On the Fundamental Solution of the Kolmogorov–Shiryaev Equation

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Summary. We derive an integral representation for the fundamental solution of the Kolmogorov forward equation

$$f_t = -((1+\mu x)f)_x + (\nu x^2 f)_{xx}$$

associated with the Shiryaev process X solving the linear SDE

$$dX_t = (1+\mu X_t)\, dt + \sigma X_t\, dB_t$$

where $\mu \in I\!R$, $\nu = \sigma^2/2 > 0$ and B is a standard Brownian motion. The method of proof is based upon deriving and inverting a Laplace transform. Basic properties of X needed in the proof are reviewed.

Key words: Shiryaev process, Kolmogorov forward equation, integral of geometric Brownian motion, parabolic partial differential equation, Laplace transform, confluent hypergeometric function, modified Bessel function, Hartman–Watson distribution, Hankel's contour integral.

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1 Introduction

We consider the Kolmogorov forward equation:

$$f_t = -((1+\mu x)f)_x + (\nu x^2 f)_{xx} \tag{1.1}$$

associated with the Shiryaev process $X = (X_t)_{t \geq 0}$ solving:

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$$dX_t = (1+\mu X_t)\,dt + \sigma X_t\,dB_t \tag{1.2}$$

with $X_0 = x_0$ in $I\!R$ where $\mu \in I\!R$, $\nu = \sigma^2/2 > 0$ and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. The problem of finding the fundamental solution $f = f(t,x)$ of (1.1) appears naturally in a number of fields (most notably in sequential analysis and financial mathematics).

The unique (strong) solution of (1.2) is given by:

$$X_t = Y_t\left(x_0 + \int_0^t \frac{1}{Y_s}\,ds\right) \tag{1.3}$$

where $Y = (Y_t)_{t\geq 0}$ is a geometric Brownian motion solving:

$$dY_t = \mu Y_t\,dt + \sigma Y_t\,dB_t \tag{1.4}$$

with $Y_0 = 1$. The unique (strong) solution of (1.4) is given by:

$$Y_t = \exp\big(\sigma B_t + (\mu-\nu)t\big). \tag{1.5}$$

Inserting (1.5) into (1.3) one obtains an explicit representation of X in terms of B.

From this representation and the invariance of B on time reversal one sees that the following identity in law is satisfied:

$$X_t \overset{\text{law}}{=} \int_0^t Y_s\,ds \tag{1.6}$$

when $x_0 = 0$. This shows that the problem of finding the fundamental solution of (1.1) when $x_0 = 0$ is equivalent to the problem of finding the distribution of the random variable $\int_0^t Y_s\,ds$. The latter problem has been intensively studied in the last 10-15 years (see [20], [4], [15] and the references therein) but none of these approaches attempts to tackle the forward equation (1.1) directly (see [14] for numerical results of a related approach).

The purpose of the present paper is to search for the fundamental solution of (1.1) by simple probabilistic and analytic means (cf. [5]). It will be seen below that this approach readily leads to the Laplace transform of $t \mapsto \int_0^x f(t,y)\,dy$ expressed in terms of confluent hypergeometric functions (modified Bessel functions) providing a link to the Hartman-Watson distribution [9]. The problem thus reduces to inverting the Laplace transform. This can be done using Hankel's contour integrals for these functions (cf. [19]) leading to representations of the solution in terms of single or double integrals. For simplicity and comparison we only treat a particular case of the equation (1.1) in complete detail. A treatment of other cases is briefly indicated and it is hoped that their study will be continued.

A disadvantage of the previous inversion approach is that the analytic expressions obtained are numerically unstable for small t. This fact was observed independently by several authors (see e.g. [2]). While this may not be such a big drawback for applications to Asian options of European type (cf. [3]), in the case of Asian options of American type such a numerical stability becomes fundamentally important (see [13]). A similar need for stable analytic expressions arises in quickest detection problems (sequential analysis) when the horizon is finite (see [8]). Further research of the Kolmogorov–Shiryaev equation (1.1) thus appears to be necessary.

The stochastic differential equation (1.2) has been derived by Shiryaev [16, Eq. (9)] in the context of quickest detection problems (sequential analysis). These problems play a prominent role in diverse applications ranging from quality control in industry to structural analysis of DNA in medicine. Applications in financial data analysis (detection of arbitrage) are recently discussed in [17]. The Kolmogorov backward and forward equation (of which (1.1) is a particular case) have been derived in [11]. In the physical literature the forward equation is often referred to as the Fokker–Planck equation (cf. [7], [12]).

2 The Shiryaev process

In this section we present basic properties of the Shiryaev process X solving (1.2). Note that the initial point x_0 of X belongs to $I\!R$ and may be negative as well.

1. The Shiryaev process X is a strong Markov process with continuous sample paths (a diffusion process). The drift of X is given by $\mu(x) = 1 - \mu x$ and the diffusion coefficient of X is given by $\sigma(x) = \sigma x$. Recall that $\mu \in I\!R$ and $\nu = \sigma^2/2 > 0$.

2. Since $\sigma(0) = 0$ we see that the state space of X splits into $(-\infty, 0]$ and $[0, \infty)$. From the representation (1.3) it is evident that:

$$\text{The point } 0 \text{ is an entrance boundary point for } [0,\infty). \tag{2.1}$$

Likewise it will be formally verified below that:

$$\text{The point } 0 \text{ is an exit boundary point for } (-\infty,0]. \tag{2.2}$$

3. The scale function of X is given by:

$$s(x) = \int_1^x z^{-\mu/\nu}\, e^{1/\nu z}\, dz \quad \text{for } x > 0 \tag{2.3}$$

$$s(x) = \int_{-x}^1 z^{-\mu/\nu}\, e^{-1/\nu z}\, dz \quad \text{for } x < 0. \tag{2.4}$$

Hence $s(0+) = -\infty$ always, and $s(\infty) = \infty$ if and only if $\mu \le \nu$. This shows that X is recurrent in $[0, \infty)$ if and only if $\mu \le \nu$. Note also that $s(-\infty) = -\infty$ if and only if $\mu \le \nu$, and $s(0-) < \infty$ always. This shows that X exists $(-\infty, 0]$

almost surely at 0 if and only if $\mu \le \nu$. We also see that X can never be recurrent in $(-\infty, 0]$.

4. The speed measure of X is given by:

$$m(dx) = \nu^{-1} x^{-2+\mu/\nu} e^{-1/\nu x} dx \quad \text{for } x > 0 \tag{2.5}$$

$$m(dx) = \nu^{-1} (-x)^{-2+\mu/\nu} e^{-1/\nu x} dx \quad \text{for } x < 0. \tag{2.6}$$

Since $\int_0^\infty m(dx) = \nu^{\mu/\nu} \Gamma(1-\mu/\nu) < \infty$ if and only if $\mu < \nu$, it follows that X has an invariant density function on $[0, \infty)$ given by:

$$f(x) = \frac{1}{\nu^{1-\mu/\nu} \Gamma(1-\mu/\nu)} \frac{1}{x^{2-\mu/\nu}} e^{-1/\nu x} \quad \text{for } x > 0 \tag{2.7}$$

if and only if $\mu < \nu$. Noting that $\int_{-\infty}^0 m(dx) = \infty$ we see that X cannot have an invariant density function on $(-\infty, 0]$ as already indicated above.

5. By the law of iterated logarithm for B one easily sees that $\int_0^\infty Y_s \, ds < \infty$ almost surely if and only if $\mu < \nu$. Hence when $\mu < \nu$ we find using (1.3) and (1.6) that:

$$X_t \xrightarrow{d} \int_0^\infty Y_s \, ds \tag{2.8}$$

as $t \to \infty$ where the density function of $\int_0^\infty Y_s \, ds$ is given by (2.7) above.

Likewise one sees that $\int_0^\infty (1/Y_s) \, ds < \infty$ almost surely if and only if $\mu > \nu$. Hence when $\mu > \nu$ we find using (1.3) that:

$$X_t \to +\infty \quad \text{if} \quad x_0 + \textstyle\int_0^\infty (1/Y_s) \, ds > 0 \tag{2.9}$$

$$X_t \to -\infty \quad \text{if} \quad x_0 + \textstyle\int_0^\infty (1/Y_s) \, ds < 0 \tag{2.10}$$

as $t \to \infty$. The probabilities of the latter two events can readily be computed upon noting that the density function of $\int_0^\infty (1/Y_s) \, ds$ is given by:

$$g(x) = \frac{1}{\nu^{\mu/\nu-1} \Gamma(\mu/\nu-1)} \frac{1}{x^{\mu/\nu}} e^{-1/\nu x} \quad \text{for } x > 0 \tag{2.11}$$

when $\mu > \nu$. This follows from the identity in law stated after (2.8) above with a new drift $\hat{\mu} = 2\nu - \mu$ and a new Brownian motion $\hat{B} = -B$. Another way to compute these probabilities is to make use of the scale function in (2.4). This gives that the probability of the event in (2.9) equals one minus the probability of the event in (2.10) which, in turn, is equal to the ratio $(S(0-)-S(x_0))/(S(0-)-S(-\infty))$.

Finally, when $\mu = \nu$ then X is recurrent in $[0, \infty)$ no matter if x_0 is positive or negative. Recall that X hits zero almost surely if $x_0 < 0$ never returning to zero again.

6. A formal verification of (2.1) and (2.2) can be made upon invoking the standard boundary classification for one-dimensional diffusions (cf. [6]).

Firstly, since $m' \in L^1((0,\infty))$ and $s\,m' \in L^1((0,\infty))$ but $s' \notin L^1((0,\infty))$ we see that (2.1) follows. Secondly, since $m' \notin L^1((-\infty,0))$ and $s'm \in L^1((-\infty,0))$ we see that (2.2) follows as claimed.

7. We will conclude this section by deriving boundary conditions which will be used in the next section. For this, let F denote the transition distribution function of X, and let f denote the transition density function of X. Since X is a time-homogeneous Markov process, it is no restriction to assume that the initial time point equals zero. We thus have:

$$F(0,x_0;t,x) = \mathsf{P}(X_t \le x \mid X_0 = x_0) \tag{2.12}$$

$$f(0,x_0;t,x) = F_x(0,x_0;t,x). \tag{2.13}$$

In the sequel we will only study the case when $x_0 \ge 0$. From the facts exposed above we then know that the state space of X equals $[0,\infty)$ and that X can only start at 0 and never arrive at it (recall (2.1) above). Hence the following boundary conditions at 0 are in agreement with what we would expect to hold:

$$f(0,x_0;t,0+) = 0 \tag{2.14}$$

$$f_x(0,x_0;t,0+) = 0. \tag{2.15}$$

In fact, all higher derivatives of f with respect to x satisfy the same zero condition, but we will only make use of the conditions (2.14) and (2.15) below.

8. A formal proof of (2.14) and (2.15) is simple. Denote X_t from (1.3) by $X_t^{x_0}$ to indicate its dependence on x_0, note that $X_t^{x_0} > 0$, and set $Z = 1/X_t^{x_0}$. Then for any $p > 0$ given and fixed we find by the Markov inequality that:

$$\begin{aligned} F(0,x_0;t,h) &= \mathsf{P}(X_t \le h \mid X_0 = x_0) = \mathsf{P}(X_t^{x_0} \le h) \\ &= \mathsf{P}(Z \ge 1/h) = \mathsf{P}(Z^p \ge 1/h^p) \le h^p\,\mathsf{E}(Z^p) \end{aligned} \tag{2.16}$$

where $\mathsf{E}(Z^p) < \infty$ by the well-known properties of B. From (2.16) we see that:

$$F(0,x_0;t,h) = O(h^p) \tag{2.17}$$

as $h \to h_0$ for $h_0 \ge 0$ whenever $p > 0$ is given and fixed. Taking $p = 3$ and using (2.17) one finds that (2.14) and (2.15) hold as claimed.

3 The fundamental solution

In this section we study the problem of finding the fundamental solution of the Kolmogorov–Shiryaev equation (1.1). For simplicity we will only examine the case when $x_0 \ge 0$ (cf. Section 2). By the fundamental solution we thus mean a non-negative solution $f = f(t,x)$ for $t > 0$ and $x > 0$, satisfying $\int_0^\infty f(t,x)\,dx = 1$ for each $t > 0$, and $f(t,x) \to \delta(x-x_0)$ weakly as $t \downarrow 0$ (where δ denotes the Dirac delta function).

1. Recall that X solving (1.2) is time-homogeneous so that there is no restriction to assume that the initial time point equals zero. We will moreover suppress the dependence on 0 and x_0 in (2.12) and (2.13) and simply write:

$$F(t,x) = \mathsf{P}(X_t \le x \mid X_0 = x_0) \tag{3.1}$$

$$f(t,x) = F_x(t,x). \tag{3.2}$$

Standard Markovian arguments (cf. [11]) imply that the transition density function (3.2) solves the equation (1.1), and thus the initial problem is equivalent to the problem of finding the transition density function (3.2).

2. Let us set:

$$g = -(1+\mu x)f + (\nu x^2 f)_x. \tag{3.3}$$

Then (1.1) can be written as:

$$f_t = g_x. \tag{3.4}$$

In view of taking the Laplace transform with respect to t and making use of the initial condition for $t=0$ we shall integrate both sides of (3.4) from 0 to x upon using that:

$$F(t,x) = \int_0^x f(t,y)\,dy. \tag{3.5}$$

Since $g(t,0+) = 0$ by (2.14) and (2.15) this gives:

$$\begin{aligned} F_t &= g(t,x) - g(t,0+) = g(t,x) = -(1+\mu x)f + (\nu x^2 f)_x \\ &= -(1+\mu x)F_x + (\nu x^2 F_x)_x = ((2\nu-\mu)x-1)F_x + \nu x^2 F_{xx}. \end{aligned} \tag{3.6}$$

Setting $\alpha = 2\nu-\mu$ we see that (3.6) reads:

$$F_t = (\alpha x-1)F_x + \nu x^2 F_{xx}. \tag{3.7}$$

3. To simplify technicalities we will assume that $x_0 = 0$ in the sequel. Then F satisfies the following initial condition:

$$F(0,x) = 1 \tag{3.8}$$

for all $x \ge 0$. Moreover, since X_t remains positive almost surely for all $t > 0$, we see that F satisfy the following boundary conditions:

$$F(t,0+) = 0 \tag{3.9}$$

$$F(t,\infty) = 1 \tag{3.10}$$

for all $t > 0$.

4. Taking the Laplace transform in (3.7) with respect to t upon setting:

$$\bar{F}(\lambda,x) = \int_0^\infty e^{-\lambda t} F(t,x)\,dt \tag{3.11}$$

we obtain the following ordinary differential equation:

$$\lambda\bar{F} - F(0,x) = (\alpha x - 1)\bar{F}_x + \nu x^2 \bar{F}_{xx}. \tag{3.12}$$

(Note that by taking the Laplace transform with respect to x, we would arrive instead to a new second-order *partial* differential equation. This is in sharp contrast with the equation studied in [5] where one has x instead of x^2 in (1.1) which makes such a transform profitable since the new partial differential equation is of the first order.) Making use of (3.8) we see that the equation (3.12) reads:

$$\nu x^2 \bar{F}_{xx} + (\alpha x - 1)\bar{F}_x - \lambda \bar{F} = -1. \tag{3.13}$$

By (3.9) and (3.10) we obtain the following boundary conditions:

$$\bar{F}(\lambda, 0+) = 0 \tag{3.14}$$

$$\bar{F}(\lambda, \infty) = 1/\lambda. \tag{3.15}$$

5. Note that a particular solution of the equation (3.13) is given by $\bar{F} \equiv 1/\lambda$. To find the general solution we need to consider the homogeneous equation which reads:

$$x^2 y'' + (Ax + B)\, y' + C\, y = 0 \tag{3.16}$$

where $A = \alpha/\nu = 2 - \mu/\nu$, $B = -1/\nu$ and $C = -\lambda/\nu$. A standard substitution for this equation (cf. (2.188) in [10, p. 447]) is given by:

$$y(x) = (1/x^p)\, z(B/x). \tag{3.17}$$

Inserting (3.17) into (3.16) one finds that $z = z(x)$ solves the *Kummer equation*:

$$x\, z'' + (b - x)\, z' - a\, x = 0 \tag{3.18}$$

where a and b are given by:

$$a = p \tag{3.19}$$

$$b = 2(p+1) - A \tag{3.20}$$

and $p > 0$ solves the quadratic equation:

$$p^2 + (1 - A)\, p + C = 0. \tag{3.21}$$

Solving (3.21) we find that:

$$a = \frac{1}{2}\left(1 - \frac{\mu}{\nu} + \sqrt{\left(1 - \frac{\mu}{\nu}\right)^2 + \frac{4\lambda}{\nu}}\,\right) \tag{3.22}$$

$$b = 1 + \sqrt{\left(1 - \frac{\mu}{\nu}\right)^2 + \frac{4\lambda}{\nu}}\,. \tag{3.23}$$

6. Two linearly independent solutions of the Kummer equation (3.18) are the *confluent hypergeometric function of the first kind*:

$$M(a,b,x) = 1 + \frac{a}{b}\, x + \frac{a(a{+}1)}{b(b{+}1)}\, \frac{x^2}{2!} + \cdots \tag{3.24}$$

and the *confluent hypergeometric function of the second kind* $U(a,b,x)$. (We refer to [1, pp. 504-510] for basic properties of these functions.) Summarizing the preceding facts about (3.16) and (3.17) it follows that the equation (3.13) has the general solution given by:

$$\bar{F}(\lambda,x) = C_1\, x^{-a}\, M(a,b,-1/\nu x) + C_2\, x^{-a}\, U(a,b,-1/\nu x) + 1/\lambda. \tag{3.25}$$

7. Letting $x \to \infty$ and using that $x^{-a}\, M(a,b,-1/\nu x) \to 0$ it follows from (3.15) that we may take $C_2 = 0$. Using the known relation (cf. (13.1.5) in [1, p. 504]):

$$x^a\, M(a,b,-x) = \frac{\Gamma(b)}{\Gamma(b{-}a)} \Big(1 + O(x^{-1})\Big) \tag{3.26}$$

as $x \to \infty$, we find that:

$$x^{-a}\, M(a,b,-1/\nu x) \to \nu^a \frac{\Gamma(b)}{\Gamma(b{-}a)} \tag{3.27}$$

as $x \downarrow 0$. Hence by (3.14) we get:

$$C_1 = -\frac{\Gamma(b{-}a)}{\lambda\, \nu^a\, \Gamma(b)}. \tag{3.28}$$

Inserting this into (3.25) upon recalling that $C_2 = 0$, we obtain the following closed-form expression for the Laplace transform (3.11) above:

$$\bar{F}(\lambda,x) = \frac{1}{\lambda}\left(1 - \frac{\Gamma(b{-}a)}{\Gamma(b)}\, (\nu x)^{-a}\, M(a,b,-1/\nu x)\right) \tag{3.29}$$

where $a = a(\lambda)$ and $b = b(\lambda)$ are given by (3.22) and (3.23) respectively.

8. By the inversion formula we have:

$$F(t,x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tz}\, \bar{F}(z,x)\, dz \tag{3.30}$$

for any $c > 0$ given and fixed. The initial problem is thus reduced to computing the complex integral (3.30). The representation (3.29) possesses a rich structure which opens various ways to tackle the inversion problem. Some of these possibilities will now be addressed.

9. By the convolution theorem we see that:

$$F(t,x) = 1 - \int_0^t G(s,x)\, ds \tag{3.31}$$

where the Laplace transform of $s \mapsto G(s,x)$ is given by:

$$\bar{G}(\lambda,x) = \int_0^\infty e^{-\lambda s}\, G(s,x)\, ds = \frac{\Gamma(b-a)}{\Gamma(b)}\, (\nu x)^{-a}\, M(a,b,-1/\nu x) \tag{3.32}$$

upon recalling that $a = a(\lambda)$ and $b = b(\lambda)$ are given by (3.22) and (3.23) respectively. The problem thus reduces to inverting the Laplace transform on the right-hand side of (3.32).

10. Consider the case when $\mu = 0$ and $\nu = 1/2$ i.e. $\sigma = 1$. Then from (3.22) and (3.23) we see that $a = (1/2)(1+\sqrt{1+8\lambda})$ and $b = 2a$ so that:

$$\bar{G}(\lambda,x) = \frac{\Gamma(a)}{\Gamma(2a)}\, (x/2)^{-a}\, M(a,2a,-2/x). \tag{3.33}$$

Using the well-known relation (cf. (13.6.3) in [1, p. 509]):

$$M(p+1/2, 2p+1, 2z) = \Gamma(1+p)\, e^z\, (z/2)^{-p}\, I_p(z) \tag{3.34}$$

where $I_p(z)$ is the modified Bessel function of the first kind (cf. [1, pp. 374-385]), together with the fact that $(-z)^{-p}\, I_p(-z) = z^{-p}\, I_p(z)$ (see (9.6.10) in [1, p. 375]), and the duplication formula for the gamma function (cf. (6.1.18) in [1, p. 256]):

$$\Gamma(2z) = (2\pi)^{-1/2}\, 2^{2z-1/2}\, \Gamma(z)\, \Gamma(z+1/2) \tag{3.35}$$

we find that the following identity holds:

$$\frac{\Gamma(a)}{\Gamma(2a)}\, (x/2)^{-a}\, M(a,2a,-2/x) = \sqrt{\frac{2\pi}{x}}\, e^{-1/x}\, I_{a-1/2}(1/x). \tag{3.36}$$

Inserting this expression into (3.32) we find that:

$$\bar{G}(\lambda,x) = \sqrt{\frac{2\pi}{x}}\, e^{-1/x}\, I_{\sqrt{1/4+2\lambda}}(1/x). \tag{3.37}$$

This provides a link to the Hartman–Watson distribution (cf. [9]).

Since by (3.37) the Laplace transform of $s \mapsto e^{-s/4}\, G(s,x)$ equals $\sqrt{2\pi/x}$ $e^{-1/x}\, I_{\sqrt{2\lambda}}(1/x)$, denoting by $\mathsf{L}_\lambda^{-1}[\,\cdot\,]$ the inverse Laplace transform in the argument λ, we see that:

$$G(s,x) = \sqrt{\frac{2\pi}{x}}\, e^{s/4-1/x}\, \mathsf{L}_\lambda^{-1}\big[I_{\sqrt{2\lambda}}(1/x)\big](s). \tag{3.38}$$

Using the classic Hankel's contour integral (see [18, Chapter XVII] for more details):

$$I_{\sqrt{2\lambda}}(y) = \frac{1}{2\pi i}\int_C e^{y\cosh(z)-(\sqrt{2\lambda})z}\, dz \tag{3.39}$$

for $y > 0$ and the well-known identity $\mathsf{L}_\lambda^{-1}[e^{-(\sqrt{2\lambda})x}](t) = (2\pi t^3)^{-1/2}\, x\, e^{-x^2/2t}$ it is possible to perform the inversion in (3.38) by expressing the result in terms of a single integral (cf. [19, pp. 86-87]):

$$\mathsf{L}_\lambda^{-1}\big[I_{\sqrt{2\lambda}}(y)\big](s) = \frac{y\, e^{\pi^2/2s}}{\sqrt{2\pi^3 s}} \int_0^\infty e^{-z^2/2s - y\cosh(z)}\, \sinh(z)\, \sin\big(\tfrac{\pi z}{s}\big)\, dz. \tag{3.40}$$

Inserting (3.40) into (3.38) and then (3.38) back into (3.31) we obtain the following expression for the distribution function (3.1) above:

$$F(t,x) = 1 - \int_0^t \frac{e^{s/4+\pi^2/2s-1/x}}{\pi\sqrt{s}\, x^{3/2}} \int_0^\infty e^{-z^2/2s-(1/x)\cosh(z)}\, \sinh(z)\, \sin\big(\tfrac{\pi z}{s}\big)\, dz\, ds \tag{3.41}$$

when $\mu = 0$ and $\nu = 1/2$. Clearly the formula (3.41) extends along the same lines to the case of general $\nu > 0$ when $\mu = 0$.

11. In the case of general $\mu \in I\!R$ and $\nu > 0$ we may proceed differently from (3.34) and exploit the following integral representation (cf. (13.2.1) in [1, p. 505]):

$$\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}\, M(a,b,z) = \int_0^1 e^{zr}\, r^{a-1}\, (1-r)^{b-a-1}\, dr. \tag{3.42}$$

Hence the right-hand side of (3.32) reads:

$$\bar{G}(\lambda,x) = \frac{(\nu x)^{-a}}{\Gamma(a)} \int_0^1 e^{-r/\nu x}\, r^{a-1}\, (1-r)^{b-a-1}\, dr. \tag{3.43}$$

To handle the term $1/\Gamma(a)$ recall the Hankel's contour integral (cf. (6.1.4) in [1, p. 255]):

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i} \int_C e^z\, z^{-a}\, dz \tag{3.44}$$

where the path of integration C starts at $-\infty$ on the real axis, circles the origin in the anticlockwise direction, and returns to the starting point. Inserting (3.44) into (3.43) and recalling (3.22) and (3.23) we find that:

$$\bar{G}(\lambda,x) = (\nu x)^{\mu/2\nu-1/2} \int_0^1 e^{-r/\nu x}\, r^{-\mu/2\nu-1/2}\, (1-r)^{\mu/2\nu-1/2}\, H(r)\, dr \tag{3.45}$$

where the function $H(r) = H(\lambda,x,\mu,\nu,r)$ is given by:

$$H(r) = \frac{1}{2\pi i} \int_C e^z\, z^{\mu/2\nu-1/2} \exp\left(-\log\left(\frac{\nu x z}{r(1-r)}\right) \sqrt{\frac{1}{4}(1-\mu/\nu)^2 + \lambda/\nu}\right) dz. \tag{3.46}$$

Recalling the well-known identity:

$$\mathsf{L}_\lambda^{-1}\big[\, e^{-w\sqrt{\alpha+\beta\lambda}}\,\big](t) = \frac{\sqrt{\beta}\, w\, e^{-\alpha t/\beta - \beta w^2/4t}}{2\sqrt{\pi t^3}} \tag{3.47}$$

that is valid for all complex numbers $w = w_1 + iw_2$ such that $\mathfrak{Re}(w) = w_1 > 0$ and $\mathfrak{Re}(w^2) = w_1^2 - w_2^2 > 0$, letting $z = re^{i\varphi}$ in (3.46) and choosing C not too close to the origin in the sense that $r \geq R$ where $R > 0$ is taken large enough, we see that it is possible to perform the inversion in (3.45) by expressing the result in terms of a double integral. A more systematic study of the expressions obtained appears worthy of further consideration.

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