Explicit Solution to an Irreversible Investment Model with a Stochastic Production Capacity

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Summary. This paper studies the problem of a company which expands its stochastic production capacity in irreversible investments by purchasing capital at a given price. The profit production function is of a very general form satisfying minimal standard assumptions. The objective of the company is to find optimal production decisions to maximize its expected total net profit in an infinite horizon. The resulting dynamic programming principle is a singular stochastic control problem. The value function is analyzed in great detail relying on viscosity solutions of the associated Bellman variational inequality: we state several general properties and in particular regularity results on the value function. We provide a complete solution with explicit expressions of the value function and the optimal control: the firm invests in capital so as to maintain its capacity above a certain threshold. This boundary can be computed quite explicitly.

Key words: singular stochastic control, viscosity solutions, Skorohod problem, irreversible investment, production.

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1 Introduction

This paper focuses on the problem of a company which wants to expand its stochastic production capacity. The investments in capital for expanding the capacity are irreversible in the sense that the company cannot recover the investment by reducing the capacity. In addition, there is a transaction cost for purchasing capital. We refer to the book by Dixit and Pindick (1994) for a review where such problems occur. There are several papers in the literature dealing with irreversible investments models. For instance, Kobila (1993) consider a model with deterministic capacity in an uncertain market and without transaction costs on buying capital. Recently, Chiarolla and Haussmann

(2003) studied an irreversible investment model in a finite horizon and obtained an explicit solution for a power type production function.

We consider a concave production function of very general form, satisfying minimal standard assumptions. The buying capital decision is modelled by a singular control. This allows for instantaneous purchase of capital of arbitrary large amounts and various other sorts of behavior. The company's objective is to maximize the expected net production profit over an infinite horizon, with choice of control of its buying. The resulting dynamic programming principle leads to a singular stochastic control problem. There is by now a number of papers on singular controls related to financial problems, see, e.g., Davis and Norman (1990) and Jeanblanc-Picqué and Shiryaev (1995).

We solve mathematically this problem by a viscosity solution approach. This contrasts with the classical approach on investment models where the principal activity is to construct by ad hoc methods a solution to the Hamilton–Jacobi–Bellman equation, and validate the optimality of the solution by a verification theorem argument for smooth functions. We, on the other hand, start by studying and deriving the general properties via the dynamic programming principle and viscosity arguments. Using the concavity property of the value function, we prove that it satisfies in fact the HJB in the classical C^2-sense. Similar approach is done in the paper by Shreve and Soner (1994) for optimal consumption models with transaction costs.

The rest of the paper goes as follows. In the next section, we give a mathematical formulation of the problem. We analyze and derive some general properties of the value function in Section 3. By means of viscosity solutions arguments, we state in Section 4 the C^2-smoothness of the value function that satisfies then in a classical sense the associated HJB equation. Section 5 is devoted to the explicit construction of the solution to this singular control problem and the optimal control.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and carrying a standard one-dimensional Brownian motion W.

We consider a firm producing some output from stochastic capacity production K_t and possibly also from other inputs. The firm can buy capital at any time t at constant price $p > 0$. The production rate process is then described by a control $L \in \mathcal{A}$, set of right-continuous with left-hand limits adapted processes, nonnegative and nondecreasing, with $L_{0^-} = 0$. Here, L_t represents the cumulative purchase of capital until time t. Given the initial capital $k \geq 0$, and control $L \in \mathcal{A}$, the firm's capacity production evolves according to the linear SDE

$$dK_t = K_t\left(-\delta dt + \gamma dW_t\right) + dL_t, \qquad K_{0^-} = k. \tag{2.1}$$

Here $\delta \geq 0$ is the depreciation rate of the capacity production and $\gamma > 0$ represents its volatility.

The instantaneous operating profit of the firm is a function $\Pi(K_t)$ of the capacity production. The production profit function Π is assumed to be continuous on $\mathbb{R}_+$, nondecreasing, concave and C^1 on $(0,\infty)$, with $\Pi(0) = 0$ and satisfying the standing usual Inada conditions :

$$\Pi'(0^+) \;:=\; \lim_{k \downarrow 0} \Pi'(k) \;=\; \infty \;\text{ and }\; \Pi'(\infty) \;:=\; \lim_{k \to \infty} \Pi'(k) \;=\; 0. \tag{2.2}$$

We define the Fenchel–Legendre transform of Π, which is finite on $(0,\infty)$ under the Inada conditions:

$$\tilde{\Pi}(z) := \sup_{k \geq 0} [\Pi(k) - kz] \;<\; \infty, \quad \forall z > 0. \tag{2.3}$$

A typical example arising from the Cobb–Douglas production function leads to a profit function of the form

$$\Pi(k) = Ck^{\alpha}, \quad \text{ with } \; C > 0,\; 0 < \alpha < 1. \tag{2.4}$$

The firm's objective is to maximize the expected profit on the infinite time horizon

$$J(k, L) = E\left[\int_0^{\infty} e^{-rt} \left(\Pi(K_t)dt \;-\; pdL_t\right)\right] \tag{2.5}$$

over all controls $L \in \mathcal{A}$. Here $r > 0$ is a fixed positive discount factor. Without loss of generality, one may consider the strategies L in $\mathcal{A}$ for which

$$E\left[\int_0^{\infty} e^{-rt} dL_t\right] < \infty, \tag{2.6}$$

Accordingly, we define the value function

$$v(k) = \sup_{L \in \mathcal{A}} J(k, L), \qquad k \geq 0. \tag{2.7}$$

Notice that since $J(k,0) \geq 0$, the value function v takes value in $[0,\infty]$.

3 Some properties of the value function

Problem (2.7) is a singular stochastic control problem and its associated Hamilton–Jacobi–Bellman equation is

$$\min \{rv - \mathcal{L}v - \Pi \;,\; -v' + p\} = 0, \tag{3.1}$$

where $\mathcal{L}$ is the second order operator

$$\mathcal{L}\varphi = \frac{1}{2}\gamma^2 k^2 \varphi'' - \delta k \varphi'$$

for any C^2-function φ.

We first state a standard comparison theorem, which says that any smooth function, being a supersolution of the HJB equation (3.1), dominates v.

To this end, we first recall in our context how Itô's formula for càdlàg semimartingales (see, e.g., [8]) is written. Let $\varphi \in C^2(0,\infty)$ and let τ be a finite stopping time, $k > 0$ and $L \in \mathcal{A}$. Then, we have:

$$e^{-r\tau}\varphi(K_\tau) = \varphi(k) + \int_0^\tau e^{-rt}\left(-r\varphi + \mathcal{L}\varphi\right)(K_t)dt + \int_0^\tau e^{-rt}\gamma K_t \varphi'(K_t)dW_t + \int_0^\tau e^{-rt}\varphi'(K_t)dL_t^c + \sum_{0\le t\le\tau} e^{-rt}\left[\varphi(K_t) - \varphi(K_{t^-})\right], \tag{3.2}$$

where

$$L_t^c = L_t - \sum_{0\le s\le t} \Delta L_s,$$

is the continuous part of L.

Proposition 3.1. *Let φ be a nonnegative C^2-function which is a supersolution on $(0,\infty)$ to (3.1), i.e.:*

$$\min\left\{r\varphi - \mathcal{L}\varphi - \Pi(k)\,,\; -\varphi' + p\right\} \ge 0, \quad k > 0. \tag{3.3}$$

Then,

$$v(k) \le \varphi(k), \quad \forall k > 0.$$

Proof. For $L \in \mathcal{A}$ define the stopping time $\tau_n = \inf\{t \ge 0 : K_t \ge n\} \wedge n$ and apply Itô's formula (3.2) between 0 and τ_n. Then, taking expectation and noting that the integrand in the stochastic integral is bounded on $[0,\tau_n)$, we get that

$$E\left[e^{-r\tau_n}\varphi(K_{\tau_n})\right] = \varphi(k) + E\left[\int_0^{\tau_n} e^{-rt}\left(-r\varphi + \mathcal{L}\varphi\right)(K_t)dt\right] + E\left[\int_0^{\tau_n} e^{-rt}\varphi'(K_t)dL_t^c\right] + E\left[\sum_{0\le t\le\tau_n} e^{-rt}\left[\varphi(K_t) - \varphi(K_{t^-})\right]\right].$$

Since $\varphi' \le p$, and $K_t - K_{t^-} = \Delta L_t$, the mean-value theorem implies that

$$\varphi(K_t) - \varphi(K_{t^-}) \le p\Delta L_t.$$

Using again the inequality $\varphi' \leq p$ in the integrals with respect to dL^c and taking into account that $-r\varphi + \mathcal{L}\varphi \leq -\Pi$, we obtain:

$$\begin{aligned} E\left[e^{-r\tau_n}\varphi(K_{\tau_n})\right] \leq \varphi(k) \;-\; & E\left[\int_0^{\tau_n} e^{-rt}\Pi(K_t)dt\right] \\ &+ E\left[\int_0^{\tau_n} e^{-rt}pdL_t^c\right] \;+\; E\left[\sum_{0\leq t\leq\tau_n} e^{-rt}p\Delta L_t\right] \\ &= \varphi(k) \;-\; E\left[\int_0^{\tau_n} e^{-rt}\Pi(K_t)dt\right] \;+\; E\left[\int_0^{\tau_n} e^{-rt}pdL_t\right], \end{aligned}$$

and so

$$E\left[\int_0^{\tau_n} e^{-rt}\left(\Pi(K_t)dt - pdL_t\right)\right] \;+\; E\left[e^{-r\tau_n}\varphi(K_{\tau_n})\right] \leq \varphi(k).$$

Since φ is nonnegative,

$$\varphi(k) \geq E\left[\int_0^{\tau_n} e^{-rt}\Pi(K_t)dt\right] \;-\; E\left[\int_0^{\infty} e^{-rt}pdL_t\right].$$

Applying Fatou's lemma we get that

$$E\left[\int_0^{\infty} e^{-rt}\left(\Pi(K_t)dt - pdL_t\right)\right] \leq \varphi(k),$$

and so, finally, $v(k) \leq \varphi(k)$ from the arbitrariness of L. □

We now give some properties on the value function v.

Lemma 3.1. *For all $k \geq 0$ and $l \geq 0$, we have:*

$$v(k) \geq -pl + v(k+l). \tag{3.4}$$

Proof. For $L \in \mathcal{A}$ we consider the control $\tilde{L}$ with $\tilde{L}_{0^-} = 0$ and $\tilde{L}_t = L_t + l$, for $t \geq 0$. Let $\tilde{K}$ be the solution of (2.1) with the control $\tilde{L}$ and initial condition $\tilde{K}_{0^-} = k$. Then, $\tilde{K}_t = K_t + l$ for $t \geq 0$, and so $\tilde{L} \in \mathcal{A}$. Thus,

$$\begin{aligned} v(k) \geq J(k, \tilde{L}) \;&=\; E\left[\int_0^{\infty} e^{-rt}\left(\Pi(\tilde{K}_t)dt - pd\tilde{L}_t\right)\right] \\ &= J(k+l, L) - pl. \end{aligned}$$

We obtain the required result from the arbitrariness of L. □

Moreover, recalling the standing assumption (2.3), we have:

Lemma 3.2. *The value function v is finite and for any $q \in [0, p]$*

$$0 \leq \; v(k) \; \leq \frac{\tilde{\Pi}((r+\delta)q)}{r} + kq, \quad k \geq 0. \tag{3.5}$$

Proof. The zero lower bound has been already noticed in Section 2. To prove the upper bound, consider for $q \in [0,p]$ the nonnegative function

$$\varphi(k) = kq + \frac{\tilde{\Pi}((r+\delta)q)}{r}.$$

Then, $\varphi' \leq p$ and

$$r\varphi - \mathcal{L}\varphi - \Pi = \tilde{\Pi}((r+\delta)q) + (r+\delta)kq - \Pi(k) \ \geq \ 0, \quad \forall\, k \geq 0,$$

by definition of $\tilde{\Pi}$ in (2.3). This implies that the nonnegative function φ is a super-solution to (3.1), and we conclude with Proposition 3.1. □

Lemma 3.3. *a) The value function v is nondecreasing, concave and continuous on $(0,\infty)$.*

b) We have the inequalities: $0 \leq v(0^+) \leq \frac{\tilde{\Pi}((r+\delta)p)}{r}$.

Proof. a) The nondecreasing monotonicity of v follows from the nondecreasing property of the process K with respect to the initial condition k given an admissible control L, and from the nondecreasing monotonicity of Π.

The proof of concavity of v is standard: it is established by considering convex combinations of initial states and controls and using the linearity of dynamics (2.1) and concavity of Π.

b) The limit $v(0^+)$ exists from the nondecreasing property of v. By taking $q = p$ in the inequality of Lemma 3.2, we obtain the required estimation on this limit. □

Since v is concave on $(0,\infty)$, it admits a right derivative $v'_+(k)$ and a left derivative $v'_-(k)$ at any $k > 0$, and $v'_+(k) \leq v'_-(k)$. Moreover, inequality (3.4) shows that

$$v'_-(k) \leq p, \quad \forall k > 0. \tag{3.6}$$

We then define the so-called no-transaction region :

$$\mathcal{NT} = \left\{k > 0 : \ v'_-(k) < p\right\}.$$

Lemma 3.4. *There exists $k_b \in [0,\infty]$ such that:*

$$\mathcal{NT} = (k_b, \infty), \tag{3.7}$$

v is differentiable on $(0,k_b)$ and

$$v'(k) = p \quad \text{on } \ \mathcal{B} = (0,k_b). \tag{3.8}$$

Proof. Put $k_b = \inf\{k \geq 0 : \ v'_+(k) < p\}$. Then $p \leq v'_+(k) \leq v'_-(k)$ if $k < k_b$. Together with (3.6), this proves (3.8). Finally, the concavity of v shows (3.7). □

Remark 3.1. We shall see later that $0 < k_b < \infty$, and the optimal strategy for the firm consists in doing nothing when it is in the region $\mathcal{NT} = (k_b,\infty)$, and in buying capital when it is below k_b in order to reach the threshold k_b. The region $\mathcal{B} = (0,k_b)$ will be then called the buy region.

4 Viscosity solutions and regularity of the value function

The concept of viscosity solutions is known to be a general power tool for characterizing the value function of a stochastic control problem, see, e.g., [4]. It is based on the dynamic programming principle which we now recall in our context.

Dynamic programming principle: Assume that v is continuous on $(0, \infty)$. Then for all $k > 0$, we have

$$v(k) = \sup_{L \in \mathcal{A}} E \left[\int_0^\theta e^{-rt} \left(\Pi(K_t)dt - pdL_t \right) \ + \ e^{-r\theta} v(K_\theta) 1_{\theta < \infty} \right], \quad (4.1)$$

where $\theta = \theta(L)$ is any stopping time, possibly depending on the control $L \in \mathcal{A}$. The precise meaning of this assertion is:

$$\begin{aligned} v(k) &= \sup_{L \in \mathcal{A}} \sup_{\tau \in \mathcal{T}} E \left[\int_0^\theta e^{-rt} \left(\Pi(K_t)dt - pdL_t \right) \ + \ e^{-r\theta} v(K_\theta) 1_{\theta < \infty} \right] \\ &= \sup_{L \in \mathcal{A}} \inf_{\tau \in \mathcal{T}} E \left[\int_0^\theta e^{-rt} \left(\Pi(K_t)dt - pdL_t \right) \ + \ e^{-r\theta} v(K_\theta) 1_{\theta < \infty} \right]. \end{aligned}$$

Here $\mathcal{T}$ denotes the set of stopping times in $[0, \infty]$. The DPP is frequently used in this form in the literature. However, many proofs cannot be considered as rigorous. Clearly, DPP holds for the case where Ω is a path space. However, it is difficult to give a precise reference which covers the situation we consider here. We use this result for granted and left the detailed discussion of this issue for further studies.

We recall the definition of viscosity solutions for a PDE of the form

$$F(x, v, D_x v, D^2_{xx} v) = 0, \quad x \in \mathcal{O}, \quad (4.2)$$

where $\mathcal{O}$ is an open subset in $\mathbb{R}^n$ and F is a continuous function and nonincreasing in its last argument (with respect to the order of symmetric matrices).

Definition 1. *Let v be a continuous function on $\mathcal{O}$. We say that v is a viscosity solution to* (4.2) *on $\mathcal{O}$ if it is*

(i) a viscosity supersolution to (4.2) *on $\mathcal{O}$: for any $x_0 \in \mathcal{O}$ and any C^2-function φ in a neighborhood of x_0 such that x_0 is a local minimum of $v - \varphi$ and $(v - \varphi)(x_0) = 0$, we have:*

$$F(x_0, \varphi(x_0), D_x\varphi(x_0), D^2_{xx}\varphi(x_0)) \geq 0;$$

(ii) a viscosity subsolution to (4.2) *on $\mathcal{O}$: for any $x_0 \in \mathcal{O}$ and any C^2-function φ in a neighborhood of x_0 such that x_0 is a local maximum of $v - \varphi$ and $(v - \varphi)(x_0) = 0$, we have:*

$$F(x_0, \varphi(x_0), D_x\varphi(x_0), D^2_{xx}\varphi(x_0)) \leq 0.$$

Theorem 4.1. *The value function v is a continuous viscosity solution of the Hamilton–Jacobi–Bellman equation* (3.1) *on* $(0,\infty)$.

Proof. The argument is based on the dynamic programming principle and Itô's formula. It is standard, but somewhat technical in this singular control context. We give it in the appendix. □

Based on the property that the value function is a concave viscosity solution of the HJB equation, we can now prove that it belongs to C^2.

Theorem 4.2. *The value function v is a classical C^2-solution on $(0,\infty)$ to the Hamilton–Jacobi–Bellman equation*

$$\min\{rv - \mathcal{L}v - \Pi(k)\ ,\ -v'(k)+p\} = 0, \qquad k > 0.$$

Proof. Step 1. We first prove that v is a C^1-function on $(0,\infty)$. Since v is concave, the left and right derivatives $v'_-(k)$ and $v'_+(k)$ exist for any $k>0$ and $v'_+(k) \le v'_-(k)$. We argue by contradiction and suppose that $v'_+(k_0) < v'_-(k_0)$ for some $k_0 > 0$. Fix some q in $(v'_+(k_0), v'_-(k_0))$ and consider the function

$$\varphi_\varepsilon(k) = v(k_0) + q(k-k_0) - \frac{1}{2\varepsilon}(k-k_0)^2,$$

with $\varepsilon > 0$. Then k_0 is a local maximum of $(v - \varphi_\varepsilon)$ with $\varphi_\varepsilon(k_0) = v(k_0)$. Since $\varphi'_\varepsilon(k_0) = q < p$ by (3.6) and $\varphi''_\varepsilon(k_0) = 1/\varepsilon$, the subsolution property for v to (3.1):

$$\min\{r\varphi(k_0) - \mathcal{L}\varphi(k_0) - \Pi(k_0)\ ,\ -\varphi'(k_0)+p\} \le 0,$$

implies that we must have the inequality

$$r\varphi(k_0) + \delta k_0 q + \frac{1}{\varepsilon} - \Pi(k_0) \le 0. \tag{4.3}$$

With ε sufficiently small, this leads to a contradiction and, hence, proves that $v'_+(k_0) = v'_-(k_0)$.

Step 2. By Lemma 3.4, v belongs to C^2 on $(0,k_b)$ and satisfies $v'(k) = p$, $k \in (0,k_b)$. From Step 1, we have $\mathcal{NT} = (k_b,\infty) = \{k>0 :\ v'(k) < p\}$. We now check that v is a viscosity solution of :

$$rv - \mathcal{L}v - \Pi = 0, \quad \text{on } (k_b,\infty). \tag{4.4}$$

Let $k_0 \in (k_b,\infty)$ and φ be a C^2-function on (k_b,∞) such that k_0 is a local maximum of $v-\varphi$, with $(v-\varphi)(k_0) = 0$. Since $\varphi'(k_0) = v'(k_0) < p$, the subsolution property for v to (3.1):

$$\min\{r\varphi(k_0) - \mathcal{L}\varphi(k_0) - \Pi(k_0)\ ,\ -\varphi'(k_0)+p\} \le 0,$$

implies the inequality

$$r\varphi(k_0) - \mathcal{L}\varphi(k_0) - \Pi(k_0) \leq 0.$$

Thus, v is a viscosity subsolution of (4.4) on (k_b, ∞). The proof of the viscosity supersolution property is similar. Now for arbitrary $k_1 \leq k_2 \in (k_b, \infty)$, consider the Dirichlet boundary problem

$$rV - \mathcal{L}V - \Pi(k) = 0, \quad \text{on } \ (k_1, k_2), \tag{4.5}$$

$$V(k_1) = v(k_1), \quad V(k_2) = v(k_2). \tag{4.6}$$

Classical results provide the existence and uniqueness of a C^2-function V on (k_1, k_2) which is a solution to (4.5)-(4.6). In particular, this smooth function V is a viscosity solution of (4.4) on (k_1, k_2). From standard uniqueness results on viscosity solutions (here for a linear PDE in a bounded domain), we deduce that $v = V$ on (k_1, k_2). From the arbitrariness of k_1, k_2, it follows that v is in C^2 on (k_b, ∞) and satisfies (4.4) in the classical sense.

Step 3. It remains to prove the C^2-condition at k_b in the case $0 < k_b < \infty$. Let $k \in (0, k_b)$. Since v is in C^2 on $(0, k_b)$ with $v'(k) = p$, the supersolution property for v to (3.1) applied at the point k and the test function $\varphi = v$:

$$\min\left\{r\varphi(k) - \mathcal{L}\varphi(k) - \Pi(k)\ ,\ -\varphi'(k) + p\right\} \geq 0,$$

implies that v satisfies (in the classical sense) the inequality:

$$rv(k) - \mathcal{L}v(k) - \Pi(k) \geq 0, \quad 0 < k < k_b.$$

The derivative of v being constant equal to p on $(0, k_b)$, this yields:

$$rv(k) + \delta kp - \Pi(k) \geq 0, \quad 0 < k < k_b,$$

and, therefore,

$$rv(k_b) + \delta k_b p - \Pi(k_b) \geq 0. \tag{4.7}$$

On the other hand, from the C^1-smooth fit at k_b, we have by sending k downwards to k_b into (4.4):

$$rv(k_b) + \delta k_b p - \Pi(k_b) = \frac{1}{2}\gamma^2 k_b^2 v''(k_b^+). \tag{4.8}$$

From the concavity of v, the right-hand side of (4.8) is nonpositive, and this fact, combined with (4.7), implies that $v''(k_b^+) = 0$. This proves that v is C^2 at k_b with $v''(k_b) = 0$. □

5 Solution of the optimization problem

5.1 Some preliminary results on an ODE

We recall some useful results on the second order linear differential equation

$$rv - \mathcal{L}v - \Pi = 0. \tag{5.1}$$

arising from the HJB equation (3.1).

It is well-known that the general solution to the ODE (5.1) with $\Pi = 0$ is given by the formula

$$\hat{V}(k) = Ak^m + Bk^n,$$

where

$$m = \frac{\delta}{\gamma^2} + \frac{1}{2} - \sqrt{\left(\frac{\delta}{\gamma^2} + \frac{1}{2}\right)^2 + \frac{2r}{\gamma^2}}, < 0$$

$$n = \frac{\delta}{\gamma^2} + \frac{1}{2} + \sqrt{\left(\frac{\delta}{\gamma^2} + \frac{1}{2}\right)^2 + \frac{2r}{\gamma^2}} > 1$$

are the roots of

$$\frac{1}{2}\gamma^2 m(m-1) + \delta m - r = 0.$$

Moreover, the ODE (5.1) admits a twice continuously differentiable particular solution on $(0, \infty)$ given, accordingly, e.g. [6], by the formula

$$\hat{V}_0(k) = J(k, 0) \; = \; E\left[\int_0^\infty e^{-rt}\Pi(\hat{K}_t^k)dt\right],$$

where $\hat{K}^k$ is the solution to the linear SDE

$$d\hat{K}_t = \hat{K}_t\left(-\delta dt + \gamma dW_t\right), \quad \hat{K}_0 = k.$$

In other words, $\hat{V}_0$ is the expected profit corresponding to the zero control $L = 0$.

Remark 5.1. The function $\hat{V}_0$ can be expressed analytically as

$$\hat{V}_0(k) = k^n G_1(k) + k^m G_2(k),$$

with

$$G_1(k) = \frac{2}{\gamma^2(n-m)}\int_k^\infty s^{-n-1}\Pi(s)ds, \quad k > 0,$$

$$G_2(k) = \frac{2}{\gamma^2(n-m)}\int_0^k s^{-m-1}\Pi(s)ds, \quad k > 0.$$

Under assumption (2.2), the limiting behavior of the derivative $\hat{V}_0'$ as k tends to zero and infinity is described as follows.

Lemma 5.1.

$$\hat{V}_0'(0^+) \;:=\; \lim_{k \downarrow 0} \hat{V}_0'(k) \;=\; \infty \;\textit{and}\; \hat{V}_0'(\infty) \;:=\; \lim_{k \to \infty} \hat{V}_0'(k) \;=\; 0.$$

Proof. We rewrite $\hat{V}_0$ as

$$\hat{V}_0(k) = E\left[\int_0^\infty e^{-rt} \Pi(kY_t)dt\right], \quad k > 0,$$

where $Y_t = e^{-\delta t} M_t$, and M is the martingale $M_t = \exp(\gamma W_t - \frac{\gamma^2}{2} t)$. It is easily checked by the Lebesgue theorem that one can differentiate the expression of $\hat{V}_0$ inside the expectation and the integral so that its derivative is given by the equality

$$\hat{V}_0'(k) = E\left[\int_0^\infty e^{-rt} Y_t \Pi'(kY_t)dt\right], \quad k > 0.$$

Using the positivity and nonincreasing monotonicity of Π', we may apply the monotone convergence theorem as k tends to zero and obtain from the Inada condition $\Pi'(0^+) = \infty$ that $\lim_{k \downarrow 0} \hat{V}_0'(k) = \infty$. On the other hand, we may also apply the dominated convergence theorem as k tends to infinity and obtain from the other Inada condition $\Pi'(\infty) = 0$ that $\lim_{k \to \infty} \hat{V}_0'(k) = 0$. □

5.2 Explicit form of the value function

Lemma 5.2. *The buying threshold satisfies the inequalities*

$$0 < \; k_b \; < \infty.$$

Proof. We first check that $k_b > 0$. If it is not the case, the buying region is empty, and we would have from Lemma 3.4 and Theorem 4.2 that

$$rv - \mathcal{L}v - \Pi = 0, \quad k > 0.$$

Hence, v would be of the form

$$v(k) = Ak^m + Bk^n + \hat{V}_0(k), \quad k > 0.$$

Since $m < 0$ and $|v(0^+)| < \infty$, this implies that $A = 0$. Now, since $n > 1$, we get that $v'(0^+) = \hat{V}_0'(0^+) = \infty$, a contradiction with the bound $v'(k) \leq p$ for all $k > 0$.

We also have $k_b < \infty$. Otherwise, v would be on the form

$$v(k) = kp + v(0^+), \quad \forall\, k > 0.$$

This contradicts to the growth condition (3.5). □

We can now explicitly determine the value function v.

Theorem 5.1. *The value function v has the following structure:*

$$v(k) = \begin{cases} kp + v(0^+), & k \le k_b, \\ Ak^m + \hat{V}_0(k), & k_b < k, \end{cases} \tag{5.2}$$

where the three constants $v(0^+)$, A and k_b are determined by the continuity, C^1- and C^2-smooth fit conditions at k_b:

$$Ak_b^m + \hat{V}_0(k_b) = k_b p + v(0^+), \tag{5.3}$$

$$mAk_b^{m-1} + \hat{V}_0'(k_b) = p, \tag{5.4}$$

$$m(m-1)Ak_b^{m-2} + \hat{V}_0''(k_b) = 0. \tag{5.5}$$

Proof. We already know from Lemma 3.4 that on the interval $(0, k_b)$, which is nonempty by Lemma 5.2, v has the structure described in (5.2). Moreover, on (k_b, ∞), the derivative $v' < p$ in virtue of Lemma 3.4. Therefore, by Theorem 4.2, v satisfies the equation $rv - \mathcal{L}v - \Pi = 0$, and so, according to Subsection 5.1, it is of the form

$$v(k) = Ak^m + Bk^n + \hat{V}_0(k), \quad k > k_b.$$

Since $m < 0$, $n > 1$, $\hat{V}_0'(k) \to 0$ as $k \to \infty$, and $\le v'(k) \le p$, we must have necessarily $B = 0$, and so v has the form written in (5.2). Finally, the three conditions resulting from the continuity, C^1- and C^2-smooth fit conditions at k_b determine the constants A, k_b and $v(0^+)$. □

Remark 5.2. By the viscosity solutions method adopted here we know the existence of a triple $(v(0^+), A, k_b) \in \mathbb{R}_+ \times \mathbb{R} \times (0, \infty)$ which is solution to the system of equations (5.3)-(5.4)-(5.5). Indeed, this results from the continuity, C^1- and C^2-properties of v at k_b that we proved to hold a priori. This contrasts with the classical verification approach where one tries to find a C^2-solution to (3.1), so of the form

$$\tilde{v}(k) = \begin{cases} kp + \tilde{v}(0^+), & k \le \tilde{k}_b, \\ \tilde{A}k^m + \hat{V}_0(k), & \tilde{k}_b < k, \end{cases} \tag{5.6}$$

and, hence, to prove the existence of a triple $(\tilde{v}(0^+), \tilde{A}, \tilde{k}_b) \in \mathbb{R}_+ \times \mathbb{R} \times (0, \infty)$ which is a solution to (5.3)-(5.4)-(5.5). By a verification argument, one then shows that $\tilde{v} = v$ proving a posteriori the C^2-property of v.

On the other hand, it is easily seen that we have uniqueness of a solution $(\hat{v}(0^+), A, k_b) \in \mathbb{R}_+ \times \mathbb{R} \times (0, \infty)$ to the system of equations (5.3) – (5.5). Indeed, otherwise we could find another smooth C^2-function $\tilde{v}$ of the form (5.6), with the linear growth condition, and solving (3.1). This contradicts the standard uniqueness results for PDE (3.1).

Remark 5.3. The value function v satisfies in (k_b, ∞) the second order ODE

$$rv(k) + \delta k v'(k) - \frac{1}{2}\gamma^2 k^2 v''(k) - \Pi(k) = 0, \quad k \in (k_b, \infty).$$

From the continuity and C^1- and C^2-conditions of v at k_b, i.e. the relations $v(k_b) = k_b p + v(0^+)$, $v'(k_b) = p$ and $v''(k_b) = 0$, we then deduce that

$$(r+\delta)k_b p + rv(0^+) = \Pi(k_b). \tag{5.7}$$

Remark 5.4. **Computation of v**
From a computational viewpoint, the constants $A, k_b, v(0^+)$ can be determined as follows. From equations (5.4)-(5.5), we obtain an equation for k_b and express A in terms of k_b :

$$F(k_b) \;:=\; (1-m)\hat{V}_0'(k_b) + k_b \hat{V}_0''(k_b) = p(1-m), \tag{5.8}$$

$$A = \frac{k_b^{1-m}}{m}\left(p - \hat{V}_0'(k_b)\right). \tag{5.9}$$

The value $v(0^+)$ is then computed from relation (5.3) or, equivalently, (5.7). Note that a straightforward calculation provides the explicit expression of F:

$$F(k) = n(n-m)k^{n-1}G_1(k) - \frac{2}{\gamma^2}\frac{\Pi(k)}{k}, \quad k > 0.$$

Example 1. **Special case of the power profit function**
We consider the case where Π is the Cobb–Douglas profit function, and we assume, without loss of generality, that $\Pi(k) = k^\alpha$ with $0 < \alpha < 1$. Then

$$\hat{V}_0(k) = Ck^\alpha, \quad \text{with} \quad C \;=\; \frac{1}{r + \alpha\delta + \frac{\gamma^2}{2}\alpha(1-\alpha)}.$$

Then, from (5.8), k_b is explicitly written as :

$$k_b = \left(\frac{p(1-m)}{\alpha C(\alpha - m)}\right)^{\frac{1}{\alpha-1}}.$$

5.3 Optimal control

We recall the following well-known Skorohod lemma, see, e.g., [7].

Lemma 5.3. *For any initial state $k \geq 0$ and given a boundary $k_b \geq 0$, there exist unique càdlàg adapted processes K^* and nondecreasing processes L^* satisfying the following Skorohod problem $\mathcal{S}(k, k_b)$:*

$$dK_t^* = K_t^*\left(-\delta dt + \gamma dW_t\right) + dL_t^*, \; t \geq 0, \qquad K_{0^-}^* = k, \tag{5.10}$$

$$K_t^* \in [k_b, \infty) \quad a.e., \quad t \geq 0, \tag{5.11}$$

$$\int_0^\infty 1_{K_u^* > k_b} dL_u^* = 0. \tag{5.12}$$

Moreover, if $k \geq k_b$, then L^* is continuous. When $k < k_b$, $L_0^* = k_b - k$, and

$K^*_0 = k_b$.

Remark 5.5. The solution K^* to the above equations is a reflected diffusion at the boundary k_b and the process L^* is the local time of K^* at k_b. Condition (5.12) means that L^* increases only when K^* hits the boundary k_b. It is also known that the r-potential of L^* is finite, i.e. $E\left[\int_0^\infty e^{-rt} dL^*_t\right] < \infty$, see Chapter X in [9], so that

$$E\left[\int_0^\infty e^{-rt} K^*_t dt\right] < \infty. \tag{5.13}$$

Theorem 5.2. *For $k \geq 0$, let (K^*, L^*) be the solution to the Skorohod problem $\mathcal{S}(k, k_b)$. Then*

$$v(k) = J(k, L^*), \quad k \geq 0.$$

Proof. 1) We first consider the case where $k \geq k_b$. Then, the processes K^*, L^* are continuous. In view of (5.11) and Theorem 4.2, we have

$$rv(K^*_t) - \mathcal{L}v(K^*_t) - \Pi(K^*_t) = 0, \ a.e. \quad t \geq 0.$$

By applying Itô's formula to $e^{-rt}v(K^*_t)$ between 0 and T, we thus get:

$$E\left[e^{-rT} v(K^*_T)\right] = \\ v(k) - E\left[\int_0^T e^{-rt} \Pi(K^*_t) dt\right] + E\left[\int_0^T e^{-rt} v'(K^*_t) dL^*_t\right]. \tag{5.14}$$

(Notice that the stochastic integral appearing in the Itô formula has zero expectation because of (5.13)). Now, in view of (5.12), we have

$$\begin{aligned} E\left[\int_0^T e^{-rt} v'(K^*_t) dL^*_t\right] &= E\left[\int_0^T e^{-rt} v'(K^*_t) 1_{K^*_t = k_b} dL^*_t\right] \\ &= E\left[\int_0^T e^{-rt} p dL^*_t\right], \end{aligned}$$

since $v'(k_b) = p$. Plugging into (5.14) yields:

$$\begin{aligned} v(k) = E&\left[e^{-rT} v(K^*_T)\right] \\ &+ E\left[\int_0^T e^{-rt} \Pi(K^*_t) dt\right] - E\left[\int_0^T e^{-rt} p dL^*_t\right]. \end{aligned} \tag{5.15}$$

From (5.13), we have that $\lim_{T\to\infty} E[e^{-rT} K^*_T] = 0$. Since v satisfies a linear growth condition in k, this implies that also

$$\lim_{T\to\infty} E[e^{-rT} v(K^*_T)] = 0.$$

By sending T to infinity into (5.15), we obtain, by the dominated convergence theorem, the required result:

$$v(k) = J(k, L^*) \;=\; E\left[\int_0^\infty e^{-rt}\left(\Pi(K_t^*) - pdL_t^*\right)\right].$$

2) If $k < k_b$, and since then $L_0^* = k - k_b$, we have:

$$\begin{aligned} J(k, L^*) &= J(k_b, L^*) - p(k - kb) \\ &= v(k_b) - p(k - kb) \;=\; v(k), \end{aligned}$$

by recalling that $v' = p$ on $(0, k_b)$. □

Conclusion. The main results of this paper in Theorems 5.1 and 5.2 provide a complete and explicit solution to our irreversible investment under uncertainty. They mathematically formulate the economic intuition that a company will invest in buying capital in order to maintain its production capacity above a threshold k_b, which can be computed quite explicitly.

Appendix : Proof of Theorem 4.1

(i) *Viscosity supersolution property*.

Fix $k_0 > 0$ and C^2-function φ such that $v(k_0) = \varphi(k_0)$ and $\varphi(k) \leq v(k)$ for all k in a neighborhood $\bar{B}_\varepsilon(k_0) = [k_0 - \varepsilon, k_0 + \varepsilon]$ of k_0 $(0 < \varepsilon < k_0)$. Consider the admissible control $L \in \mathcal{A}$ defined by

$$L_t = \begin{cases} 0, \, t = 0 \\ \eta, \, t \geq 0, \end{cases}$$

where $0 \leq \eta < \varepsilon$. Define the exit time $\tau_\varepsilon = \inf\{t \geq 0 : \; K_t \notin \bar{B}_\varepsilon(x_0)\}$. Here K is the capacity production starting from k_0 and controlled by L above. Notice that K has at most one jump at $t = 0$ and is continuous on $(0, \tau_\varepsilon]$. By the dynamic programming principle (4.1) with $\theta = \tau_\varepsilon \wedge h$, $h > 0$, we have :

$$\begin{aligned} \varphi(k_0) \;=\; v(k_0) &\geq E\left[\int_0^{\tau_\varepsilon \wedge h} e^{-rt}(\Pi(K_t)dt - pdL_t) + e^{-r(\tau_\varepsilon \wedge h)} v(K_{\tau_\varepsilon \wedge h})\right] \\ &\geq E\left[\int_0^{\tau_\varepsilon \wedge h} e^{-rt}(\Pi(K_t)dt - pdL_t) + e^{-r(\tau_\varepsilon \wedge h)} \varphi(K_{\tau_\varepsilon \wedge h})\right]. \qquad (5.16) \end{aligned}$$

Applying Itô's formula to the process $e^{-rt}\varphi(K_t)$ between 0 and $\tau_\varepsilon \wedge h$, and taking the expectation, we obtain similarly as in the proof of Proposition 3.1 by noting also that $dL_t^c = 0$:

$$E[e^{-r(\tau_\varepsilon\wedge h)}\varphi(K_{\tau_\varepsilon\wedge h})] = \varphi(k_0) + E\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt}\left(-r\varphi + \mathcal{L}\varphi\right)(K_t)dt\right] + E\left[\sum_{0\le t\le\tau_\varepsilon\wedge h} e^{-rt}\left[\varphi(K_t)-\varphi(K_{t-})\right]\right]. \quad (5.17)$$

Combining relations (5.16) and (5.17), we see that

$$E\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt}\left(r\varphi - \mathcal{L}\varphi - \Pi\right)(K_t)dt\right] + E\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt} p dL_t\right] - E\left[\sum_{0\le t\le\tau_\varepsilon\wedge h} e^{-rt}\left[\varphi(K_t)-\varphi(K_{t-})\right]\right] \ge 0. \quad (5.18)$$

⋆ Taking first $\eta = 0$, i.e. $L = 0$, we see that K is continuous, and only the first term in the left-hand side of (5.18) is non zero. By dividing the above inequality by h with $h \to 0$, we conclude by the dominated convergence theorem:

$$r\varphi(k_0) - \mathcal{L}\varphi(k_0) - \Pi(k_0) \ge 0. \quad (5.19)$$

⋆ Now, by taking $\eta > 0$ in (5.18), and noting that L and K jump only at $t = 0$ with the jump size η, we get that

$$E\left[\int_0^{\tau_\varepsilon\wedge h} e^{-rt}\left(r\varphi - \mathcal{L}\varphi - \Pi\right)(K_t)dt\right] + p\eta - \varphi(k_0+\eta) + \varphi(k_0) \ge 0. \quad (5.20)$$

Taking $h \to 0$, then dividing by η and letting $\eta \to 0$, we obtain the inequality

$$p - \varphi'(k_0) \ge 0. \quad (5.21)$$

This proves the required viscosity supersolution property:

$$\min\left\{r\varphi(k_0) - \mathcal{L}\varphi(k_0) - \Pi(k_0), -\varphi'(k_0) + p\right\} \ge 0. \quad (5.22)$$

(ii) *Viscosity sub-solution property.*

We prove this part by contradiction. Suppose the claim is not true. Then, there is $k_0 > 0$, $\varepsilon \in (0, k_0)$, a φ C^2-function with $\varphi(k_0) = v(k_0)$ and $\varphi \ge v$ in $\bar{B}_\varepsilon(k_0) = [k_0 - \varepsilon, k_0 + \varepsilon]$, and $\nu > 0$ such that for all $k \in \bar{B}_\varepsilon(k_0)$ we have:

$$r\varphi(k) - \mathcal{L}\varphi(k) - \Pi(k) \ge \delta, \quad (5.23)$$

$$\varphi'(k) \le p - \nu. \quad (5.24)$$

For a control $L \in \mathcal{A}$, consider the exit time $\tau_\varepsilon = \inf\{t \geq 0 : \ K_t \notin \bar{B}_\varepsilon(x_0)\}$. (Here K is the capacity production starting from k_0 and controlled by L). By applying Itô's formula to $e^{-rt}\varphi(K_t)$, we get :

$$
\begin{aligned}
E\left[e^{-r\tau_\varepsilon}\varphi(K_{\tau_\varepsilon^-})\right] &= \varphi(k_0) \ + \ E\left[\int_0^{\tau_\varepsilon} e^{-rt}\left(-r\varphi + \mathcal{L}\varphi\right)(K_t)dt\right] \\
&\quad + E\left[\int_0^{\tau_\varepsilon} e^{-rt}\varphi'(K_t)dL_t^c\right] \\
&\quad + E\left[\sum_{0\leq t<\tau_\varepsilon} e^{-rt}\left[\varphi(K_t) - \varphi(K_{t^-})\right]\right]. \qquad (5.25)
\end{aligned}
$$

Notice that for all $t \in [0, \tau_\varepsilon)$, $K_t \in \bar{B}_\varepsilon(k_0)$. Then, from Taylor's formula and (5.24), noting that $\Delta K_t = \Delta L_t$, we obtain for $t \in [0, \tau_\varepsilon)$:

$$
\begin{aligned}
\varphi(K_t) - \varphi(K_{t^-}) &= \Delta K_t \int_0^1 \varphi'(K_t + z\Delta K_t)dz \\
&\leq (p-\nu)\Delta L_t. \qquad (5.26)
\end{aligned}
$$

Due to relations (5.23) – (5.26), we thus obtain:

$$
\begin{aligned}
&E\left[e^{-r\tau_\varepsilon}\varphi(K_{\tau_\varepsilon^-})\right] \\
&\leq \varphi(k_0) \ + \ E\left[\int_0^{\tau_\varepsilon} e^{-rt}\left(-\Pi - \nu\right)(K_t)dt\right] \\
&\quad + E\left[\int_0^{\tau_\varepsilon^-} e^{-rt}(p-\nu)dL_t\right] \\
&= \varphi(k_0) \ + \ E\left[\int_0^{\tau_\varepsilon} e^{-rt}\left(-\Pi(K_t)dt + pdL_t\right)\right] \ - \ E\left[e^{-r\tau_\varepsilon}p\Delta L_{\tau_\varepsilon}\right] \\
&\quad -\nu\left\{E\left[\int_0^{\tau_\varepsilon} e^{-rt}dt\right] + E\left[\int_0^{\tau_\varepsilon^-} e^{-rt}dL_t\right]\right\}. \qquad (5.27)
\end{aligned}
$$

Notice that while $K_{\tau_\varepsilon^-} \in \bar{B}_\varepsilon(k_0)$, K_{τ_ε} is either on the boundary $\partial B_\varepsilon(k_0)$ or out of $\bar{B}_\varepsilon(k_0)$. However, there is some random variable α taking values in $[0,1]$ such that

$$
\begin{aligned}
k_\alpha &:= K_{\tau_\varepsilon^-} + \alpha\Delta K_{\tau_\varepsilon} \\
&= K_{\tau_\varepsilon^-} + \alpha\Delta L_{\tau_\varepsilon} \ \in \ \partial\bar{B}_\varepsilon(k_0) \ = \ \{k_0 - \varepsilon, k_0 + \varepsilon\}.
\end{aligned}
$$

Then, similarly as in (5.26), we have :

$$
\varphi(k_\alpha) - \varphi(K_{\tau_\varepsilon^-}) \leq \alpha(p-\nu)\Delta L_{\tau_\varepsilon}. \qquad (5.28)
$$

Notice that $K_{\tau_\varepsilon} = k_\alpha + (1-\alpha)\Delta L_{\tau_\varepsilon}$, and so from Lemma 3.1 we have:

$$v(k_\alpha) \geq -p(1-\alpha)\Delta L_{\tau_\varepsilon} + v(K_{\tau_\varepsilon}). \tag{5.29}$$

Recalling that $\varphi(k_\alpha) \geq v(k_\alpha)$, inequalities (5.28), (5.29) imply:

$$\varphi(K_{\tau_\varepsilon^-}) \geq v(K_{\tau_\varepsilon}) - (p - \alpha\nu)\Delta L_{\tau_\varepsilon}.$$

Plugging the last inequality into (5.27) and recalling that $\varphi(k_0) = v(k_0)$, we obtain:

$$\begin{aligned} v(k_0) &\geq E\left[\int_0^{\tau_\varepsilon} e^{-rt}\left(\Pi(K_t)dt - pdL_t\right) + v(K_{\tau_\varepsilon})\right] \\ &+ \nu\left\{E\left[\int_0^{\tau_\varepsilon} e^{-rt}dt\right] + E\left[\int_0^{\tau_\varepsilon^-} e^{-rt}dL_t\right] \ + \ E\left[e^{-r\tau_\varepsilon}\alpha\Delta L_{\tau_\varepsilon}\right]\right\}. \end{aligned} \tag{5.30}$$

$\star$ We now claim that there is a constant $g_0 > 0$ such that for all $L \in \mathcal{A}$:

$$E\left[\int_0^{\tau_\varepsilon} e^{-rt}dt\right] + E\left[\int_0^{\tau_\varepsilon^-} e^{-rt}dL_t\right] \ + \ E\left[e^{-r\tau_\varepsilon}\alpha\Delta L_{\tau_\varepsilon}\right] \geq g_0. \tag{5.31}$$

Indeed, one can always find some constant $G_0 > 0$ such that the C^2-function

$$\psi(k) = G_0((k-k_0)^2 - \varepsilon^2),$$

satisfies the relations

$$\begin{aligned} \min\left\{r\psi - \mathcal{L}\psi + 1, 1 - |\psi|\right\} \geq 0, &\quad \text{on } \bar{B}_\varepsilon(k_0), \\ \psi = 0, &\quad \text{on } \partial\bar{B}_\varepsilon(k_0). \end{aligned}$$

For instance, we can choose:

$$G_0 = \min\left\{\frac{1}{r\varepsilon^2 + 2\varepsilon\delta(k_0+\varepsilon) + \gamma^2(k_0+\varepsilon)^2}, \frac{1}{2\varepsilon}\right\} \ > \ 0.$$

By applying again Itô's lemma, we get that

$$E\left[e^{-r\tau_\varepsilon}\psi(K_{\tau_\varepsilon^-})\right] \leq \psi(k_0) + E\left[\int_0^{\tau_\varepsilon} e^{-rt}dt\right] + E\left[\int_0^{\tau_\varepsilon^-} e^{-rt}dL_t\right] \tag{5.32}$$

Since $\psi'(k) \geq -1$, we have:

$$\psi(K_{\tau_\varepsilon^-}) - \psi(k_\alpha) \geq -\left(K_{\tau_\varepsilon^-} - k_\alpha\right) \ = \ \alpha\Delta L_{\tau_\varepsilon} \ \geq \ 0.$$

Plugging into (5.32) yields:

$$\begin{aligned} &E\left[\int_0^{\tau_\varepsilon} e^{-rt}dt\right] + E\left[\int_0^{\tau_\varepsilon^-} e^{-rt}dL_t\right] \\ &\geq E\left[e^{-r\tau_\varepsilon}\psi(k_\alpha)\right] - \psi(k_0) \ = \ -\psi(k_0) \ = \ G_0\varepsilon^2. \end{aligned} \tag{5.33}$$

Hence, the claim (5.31) holds with $g_0 = G_0\varepsilon^2$.

$\star$ Finally, by taking supremum over all $(L, M) \in \mathcal{A}$ in (5.30), and invoking the dynamic programming principle (4.1), we have that $v(k_0) \geq v(k_0) + \nu g_0$, which is the required contradiction.

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