Optimal Time to Invest under Tax Exemptions

Vadim I. ARKIN[1] and Alexander D. SLASTNIKOV[1]

Central Economics & Mathematics Institute,
Moscow, Nakhimovskii pr. 47, Moscow, Russia.
arkin@cemi.rssi.ru
slast@cemi.rssi.ru

Summary. We develop a model of the behavior of an agent acting under uncertainty and in a fiscal environment who wants to invest into a creation of new firm and faces a timing problem. The presence of tax exemptions for newly created firms reduces the investor planning to the optimal stopping problem for bivariate diffusion process with a non-linear homogeneous reward function. We find a closed-form formula for optimal stopping time and prove that under certain conditions it gives indeed the optimal solution to the investment timing problem.

Key words: real options, tax exemptions, optimal stopping time, bivariate geometric Brownian motion, homogeneous reward function

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1 Introduction

Uncertainty and irreversibility have long been recognized as main determinants of investment. As argued in [6], most investment decisions feature three important characteristics: investment irreversibility, uncertainty, and the ability to choose the optimal timing of investment. In contrast with the traditional investment theory based on the Net Present Value Criterion and Now-or-Never Principle, the real option literature has been focused around the delay in investment decisions (see, e.g., [6], [17] as well as the seminal paper [11]). This flexibility in the investment timing gives the option to wait for new information.

In the real option framework the optimal investment policy can be obtained as the solution to an optimal stopping problem. In the simple case of a project with constant (over time) investments the underlying problem is an optimal stopping for one-dimensional process of the present value of the project, which is usually assumed to be a geometric Brownian motion without/with jumps (see [6], [11], [12]). In a more symmetric case, when both the present value and the investment required for launching the project evolve as stochastic processes, the underlying problem will be an optimal stopping for bivariate stochastic process (usually, of a geometric Brownian type) and reward function which is the expected discounted difference between the present value and the investment cost. One of the first results in this direction was obtained by McDonald and Siegel [11] who gave a closed-form solution for the case of bivariate correlated geometric Brownian motion. However, they did not set the precise conditions needed for the validity of their result. The rigorous proof of optimality in the McDonald–Siegel formula for optimal stopping time as well as the relevant conditions was given only a decade later by Hu and Øksendal [8]. Moreover, they treated a multi-dimensional case where the investment cost is a sum of correlated geometric Brownian motions.

Another source of multi-dimensional optimal stopping problems is a valuation of American options on multiple assets — see, e.g. [5], [7]. The Russian option introduced by Shepp and Shiryaev [14], also can be viewed as an optimal stopping problem for a bivariate Markov process whose components are processes of stock prices and maximal historical (up to the current time) stock prices.

Although the theory provides general rules for finding an optimal stopping time (see, e.g., Shiryaev's monographs [15], [16]), the obtaining of closed form formulas is a hard problem for multi-dimensional processes. Most of results in this direction (for multivariate case) are related to geometric Brownian motion and linear reward function. A rare exception is the paper by Gerber and Shiu [7], who derived a closed-form formula for bivariate correlated geometric Brownian motion and homogeneous reward function. Their case covers such perpetual (without the expiration date) American options on two stocks as Margrabe exchange option, maximum option and some others. They used first-order conditions to determine the optimal stopping boundaries, but did not verify whether the relevant solution is indeed the optimal one to the underlying problem.

In the present paper we demonstrate that multivariate optimal stopping problem with non-linear reward function arises in a natural way for the models of creation of new firms in a fiscal environment (including both taxes and tax exemptions for new firms). Namely, some not restrictive assumptions about the structure of investor's cash flow and tax holidays for newly created enterprizes lead to an optimal investment timing problem with non-linear (relatively to the underlying processes) reward function. We derive a closed-form formula for optimal investment time and prove that under certain conditions it gives indeed the optimal solution to the investment timing problem.

The paper is organized as follows. Section 2 describes the behavior of an investor (under uncertainty and in a fiscal environment) who is interested in investing into the project aimed at creating a new firm and faces the investment timing problem. A solution to this problem, an optimal investment rule, is described in Section 3. As we show in 3.3, the problem under consideration is reduced to an optimal stopping problem for bivariate diffusion process and homogeneous (of degree 1) reward function. The closed-form formula for optimal investment time described in Theorem 1 is proved in Section 4.

2 The basic model

Before to proceed with the model description, we compare our model with some closely related contributions.

The model is connected with an investment project directed to the creation of a new industrial firm (enterprize). An important feature of the considered model is the assumption that, at any moment, a decision-maker (investor) may either *accept* the project and proceed with the investment or *delay* the decision until he obtains a new information on the environment (prices of the product and resources, the demand etc.). Thus, the main goal of the decision-maker in this situation is to find, using the available information, a "good" time for investing the project (investment timing problem).

The real options theory is a convenient and adequate tool for modelling the process of firm creation since it allows us to study the effects connected with a delay in the investment (investment waiting). As in the real options literature, we model investment timing problem as an optimal stopping problem for present values of the created firm (see, e.g. [6], [11]).

A creation of an industrial enterprize is usually accompanied by certain tax benefits (in particular, the new firm can be exempted from the profit taxes during a certain period). The distinguishing feature of our settings is the representation of the firm present value as an integral of the profit flow. Considerations of this type allows us to take into account in an explicit form some peculiarities of a corporate profit taxation system, including the tax exemption. Such an approach was applied by the authors in a detailed modelling of investment project under taxation (but without tax exemptions) in [3], and for finding the optimal depreciation policy in [1].

Uncertainty in an economic system is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbf{F} = (\mathcal{F}_t,\ t \geq 0)$. The σ-algebra $\mathcal{F}_t$ can be interpreted as the observable information about the system up to the time t.

An infinitely-lived investor faces a problem to choose when to invest in a project aimed to launch a new firm.

The cost of investment required to create firm at time t is I_t. Investment are considered to be instantaneous and irreversible so that they cannot be withdrawn from the project any more and used for other purposes (sunk cost). We assume that $(I_t,\ t \geq 0)$ is an adapted random process.

Let us suppose that investment into creating a firm is made at time $\tau \geq 0$.

Let $\pi^{\tau}_{\tau+t}$ be the flow of profit from the firm at time $t+\tau$, i.e. gross income minus production cost except depreciation charges, and $D^{\tau}_{t+\tau}$ denotes the flow of depreciation at the same time. $\pi^{\tau}_{\tau+t}$ and $D^{\tau}_{t+\tau}$ are assumed to be $\mathcal{F}_{t+\tau}$-measurable random variables $(t, \tau \geq 0)$.

If γ is the corporate profit tax rate, then after-tax cash flow of the firm at time $t+\tau$ is equal to

$$\pi^\tau_{\tau+t} - \gamma(\pi^\tau_{\tau+t} - D^\tau_{t+\tau}) = (1-\gamma)\pi^\tau_{\tau+t} + \gamma D^\tau_{t+\tau}. \tag{2.1}$$

Creating a new firm in the real economy is usually accompanied by certain tax benefits. One of the popular incentives tools is tax holidays, when the new firm is exempted from the profit tax during a payback period. According to the accepted definitions, the payback period is specified as the minimal interval (following the time of firm's creation) during which the accumulated discounted expected profits exceed the initial investment required for creating the firm.

For the firm created at time τ, we define the payback period ν_τ as follows:

$$\nu_\tau = \inf\left\{\nu \ge 0 : \ \mathbf{E}\Big(\int_0^\nu \pi^\tau_{\tau+t} e^{-\rho t} dt \ \Big| \mathcal{F}_\tau\Big) \ge I_\tau\right\} \tag{2.2}$$

where ρ is discount rate.

Note that ν_τ is an $\mathcal{F}_\tau$-measurable random variable not necessarily finite a.s. Further we will often refer to the set of finite payback periods:

$$\Omega_\tau = \{\omega \in \Omega : \ \nu_\tau < \infty\}. \tag{2.3}$$

For simplicity we assume that the firm begins to generate profits right after the investment is made. Then, accordingly to the cash flow (2.1) and tax holidays (2.2), the present value of the firm V_τ (discounted to the investment time τ) can be expressed by the following formula:

$$V_\tau = \mathbf{E}\left(\int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt + \chi_{\Omega_\tau} \int_{\nu_\tau}^\infty [(1-\gamma)\pi^\tau_{\tau+t} + \gamma D^\tau_{t+\tau}] e^{-\rho t}\, dt \,\middle|\, \mathcal{F}_\tau\right), \tag{2.4}$$

where $\chi_{\Omega_\tau}(\omega)$ is the indicator function of the event Ω_τ defined in (2.3).

The behavior of the agent is assumed to be rational. This means that he solves *the investment timing problem*: at any time τ prior to the investment he chooses whether to pay I_τ and earn the present value V_τ, or to delay further his investment. So, the investor's decision problem is to find such a stopping time τ (investment rule), that maximizes the expected net present value (NPV) from the future activity:

$$\mathbf{E}\,(V_\tau - I_\tau)\, e^{-\rho\tau} \to \max_\tau, \tag{2.5}$$

where the maximum is taken over all Markov times τ and V_τ is defined in (2.4).

3 Solution of the investment timing problem

3.1 Main assumptions

Let (w_t^1), (w_t^2) be two independent standard Wiener processes on the given stochastic basis. They are thought as underlying processes modelling Economic Stochastics. We assume that σ-algebra $\mathcal{F}_t$ is generated by these processes up to t, i.e. $\mathcal{F}_t = \sigma\{(w_s^1, w_s^2),\ s \le t\}$.

The process of *profits* $\pi_{\tau+t}^{\tau}$ is represented as follows:

$$\pi_{\tau+t}^{\tau} = \pi_{\tau+t}\xi_{\tau+t}^{\tau}, \quad t, \tau \ge 0, \tag{3.6}$$

where (π_t) is geometric Brownian motion, specified by the stochastic equation

$$\pi_t = \pi_0 + \alpha_1 \int_0^t \pi_s\, ds + \sigma_1 \int_0^t \pi_s\, dw_s^1 \qquad (\pi_0 > 0,\ \sigma_1 \ge 0), \quad t \ge 0, \tag{3.7}$$

and $(\xi_{\tau+t}^{\tau},\ t \ge 0)$ is a family of non-negative diffusion processes, homogeneous in $\tau \ge 0$, defined by the stochastic equations

$$\xi_{\tau+t}^{\tau} = 1 + \int_\tau^{t+\tau} a(s-\tau, \xi_s^{\tau})\, ds + \int_\tau^{t+\tau} b(s-\tau, \xi_s^{\tau})\, dw_s^1, \quad t, \tau \ge 0, \tag{3.8}$$

with given functions $a(t,x)$, $b(t,x)$ (satisfying the standard conditions for the existence of the unique strong solution in (3.8) – see, e.g. [13, Ch.5]).

The process π_t in representation (3.6) can be related to the exogeneous prices of produced goods and consumed resources (external uncertainty), whereas fluctuations $\xi_{\tau+t}^{\tau}$ can be generated by the firm created at time τ (firm's uncertainty). Obviously, $\pi_\tau^\tau = \pi_\tau$ for any $\tau \ge 0$.

The cost of the required investment I_t is also described by the geometric Brownian motion as

$$I_t = I_0 + \alpha_2 \int_0^t I_s\, ds + \int_0^t I_s(\sigma_{21}\, dw_s^1 + \sigma_{22}\, dw_s^2), \quad (I_0 > 0) \quad t \ge 0, \tag{3.9}$$

where $\sigma_{21},\ \sigma_{22} \ge 0$. To avoid a degenerate case we assume that $\sigma_{22} > 0$. Then the linear combination $\sigma_{21}w_t^1 + \sigma_{22}w_t^2$ has the same distribution as $(\sigma_{21}^2 + \sigma_{22}^2)^{1/2}\tilde{w}_t$, where $\tilde{w}_t$ is a Wiener process correlated with w_t^1 and the correlation coefficient is equal to $\sigma_{21}(\sigma_{21}^2 + \sigma_{22}^2)^{-1/2}$.

Depreciation charges at the time $t+\tau$ (for the firm created at the time τ) will be represented as:

$$D_{\tau+t}^{\tau} = I_\tau a_t, \quad t \ge 0, \tag{3.10}$$

where (a_t) is the "depreciation density" (per unit of investment), characterizing a depreciation policy, i.e. a non-negative function $a : R_+^1 \to R_+^1$ such that $\int a_t\,dt = 1$. Such a scheme covers various depreciation models, accepted by the modern tax laws (more exactly, their variants in continuous time). For example, the well-known Declining Balance Depreciation Method can be described by the exponential density $a_t = \eta e^{-\eta t}$, where $\eta > 0$ is the DB-depreciation rate.

3.2 Derivation of the present value

The above assumptions allow us to obtain formulas for the present value of the future firm.

In order all values in the model were well-defined, we suppose that the profits $\pi^{\tau}_{\tau+t}$ have finite expectations for all $t, \tau \geq 0$.

We need the following assertion.

Lemma 3.1. *Let τ be a finite (a.s.) Markov time. Then for all $t \geq 0$*

$$\mathbf{E}(\pi^{\tau}_{\tau+t}|\mathcal{F}_{\tau}) = \pi_{\tau} B_t, \quad \textit{where} \;\; B_t = \mathbf{E}(\pi_t \xi^0_t)/\pi_0.$$

Proof. Recall that the process $\widehat{w}_t = w^1_{t+\tau} - w^1_{\tau},\ t \geq 0$ is a Wiener process independent on $\mathcal{F}_{\tau}$. Using the explicit formula for the geometric Brownian motion one can rewrite relation (3.6) as follows:

$$\pi^{\tau}_{\tau+t} = \pi_{\tau} \mathit{\Pi}^{\tau}_{t+\tau}, \quad \text{where } \mathit{\Pi}^{\tau}_{t+\tau} = \exp\{(\alpha_1 - \tfrac{1}{2}\sigma_1^2)t + \sigma_1 \widehat{w}_t\}\xi^{\tau}_{\tau+t}.$$

Homogeneity of the family (3.8) in τ implies that the process $\xi^{\tau}_{\tau+t}$ coincides (a.s.) with the unique (in the strong sense) solution of the stochastic equation

$$\xi_t = 1 + \int_0^t a(s,\xi_s)\,ds + \int_0^t b(s,\xi_s)\,d\widehat{w}_s.$$

Since $(\xi_t,\ t \geq 0)$ is independent on $\mathcal{F}_{\tau}$, the process $\mathit{\Pi}^{\tau}_{t+\tau}$ is independent also. Moreover, $\mathit{\Pi}^{\tau}_{t+\tau}$ has the same distribution as $\exp\{(\alpha_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 \widehat{w}_t\}\xi_t$, or as $(\pi_t/\pi_0)\xi^0_t$. Therefore, $\mathbf{E}(\pi^{\tau}_{\tau+t}|\mathcal{F}_{\tau}) = \pi_{\tau}\mathbf{E}\mathit{\Pi}^{\tau}_{t+\tau} = \pi_{\tau}\mathbf{E}(\pi_t\xi^0_t)/\pi_0$. □

Let us assume that the following condition holds:

$$B = \int_0^{\infty} B_t e^{-\rho t}\,dt < \infty,$$

where the function B_t is defined in Lemma 1.

We will denote the conditional expectation with respect to $\mathcal{F}_{\tau}$ as $\mathbf{E}_{\tau}$.

The above relations and Lemma 1 give the following formulas for the present value (2.4).

Let τ be a finite (a.s.) Markov time.

If payback period $\nu_\tau < \infty$ (i.e. $\omega \in \Omega_\tau$, see (2.3)), then

$$\begin{aligned} V_\tau &= I_\tau + (1-\gamma)\left(\mathbf{E}_\tau \int_0^\infty \pi^\tau_{\tau+t} e^{-\rho t} dt - \mathbf{E}_\tau \int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt\right) + \gamma I_\tau A(\nu_\tau) \\ &= I_\tau(1+\gamma A(\nu_\tau)) - (1-\gamma)\left(I_\tau - \pi_\tau \int_0^\infty B_t e^{-\rho t} dt\right) \\ &= \gamma I_\tau (1 + A(\nu_\tau)) + (1-\gamma)\pi_\tau B, \end{aligned} \tag{3.11}$$

where the function $A(\cdot)$ is defined as

$$A(\nu) = \int_\nu^\infty a_t e^{-\rho t} dt, \quad \nu \geq 0. \tag{3.12}$$

According to (2.2) on the set Ω_τ we have:

$$I_\tau = \mathbf{E}_\tau \int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt = \pi_\tau \int_0^{\nu_\tau} B_t e^{-\rho t} dt. \tag{3.13}$$

Let us define the function

$$\nu(p) = \min\{\nu > 0 : \int_0^\nu B_t e^{-\rho t} dt \geq p^{-1}\}, \quad p > 0 \tag{3.14}$$

(we put $\nu(p) = \infty$ if min in (3.14) is not attained).

Then (3.13) implies that $\nu_\tau = \nu(\pi_\tau / I_\tau)$ for $\omega \in \Omega_\tau$. It is easy to see that $\Omega_\tau = \{\nu_\tau < \infty\} = \{\nu(\pi_\tau/I_\tau) < \infty\}$.

If $\nu_\tau = \infty$ (i.e. $\omega \notin \Omega_\tau$), then

$$V_\tau = \mathbf{E}_\tau \int_0^\infty \pi^\tau_{\tau+t} e^{-\rho t} dt = \pi_\tau B. \tag{3.15}$$

Combining (3.11) and (3.15) we can write the following formula for the present value of the created firm:

$$V_\tau = \begin{cases} \gamma I_\tau (1 + A(\nu(\pi_\tau/I_\tau))) + (1-\gamma)\pi_\tau B, & \text{if } \nu(\pi_\tau/I_\tau) < \infty \\ \pi_\tau B, & \text{if } \nu(\pi_\tau/I_\tau) = \infty, \end{cases} \tag{3.16}$$

where the function $\nu(\cdot)$ is defined in (3.14).

3.3 Optimal investment timing

As it was pointed out at previous section the problem faced by the investor (2.5) can be considered as an optimal stopping problem:

$$\mathbf{E}(V_\tau - I_\tau)e^{-\rho\tau} \to \max_{\tau\in\mathcal{M}}, \tag{3.17}$$

where $\mathcal{M}$ is the class of all Markov times with values in $R_+ \cup \{\infty\}$.

Let us define the following function: for $p \geq 0$

$$\widehat{A}(p) = \begin{cases} A(\nu(p)), & \text{if } \nu(p) < \infty, \\ 0, & \text{if } \nu(p) = \infty, \end{cases}$$

where $\nu(p)$ is specified in (3.14), and put

$$g(\pi, I) = (1-\gamma)(\pi B - I) + \gamma I \widehat{A}(\pi/I). \tag{3.18}$$

Obviously, $g(\pi, I)$ is a homogeneous, i.e. $g(\lambda\pi, \lambda I) = \lambda g(\pi, I)$ for all $\lambda \geq 0$, but non-linear, function. It follows from (3.16) that $V_\tau - I_\tau \leq g(\pi_\tau, I_\tau)$. More precisely, $V_\tau - I_\tau = g(\pi_\tau, I_\tau)$ if $\nu(\pi_\tau/I_\tau) < \infty$, and $V_\tau - I_\tau < g(\pi_\tau, I_\tau)$ if $\nu(\pi_\tau/I_\tau) = \infty$.

Consider the optimal stopping problem for the bivariate process (π_τ, I_τ):

$$\mathbf{E}g(\pi_\tau, I_\tau)e^{-\rho\tau} \to \max_{\tau\in\mathcal{M}}. \tag{3.19}$$

A relation between the solutions to the problems (3.17) and (3.19) is described by the lemma below.

Lemma 3.2. *Let τ^* be a finite (a.s.) stopping time solving the problem* (3.19)*. If $\nu(\pi_{\tau^*}/I_{\tau^*}) < \infty$ (a.s.), then τ^* is the optimal investment time for the investor problem* (3.17)*.*

Proof. Obviously,

$$\max_\tau \mathbf{E}(V_\tau - I_\tau)e^{-\rho\tau} \leq \max_\tau \mathbf{E}g(\pi_\tau, I_\tau)e^{-\rho\tau} = \mathbf{E}g(\pi_{\tau^*}, I_{\tau^*})e^{-\rho\tau^*}.$$

On the other hand, since $\nu(\pi_{\tau^*}/I_{\tau^*}) < \infty$ a.s., then

$$\max_\tau \mathbf{E}(V_\tau - I_\tau)e^{-\rho\tau} \geq \mathbf{E}(V_{\tau^*} - I_{\tau^*})e^{-\rho\tau^*} = \mathbf{E}g(\pi_{\tau^*}, I_{\tau^*})e^{-\rho\tau^*}.$$

Therefore,

$$\max_\tau \mathbf{E}(V_\tau - I_\tau)e^{-\rho\tau} = \mathbf{E}g(\pi_{\tau^*}, I_{\tau^*})e^{-\rho\tau^*} = \mathbf{E}(V_{\tau^*} - I_{\tau^*})e^{-\rho\tau^*},$$

i.e. τ^* is an optimal stopping time for the problem (3.17). □

So, the investment timing problem is reduced to an optimal stopping problem for bivariate geometric Brownian motion and homogeneous reward function. Similar problem was considered by Gerber and Shiu [7] in the framework of perpetual American options on two assets. Their study was focused on the derivation of optimal continuation regions by the smooth pasting method, but they did not state precise conditions for the validity of their results.

We set below the formula for optimal stopping time for such a problem, and prove rigorously that under some additional conditions it gives indeed an optimal solution to the investment timing problem.

Let β be a positive root of the quadratic equation

$$\tfrac{1}{2}\tilde{\sigma}^2\beta(\beta-1)+(\alpha_1-\alpha_2)\beta-(\rho-\alpha_2)=0, \tag{3.20}$$

where $\tilde{\sigma}^2=(\sigma_1-\sigma_{21})^2+\sigma_{22}^2>0$ (since $\sigma_{22}>0$) is the "total" volatility of investment project. It is easy to see that $\beta>1$ whenever $\rho>\max(\alpha_1,\alpha_2)$.

Let us denote $f(p)=g(p,1)$, where function g is defined in (3.18), and

$$h(p)=f(p)p^{-\beta}, \quad p>0. \tag{3.21}$$

As one can see, $h(p)<0$ if $p<B^{-1}$ (and $\nu(p)=\infty$), $h(p)>0$ if $p>B^{-1}$, and $h(p)\to 0$ when $p\to\infty$. Since g is continuous function, $h(p)$ attains maximum at some point $p^*>B^{-1}$. Remind that p^* is called a strict maximum point for the function $h(p)$ if $h(p^*)>h(p)$ for any $p\neq p^*$.

The next theorem characterizes completely the optimal investment time.

Theorem 3.1. *Let the processes of profits and required investments be described by relations* (3.6)–(3.9)*. Assume that* $\rho>\max(\alpha_1,\alpha_2)$ *and the following condition is satisfied:*

$$\alpha_1-\frac{1}{2}\sigma_1^2\geq\alpha_2-\frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2). \tag{3.22}$$

Let $a_t, B_t\in C^1(R_+)$, p^* *be the strict maximum point for the function* $h(p)$, *and*

$$f'(p)p^{-\beta+1} \quad \text{decrease for} \quad p>p^*. \tag{3.23}$$

Then the optimal investment time for the problem (3.17) *is*

$$\tau^*=\min\{t\geq 0:\ \pi_t\geq p^*I_t\}.$$

The proof of this theorem one can find in the next section.

4 The proof

As we have seen above the investor's problem (3.17) is reduced to the optimal stopping problem (3.19) for bivariate process (π_t, I_t) specified by formulas (3.7) and (3.9).

For proving the Theorem 3.3 we will use the variational approach to optimal stopping problems for multi-dimensional diffusion processes described in [2], [3]. Besides the formal proof we demonstrate also an approach to obtain a formula for the optimal stopping time different from the smooth pasting method.

It is convenient to introduce the "homogeneous" notations $X_t^1 = \pi_t,\ X_t^2 = I_t$. The process $X_t = (X_t^1, X_t^2)$, is a bivariate geometric Brownian motion with correlated components:

$$\begin{aligned} dX_t^1 &= X_t^1(\alpha_1 dt + \sigma_1 dw_t^1), \\ dX_t^2 &= X_t^2[\alpha_2 dt + (\sigma_{21}\, dw_t^1 + \sigma_{22}\, dw_t^2)], \end{aligned} \tag{4.24}$$

and initial state $(X_0^1, X_0^2) = (x_1, x_2)$.

Let us consider a family of regions in $R_{++}^2 = \{(x_1, x_2) : x_1, x_2 > 0\}$ of the following type

$$G_p = \{(x_1, x_2) \in R_{++}^2 :\ x_1 < px_2\}, \quad p > 0.$$

For the process $X_t = (X_t^1, X_t^2)$, described by the system (4.24) with initial state $x = (x_1, x_2) \in R_{++}^2$, we denote $\tau_p(x)$ the exit time from the region G_p:

$$\tau_p(x) = \min\{t \ge 0 :\ X_t \notin G_p\} = \min\{t \ge 0 :\ X_t^1 \ge pX_t^2\}.$$

For $x \in R_{++}^2$ and homogeneous function $g(x)$ (see (3.18)) define

$$F_p(x) = \mathbf{E}^x e^{-\rho\tau_p(x)} g(X_{\tau_p(x)})$$

(here and below the upper index at the expectation $\mathbf{E}^x$ denotes the initial state x of the process X_t).

If $x \notin G_p$, then $\tau_p(x) = 0$ and, hence, $F_p(x) = g(x)$ for $x \in R_{++}^2 \backslash G_p$. If $x \in G_p$, then $\tau_p(x) > 0$ a.s. due to continuity of diffusion process.

Lemma 4.3. *If* (3.22) *holds, then* $\tau_p(x) < \infty$ *a.s. for any* $x \in R_{++}^2$ *and* $p > 0$.

Proof. It follows the explicit formulas for X_t^1 and X_t^2 that

$$\begin{aligned} \frac{X_t^1}{X_t^2} &= \frac{x_1}{x_2} \exp\left\{ \left(\alpha_1 - \alpha_2 + \frac{\sigma_{21}^2 + \sigma_{22}^2 - \sigma_1^2}{2} \right) t + (\sigma_1 - \sigma_{21}) w_t^1 - \sigma_{22} w^2 \right\} \\ &= \frac{x_1}{x_2} \exp\left\{ \left(\alpha_1 - \alpha_2 + \frac{\sigma_{21}^2 + \sigma_{22}^2 - \sigma_1^2}{2} \right) t + \tilde{\sigma}\tilde{w}_t \right\}, \end{aligned} \tag{4.25}$$

where $\tilde{w}_t = \dfrac{\sigma_1 - \sigma_{21}}{\tilde{\sigma}} w_t^1 - \dfrac{\sigma_{22}}{\tilde{\sigma}} w^2$ is a standard Wiener process. According to the law of iterated logarithm for Wiener process

$$\limsup_{t\to\infty} |\tilde{w}_t| / \sqrt{2t \log\log t} = 1 \quad \text{a.s.}$$

and (4.25) implies $\limsup_{t\to\infty} X_t^1/X_t^2 = \infty$ a.s. if $\alpha_1-\alpha_2+\frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2-\sigma_1^2) \geq 0$ (condition (3.22)). Therefore, $\tau_p(x) = \min\{t \geq 0: \ X_t^1/X_t^2 \geq p\} < \infty$ a.s. for any $x \in R_{++}^2$ and $p > 0$. □

Now we can derive the functional $F_p(x)$.

Lemma 4.4. *If* $\rho > \max(\alpha_1, \alpha_2)$ *and* (3.22) *holds, then*

$$F_p(x_1, x_2) = \begin{cases} h(p)x_1^{\beta}x_2^{1-\beta}, & \text{if } x_1 < px_2 \\ g(x_1, x_2), & \text{if } x_1 \geq px_2 \end{cases},$$

where $h(p)$ *is defined in* (3.21).

Proof. At first, show that $F_p(x)$ is a homogeneous (of degree 1) function.

Since $\tau_p(x)$ is the first exit time over the level p for the process X_t^1/X_t^2, formula (4.25) implies that the function $\tau_p(x)$ is homogeneous of degree 0 in $x = (x_1, x_2)$, i.e. $\tau_p(\lambda x) = \tau_p(x)$ for all $\lambda > 0$. The homogeneity properties of the process X_t (in initial state) and the function g imply:

$$\begin{aligned} F_p(\lambda x) &= \mathbf{E}^{\lambda x} e^{-\rho\tau_p(\lambda x)} g(X_{\tau_p(\lambda x)}) = \mathbf{E}^{\lambda x} e^{-\rho\tau_p(x)} g(X_{\tau_p(x)}) \\ &= \mathbf{E}^{x} e^{-\rho\tau_p(x)} g(\lambda X_{\tau_p(x)}) = \lambda F_p(x). \end{aligned}$$

It is known that $F_p(x)$ is the solution of Dirichlet boundary problem:

$$\mathcal{L}F(x) = \rho F(x), \quad x \in G_p, \tag{4.26}$$

$$F(x) \to g(a), \quad \text{when } x \to a, \ x \in G_p, \ a \in \partial G_p, \tag{4.27}$$

where $\mathcal{L}$ is the generator of the process X_t (variants of a more general statement usually referred to as the Feynman–Kac formula one can find in [9], [10], [13]).

As one can see, the generator of the process (4.24) is

$$\begin{aligned} \mathcal{L}F(x_1, x_2) = \alpha_1 x_1 \frac{\partial F}{\partial x_1} + \alpha_2 x_2 \frac{\partial F}{\partial x_2} + \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 F}{\partial x_1^2} + \sigma_1\sigma_{21} x_1 x_2 \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ + \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2) x_2^2 \frac{\partial^2 F}{\partial x_2^2}. \end{aligned} \tag{4.28}$$

The homogeneous function $F_p(x)$ can be represented as $F_p(x_1, x_2) = x_2 Q(y)$ where $y = x_1/x_2, \ Q(y) = F_p(y, 1)$. This and formula (4.28) for the elliptic operator $\mathcal{L}$ transforms PDE (4.26) to the ordinary differential equation

$$\frac{1}{2} y^2 Q''(y)\tilde{\sigma}^2 + yQ'(y)(\alpha_1 - \alpha_2) - Q(y)(\rho - \alpha_2) = 0. \tag{4.29}$$

The general solution of equation (4.29) for $0 < y < p$ is of the form $Q(y) = C_1 y^{\beta_1} + C_2 y^{\beta_2}$, where $\beta_1 > 0, \ \beta_2 < 0$ are the roots of quadratic equation (3.22). Returning to initial function we have

$$F_p(x_1,x_2)=C_1x_1^{\beta_1}x_2^{1-\beta_1}+C_2x_1^{\beta_2}x_2^{1-\beta_2}, \quad 0<x_1<px_2. \tag{4.30}$$

Since the homogeneous function g, defined in (3.18), is bounded by some linear function, i.e. $g(x_1,x_2)\le C(x_1+x_2)$, were $C=\max\limits_{0\le y\le 1} g(y,1-y)$,

$$F_p(x_1,x_2)\le C\max_\tau \mathbf{E}(X_\tau^1+X_\tau^2)e^{-\rho\tau}$$

where max is taken over all Markov times τ. Standard martingale arguments and the condition $\rho>\max(\alpha_1,\alpha_2)$ imply that

$$\mathbf{E}X_\tau^1e^{-\rho\tau}=x_1\mathbf{E}e^{-(\rho-\alpha_1)\tau}e^{\sigma_1w_\tau^1-\sigma_1^2\tau/2}\le x_1\mathbf{E}e^{\sigma_1w_\tau^1-\sigma_1^2\tau/2}=x_1.$$

Similarly, $\mathbf{E}X_\tau^2e^{-\rho\tau}\le x_2$. Therefore, $F_p(x_1,x_2)$ is also bounded by the linear function $C(x_1+x_2)$.

This fact implies that $C_2=0$ in representation (4.30) (otherwise $F_p(x_1,x_2)$ would be unbounded when $x_1\to 0,\ x_1<px_2$). The constant C_1 can be found from the boundary condition (4.27) at the line $\{x_1=px_2\}$, namely, $F_p(px_2,x_2)=C_1x_2p^{\beta_1}=g(px_2,x_2)=x_2f(p)$, i.e. $C_1=f(p)p^{-\beta_1}=h(p)$, see (3.21). □

Let $\mathcal{M}_1(x)=\{\tau_p(x),\ p>0\}\subset\mathcal{M}$ be the class of first exit times from the sets G_p for the process X_t (starting from the state $x=(x_1,x_2)$). Consider the restriction of the optimal stopping problem (3.19) to the class $\mathcal{M}_1(x)$:

$$\mathbf{E}^x g(X_\tau)e^{-\rho\tau}\to\max_{\tau\in\mathcal{M}_1(x)}. \tag{4.31}$$

Obviously, this problem is equivalent to the following extremal problem

$$F_p(x_1,x_2)\to\max_{p>0}. \tag{4.32}$$

The explicit form of the functional F_p from Lemma 4.2 allows us to find the solution to the problem (4.32) and, therefore, the solution to the optimal stopping problem (4.31).

Lemma 4.5. *Let the conditions of Lemma 4.2 hold, p^* be a strict maximum point of the function $h(p)$ (defined in* (3.21)*), and $h(p)$ decrease for $p>p^*$. Then the following statements hold:*

1) $\tau^*=\min\{t\ge 0:\ X_t^1\ge p^*X_t^2\}$ is the optimal stopping time for the problem (4.31) *for all $x\in R_{++}^2$;*

2) If, in addition, $\tau_{\widehat{p}}(x)>0$ a.s. for some $x\in R_{++}^2$, $\widehat{p}>0$, and $h(p)$ strictly decreases for $p>p^*$, then $\tau_{\widehat{p}}(x)$ is the optimal stopping time for the problem (4.31) *if and only if $\widehat{p}=p^*$;*

3) The optimal value of the functional in the problem (4.31) *is*

$$\Phi(x_1,x_2)=\begin{cases} h(p^*)x_1^\beta x_2^{1-\beta}, & \text{if } x_1<p^*x_2 \\ g(x_1,x_2), & \text{if } x_1\ge p^*x_2\end{cases}. \tag{4.33}$$

Proof. 1) Let us check that $F_p(x) \le F_{p^*}(x)$ for all $p > 0$ and $x \in R^2_{++}$.

By the definition of p^* we have for the homogeneous function g:

$$g(x) = x_2 f(x_1/x_2) = h(x_1/x_2) x_1^\beta x_2^{1-\beta} \le h(p^*) x_1^\beta x_2^{1-\beta}.$$

Let $p<p^*$. Then Lemma 4.2 gives: if $x_1 \ge p^* x_2$ then $F_p(x)=g(x)=F_{p^*}(x)$; if $px_2 \le x_1 < p^* x_2$ then $F_p(x)=g(x) \le h(p^*) x_1^\beta x_2^{1-\beta} = F_{p^*}(x)$; and if $x_1 < px_2$ then

$$F_p(x) = h(p) x_1^\beta x_2^{1-\beta} < h(p^*) x_1^\beta x_2^{1-\beta} = F_{p^*}(x). \tag{4.34}$$

For $p > p^*$ we have: if $x_1 \ge px_2$ then $F_p(x)=g(x)=F_{p^*}(x)$; if $p^* x_2 \le x_1 < px_2$ then $F_p(x)=h(p)x_1^\beta x_2^{1-\beta} \le h(x_1/x_2) x_1^\beta x_2^{1-\beta} = g(x) = F_{p^*}(x)$ due to monotonicity of $h(p)$ for $p > p^*$; and if $x_1 < p^* x_2$ then

$$F_p(x) = h(p) x_1^\beta x_2^{1-\beta} < h(p^*) x_1^\beta x_2^{1-\beta} = F_{p^*}(x). \tag{4.35}$$

Thus, $F_p(x) \le F_{p^*}(x)$ for all $x \in R^2_{++}$ and $p > 0$. Hence, maximum at the problem (4.32) is attained at $p = p^*$. From this and the definition of class $\mathcal{M}_1(x)$ follows statement 1).

2) Since $\tau_{\widehat{p}}(x) > 0$ a.s., $x_1 < \widehat{p} x_2$. Let us show that the optimality of $\tau_{\widehat{p}}(x)$ implies that $\widehat{p} = p^*$.

Assume that $\widehat{p} < p^*$. Then we have inequality (4.34) with $p = \widehat{p}$, that contradicts to the optimality of $\tau_{\widehat{p}}(x)$. Assume now that $\widehat{p} > p^*$. For $x_1 < p^* x_2$ we have (4.35) with $p = \widehat{p}$, i.e. the contradiction with the optimality. And if $p^* x_2 \le x_1 < \widehat{p} x_2$, then $F_{\widehat{p}}(x) = h(\widehat{p}) x_1^\beta x_2^{1-\beta} < h(x_1/x_2) x_1^\beta x_2^{1-\beta} = g(x) = F_{p^*}(x)$ due to strict decreasing of $h(p)$ for $p > p^*$. So, $\widehat{p} = p^*$ that proves (together with the optimality of p^*) statement 2) of the lemma.

Statement 3) follows directly from Lemma 4 for $p = p^*$. □

Let us emphasize that the region of optimal stopping

$$G_{p^*} = \{(x_1, x_2) \in R^2_{++} : x_1 \ge p^* x_2\}$$

does not depend on the initial state of the process X_t.

Proof of Theorem 3.3. In order to prove that the stopping time τ^*, defined in Lemma 4.3, will be optimal for the initial problem

$$\mathbf{E}^x g(X_\tau) e^{-\rho\tau} \to \max_{\tau \in \mathcal{M}} \tag{4.36}$$

(over all Markov times $\mathcal{M}$) we use the following "verification theorem", based on variational inequalities method (see, e.g. [4], [13]). Below we formulated it for our case.

Theorem 4.2 (Øksendal [13], Hu, Øksendal [8]). *Suppose, there exists a function $\Phi :\ R^2_{++} \to R$, satisfying the following conditions:*
1) $\Phi \in C^1(R^2_{++}) \cap C^2(R^2_{++} \setminus \partial G)$ where $G = \{x \in R^2_{++} :\ \Phi(x) > g(x)\}$;

2) ∂G *is locally the graph of Lipschitz function and* $\mathbf{E}^x \int_0^\infty \chi_{\partial G}(X_t)\, dt = 0$ *for all* $x \in R^2_{++}$*;*
3) $\Phi(x) \geq g(x)$ *for all* $x \in R^2_{++}$*;*
4) $\mathcal{L}\Phi(x) = \rho\Phi(x)$ *for all* $x \in G$*;*
5) $\mathcal{L}\Phi(x) \leq \rho\Phi(x)$ *for all* $x \in R^2_{++} \setminus \bar{G}$ *(*$\bar{G}$ *is a closure of the set* G*);*
6) $\bar{\tau} = \inf\{t \geq 0 : \ X_t \notin G\} < \infty$ *a.s. for all* $x \in R^2_{++}$*;*
7) the family $\{g(X_\tau)e^{-\rho\tau}, \ \mathcal{M} \ni \tau \leq \bar{\tau}\}$ *is uniformly integrable for all* $x \in G$*.*

Then $\bar{\tau}$ *is the optimal stopping time for the problem* (4.36)*, and* $\Phi(x)$ *is the correspondent optimal value of the functional in* (4.36)*.*

As a candidate we try the function $\Phi(x_1, x_2)$, defined in (4.33). It is easy to see that $\Phi{\in}C^1(R^2_{++})$ due to first-order condition for the maximum point p^*: $\quad \beta h(p^*)(p^*)^{\beta-1} = f'(p^*)$.

For $x = (x_1, x_2) \in R^2_{++}$ let us denote $p(x) = x_1/x_2$.

Since $h(p^*){>}h(p)$ for all $p \neq p^*$, then on the set $\{(x_1, x_2){\in}R^2_{++} : x_1{<}p^* x_2\}$ we have

$$\begin{aligned}\Phi(x_1, x_2) &= h(p^*)x_1^\beta x_2^{1-\beta} > h(p(x))x_2\,(x_1/x_2)^\beta \\ &= x_2 f\,(x_1/x_2)\,(x_1/x_2)^{-\beta}\,(x_1/x_2)^\beta = g(x_1, x_2)\end{aligned}$$

(the latter equality follows from the homogeneity of the function g).

Therefore, $\Phi(x) \geq g(x)$ for all $x \in R^2_{++}$, and the domain $G = \{x \in R^2_{++} : \Phi(x) > g(x)\}$ coincides with $\{x_1 < p^* x_2\} = \{(x_1, x_2) : \ 0 \leq p(x) < p^*\}$. So, $\partial G = \{(x_1, x_2) : \ x_1 = p^* x_2\}$.

The property $\Phi{\in}C^2(R^2_{++}{\setminus}\partial G)$ follows from the twice differentiability of $g(x_1, x_2)$ on the set $\{(x_1, x_2){\in}R^2_{++} : Bx_1{>}x_2\}$, due to the conditions $a_t, B_t{\in}C^1(R_+)$.

Condition 2) of Theorem 4.4 follows from local properties of geometric Brownian motion. Condition 4) follows immediately from the construction of the function $\Phi = F_{p^*}$ (see (4.26) in the proof of Lemma 4.2).

Furthermore, $\bar{\tau} = \inf\{t \geq 0 : \ X_t \notin G\} = \inf\{t \geq 0 : \ X^1_t \geq p^* X^2_t\} < \infty$ a.s. for all $x \in R^2_{++}$ due to Lemma 4.1, i.e. 6) holds.

Let us show that condition 7) of Theorem 4.4 holds if $\rho > \alpha_2$. Indeed, if $\tau \leq \bar{\tau}$ then $X^1_\tau \leq p^* X^2_\tau$ and, therefore,

$$\Phi(X_\tau)e^{-\rho\tau}{=}h(p^*)X^2_\tau \left(\frac{X^1_\tau}{X^2_\tau}\right)^\beta e^{-\rho\tau}{\leq}h(p^*)(p^*)^\beta X^2_\tau e^{-\rho\tau}{=}CX^2_\tau e^{-\rho\tau},$$

where $C = h(p^*)(p^*)^\beta$.

Let us denote $\sigma_2^2 = \sigma_{21}^2 + \sigma_{22}^2$. Then $\bar{w}_t = (\sigma_{21}^2 w^1_t + \sigma_{22}^2 w^2_t)/\sigma_2$ is the standard Wiener process. Hence, from the explicit formula for geometric Brownian motion using martingale arguments we have:

$$\begin{aligned}\mathbf{E}^x[\Phi(X_\tau)e^{-\rho\tau}]^k &\le C^k x_2^k \mathbf{E}^x \exp\{[-\rho\tau + (\alpha_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2\bar{w}_\tau]k\} \\ &= C^k x_2^k \mathbf{E}^x \exp\{-[\rho - \alpha_2 - \frac{1}{2}\sigma_2^2(k-1)]k\tau + k\sigma_2\bar{w}_\tau - \frac{1}{2}k^2\sigma_2^2\tau\} \\ &\le C^k x_2^k \mathbf{E}^x \exp\{k\sigma_2\bar{w}_\tau - \frac{1}{2}k^2\sigma_2^2\tau\} = C^k x_2^k,\end{aligned}$$

if $k > 1$ is chosen such that $\rho - \alpha_2 - \frac{1}{2}\sigma_2^2(k-1) \ge 0$. Thus, the uniform integrability of the family $\{g(X_\tau)e^{-\rho\tau},\ \tau \le \bar{\tau}\}$ holds (since $g(x) \le \Phi(x)$) .

It is remained to check the condition 5) of Theorem 4.4. Let us take $x = (x_1, x_2) \notin \bar{G}$, i.e. $x_1 > p^* x_2$. For this case $p(x) > p^*$ and $\Phi(x_1, x_2) = g(x_1, x_2) = x_2 f(p(x))$. Repeating arguments, similar to those in the proof of Lemma 4.2, we have:

$$\begin{aligned}\mathcal{L}g(x) - \rho g(x) = x_2 \Big[&\frac{1}{2}p^2(x)f''(p(x))\tilde{\sigma}^2 + p(x)f'(p(x))(\alpha_1 - \alpha_2) \\ &- f(p(x))(\rho - \alpha_2)\Big].\end{aligned}$$

The condition (3.23) is equivalent to the inequality $pf''(p) \le (\beta - 1)f'(p)$ for $p > p^*$. Integrating both sides of the latter relation from p^* to p one can obtain that $pf'(p) \le p^* f'(p^*) - \beta f(p^*) + \beta f(p) = \beta f(p)$, since $h'(p^*) = 0$. These inequalities imply:

$$\begin{aligned}\frac{\mathcal{L}g(x) - \rho g(x)}{x_2} &= \frac{1}{2}p^2 f''(p)\tilde{\sigma}^2 + pf'(p)(\alpha_1 - \alpha_2) - f(p)(\rho - \alpha_2) \\ &\le \frac{1}{2}p^2 f''(p)\tilde{\sigma}^2 + pf'(p)\left[\alpha_1 - \alpha_2 - \frac{1}{\beta}(\rho - \alpha_2)\right] \\ &= \frac{1}{2}p^2 f''(p)\tilde{\sigma}^2 - pf'(p)\frac{1}{2}\tilde{\sigma}^2(\beta - 1) \le 0, \quad \text{where} \quad p = p(x)\end{aligned}$$

(here we use the fact that β is a root of equation (3.22)). Thus, all the conditions of Theorem 4.4 hold and, therefore, $\bar{\tau} = \inf\{t \ge 0 :\ X_t^1 \ge p^* X_t^2\} = \tau^*$ is the finite (a.s.) optimal stopping time for the problem (4.34).

As it is shown before the formulation of Theorem 3.3, $p^* > 1/B$. Hence $\nu(p^*) = \nu(X_{\tau^*}^1 / X_{\tau^*}^2) < \infty$, and, due to Lemma 3.2, τ^* is the optimal stopping time for the investor's problem (3.17). □

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