Multiparameter Generalizations of the Dalang–Morton–Willinger Theorem

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Summary. We investigate possible generalizations of Dalang–Morton–Willinger theorem in the context of Cairoli– Walsh theory of random fields on the discrete rectangle.

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1 Introduction.

The classical Dalang–Morton–Willinger theorem [2] says that in the standard discrete time finite-horizon model of a frictionless financial market there are no arbitrage opportunities if and only if there exists an equivalent martingale measure with bounded density. In the probabilistic language this theorem can be formulated as follows.

We are given an $\mathbf{R}^{d+1}$-valued adapted process

$$\bar{S} = (S_t^0, S_t) = (S_t^0, S_t^1, ..., S_t^d)$$

where $t = 0, 1, ..., T$.

With any $\mathbf{R}^{d+1}$-valued adapted process $\bar{\varphi} = (\varphi_t^0, \varphi_t)$ with $\bar{\varphi}_0 = 0$ we associate the scalar process $V_t = \bar{\varphi}_t \bar{S}_t = \varphi_t^0 S_t^0 + \varphi_t S_t$. In financial modelling $\bar{S}$ is the price process, $\bar{\varphi}$ is the strategy, representing holdings in various assets (in nominal units), and V is the corresponding value process of the portfolio.

For a specified class K of strategies we define the set of random variables $R_T^K := \{\bar{\varphi}_T \bar{S}_T : \ \bar{\varphi} \in K\}$. We shall say that the $NA(K)$-*property* holds if $R_T^K \cap L_+^0 = \{0\}$.

In the standard model $S_t^0 = 1$ identically, i.e. the corresponding asset (usually called bank account) is the *numéraire*, and K is the class of self-financing strategies described as follows: the process $\bar\varphi$ is predictable (in symbols: $\bar\varphi \in \mathcal{P}$) and

$$\Delta\varphi_t^0 + S_{t-1}\Delta\varphi_t = 0, \qquad t = 1, ..., T, \tag{1}$$

with the usual definition $\Delta X_t = X_t - X_{t-1}$. The above relation can be written also as $\bar S_{t-1}\Delta\bar\varphi_t = 0$. Thus, by the product formula, for the strategies from this class we have

$$\Delta(\bar S_t \bar\varphi_t) = \bar S_{t-1}\Delta\bar\varphi_t + \bar\varphi_t \Delta\bar S_t = \varphi_t \Delta S_t$$

and, therefore, $R_T^K = R_T := \{\varphi \cdot S_T : \ \varphi \in \mathcal{P}\}$, i.e. the set of the resulting random variables is just the set of discrete time integrals $\varphi \cdot S_T := \sum_{t=1}^T \varphi_t \Delta S_t$ where φ is an arbitrary d-dimensional predictable process without any constraints. With this $A_T := R_T - L_+^0$ is the set of hedgeable claims. We consider also the subset $R_T(t)$ of R_T corresponding to strategies which are zero except the date t, that is $R_T(t) = \{\varphi_t \Delta S_t : \ \varphi_t \in \mathcal{F}_{t-1}\}$. The notation $A_T(t)$ is clear.

The condition $R_T \cap L_+^0 = 0$ (obviously equivalent to $A_T \cap L_+^0 = 0$) is referred to as the *NA-property*.

The introduced concepts serve to model the situation when an agent revise the portfolio between the trading days $t-1$ and t using the information available (φ_t is $\mathcal{F}_{t-1}$-measurable) without retracting or adding funds (the relation (1) is a "fund conservation law"); in this case, R_T^K is the set of all possible "results" achieved from zero initial endowment and absence of non-risky profits corresponds to the absence of arbitrage opportunities on the market.

The extended formulation of the Dalang–Morton–Willinger theorem is a long list of equivalent conditions but we retain only four here:

(a) $A_T \cap L_+^0 = \{0\}$ *(NA);*

(b) $A_T \cap L_+^0 = \{0\}$ *and* $A_T = \bar A_T$ *(closure in probability);*

(c) $A_T(t) \cap L_+^0 = \{0\}$ *for all* $t \le T$ *(NA for all one-step models);*

(d) there is a probability measure $\tilde P \sim P$ *with* $d\tilde P/dP \in L^\infty$ *such that* S *is a* $\tilde P$*-martingale.*

The DMW theorem is widely recognized as one of the most important results in the arbitrage pricing theory and we have no need to discuss its various aspects. It is a (deep!) generalization of the pioneering Harrison–Pliska theorem which has exactly the same formulation but under hypothesis that Ω is finite. Of course, in the latter case the property (b) coincides with (a) (A_T is polyhedral cone) and (d) sounds simpler as all random variables are bounded.

These result are the starting points of intensive mathematical studies and their numerous generalizations and ramifications are known, see, e.g. the survey [6] with further references therein and more recent papers [3], [4], [5], [7],

[9], [10]. In the present note we make an attempt to explore relationships between possible versions of the above conditions in the setting of random fields. To our knowledge, the syntheses of both theories is not done yet.

A specific feature of random fields is that there are several rather natural definitions of the "past" and consequently, several definitions of the martingale property. We shall investigate analogs of NA criteria in the standard framework of Cairoli–Walsh, using an appropriate techniques which sometimes is quite different from that of one-parameter processes.

First, recall the basic definitions.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_{\mathbf{t}})_{\mathbf{t}\in\mathbb{T}}, P)$ be a stochastic basis where $\mathbb{T}$ stands for the rectangle $[0, \mathbf{T}] := \{0, 1, ..., T_1\} \times \{0, 1, ..., T_2\}$ of the integer lattice $\mathbb{Z}^2$; the notation $]0, \mathbf{T}] := \{1, ..., T_1\} \times \{1, ..., T_2\}$ also will be used. We shall suppose that the σ-algebras of the axes are trivial: $\mathcal{F}_{i0} = \mathcal{F}_{0k} = \{\emptyset, \Omega\}$.

Put $\mathbf{i} := (1, 0)$, $\mathbf{j} := (0, 1)$, and $\mathbf{1} := \mathbf{i} + \mathbf{j} = (1, 1)$.

Let $X = (X_{\mathbf{t}})_{\mathbf{t}\in\mathbb{T}}$ be a random field. We shall use the following notations:

$$\Delta^1 X_{\mathbf{t}} := X_{\mathbf{t}} - X_{\mathbf{t}-\mathbf{i}}, \quad \Delta^2 X_{\mathbf{t}} := X_{\mathbf{t}} - X_{\mathbf{t}-\mathbf{j}}, \quad \Delta X_{\mathbf{t}} = X_{\mathbf{t}} - X_{\mathbf{t}-\mathbf{i}} - X_{\mathbf{t}-\mathbf{j}} + X_{\mathbf{t}-\mathbf{1}}.$$

Also $X^{-\mathbf{i}} := (X_{\mathbf{t}-\mathbf{i}})$ and, in the same spirit, $X^{-\mathbf{j}}$, $X^{-\mathbf{1}}$.

Clearly, knowing the field X on the axes as well as the elementary "areas" $\Delta X_{\mathbf{t}}$, one can recover X on the whole rectangle $\mathbb{T}$.

Define the σ-algebras $\widehat{\mathcal{F}}_{\mathbf{t}} := \mathcal{F}_{\mathbf{t}+\mathbf{i}} \vee \mathcal{F}_{\mathbf{t}+\mathbf{j}}$ and also $\tilde{\mathcal{F}}^1_{\mathbf{t}} := \mathcal{F}_{t_1,T_2} \vee \mathcal{F}_{\mathbf{t}+\mathbf{i}}$, $\tilde{\mathcal{F}}^2_{\mathbf{t}} := \mathcal{F}_{\mathbf{t}+\mathbf{j}} \vee \mathcal{F}_{T_1,t_2}$ (the parentheses in subscripts are omitted).

Definition 1. *An integrable adapted field X constant on the coordinate axes is called:*

1) strong martingale if $E(\Delta X_{\mathbf{t}}|\widehat{\mathcal{F}}_{\mathbf{t}-\mathbf{1}}) = 0$*;*
2) weak martingale if $E(\Delta X_{\mathbf{t}}|\mathcal{F}_{\mathbf{t}-\mathbf{1}}) = 0$*;*
3_1) 1-martingale if $E(\Delta X_{\mathbf{t}}|\mathcal{F}_{\mathbf{t}-\mathbf{i}}) = 0$*;*
3_2) 2-martingale if $E(\Delta X_{\mathbf{t}}|\mathcal{F}_{\mathbf{t}-\mathbf{j}}) = 0$.

Definition 2. *The filtration $(\mathcal{F}_{\mathbf{t}})$ satisfies the Cairoli–Walsh condition (F_4 of [1]) if for any $\mathcal{F}$-measurable integrable random variable Z and for any* $\mathbf{t} = (t_1, t_2) \in \mathbb{T}$

$$E(E(Z|\mathcal{F}_{t_1,T_2})|\mathcal{F}_{T_1,t_2}) = E(E(Z|\mathcal{F}_{T_1,t_2})|\mathcal{F}_{t_1,T_2}) = E(Z|\mathcal{F}_{t_1,t_2}).$$

Definition 3. *We say that a random field H is:*

1) weakly predictable if $H_{\mathbf{t}+\mathbf{1}} \in \widehat{\mathcal{F}}_{\mathbf{t}}$, $\mathbf{t} + \mathbf{1} \in \mathbb{T}$*;*
2) predictable if $H_{\mathbf{t}+\mathbf{1}} \in \mathcal{F}_{\mathbf{t}}$, $\mathbf{t} + \mathbf{1} \in \mathbb{T}$.

Let X and Y be two random fields constant on the coordinate axes. We define two lattice integrals as

$$X \cdot Y_{\mathbf{t}} := \sum_{\mathbf{s}\in]0,\mathbf{t}]} X_{\mathbf{s}} \Delta Y_{\mathbf{s}}, \qquad X * Y_{\mathbf{t}} := \sum_{\mathbf{s}\in]0,\mathbf{t}]} [\Delta^2 X_{\mathbf{s}-\mathbf{i}} \Delta^1 Y_{\mathbf{s}} + \Delta^1 X_{\mathbf{s}-\mathbf{j}} \Delta^2 Y_{\mathbf{s}}]$$

with the convention that they are equal to zero when $\mathbf{t}$ belongs to the axes. It is easy to see that $\Delta(X \cdot Y)_{\mathbf{t}} = X_{\mathbf{t}} \Delta Y_{\mathbf{t}}$ and the following product formula holds:

$$X_{\mathbf{t}} Y_{\mathbf{t}} = X^{-\mathbf{1}} \cdot Y_{\mathbf{t}} + X * Y_{\mathbf{t}} + Y \cdot X_{\mathbf{t}}. \tag{2}$$

We fix an $\mathbf{R}^d$-valued adapted random field S which components on the coordinate axes are equal to the unit and put $\bar{S} := (1, S)$, i.e. we add to S one more component identically equal to the unit everywhere. With any $\mathbf{R}^{d+1}$-valued adapted random field $\bar{\varphi} = (\varphi^0, \varphi)$ we associate a scalar field

$$V_{\mathbf{t}} = \bar{\varphi}_{\mathbf{t}} \bar{S}_{\mathbf{t}} = \varphi^0_{\mathbf{t}} + \varphi_{\mathbf{t}} S_{\mathbf{t}}.$$

By analogy with the one-parameter case we shall call *strategy* the field $\bar{\varphi}$ vanishing on the axes and V its *value field*.

For a class K of strategies define the set of random variables

$$R^K_{\mathbf{T}} := \{\bar{\varphi}_{\mathbf{T}} \bar{S}_{\mathbf{T}} : \ \bar{\varphi} \in K\}.$$

We say that the $NA(K)$-*property* holds if $R^K_{\mathbf{T}} \cap L^0_+ = \{0\}$, or, equivalently, $A^K_{\mathbf{T}} \cap L^0_+ = \{0\}$ with $A^K_T = R^K_T - L^0_+$.

2 Strong martingale, weakly predictable strategies

We say that a weakly predictable strategy $\bar{\varphi}$ satisfies the strong SF-property if

$$\bar{S}_{\mathbf{t}-\mathbf{1}} \Delta \bar{\varphi}_{\mathbf{t}} + \Delta^2 \bar{S}_{\mathbf{t}-\mathbf{i}} \Delta^1 \bar{\varphi}_{\mathbf{t}} + \Delta^1 \bar{S}_{\mathbf{t}-\mathbf{j}} \Delta^2 \bar{\varphi}_{\mathbf{t}} = 0 \qquad \forall\, \mathbf{t}. \tag{1}$$

This relation plays the role of (1): in this case from the product formula (2) we have that $V_{\mathbf{t}} = \varphi \cdot S_{\mathbf{t}}$ for all $\mathbf{t} \in \mathbb{T}$.

In this section we fix as K the class of weakly predictable strategies satisfying the strong SF-property abbreviated as SSF.

It is easily seen that if φ is a weakly predictable d-dimensional field, then it is the component of a certain strategy $\bar{\varphi} = (\varphi^0, \varphi)$ from SSF. Indeed, suppose that $\bar{\varphi}$ is already known outside of the rectangle $[\mathbf{t}, \mathbf{T}]$. We use the self-financing condition (1) to define $\varphi^0_{\mathbf{t}} \in \widehat{\mathcal{F}}_{\mathbf{t}-\mathbf{1}}$ and get that

$$\varphi^0_{\mathbf{t}} = \varphi^0_{\mathbf{t}-\mathbf{i}} + \varphi^0_{\mathbf{t}-\mathbf{j}} - \varphi^0_{\mathbf{t}-\mathbf{1}} - S_{\mathbf{t}-\mathbf{1}} \Delta \varphi_{\mathbf{t}} - \Delta^2 S_{\mathbf{t}-\mathbf{i}} \Delta^1 \varphi_{\mathbf{t}} - \Delta^1 S_{\mathbf{t}-\mathbf{j}} \Delta^2 \varphi_{\mathbf{t}}.$$

Let us consider the point $\mathbf{t} + \mathbf{i}$. Since $\bar{\varphi}$ is already defined at the "preceding" points $\mathbf{t}$, $\mathbf{t}+\mathbf{i}-\mathbf{j}$, $\mathbf{t}+\mathbf{i}-\mathbf{1}$ and $\varphi_{\mathbf{t}+\mathbf{i}}$ is known, the relation (1) corresponding to the point $\mathbf{t} + \mathbf{i}$ serves as an equation to define the remaining component $\varphi^0_{\mathbf{t}+\mathbf{i}}$. These arguments can be repeated also for $\mathbf{t} + 2\mathbf{i}$, $\mathbf{t} + 3\mathbf{i}$, and so on, allowing us to define the SSF-strategy $\bar{\varphi}$ outside of the rectangle $[\mathbf{t} + \mathbf{j}, \mathbf{T}]$. By symmetry, we have the same recurrent structure along the y-axis. As a result, we obtain the weakly predictable strategy $\bar{\varphi}$ satisfying the strong SF-property on the whole rectangle $[0, \mathbf{T}]$.

Since the d-dimensional weakly predictable field φ can be chosen arbitrarily, we have the following

Proposition 1. *Assume that the $NA(SSF)$-property holds. Let $\alpha \in \widehat{\mathcal{F}}_{\mathbf{t-1}}$ and $\alpha \Delta S_{\mathbf{t}} \geq 0$. Then $\alpha \Delta S_{\mathbf{t}} = 0$.*

Remark 1. Note that this does not require any additional assumption on the filtration and the probability space. In particularly, we do not use the Cairoli–Walsh condition.

The next result is an analog of the Harrison–Pliska theorem and its proof is exactly the same as the latter.

Proposition 2. *Let Ω be finite. Then the following conditions are equivalent:*
(a) the $NA(SSF)$-property holds;
(b) there exists a probability measure $\tilde{P} \sim P$ such that S is a strong martingale with respect to $\tilde{P}$.

Proposition 1 asserts that the $NA(SSF)$-property implies the $NA(SSF)$-property for the increments (i.e., for all "one-step models"). Surprisingly, the inverse implication fails to be true. We present an example where the NA property does not hold though there is no-arbitrage for the increments, i.e. the situation is similar to the observed already in models with restricted information, [8].

Example. It is very simple: the field S is one-dimensional, $T_1 = T_2 = 2$, and the probability space consists only of five points. The filtration is natural. The values of the field are given by the following table:

	S_{11}	S_{12}	S_{21}	S_{22}
ω_1	5/6	1/2	5/3	4/3
ω_2	5/6	2/3	7/6	1
ω_3	5/6	4/3	1/2	1
ω_4	7/6	1	7/6	1
ω_5	7/6	4/3	7/6	4/3

Recall that S equals 1 on the axes. Note that the values of S^2_{22} are chosen to get the identity $\Delta S^2_{22} = 0$, that is $S^2_{22} = S^2_{12} + S^2_{21} - S^2_{11}$.

Let us show that the constant strategy $\bar{\varphi} = (-1, 1)$ (obviously, weakly predictable and strongly SF) is an arbitrage opportunity in our sense.

We have $V_{22} = \bar{\varphi}_{22}\bar{S}_{22} = \varphi_{22}S_{22} - 1$ and, hence,

$$V_{22}(\omega_1) = V_{22}(\omega_5) = \frac{1}{3}, \quad V_{22}(\omega_2) = V_{22}(\omega_3) = V_{22}(\omega_4) = 0.$$

It remains to verify that for each point $\mathbf{t} = (1, 2)$, $\mathbf{t} = (2, 1)$, and $\mathbf{t} = (2, 2)$ the relation $\alpha \Delta S_{\mathbf{t}} \geq 0$ with $\alpha \in \widehat{\mathcal{F}}_{\mathbf{t-1}}$ may hold only if $\alpha \Delta S_{\mathbf{t}} = 0$.

Note that $\widehat{\mathcal{F}}_{00} = \mathcal{F}_{00}$, $\widehat{\mathcal{F}}_{10} = \mathcal{F}_{11}$, $\widehat{\mathcal{F}}_{01} = \mathcal{F}_{11}$,
$\triangle S_{11} = S_{11} - S_{00}$, $\triangle S_{21} = S_{21} - S_{11}$, $\triangle S_{12} = S_{12} - S_{11}$.

We want to prove that for $\alpha \in \mathcal{F}_{00}$, $\beta \in \mathcal{F}_{11}$, $\gamma \in \mathcal{F}_{11}$ the inequalities

$$\alpha(S_{11} - S_{00}) \geq 0, \qquad \beta(S_{21} - S_{11}) \geq 0, \qquad \gamma(S_{12} - S_{11}) \geq 0,$$

may hold only as the equalities

$$\alpha(S_{11} - S_{00}) = 0, \qquad \beta(S_{21} - S_{11}) = 0, \qquad \gamma(S_{12} - S_{11}) = 0.$$

But this is obvious: on each atom the increments take values of different signs.

The next proposition is a technical one. It deals with the case of SSF-strategies measurable with respect to a wider σ-algebra.

Proposition 3. *Let K be the class of d-dimensional fields $\varphi = (\varphi_{\mathbf{t}})$ such that $\varphi_{\mathbf{t}} \in \tilde{\mathcal{F}}^1_{\mathbf{t-1}}$. Then the following conditions are equivalent:*

(i) $A^K_{\mathbf{T}} \cap L^0_+ = \{0\}$;
(ii) $A^K_{\mathbf{T}} \cap L^0_+ = \{0\}$, $A^K_{\mathbf{T}} = \bar{A}^K_{\mathbf{T}}$;
(iii) The relation $\alpha \Delta S_{\mathbf{t}} \geq 0$ for $\mathbf{t} \in \mathbb{T}$ and $\alpha \in \tilde{\mathcal{F}}^1_{\mathbf{t-1}}$ holds only if $\alpha \Delta S_{\mathbf{t}} = 0$;
(iv) There exists a probability measure $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that $\Delta S_{\mathbf{t}} \in L^1(\tilde{P})$ and $\tilde{E}(\Delta S_{\mathbf{t}} | \tilde{\mathcal{F}}^1_{\mathbf{t-1}}) = 0$ for all $\mathbf{t} \in \mathbb{T}$ (i.e. S is a strong martingale with respect to the filtration $(\tilde{\mathcal{F}}^1_{\mathbf{t}})$ and $\tilde{P}$).

This result is easily reduced to the DMW-theorem. To see this we define the bijection L of $]0, \mathbf{T}]$ onto the set $\{1, 2, ..., T_1 T_2\}$ by the formula

$$L : \mathbf{t} \mapsto (t_1 - 1)T_2 + t_2.$$

The one-parametric process $W_n := \sum_{k \leq n} \xi_k$ where $\xi_k = \Delta S_{L^{-1}k}$ is adapted with respect to the filtration formed by the σ-algebras $\mathcal{F}_n := \tilde{\mathcal{F}}^1_{L^{-1}n}$. The conditions of the above proposition are those of the DMW-theorem for W.

3 Weak martingales, predictable strategies

We say that a predictable strategy $\bar{\varphi}$ satisfies *the weak SF-property* if

$$\bar{S}_{\mathbf{t-1}} \Delta \bar{\varphi}_{\mathbf{t}} = 0 \qquad \forall \, \mathbf{t}. \tag{1}$$

In this case the value field is given by the formula

$$V_{\mathbf{t}} = \bar{\varphi} \cdot \bar{S}_{\mathbf{t}} + \bar{\varphi} * \bar{S}_{\mathbf{t}}.$$

For the no-arbitrage property in this case we shall use the notation NA(WSF). The latter implies the no-arbitrage property for he increments. Namely, we have

Proposition 1. *Assume that the NA(WSF)-property holds. Let $\alpha \in \mathcal{F}_{\mathbf{t-1}}$ be such that $\alpha \Delta S_{\mathbf{t}} \geq 0$. Then $\alpha \Delta S_{\mathbf{t}} = 0$.*

Proof. Suppose that the claim fails and there is $\alpha \in \mathcal{F}_{\mathbf{t-1}}$ such that the probability $P(\alpha \Delta S_{\mathbf{t}} > 0)$ is strictly positive. We come to a contradiction by constructing a predictable strategy $\bar{\varphi}$ satisfying (1) and such that the end value $V_{\mathbf{T}} = \bar{\varphi}_{\mathbf{T}} \bar{S}_{\mathbf{T}} = \alpha \Delta S_{\mathbf{t}}$. The φ-component of $\bar{\varphi}$ will be zero except the point $\mathbf{t}$ where it coincides with $-\alpha$. To this aim, we put $\bar{\varphi}$ equal to zero outside of $[\mathbf{t}, \mathbf{T}]$. We use the self-financing condition (1) to define $\varphi^0_{\mathbf{t}}$ and get that

$$\varphi^0_{\mathbf{t}} = \alpha S_{\mathbf{t-1}} \in \mathcal{F}_{\mathbf{t-1}}.$$

Let us consider the point $\mathbf{t+i}$. Since that $\bar{\varphi}$ is already defined at the points $\mathbf{t}$, $\mathbf{t+i-j}$, $\mathbf{t+i-1}$ and we have $\varphi_{\mathbf{t+i}} = 0$, the relation (1) corresponding to the point $\mathbf{t+i}$ takes the form:

$$\varphi^0_{\mathbf{t+i}} - \varphi^0_{\mathbf{t}} + S_{\mathbf{t-j}} \Delta \varphi_{\mathbf{t+i}} = 0$$

which suggests us to define

$$\varphi^0_{\mathbf{t+i}} = -\alpha (S_{\mathbf{t-j}} - S_{\mathbf{t-1}}) = -\alpha \Delta^1 S_{\mathbf{t-j}} \in \mathcal{F}_{\mathbf{t-j}}.$$

Similar observations for the point $\mathbf{t+j}$ lead us to define

$$\varphi^0_{\mathbf{t+j}} = -\alpha (S_{\mathbf{t-i}} - S_{\mathbf{t-1}}) = -\alpha \Delta^2 S_{\mathbf{t-i}} \in \mathcal{F}_{\mathbf{t-i}}.$$

Next we consider the condition (1) at the point $\mathbf{t+1}$. We get

$$\Delta \varphi^0_{\mathbf{t+1}} + S_{\mathbf{t}} \Delta \varphi_{\mathbf{t+1}} = 0,$$

or

$$\varphi^0_{\mathbf{t+1}} - \varphi^0_{\mathbf{t+j}} - \varphi^0_{\mathbf{t+1}} + \varphi^0_{\mathbf{t}} + S_{\mathbf{t}} \varphi_{\mathbf{t}} = 0,$$

With the already defined values of the strategy φ, we come to the following expression for $\varphi^0_{\mathbf{t+1}}$:

$$\varphi^0_{\mathbf{t+1}} = \alpha \Delta S_{\mathbf{t}} \in \mathcal{F}_{\mathbf{t}}.$$

Now with such a strategy φ we get at the point $\mathbf{t+1}$ the following expression for the value field

$$V_{\mathbf{t+1}} = \bar{\varphi}_{\mathbf{t+1}} \bar{S}_{\mathbf{t+1}} = \alpha \Delta S_{\mathbf{t}}.$$

It is left to finalize our construction by setting

$$\varphi^0_{\mathbf{t}+m\mathbf{i}} = \varphi^0_{\mathbf{t+i}}, \quad m = 2, \ldots, T_1 - t_1,$$

$$\varphi^0_{\mathbf{t}+m\mathbf{j}} = \varphi^0_{\mathbf{t+j}}, \quad m = 2, \ldots, T_2 - t_2,$$

and

$$\varphi^0_{\mathbf{t}+m\mathbf{i}+l\mathbf{j}} = \varphi^0_{\mathbf{t+1}}, \quad m = 2, \ldots, T_1 - t_1, \quad l = 2, \ldots, T_2 - t_2.$$

In such a way we obtain a predictable strategy satisfying WSF-property such that $V_{\mathbf{T}} = \bar{\varphi}_{\mathbf{T}} \bar{S}_{\mathbf{T}} = \alpha \Delta S_{\mathbf{t}}$. Since $\alpha \Delta S_{\mathbf{t}} \neq 0$ we obtain an arbitrage opportunity, that is the contradiction. □

Remark 2. The same example as in the previous section demonstrates that the inverse implication is not true.

Introduce the notations: $\mathbf{t}^1_T := (T_1, t_2)$, $\mathbf{t}^2_T := (t_1, T_2)$, and $Z := d\tilde{P}/dP$.

Proposition 2. *(a) Suppose that there is a measure $\tilde{P} \sim P$ with $Z \in L^\infty$ such that S is a weak $\tilde{P}$-martingale and the Cairoli–Walsh commutation condition is fulfilled for $\tilde{P}$. Then the inequality*

$$\sum_{\mathbf{t}\in[0,\mathbf{T-1}]} \alpha_{\mathbf{t}} E(\Delta S_{\mathbf{t+1}}\xi_t|\mathcal{F}_{\mathbf{t+i}}) \geq 0$$

with $\alpha_{\mathbf{t}} \in \mathcal{F}_{\mathbf{t}^2_T}$ and $\xi_{\mathbf{t}} = Z/E(Z|\mathcal{F}_{\mathbf{t+i}})$ may hold only as the equality.

(b) Suppose that the inequality

$$\sum_{\mathbf{t}\in[0,\mathbf{T-1}]} \alpha_{\mathbf{t}} E(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t+i}}) \geq 0$$

with $\alpha_{\mathbf{t}} \in \mathcal{F}_{\mathbf{t}^2_T}$ may hold only as the equality. Then there is $\tilde{P} \sim P$ with $Z \in L^\infty$ such that $\tilde{E}(E(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t+i}})|\mathcal{F}_{\mathbf{t}^2_T}) = 0$ for all $\mathbf{t} \in [0, T-1]$. If, in addition, the Cairoli–Walsh condition is fulfilled for $\tilde{P}$, then $\tilde{E}(\Delta S_{\mathbf{t+1}}\hat{\xi}_t|\mathcal{F}_{\mathbf{t}}) = 0$, where $\hat{\xi}_t = Z^{-1}/E(Z^{-1}|\mathcal{F}_{\mathbf{t+i}})$.

Proof. (a) We have that $\tilde{E}(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t}}) = 0$. Thus, for any $\alpha_{\mathbf{t}} \in \mathcal{F}_{\mathbf{t}^2_T}$ we get, taking into account the Cairoli–Walsh, that

$$\tilde{E}\left(\sum_{\mathbf{t}\in[0,\mathbf{T-1}]} \alpha_{\mathbf{t}} \tilde{E}(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t+i}})\middle|\mathcal{F}_{\mathbf{t}^2_T}\right) = 0.$$

The proof follows now immediately from DMW theorem and the identity

$$\tilde{E}(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t+i}}) = E(\Delta S_{\mathbf{t+1}}\xi_t|\mathcal{F}_{\mathbf{t+i}}).$$

(b) We have, in particular, that the inequality

$$\sum_{\mathbf{t}\in[0,\mathbf{T-1}]} \alpha_{\mathbf{t}} E(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t+i}}) \geq 0$$

with $\alpha_{\mathbf{t}} \in \mathcal{F}_{\mathbf{t}^2_T}$ may hold only as the equality. In this case DMW theorem guarantees that there exists $\tilde{P} \sim P$ with $Z \in L^\infty$ such that

$$\sum_{\mathbf{t}\in[0,\mathbf{T-1}]} \alpha_{\mathbf{t}} \tilde{E}(E(\Delta S_{\mathbf{t+1}}|\mathcal{F}_{\mathbf{t+i}})|\mathcal{F}_{\mathbf{t}^2_T}) = 0.$$

The last step is obvious. □

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