{H}), \ y \geq x,$$

for $\xi \in \Xi^{\theta+x}$ in Proposition 2.1.

Proposition 2.2 and Lemma 2.1 remain almost unchanged except for replacing Ξ^x by $\Xi^{x+\theta}$. The estimation of Lemma 2.2 is slightly modified due to (2.13), Lemma 2.3 remains practically the same.

Instead of Proposition 2.3 one has to establish the following:

Proposition 2.5. *There is a function $G : [0,\infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$ such that $G(\theta + y)$ is a version of*

$$\text{ess.}\sup_{\xi \in \Xi^{\theta+y}} E(V(\theta + y + \langle \xi, Y\rangle)|\mathcal{H})$$

for each fixed $y \in [0,\infty)$; $G(x,\omega) = -\infty$ if $x < \theta(\omega)$, $G(\cdot,\omega)$ is a nondecreasing, concave, continuous function on $[\theta(\omega),\infty)$ and finite-valued on $(\theta(\omega),\infty)$, for almost all ω.

In Lemma 2.4 the approximating sequence should be constructed on the random interval (θ,∞). Then along the same arguments we finally get:

Proposition 2.6. *There exists a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{H}$-measurable function $\tilde{\xi}$ such that for any $\mathcal{H}$-measurable random variable $H \geq \theta$ we have $\tilde{\xi}(H) \in \Xi^H$ and*

$$G(H) = E(V(H + \langle \tilde{\xi}(H), Y\rangle)|\mathcal{H}) = \text{ess.}\sup_{\xi \in \Xi^H} E(V(H + \langle \xi, Y\rangle)|\mathcal{H}),$$

almost surely.

3 Dynamic programming

From now on we suppose that

$$U(1) = 0. \tag{3.1}$$

This is to assure (2.2), which plays a role in Lemma 2.2. Obviously there is no loss of generality here: by adding a constant to the utility function one may always have (3.1) without changing the optimal strategy.

Define by recursion the following random functions. The existence of the conditional expectations will be shown in Proposition 3.1 below. Set

$$U_T(x, \omega) := U(x), \quad x \in [0, \infty), \quad \omega \in \Omega, \tag{3.2}$$

and, for $t < T$,

$$U_t(x, \omega) := \text{ess.} \sup_{\xi \in \Xi_t^x} E(U_{t+1}(x + \langle \xi, \Delta S_{t+1} \rangle) | \mathcal{F}_t)(\omega), \ x \in [0, \infty), \ \omega \in \Omega; \tag{3.3}$$

later on we shall omit the dependence on ω in notations. Set $U_t(x) := -\infty$, $x < 0$.

Proposition 3.1. *The functions U_t, $0 \le t \le T$, have versions which are almost surely nondecreasing, concave and continuous on $[0, \infty)$, finite-valued on $(0, \infty)$ and*

$$U_t(1) \ge 0, \ 0 \le t \le T, \tag{3.4}$$

$$\text{ess.} \sup_{\xi \in \Xi_{t-1}^x} E(U_t(x + \langle \xi, \Delta S_t \rangle) | \mathcal{F}_{t-1}) < \infty, \ x \in [0, \infty), \ 1 \le t \le T, \tag{3.5}$$

where the expectations are well-defined. There exist $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_t$-measurable functions $\tilde{\xi}_{t+1}$, $0 \le t \le T-1$, such that for all $x \in (0, \infty)$

$$U_t(x, \omega) = E(U_{t+1}(x + \langle \tilde{\xi}_{t+1}(x), \Delta S_{t+1} \rangle) | \mathcal{F}_t). \tag{3.6}$$

Proof. Going backwards from T to 0, we apply Lemma 2.5 with $V := U_t$, $\mathcal{H} = \mathcal{F}_{t-1}$, $D := D_t$, $Y := \Delta S_t$.

We need to verify the conditions of Section 2: D is a random subspace by Propositions 1.1 and 4.1; (2.1) follows from (1.6); (2.2) and (2.3) will come from (3.4) and (3.5). We will check (3.4) and (3.5) in a little while.

Expectations exist by (2.3), a good version for U_t is provided by Proposition 2.3. Denote the resulting $\tilde{\xi}$ of Lemma 2.5 by $\tilde{\xi}_t$, $1 \le t \le T$; it certainly satisfies (3.6).

It remains to establish (3.4) and (3.5). The first statement is true, since

$$U_t(x) \ge E(U_{t+1}(x) | \mathcal{F}_t) \ge \cdots \ge E(U_T(x) | \mathcal{F}_t) = U(x), \tag{3.7}$$

and $U(1) = 0$ by Assumption 1.1. As to the second statement, it holds for $t = T$ by (1.5). For $t = T - 1$ consider

$$U_{T-1}(x + \langle \xi, \Delta S_{T-1} \rangle) =$$
$$E(U_T(x + \langle \xi, \Delta S_{T-1} \rangle + \langle \tilde{\xi}_{T-1}(x + \langle \xi, \Delta S_{T-1} \rangle), \Delta S_T \rangle)|\mathcal{F}_{T-1}),$$

so the statement holds by (1.5) again. For other values of t the notation gets more and more complicated but the same argument applies.

Now set $\varphi_1^*(c) := \tilde{\xi}_1(c)$ and define recursively:

$$\varphi_{t+1}^*(c) := \tilde{\xi}_{t+1}(c + \sum_{j=1}^{t} \langle \varphi_j^*, \Delta S_j \rangle),\ 1 \le t \le T-1.$$

Joint measurability of $\tilde{\xi}_t$ assures that $\varphi^* = \varphi^*(c)$ is a predictable process with respect to the given filtration.

Proposition 3.2. *We have $\varphi^* \in \mathcal{A}(c)$ and for any strategy $\varphi \in \mathcal{A}(c)$*

$$E(U(V_T^{c,\varphi})|\mathcal{F}_0) \le E(U(V_T^{c,\varphi^*})|\mathcal{F}_0) = U_0(c). \tag{3.8}$$

Proof. Notice that $\varphi_t^* \in \Xi_{t-1}(V_{t-1}^{c,\varphi^*})$, so $\varphi^* \in \mathcal{A}(c)$. Remembering $U_T = U$ and using Proposition 2.4, we may rewrite the right-hand side of (3.8) as follows:

$$E(U_T(V_T^{c,\varphi^*})|\mathcal{F}_0) = E(E(U_T(V_{T-1}^{c,\varphi^*} + \langle \varphi_T^*, \Delta S_T \rangle)|\mathcal{F}_{T-1})|\mathcal{F}_0) =$$
$$= E(U_{T-1}(V_{T-1}^{c,\varphi^*})|\mathcal{F}_0).$$

Continuing the procedure, we finally arrive at $\varphi^* \in \mathcal{A}(c)$ and

$$E(U(V_T^{c,\varphi^*})|\mathcal{F}_0) = E(U_1(V_1^{c,\varphi^*})|\mathcal{F}_0) = E(U_1(c + \langle \varphi_1^*, \Delta S_1 \rangle)|\mathcal{F}_0) = U_0(c). \tag{3.9}$$

We remark that all conditional expectations below exist by Proposition 3.1. By the definition of U_{T-1} and $\varphi \in \mathcal{A}(c)$ one has $\varphi_T \in \Xi_{T-1}(V_{T-1}^{c,\varphi})$ and

$$E(U_T(V_T^{c,\varphi})|\mathcal{F}_{T-1}) = E(U_T(V_{T-1}^{c,\varphi} + \langle \varphi_T, \Delta S_T \rangle)|\mathcal{F}_{T-1}) \le U_{T-1}(V_{T-1}^{c,\varphi}) \text{ a.s.}$$

Iterate the same argument and obtain

$$E(U(V_T^{c,\varphi})|\mathcal{F}_0) \le U_0(c) \text{ a.s.} \tag{3.10}$$

Putting (3.9) and (3.10) together, one gets exactly (3.8).

Proof (of Theorem 1.1). Proposition 3.2 shows that $u(c) = EU_0(c)$ and the φ^* constructed in the last two sections is a maximizer such that $\varphi_t^* \in D_t$.

Remark 3.1. We indicate how Theorem 1.1 can be generalized. Let $B \ge 0$ be a bounded random variable, interpreted as a contingent claim. Define recursively the superhedging prices as follows:

$$\pi_T(B) := B,$$
$$\pi_t(B) := \text{ess.}\inf\{X:\ \sigma(X) \subset \mathcal{F}_t,\ \exists \varphi \in \Xi_t\ X + \langle \varphi, \Delta S_{t+1}\rangle \geq \pi_{t+1}(B) \text{ a.s.}\},$$

for $0 \leq t \leq T-1$.

Define for $c > \pi_0(B)$

$$\mathcal{A}(B,c) := \{\varphi \in \Phi:\ V_t^{c,\varphi} \geq \pi_t(B) \text{ a.s.},\ 0 \leq t \leq T\},$$

and redefine for each $\mathcal{F}_t$-measurable $H \geq \pi_t(B)$

$$\Xi_t(H) := \{\xi \in \Xi_t:\ H + \langle \xi, \Delta S_{t+1}\rangle \geq \pi_{t+1}(B) \text{ a.s.}\}$$

Theorem 3.1. *Suppose that the conditions of Theorem 1.1 hold and $\mathcal{F}_0$ is trivial. Then for all $c > \pi_0(B)$*

$$u(B,c) := \sup_{\varphi \in \mathcal{A}(B,c)} EU(V_T^{c,\varphi} - B) < \infty, \tag{3.11}$$

and there exists $\varphi^*(c) \in \mathcal{A}(B,c)$ such that

$$u(B,c) = EU(V_T^{c,\varphi^*(c)} - B).$$

Proof. As $\mathcal{F}_0$ is trivial, $\pi_0(B)$ is a constant; (3.11) follows from (1.5) and the boundedness of B. Since B is bounded, by Assumption 1.1 there exists $F > 0$ such that $U_T(F) \geq 0$, and this will remain true for each U_t by (3.7).

Replace (3.2) by

$$U_T(x,\omega) := U(x - B(\omega)),\ x \geq B(\omega),\ U_T(x,\omega) = -\infty,\ x < B(\omega),$$

set for $y \in [0,\infty)$

$$U_t(\pi_t(B) + y, \omega) := \text{ess.} \sup_{\xi \in \Xi_t(\pi_t(B)+y)} E(U_{t+1}(\pi_t(B) + y + \langle \xi, \Delta S_{t+1}\rangle)|\mathcal{F}_t),$$

and

$$U_t(x,\omega) = -\infty, \quad x < \pi_t(B)(\omega),$$

instead of (3.3) and follow the argument of this section. Use the extended setting of section 2 as explained in Remark 2.2. Apparently, Θ, θ will correspond to $\pi_{t+1}(B), \pi_t(B)$ in the backward induction. The rest of the argument is essentially unchanged.

4 Auxiliary results

We shall often rely on the measurable selection theorem, see III. 70-73 of [2]. Let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra containing P-null sets. An $\mathcal{H}$-measurable *random set* or *measurable multifunction* A is an element of $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sets of $\mathbb{R}^d$. A *random affine subspace* A is an $\mathcal{H}$-measurable random set such that $A(\omega)$ is an affine subspace of $\mathbb{R}^d$ for each ω.

Let Y be a d-dimensional random variable and $\mu(\cdot,\omega) := P(Y \in \cdot|\mathcal{H})$ a regular version of its conditional distribution. Let $D(\omega)$ be the smallest affine subspace of $\mathbb{R}^d$ containing the support of $\mu(\cdot,\omega)$.

Proposition 4.1. *D is an $\mathcal{H}$-measurable random affine subspace.*

Proof. We begin by showing that supp $\mu(\cdot,\omega)$ or, equivalently, its complement $\mathrm{supp}^C\mu(\cdot,\omega)$ is a random set. Let $\mathcal{G}$ be a countable base for the topology of $\mathbb{R}^d$. Then

$$\mathrm{supp}^C\mu(\cdot,\omega) := \bigcup\{G \in \mathcal{G}:\ \mu(G,\omega) = 0\},$$

which proves the assertion. Actually, $Z(\omega) := \mathrm{conv}(\mathrm{supp}\mu(\cdot,\omega))$ is a random set, where $\mathrm{conv}(\cdot)$ denotes closed convex hull, this follows from Theorem III. 40 on p. 87 of [1].

Take a measurable selector $\nu(\omega)$ of $Z(\omega)$; $Z - Z$ contains the origin in its relative interior and

$$\left[\bigcup_{n\in\mathbb{N}}\{nz:\ z \in Z(\omega) - Z(\omega)\}\right] + \nu(\omega),$$

equals $D(\omega)$, which proves the proposition.

Proposition 4.2. *Fix $x > 0$. There exists $M(x) \in \mathcal{H}\otimes\mathcal{B}(\mathbb{R}^d)$ which is convex, compact (a.s.) and*

$$\xi \in \Xi^x \text{ and } \xi \in D \text{ a.s.} \iff \xi \in M(x) \text{ a.s.}$$

Proof. Take a sequence of $\mathcal{H}$-measurable random variables σ_i such that for almost all ω the sequence $\sigma_i(\omega),\ i \in \mathbb{N}$, is dense in $\mathrm{supp}\mu(\cdot,\omega)$. Such a sequence exists by Theorem III. 22 on p. 74 of [1]. Define the convex closed random set

$$\tilde{M}(x) := \bigcap_{i\in\mathbb{N}}\{(\omega,p):\ x + \langle p, \sigma_i(\omega)\rangle \geq 0\}.$$

The following series of equivalences is clear:

$$\begin{aligned}\xi \in \Xi^x \iff P(x + \langle \xi, Y\rangle \geq 0) = 1 \iff P(x + \langle \xi, Y\rangle \geq 0|\mathcal{H}) = 1, \text{ a.s.}\\ \iff \mu(\{y \in \mathbb{R}^d:\ x + \langle \xi(\omega), y\rangle \geq 0\}, \omega) = 1 \text{ a.s.} \iff\\ \{y \in \mathbb{R}^d:\ x + \langle \xi(\omega), y\rangle \geq 0\} \supseteq \mathrm{supp}\mu(\cdot,\omega) \text{ a.s.} \iff\\ \{y \in \mathbb{R}^d:\ x + \langle \xi(\omega), y\rangle \geq 0\}\sigma_i(\omega) \text{ a.s.},\ i \in \mathbb{N},\end{aligned}$$

and this last one means precisely $\xi \in \tilde{M}(x)$ a.s. Define $M(x) := \tilde{M}(x) \cap D$. The argument of Lemma 2.1 implies that $M(x)$ is compact, almost surely, so $M(x)$ is as desired.

Let ri $M(x,\omega)$ denote the relative interior of $M(x,\omega)$ and let $R = R(x,\omega)$ denote the linear span of $M(x,\omega)$.

Proposition 4.3. *Both* ri $M(x)$ *and $R(x)$ are $\mathcal{H}$-measurable random sets.*

Proof. The set $M - M$ contains zero in its relative interior, hence

$$R = \bigcup_{n \in \mathbb{N}} \{nz : \ z \in M - M\}$$

and this is indeed a random set. Take $\mathcal{H}$-measurable random variables $\zeta_i(\omega),\ 1 \leq i \leq d$, which are orthogonal and generate $R(x)$ (some of them might be 0), this follows easily from the measurable selection theorem. The function

$$[\dim\ R(x)](\omega) := \sum_{j=1}^{d} I_{\{\zeta_j \neq 0\}}(\omega)$$

is $\mathcal{H}$-measurable. It suffices to prove the proposition separately on the events

$$\{\omega : \ \dim\ R(x, \omega) = m\} \in \mathcal{H},$$

for each $m \leq d$. The case $m = 0$ is trivial, so we suppose, without loss of generality, that dim $R(x, \omega) = m \geq 1$ for a fixed m. We may assume that $\zeta_i(\omega),\ 1 \leq i \leq m$ is an orthonormed basis of $R(x, \omega)$.

The interior points are precisely those, around which a little cube can be drawn in $R(x)$ which still belongs to $M(x)$. As $M(x)$ is convex, this is equivalent to the fact that the edges of that cube belong to $M(x)$. Hence

$$\text{ri}\ M(x) = \bigcup_{n \in \mathbb{N}} \left\{ (\omega, p) : \ p + \frac{1}{n} \sum_{j=0}^{m} i(j) \zeta_j(\omega) \in M(\omega, x),\ \forall i \in \{-1, +1\}^m \right\},$$

which is clearly a measurable multifunction.

Lemma 4.1. *Let $a, b \in \mathbb{R},\ a < b$. Let $\eta_n : [a, b] \times \Omega \to \mathbb{R}^d$ be a sequence of $\mathcal{B}([a, b]) \otimes \mathcal{H}$-measurable functions such that for almost all ω*

$$\forall x \ \liminf_{n \to \infty} |\eta_n(x, \omega)| < \infty.$$

Then there is a sequence n_k of $\mathcal{B}([a, b]) \otimes \mathcal{H}$-measurable $\mathbb{N}$-valued functions, $n_k < n_{k+1},\ k \in \mathbb{N}$, such that $\tilde{\eta}_k(x, \omega) := \eta_{n_k}(x, \omega)$ converges for all x to some $\tilde{\eta}(x, \omega)$ as $k \to \infty$, for almost all ω. To put it more concisely, there is a convergent random subsequence.

Proof. This is just a variant of Lemma 2 in [5].

5 Conclusions

Finally, we present a concrete model class where there exists an optimal investment strategy. Let $\mathcal{W}$ denote the family of random variables with finite moments of all orders.

Proposition 5.1. *Let U satisfy Assumption 1.1. Let* $|S_t| \in \mathcal{W}$, $0 \leq t \leq T$, *and supppose that* (1.6) *holds with* $1/\beta_t \in \mathcal{W}$, $0 \leq t \leq T-1$. *Then* (1.5) *holds and the assertion of Theorem 1.1 is true.*

Proof. For notational simplicity let $\xi \Delta S_t$ denote scalar product. We shall show by backward induction that there exists $J_t \in \mathcal{W}$ such that

$$U_t(x) \leq J_t x < \infty, \quad x \in (0,\infty), \quad 0 \leq t \leq T.$$

Indeed, for $t = T$ this is true with $J_T := U'(1)$. Now suppose that this statement has been established for $s \geq t+1$. Proposition 2.2 and Lemma 2.1 imply that

$$\begin{aligned}\text{ess.}\sup_{\xi \in \Xi_t^x} E(U_{t+1}(x + \xi \Delta S_{t+1})|\mathcal{F}_t) &= \text{ess.}\sup_{\xi \in \Xi_t^x, \xi \in D_{t+1}} E(U_{t+1}(x + \xi \Delta S_{t+1})|\mathcal{F}_t) \\ &\leq E(U_{t+1}(x + |\Delta S_{t+1}|x/\beta_t)|\mathcal{F}_t) \leq E(J_{t+1}x + J_{t+1}x|\Delta S_{t+1}|/\beta_t|\mathcal{F}_t),\end{aligned}$$

so we may set $J_t := E(J_{t+1}(1 + |\Delta S_{t+1}|/\beta_t)|\mathcal{F}_t)$. Finally we arrive at the bound $U_0(x) \leq J$ almost surely, where $J \in \mathcal{W}$ so we get for all $x > 0$

$$u(x) = EU_0(x) < \infty,$$

i.e. (1.5) holds true. The proof of Theorem 1.1 shows that there exists an optimal φ^*.

Remark 5.1. The previous proposition applies, in particular, when $\beta_t = \beta$ is a deterministic constant. The hypothesis that (1.6) holds with deterministic β is called *uniform no-arbitrage condition*. This assumption has been introduced in [8].

Remark 5.2. One may consider concave nondecreasing functions $U : \mathbb{R} \to \mathbb{R}$. Under (NA), (1.5) and additional hypotheses on U there exists an optimal strategy in Φ, see [7]. We may also look at "tame" portfolios, i.e. φ such that there exists $a \in \mathbb{R}$ satisfying

$$V_t^{c,\varphi} \geq a \text{ a.s.}, \quad 0 \leq t \leq T. \tag{5.1}$$

Theorem 1.1 of the present paper immediately implies that (under (NA) and (1.5)) there exists an optimal strategy among φ satisfying (5.1) with a fixed a. It is an intriguing question under what kind of conditions there is an optimal control among all tame strategies.

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