Tail Distributions of Supremum and Quadratic Variation of Local Martingales

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Summary. We extend some known results concerning the tail distribution of supremum and quadratic variation of a continuous local martingale to the case of locally square integrable martingales with bounded jumps. The predictable and optional quadratic variations are involved in the main result.

Key words: tail distribution, martingale supremum, quadratic variation

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1 Introduction and main result

Let $M = (M_t)_{t\geq 0}$ be a local martingale starting from zero and with paths in the Skorohod space $\mathbb{D}_{[0,\infty)}$. We assume that it is defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with usual conditions. We shall use the standard notation $\mathcal{M}_{\rm loc}$ for the class of local martingales and $\mathcal{M}^2_{\rm loc}$ $\mathcal{M}^c$, $\mathcal{M}$, $\mathcal{M}^2$ for its subclasses.

Recall that a adapted process X with paths in $\mathbb{D}_{[0,\infty)}$ defined on this stochastic basis belongs to the class $\mathcal{D}$ if the family $(X_\tau, \tau \in \mathcal{T})$, where $\mathcal{T}$ is the set of stopping times τ, is uniformly integrable.

Henceforth $\triangle M_t := M_t - M_{t-}$, $\langle M\rangle_t$ and $[M,M]_t$ denote the jumps, predictable quadratic variation and optional quadratic variation of M.

It is well-known (see, e.g., [9], [7] and references therein) that for any $M \in \mathcal{M}^2_{\rm loc}$:

$$\langle M\rangle_\infty < \infty \text{ a.s.} \Rightarrow \begin{cases} [M,M]_\infty < \infty \text{ a.s.} \\ \lim\limits_{t\to\infty} M_t = M_\infty \in \mathbb{R} \text{ a.s.} \end{cases} \tag{1.1}$$

There are many other remarkable relations between M_∞ and $\langle M\rangle_\infty$ (e.g., Burkholder–Gundy–Davis's inequalities, law of large numbers for martingales, etc.). For $M \in \mathcal{M}\cap\mathcal{D}$ we have the Wald equality

$$EM_\infty = 0,$$

which plays a fundamental role in many applications of the stochastic calculus.

Recall that the condition $E\langle M\rangle_\infty < \infty$ implies that $M \in \mathcal{M}^2$ and notice that $\langle M\rangle_\infty < \infty \not\Rightarrow M \in \mathcal{M}$. However, the condition $\langle M\rangle_\infty < \infty$, implying the existence of the limit value M_∞ (see, (1.1)), jointly with $EM_\infty = 0$ ensures $M \in \mathcal{M}$. One may ask which condition on $\langle M\rangle_\infty$ can provide the equality $EM_\infty = 0$? A positive answer for $M \in \mathcal{M}^c_{\rm loc}$ with $\langle M\rangle_\infty < \infty$ is known from Novikov, [10], and Elworthy, Li and Yor, [2], under the additional assumption: $Ee^{\varepsilon M^+_\infty} < \infty$ for sufficiently small $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = 0.$$

More precisely, the following statement is valid.

Theorem. ([10]) *Let $M \in \mathcal{M}^c_{\rm loc}$ and $\langle M\rangle_\infty < \infty$. Assume $\sup_{t>0} Ee^{\varepsilon M_t} < \infty$ for some sufficiently small $\varepsilon > 0$. Then:*

$$0 \le EM_\infty \le EM^+_\infty < \infty,$$
$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = \sqrt{\frac{2}{\pi}}EM_\infty.$$

For related topics see Azéma, Gundy and Yor [1], Gundy [5], Galtchouk and Novikov [6], Takaoka, [14], Peskir and Shiryaev [13], and Vondraček [15]).

The aim of this paper is to extend the statement of this Theorem for local martingales with bounded jumps.

Theorem 1.1. *Let $M \in \mathcal{M}^2_{\rm loc}$, $\langle M\rangle_\infty < \infty$ and $M^+ \in \mathcal{D}$. Then*

(i) *$M_\infty = \lim_{t\to\infty} M_t$ possesses the following properties*:

$$0 \le EM_\infty \le EM^+_\infty < \infty;$$

(ii) *the uniform integrability of $(|\triangle M_t|)_{t>0}$ and (i) imply*

$$\lim_{\lambda\to\infty} \lambda P\big(\sup_{t\ge 0} M^-_t > \lambda\big) = EM_\infty;$$

(iii) *$|\triangle M| \le K$ and $Ee^{\varepsilon M_\infty} < \infty$ for some $K > 0$ and sufficiently small $\varepsilon > 0$ imply*

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle^{1/2}_\infty > \lambda\big) = \lim_{\lambda\to\infty} \lambda P\big([M,M]^{1/2}_\infty > \lambda\big) = \sqrt{\frac{2}{\pi}}EM_\infty.$$

For $M^+ \in \mathcal{D}$, Theorem 1.1 gives necessary and sufficient conditions for $M \in \mathcal{M}$ expressed in terms of $\sup_{t\geq 0} M_t^-$, $\langle M\rangle_\infty$, and $[M,M]_\infty$. Concerning an effectiveness of these conditions see Jacod and Shiryaev [8].

Corollary 1.1. *Under the assumptions of Theorem 1.1, the process $M \in \mathcal{M}$ iff any of the following conditions hold*:

$$\lim_{\lambda\to\infty} \lambda P\big(\sup_{t\geq 0} M_t^- > \lambda\big) = 0,$$
$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big) = 0,$$
$$\lim_{\lambda\to\infty} \lambda P\big([M,M]_\infty^{1/2} > \lambda\big) = 0.$$

The proofs of statements **(i)** and **(ii)** of Theorem 1.1 are obvious and might even be known. The proof of **(iii)** exploits a combination of techniques:

"Stochastic exponential + Tauberian theorem"

used by Novikov in [11] and [12].

The necessary information on the stochastic exponential is gathered in Section 2. The proof of Theorem 1.1 is given in Section 3. We mention also a result, formulating in Theorem 3.1 (Section 3), presenting conditions alternative to $|\triangle M| \leq K$.

2 Stochastic exponential

We start with recalling necessary notions and objects (for details see, e.g., [9] or [7]).

For any $M \in \mathcal{M}^2_{\mathrm{loc}}$ we have the decomposition $M = M^c + M^d$ where $M^c, M^d \in \mathcal{M}^2_{\mathrm{loc}}$ are continuous and purely discontinuous martingales, respectively. Since $\langle M\rangle = \langle M^c\rangle + \langle M^d\rangle$, the assumption $\langle M\rangle_\infty < \infty$ implies $\langle M^c\rangle_\infty < \infty$, $\langle M^d\rangle_\infty < \infty$. The jump process $\triangle M \equiv \triangle M^d$ generates the integer-valued measure $\mu = \mu(dt,dz)$ with $\mu((0,t]\times A) = \sum_{s\leq t} I(\triangle M_s \in A)$. We denote by $\nu = \nu(dt,dz)$ the compensator of μ. The condition $|\triangle M| \leq K$ guarantees the existence of a version ν such that $\nu(\mathbb{R}_+ \times \{|z| > K\}) = 0$. This version of ν is used in the sequel.

The purely discontinuous martingale M^d can be represented as the Itô integral with respect to $\mu - \nu$:

$$M_t^d = \int_0^t \int_{|z|\leq K} z\big(\mu(ds,dz) - \nu(ds,dz)\big).$$

Recall that $\int_{|z|\leq K} z\nu(\{t\},dz) = 0$ and, so that,

$$\langle M^d\rangle_t = \int_0^t \int_{|z|\le K} z^2 \nu(ds,dz) < \infty,\ t > 0.$$

Hence, $\langle M\rangle_\infty < \infty$ implies $\int_0^\infty \int_{|z|\le K} z^2\nu(ds,dz) < \infty$ and the existence of the cumulant process (for $\lambda \in \mathbb{R}$)

$$G_t(\lambda) = \int_0^t \int_{|z|\le K} \big(e^{\lambda z} - 1 - \lambda z\big)\nu(ds,dz),$$

$$\triangle G_t(\lambda) = \int_{|z|\le K} \big(e^{\lambda z} - 1 - \lambda z\big)\nu(\{t\},dz).$$

We emphasize that $G_t(\lambda)$ increases in $t \uparrow$ to $G_\infty(\lambda) := \lim_{t\to\infty} G_t(\lambda) < \infty$ and $\triangle G_t(\lambda) \ge 0$.

The process

$$\mathcal{E}_t(\lambda) = \exp\Big(\frac{\lambda^2}{2}\langle M^c\rangle_t + G_t(\lambda)\Big) \prod_{0<s\le t} \big(1 + \triangle G_s(\lambda)\big)e^{-\triangle G_s(\lambda)}$$

is called "stochastic exponential" for the martingale M. Since $\triangle G(\lambda) \ge 0$, the stochastic exponential is nonnegative. A remarkable property of $\mathcal{E}_t(\lambda)$ is that the process

$$\mathfrak{z}_t(\lambda) = e^{\lambda M_t - \log \mathcal{E}_t(\lambda)} \tag{2.1}$$

is a positive local martingale with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$. This property is readily verified with the help of Itô's formula applied to (2.1):

$$d\mathfrak{z}_t(\lambda) = \lambda \mathfrak{z}_t(\lambda) dM_t^c + \int_{|z|\le K} \mathfrak{z}_{t-}(\lambda) \frac{\big(e^{\lambda z} - 1\big)}{1 + \triangle G_t(\lambda)} (\mu - \nu)(dt,dz).$$

As any nonnegative local martingale, $\mathfrak{z}_t(\lambda)$ is also a supermartingale (see, e.g., Problem 1.4.4 in Liptser and Shiryaev [9]) and, therefore, has a finite limit at infinity

$$\mathfrak{z}_\infty(\lambda) := \lim_{t\to\infty} \mathfrak{z}_t(\lambda) \in \mathbb{R}_+$$

and $E\mathfrak{z}_\tau(\lambda) \le 1$ for any stopping time τ. In particular, $E\mathfrak{z}_\infty \le 1$.

Proposition 2.1. *Under the conditions from statement* **(iii)** *of Theorem 1.1 we have*:

1) $E\mathfrak{z}_\infty(\lambda) = 1$.
2) $\mathcal{E}_\infty(\lambda) = \lim_{t\to\infty} \mathcal{E}_t(\lambda) \in (0,\infty)$.

Proof. 1) Let (τ_n) be a sequence of stopping times increasing to infinity and such that $(M_{t\wedge\tau_n})_{t\ge 0}$ and $(\mathfrak{z}_{t\wedge\tau_n}(\lambda))_{t\ge 0}$ are uniformly integrable martingales for any n. Then $E\mathfrak{z}_{\tau_n}(\lambda) \equiv 1$. By Jensen's inequality,

$$E\big(e^{\lambda M_\infty^+} | \mathcal{F}_{\tau_n}\big) \ge e^{\lambda E(M_\infty^+ | \mathcal{F}_{\tau_n})} \ge e^{\lambda M_{\tau_n}^+} \ge \mathfrak{z}_{\tau_n}(\lambda).$$

In other words, the martingale $\big(\mathfrak{z}_{\tau_n}(\lambda), \mathcal{F}_{\tau_n}\big)_{n\geq 1}$ is majorized by the uniformly integrable martingale $\big(E\big(e^{\lambda M_\infty^+}|\mathcal{F}_{\tau_n}\big), \mathcal{F}_{\tau_n}\big)_{n\geq 1}$, that is, $\big(\mathfrak{z}_{\tau_n}(\lambda), \mathcal{F}_{\tau_n}\big)_{n\geq 1}$ is the uniformly martingale itself. Consequently, $1 = \lim_{n\to\infty} E\mathfrak{z}_{\tau_n}(\lambda) = E\mathfrak{z}_\infty(\lambda)$.

2) Notice that $|M_\infty| < \infty$, $\mathcal{E}_\infty(\lambda) < \infty$ and $\mathfrak{z}_\infty(\lambda) = e^{\lambda M_\infty - \log \mathcal{E}_\infty(\lambda)}$ imply that

$$1 \geq EI(\mathcal{E}_\infty(\lambda) = 0)\mathfrak{z}_\infty(\lambda) \geq NP(\mathcal{E}_\infty(\lambda) = 0)$$

for any $N > 0$.

Hence, $P(\mathcal{E}_\infty(\lambda) = 0) = 0$.

3 The proof of Theorem 1.1

3.1 The proof of (i) and (ii)

(i) Let $(\tau_n)_{n\geq 1}$ be an increasing sequence of stopping times with tending to infinity and such that $(M_{\tau_n})_{n\geq 1} \in \mathcal{M}$. Therefore, $EM_{\tau_n}^- - EM_{\tau_n}^+ = 0, n \geq 1$. By $M^+ \in \mathcal{D}$, we have $\lim\limits_{n\to\infty} EM_{\tau_n}^+ = EM_\infty^+ < \infty$. Further, by the Fatou lemma $\underline{\lim}_{n\to\infty} EM_{\tau_n}^- \geq EM_\infty^-$, so that $EM_\infty^+ - EM_\infty^- \geq 0$.

Hence, $EM_\infty = (EM_\infty^+ - EM_\infty^-) \geq 0$.

(ii) Notice that $\{\sup_{t\geq 0} M_t^- > \lambda\} = \{S_\lambda < \infty\}$, where

$$S_\lambda = \inf\{t : M_t^- \geq \lambda\}, \quad \inf\{\varnothing\} = \infty.$$

Since $(|\triangle M_t|)_{t>0}$ is uniformly integrable process and $M^+ \in \mathcal{D}$, we have $(M_{t\wedge S_\lambda})_{t\geq 0} \in \mathcal{M}$, that is,

$$0 = EM_{S_\lambda} = EM_\infty I_{\{S_\lambda = \infty\}} + EM_{S_\lambda} I_{\{S_\lambda < \infty\}}.$$

We derive the desired statement from the relations

$$\begin{aligned} &\lim_{\lambda\to\infty} EM_\infty I_{\{S_\lambda=\infty\}} = EM_\infty, \\ &\lim_{\lambda\to\infty} EM_{S_\lambda} I_{\{S_\lambda<\infty\}} = -\lambda P\Big(\sup_{t\geq 0} M_t^- > \lambda\Big). \end{aligned} \tag{3.1}$$

By **(i)**, $EM_\infty^- \leq EM_\infty^+ < \infty$. Consequently, $M_\infty^- < \infty$ and, therefore, we have $\lim_{\lambda\to\infty} S_\lambda = \infty$. The first part of (3.1) is implied by the inequality

$$\big|EM_\infty I_{\{S_\lambda=\infty\}} - EM_\infty\big| \leq E|M_\infty| I_{\{S_\lambda<\infty\}}$$

and the Lebesgue dominated theorem. The second part in (3.1) follows from $M_{S_\lambda} I_{\{S_\lambda<\infty\}} = -\lambda I_{\{S_\lambda<\infty\}} + (M_{S_\lambda} + \lambda) I_{\{S_\lambda<\infty\}}$ since

$$E|M_{S_\lambda} + \lambda| I_{\{S_\lambda<\infty\}} \leq E|\triangle M_{S_\lambda}| I_{\{S_\lambda<\infty\}} \leq KP(S_\lambda < \infty) \xrightarrow[\lambda\to\infty]{} 0.$$

3.2 Proof of (iii)

Auxiliary lemmas

Lemma 3.1. *Under assumptions from the statement* ***(iii)*** *of Theorem 1.1,*

$$\lim_{\lambda\downarrow 0} E\frac{1}{\lambda}\Big(1-e^{-\log \mathcal{E}_\infty(\lambda)}\Big)=EM_\infty.$$

Proof. With $\lambda\le\varepsilon$ for ε involved in **(iii)**, by Proposition 2.1 we have the equality $E\mathfrak{z}_\infty(\lambda)=1$. Hence,

$$\begin{aligned}E\frac{1}{\lambda}\Big(1-e^{-\log \mathcal{E}_\infty(\lambda)}\Big)&=E\frac{1}{\lambda}\Big(\mathfrak{z}_\infty(\lambda)-e^{-\log \mathcal{E}_\infty(\lambda)}\Big)\\&=E\frac{1}{\lambda}\Big(e^{\lambda M_\infty}-1\Big)e^{-\log \mathcal{E}_\infty(\lambda)}.\end{aligned}$$

The required statement follows from the relations

$$\lim_{\lambda\downarrow 0}\frac{1}{\lambda}e^{-\log \mathcal{E}_\infty(\lambda)}\big(e^{\lambda M_\infty}-1\big)=M_\infty,$$

$$\frac{1}{\lambda}e^{-\log \mathcal{E}_\infty(\lambda)}\big|e^{\lambda M_\infty}-1\big|\le e^{\varepsilon M_\infty}$$

and $Ee^{\varepsilon M_\infty}<\infty$ by the Lebesgue dominated theorem.

Lemma 3.2. *Under assumptions from the statement* **(iii)** *of Theorem 1.1,*

$$\lim_{\lambda\downarrow 0} E\frac{1}{\lambda}\Big(1-e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big)=EM_\infty.$$

Proof. According to Lemma 3.1, it suffices to show that

$$\lim_{\lambda\downarrow 0} E\frac{1}{\lambda}\Big|e^{-\log \mathcal{E}_\infty(\lambda)}-e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big|=0. \tag{3.2}$$

The verification of (3.2) uses the following estimates: for some $C>0$ and sufficiently small $\lambda>0$,

$$0<\big[1-C\lambda\big]\frac{\lambda^2}{2}\langle M\rangle_\infty\le\log\mathcal{E}_\infty(\lambda)\le\big[1+C\lambda\big]\frac{\lambda^2}{2}\langle M\rangle_\infty. \tag{3.3}$$

The estimate from above is implied by $\log\mathcal{E}_\infty(\lambda)\le\frac{\lambda^2}{2}\langle M^c\rangle_\infty+G_\infty(\lambda)$ and the property of $\nu(dt,dz)$ to be supported, in z, on $[-K,K]$.

The estimate from below is determined in the following way. Denote by $\Phi(\lambda,K)=1-\lambda Ke^{\lambda K}$ and

$$G^c_\infty(\lambda)=\int_0^\infty\int_{|z|\le K}\big(e^{\lambda z}-1-\lambda z\big)\nu^c(dt,dz),$$

where $\nu^c(dt, dz) := \nu(dt, dz) - \nu(\{t\}, dz)$. Write

$$\begin{aligned}\log \mathcal{E}_\infty(\lambda) &= \frac{\lambda^2}{2}\langle M^c\rangle_\infty + G^c_\infty(\lambda) + \sum_{t>0} \log\big(1 + \triangle G_t(\lambda)\big) \\ &\geq \frac{\lambda^2}{2}\langle M^c\rangle_\infty + \varPhi(\lambda, K)\int_0^\infty \int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu^c(dt, dz) \\ &\quad + \sum_{t>0} \log\left(1 + \varPhi(\lambda, K)\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz)\right).\end{aligned} \tag{3.4}$$

We choose λ so small to keep $1 - \lambda K e^{\lambda K} > 0$ and estimate from below the "$\sum_{t>0} \log$" in the last line of (3.4) by applying $\log(1 + x) \geq x - \frac{1}{2}x^2,\ x \geq 0$. This gives the lower bound

$$\begin{aligned}&\sum_{t>0} \log\left(1 + \varPhi(\lambda, K)\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz)\right) \\ &\geq \varPhi(\lambda, K)\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz) - \frac{1}{2}\varPhi^2(\lambda, K)\left(\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz)\right)^2.\end{aligned}$$

Taking into account $\nu(\{t\}, |z| \leq K) \leq 1$, by the Cauchy–Schwarz inequality we find the upper bound

$$\begin{aligned}&\left(\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz)\right)^2 \\ &\leq \frac{\lambda^4}{4}\int_{|z|\leq K} z^4 \nu(\{t\}, dz) \leq \frac{\lambda^4 K^2}{4}\int_{|z|\leq K} z^2 \nu(\{t\}, dz)\end{aligned}$$

providing the inequality

$$\begin{aligned}&\sum_{t>0} \log\left(1 + \varPhi(\lambda, K)\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz)\right) \\ &\geq \left(\varPhi(\lambda, K) - \frac{\lambda^2}{8}K^2\varPhi^2(\lambda, K)\right)\int_{|z|\leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz).\end{aligned}$$

We choose λ so small to keep

$$\varPhi(\lambda, K) - \frac{\lambda^2}{8}K^2\varPhi^2(\lambda, K) \geq 1 - \lambda c > 0$$

for some constant $c > 0$.

Now, we may choose a positive constant C such that (3.3) is valid for both the upper and lower bounds.

From (3.3), we derive that

$$\frac{1}{\lambda}\Big|e^{-\log \mathcal{E}_\infty(\lambda)} - e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big| \le C\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty} \xrightarrow[\lambda\to 0]{} 0.$$

and, due to $xe^{-x} \le e^{-1}$, it remains to apply the Lebesgue dominated theorem.

Lemma 3.3. *Under assumptions from the statement* **(iii)** *of Theorem 1.1,*

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big) = \boldsymbol{\vartheta} \Leftrightarrow \lim_{\lambda\to\infty} \lambda P\big([M,M]_\infty^{1/2} > \lambda\big) = \boldsymbol{\vartheta}.$$

Proof. Obviously, the desired result holds true if

$$\begin{aligned} \overline{\lim_{\lambda\to 0}} \frac{P\big([M,M]_\infty^{1/2} > \lambda\big)}{P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \le 1, \\ \underline{\lim}_{\lambda\to 0} \frac{P\big([M,M]_\infty^{1/2} > \lambda\big)}{P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \ge 1. \end{aligned} \tag{3.5}$$

Denote $L = [M,M] - \langle M\rangle$ and notice that $[M,M]_\infty \le \langle M\rangle_\infty + \sup_{t\ge 0}|L_t|$. By an obvious inequality $(c+d)^{1/2} \le c^{1/2} + d^{1/2}$, we obtain that

$$\begin{aligned} P\big([M,M]_\infty^{1/2} > \lambda\big) &\le P\big([\langle M\rangle_\infty + \sup_{t\ge 0}|L_t|]^{1/2} > \lambda\big) \\ &\le P\big(\langle M\rangle_\infty^{1/2} + \sup_{t\ge 0}|L_t|^{1/2} > \lambda\big) \\ &\le P\big(\langle M\rangle_\infty^{1/2} > (1-a)\lambda\big) + P\big(\sup_{t\ge 0}|L_t| > a\lambda\big),\ a\in(0,1). \end{aligned}$$

With $\lambda_a = (1-a)\lambda$, the resulting bound can be rewritten as:

$$\lambda P\big([M,M]_\infty^{1/2} > \lambda\big) \le (1-a)^{-1}\lambda_a P\big(\langle M\rangle_\infty^{1/2} > \lambda_a\big) + \lambda P\big(\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\big). \tag{3.6}$$

Now, we evaluate from from above $P\big(\sup_{t\ge 0}|L_t|^{1/2} > a\lambda\big)$. A helpful tool here is the inequality: for some $C > 0$, any stopping time τ and K being a bound for $|\triangle M|$,

$$E\sup_{t\le\tau}|L_t|^2 \le CK^2 E\langle M\rangle_\tau. \tag{3.7}$$

In order to establish (3.7), we use the following facts:

- L is the purely discontinuous local martingale with

$$\begin{aligned} [L,L]_t &= \sum_{s\le t}(\triangle L_s)^2 = \sum_{s\le t}\big((\triangle M_s)^2 - \triangle\langle M\rangle_s\big)^2 \\ &= \sum_{s\le t}\left(\int_{|z|\le K} z^2(\mu(\{s\},dz) - \nu(\{s\},dz)\right)^2, \end{aligned}$$

- $\langle L\rangle_t = \int_0^t \int_{|z|\le K} z^4(\nu(ds,dz) - \sum_{s\le t}\Big(\int_{|z|\le K} z^2\nu(\{s\},dz)\Big)^2,$

- $\langle L\rangle_t \le \int_0^t \int_{|z|\le K} z^4\nu(ds,dz) \le K^2 \int_0^t \int_{|z|\le K} z^2\nu(\{ds,dz) \le K^2\langle M\rangle_t$,
- $K^2\langle M\rangle - \langle L\rangle$ is the increasing process.

Now, we refer to the Burkholder–Gundy inequality (see, e.g., Theorem 1.9.7 in [9]): for any stopping time τ,

$$E\sup_{t\le\tau}|L_t|^2 \le CE[L,L]_\tau.$$

Due to the relations $E[L,L]_\tau = E\langle L\rangle_\tau$ and $K^2\langle M\rangle_\tau \ge \langle L\rangle_\tau$ (recall that $K^2\langle M\rangle \ge \langle L\rangle$), we have $E\langle L\rangle_\tau \le K^2E\langle M\rangle_\tau$, that is, (3.7) is valid.

By (3.7) and the fact that $\langle M\rangle$ is a predictable process, the Lenglart–Rebolledo inequality (see, e.g., Theorem 1.9.3 in [9]) is applicable (notice that $\{\sup_{t\ge0}|L_t|^{1/2} > a\lambda\} \equiv \{\sup_{t\ge0}|L_t| > a^2\lambda^2\}$), so that,

$$\begin{aligned} P\Big(\sup_{t\ge0}|L_t|^{1/2} > a\lambda\Big) &\le \frac{\lambda^{5/2}}{a^4\lambda^4} + P\big(CK^2\langle M\rangle_\infty > \lambda^{5/2}\big) \\ &= \frac{\lambda^{5/2}}{a^4\lambda^4} + P\Big(\langle M\rangle_\infty^{1/2} > \lambda^{5/4}/(C^{1/2}K)\Big). \end{aligned}$$

Hence, with $r = 1/(C^{1/2}K)$ and $\lambda_r = r\lambda^{5/4}$,

$$\lambda P\Big(\sup_{t\le T_x}|L_t|^{1/2} > a\lambda\Big) \le \frac{1}{a^4\lambda^{1/2}} + \frac{1}{r\lambda^{1/4}}\lambda_r P\Big(\langle M\rangle_\infty^{1/2} > \lambda_r\Big). \tag{3.8}$$

Now, (3.6) and (3.8) imply the inequality

$$\begin{aligned} &\lambda P\Big([M,M]_\infty^{1/2} > \lambda\Big) \\ &\le (1-a)^{-1}\lambda_a P\big(\langle M\rangle_\infty^{1/2} > \lambda_a\Big) + \frac{1}{a^4\lambda^{1/2}} + \frac{r}{\lambda^{1/4}}\lambda_r P\big(\langle M\rangle_\infty^{1/2} > \lambda_r\big). \end{aligned}$$

If $\boldsymbol{\vartheta} > 0$, by

$$\begin{aligned} \frac{P\big([M,M]_\infty^{1/2} > \lambda\big)}{P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} &\le \frac{(1-a)^{-1}\lambda_a P\big(\langle M\rangle_\infty^{1/2} > \lambda_a\big)}{\lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \\ &+ \frac{\frac{1}{a^4\lambda^{1/2}} + \frac{r}{\lambda^{1/4}}\lambda_r P\big(\langle M\rangle_\infty^{1/2} > \lambda_r\big)}{\lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \xrightarrow[\lambda\to\infty]{} \frac{1}{1-a} \xrightarrow[a\to0]{} 1 \end{aligned}$$

and the first part from (3.5) is valid. The second part from (3.5) is established similarly and we give only a sketch of the proof. The use of the bound

$$P\big(\langle M\rangle^{1/2} > \lambda\big) \le P\big([M,M]^{1/2} > (1-a)\lambda\big) + P\big(\sup_{t\ge0}|L_t| > a\lambda\big),\ a \in (0,1),$$

implies that

$$\frac{P\big([M,M]\rangle_\infty^{1/2} > (1-a)\lambda\big)}{P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)} \ge 1 - \frac{P\big(\sup_{t\ge 0}|L_t| > a\lambda\big)}{P\big(\langle M\rangle_\infty^{1/2} > \lambda\big)}$$

and we get the result.

If $\boldsymbol{\vartheta} = 0$, we replace M by $M + \delta M'$, where $\delta > 0$ and $M' \in \mathcal{M}^c$ with $\langle M'\rangle_\infty < \infty$ possessing $\lim_{\lambda\to\infty} \lambda P\big(\langle M'\rangle_\infty^{1/2} > \lambda\big) = \boldsymbol{\vartheta'} > 0$, is independent of M^c. Therefore, by $\langle M + \delta M'\rangle = \langle M\rangle + \delta^2\langle M'\rangle$, we have

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M + \delta M'\rangle_\infty^{1/2} > \lambda\big) = \delta^2\boldsymbol{\vartheta'} > 0.$$

Hence, by using the result already proved, it holds

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M + \delta M'\rangle_\infty^{1/2} > \lambda\big) = \delta^2\boldsymbol{\vartheta'} \\ \Leftrightarrow \lim_{\lambda\to\infty} \lambda P\big([M + \delta M', M + \delta M']_\infty^{1/2} > \lambda\big) = \delta^2\boldsymbol{\vartheta'}$$

and, by the arbitrariness of δ,

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M > \lambda\big) = 0 \Leftrightarrow \lim_{\lambda\to\infty} \lambda P\big([M,M]_\infty^{1/2} > \lambda\big) = 0.$$

Final part of the proof for (iii)

We refer to the Tauberian theorem.

Theorem. (Feller, [4], XIII.5, Example (c)) *Let X be a nonnegative random variable such that* $\lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}X}\Big) \in \mathbb{R}$.

Then,

$$\sqrt{\frac{2}{\pi}} \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}X}\Big) = \lim_{\lambda\to\infty} \lambda P(X^{1/2} > \lambda).$$

Letting $X = \langle M\rangle_\infty$, we find that

$$\sqrt{\frac{2}{\pi}} \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\big(1 - Ee^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\big) = \lim_{\lambda\to\infty} \lambda P(\langle M\rangle_\infty^{1/2} > \lambda),$$

while, by Lemmas 3.1, 3.2 and 3.3,

$$\lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(1 - Ee^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big) = \sqrt{\frac{2}{\pi}} EM_\infty,$$

$$\lim_{\lambda\to\infty} \lambda P\big([M,M]_\infty^{1/2} > \lambda\big) = \sqrt{\frac{2}{\pi}} EM_\infty.$$

3.3 Supplement

The condition $|\triangle M| \leq K$ might be too restrictive to be valid for serving some examples. Following [10], we show that this condition can be replaced by one seems to be more suitable for applications.

Theorem 3.1. *Assume conditions for the statement* **(iii)** *of Theorem 1.1 are valid except the boundedness* $|\triangle M| \leq K$ *replaced by the two inequalities*

$$\frac{\lambda^2}{2}\langle M\rangle_\infty(1-|\lambda|\zeta_1)^+ \leq \log \mathcal{E}_\infty(\lambda) \leq \frac{\lambda^2}{2}\langle M\rangle_\infty(1+|\lambda|\zeta_2) \qquad (3.9)$$

with sufficiently small $\lambda > 0$ *and nonnegative integrable random variables* ζ_1, ζ_2.

Then

$$\lim_{\lambda\to\infty} \lambda P\big(\langle M\rangle_\infty^{1/2} > \lambda\big) = \sqrt{\frac{2}{\pi}} E M_\infty.$$

Proof. Since (3.2) has to be verified only, by (3.9) we have

$$\begin{aligned}\frac{1}{\lambda}\Big|e^{-\log \mathcal{E}_\infty(\lambda)} - e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}\Big| &\leq \Big(\zeta_2 \vee \frac{|1-(1-\zeta_1\lambda)^+|}{\lambda}\Big)\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty} \\ &\leq \big(\zeta_2\vee\zeta_1\big)\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-\frac{\lambda^2}{2}\langle M\rangle_\infty}.\end{aligned}$$

The right-hand side of this inequality converges to zero, as $\lambda \to 0$, and is bounded by $e^{-1}(\zeta_2 \vee \zeta_1)$. So, (3.2) holds by the Lebesgue dominated theorem.

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