A Consumption–Investment Problem with Production Possibilities

Yuri KABANOV[1] * and Masaaki KIJIMA[2]

[1] Université de Franche-Comté, 16 Route de Gray,
F-25030 Besançon Cedex, France,
Central Economics and Mathematics Institute, Moscow, Russia.
kabanov@math.univ-fcomte.fr

[2] Daiwa Chair, Graduate School of Economics, Kyoto University,
Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan.
kijima@econ.kyoto-u.ac.jp

Summary. We investigate a consumption-investment problem in the setting of corporate finance considering a single agent disposing production possibilities. He can invest funds into both manufacturing and financial assets diversifying the income. The agent, endowed with an initial fund as well as initial production assets, strives to maximize the total expected utility from consumption over the finite time horizon. We establish for this problem a separation theorem. Namely, it can be solved by a two-stage procedure. The first stage is an independent optimization problem for the manufacturing arm and the second one is a standard Merton consumption-investment (portfolio selection) problem. The input parameter of the latter, the initial budget, is determined by the optimal value of the manufacturing problem for which the Bismut stochastic maximum principle is the necessary and sufficient condition of optimality. In the case of deterministic coefficients and absence of random fluctuations the first problem is a classical deterministic problem which can be analyzed by the classical Pontriagin maximum principle. In particular examples we obtain closed form solutions and show that in certain cases the optimal production trajectories exhibit a turnpike behavior.

Key words: Consumption–investment problem, portfolio, production, stochastic equation, martingale, backward stochastic differential equation, Bismut stochastic maximum principle, Pontriagin maximum principle, turnpike

Mathematics Subject Classification (2000): 60G44

*This research was done during the stay of the author at Daiwa Chair of Graduate School of Economics, Kyoto University.

It is a pleasure to start this paper by a short historical comment relevant to our anniversary volume. The mathematical tools used in the note below are common nowadays but in the early seventies they were the newest “hot” topics of the seminar leaded by Albert Shiryaev and their development, to great extent, was inspired by him. In this period, the seminar, due to his inexhaustible energy and charisma, became one of the world centers in stochastic calculus and control. We can only admire Shiryaev’s intuition to concentrate efforts on the directions which were later recognized as the most important in the theory of random processes and its applications, in particular, in mathematical finance. He was one of the first who understood the importance of the predictable representation theorem due to J.M.C. Clark (1971), related, as we know now, with the fundamental concept of market completeness. He suggested me, as the subject of my diploma project, to find an easier proof of this theorem and extend it to jump processes. It was the beginning of my studies as a mathematician. Another area of his interests was the Girsanov theorem and problems of absolute continuity. Shiryaev and his collaborators (many of are authors of this book) published a number of papers on this subject which constitutes an accomplished theory. Experience in these fields which form the heart of modern stochastic finance was very useful in subsequent studies in arbitrage theory. Optimal control was another preferable topic of the seminar. I remember our excitement when Shiryaev brought from France the first preprints by Bismut on backward stochastic equations and stochastic maximum principle. He explained the importance of new concepts and inspired members of the seminar to make research in this field (several papers by Arkin, Saksonov and myself were published more when a decade before the revival of the interest to BSDEs elsewhere).

Yuri Kabanov

1 Introduction

We consider here a consumption–investment decision problem for a single “small” economic agent which can be viewed as a firm having production and financial arms. The initial endowment is in both assets. The problem is to maximize the total expected utility of the consumption rate over a finite time interval $[0, T]$ investing into the production as well as in the financial assets. It is assumed that the agent has an access to a frictionless security market with $d + 1$ assets, one of which is riskless and the others are risky. The market model is fairly standard: it is of the same type as in Karatzas et al. [10], see also Cox and Huang [4] and the expository paper [9]. Allocating the resources, the agent may invest funds into m production assets. This type of assets has features different from that of financial assets in the following two points. The investments into the manufacturing arm are irreversible. The profit flow from the production at time t is $R(t, K_t)$ where $K_t = (K_t^1, ..., K_t^m)$

is the capital accumulation. The latter subjects random depreciations and, eventually, fluctuations due to external factors. The production assets cannot be cashed back before the terminal date T when the production arm can be sold at the price $Q(K_T)$. A similar problem was considered by Hirayama and Kijima in [8].

The agent in this model may be an owner of a small firm that produced some production goods. The consumption in this case can be interpreted as the dividend flow from the firm. The owner does not want to sell the business, since the ownership for him is very important (this is rather typical, especially, in such country as Japan). The role of the owner is to maximize the total utility from dividend. To do so, the owner may want to invest the limited fund in the production assets as much as possible to earn higher profits. But, since there is a financial market, he may also allocate a part of his wealth in securities. The problem for the owner is to decide portfolio strategy, dividend strategy, and production strategy so as to maximize the objective.

As we mentioned already, without the production arm, our model is reduced to the mainstream continuous-time portfolio optimization problem started in the famous papers by Merton [15], [16] and developed further in numerous publications (see, e.g., [4], [9], [10], [11], [17] and references therein). Production models were considered in [14] but without financial investments while the equilibrium approach to production economies was discussed in [19]. In real economies, firms invest their surplus funds in financial assets. It seems of interest to study optimal strategies in this more general context.

In our presentation we try to avoid technicalities. That is why we work with the easily treated hypotheses, preferring, e.g., the boundedness assumption on coefficients to that of integrability. Our main message is that for the linear model with concave utility and production functions the problem can be split into two separate stages. First, the optimal production investment process $I^o = (I^o_t)$ can be found independently of the other counterparts of the optimal control as the optimal solution of a certain auxiliary control problem. Finding I^o, we have to solve, as the second stage, a classical portfolio problem which, as well-known, consists itself of two separate parts: a search for the optimal consumption and a search for the optimal investment (that is why we can say also that the whole problem has three stages).

This separation principle is the main feature of the considered model. It is quite understandable because in the case of a complete market a suitably integrable stochastic income (from the production, in our case) leads only to a change of the initial endowment of the Merton problem. This fact (used already in [8]) is now well-known, see, e.g., the paper [5] where the stochastic income is bounded. Our hypothesis and the definition of admissible strategies ensures the applicability of this principle.

We prove the needed existence of the optimal solution for the auxiliary problem (using the Komlós theorem) and derive necessary and sufficient conditions of optimality in the form of the Bismut maximum principle providing a self-contained exposition of the latter for the considered case.

We investigate in more details a particular case of the model where the production block is not directly influenced by random perturbations. In this case the first stage is a deterministic control problem, still interesting, which can be analyzed on the basis of the Pontryagin maximum principle. We give examples where the optimal production policy is of the bang–bang type. We provide also an example showing that in a long-run the optimal production trajectories follow a "turnpike". This means that there exists a function, independent on the initial endowment and the terminal (liquidation) cost, with which the optimal production trajectory coincides except its first part (depending on the "starting point") and its final part (depending on the "destination", i.e. of the terminal cost functional).

We use vector notations; in particular, xy stands for the scalar product and $\operatorname{diag} x$ denotes the diagonal operator corresponding to the vector x.

2 Model Description

We shall work in the standard probabilistic framework assuming that the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$ is fixed and the filtration is spanned by a d-dimensional Wiener process W. The time horizon T is finite.

First, we describe the production arm of the firm. It disposes m assets and if $K \in \mathbf{R}^m_+$ is a vector of values of these assets, the rate of the profit flow at time t is $R(t, K)$. The production asset i is depreciated with the rate λ^i which is, in general, a non-negative bounded predictable process. Its value also may fluctuate due to external factors. The capital accumulation evolves according to the stochastic differential equations

$$dK^i_t = (I^i_t - \lambda^i_t K^i_t)dt + K^i_t dL^i_t, \qquad K^i_0 = k^i, \tag{2.1}$$

where L is a martingale with

$$dL^i_t = \sum_{j=1}^{d} \sigma^{ij}_t dW^j_t, \qquad i \le m,$$

for some bounded predictable matrix-valued process σ.

The investments are assumed to be irreversible, i.e. the capital accumulation may decrease only by depreciation and by random fluctuations (if $\sigma = 0$, the latter are not taken into account). The production strategy I is a predictable process with values in a compact convex subset Γ of $\mathbf{R}^m_+$. It follows (by a standard arguments based on the Gronwall–Bellman lemma) that the sup norm of the capital accumulation process are bounded by a square integrable random variable.

The production assets cannot be sold before T, but they can be liquidated at the price $Q(K_T)$ at the terminal date. It is natural to assume that in the variable K the functions R and Q are concave and increasing (componentwise).

Since the concave function is dominated by a linear one, the family of random variables $Q(K_T)$, K is a capital accumulation process, is dominated by a random variable from L^2. The same property holds for the family of random variables $\int_0^T R(s,K_s)ds$ when

$$R(s,K) \le f(s)(1+lK),$$

where $l \in \mathbf{R}^m$ and f is a function integrable on the interval $[0,1]$; we assume that this condition is always fulfilled.

Thus, our set of assumptions ensures the following important property:

$$\int_0^T R(s,K_s)ds + Q(K_T) \le \zeta \in L^2. \tag{2.2}$$

The agent also has an access to a frictionless financial market of the Black–Scholes type with $d+1$ securities. One of them is non-risky ("bond" or "bank account") and has the price evolving as

$$\frac{dP_t^0}{P_t^0} = r_t dt, \qquad P_0^0 = p^0 = 1. \tag{2.3}$$

For simplicity, mainly, notational, we suppose from the very beginning that $r = 0$, i.e. bond is the numéraire and all investments are measured in its units.

The prices of remaining assets, (risky) stocks, are modelled by the stochastic equations

$$\frac{dP_t^i}{P_t^i} = b_t^i dt + dM_t^i, \qquad P_0^i = p^i, \tag{2.4}$$

where M is a square integrable martingale generating our basic filtration $\mathbf{F}$ (of the Wiener process W). We assume more specifically that

$$dM_t^i = \sum_{j=1}^d \Sigma_t^{ij} dW_t^j, \qquad i \le d.$$

The vector of instantaneous rate of returns b and the (non-degenerate) volatility matrix Σ and its inverse Σ^{-1} are assumed to be bounded predictable processes.

The agent's portfolio at date t contains n_t^i units of the asset i. His holdings in risky assets of the financial market $\pi_t^i = n_t^i P_t^i$, $1 \le i \le d$, are predictable processes such that

$$\int_0^T |\pi_t|^2 dt < \infty.$$

The agent consumption intensity is a predictable non-negative process $c = (c_t)$ with

$$\int_0^T c_t dt < \infty.$$

The triplet of the investment processes and consumption $u = (\pi, I, c)$ is the control strategy. The optimization problem can be formulated as:

$$E \int_0^T e^{-\beta t} U(c_t) dt \to \max, \tag{2.5}$$

with the controlled dynamics of the total fund given by the following stochastic differential equation where $\mathbf{1} := (1, ..., 1)$:

$$dX_t = (R(t, K_t) - \mathbf{1} I_t - c_t)dt + \pi_t(b_t dt + dM_t), \qquad X_0 = x. \tag{2.6}$$

To avoid technicalities, we suppose that the utility function $U : \mathbf{R}_+ \to \mathbf{R}_+$ in (2.5) is a concave increasing function vanishing at zero with $U'(0) = \infty$ and $U'(\infty) = 0$ (note that U is differentiable everywhere except at most a countable number of points).

In addition to the constraints indicated above we impose a constraint on the controls which prevents a "bankruptcy" before the date T. Namely, we shall consider as **admissible** only the controls u such that

$$V_t := X_t + \tilde{E}\left[\int_t^T R(s, K_s)ds + Q(K_T)|\mathcal{F}_t\right] \geq 0, \qquad \forall\, t \leq T. \tag{2.7}$$

The symbol $\tilde{E}$ indicates that the expectation is taken with respect to the (unique) martingale measure $\tilde{P}$. The corresponding term can be interpreted as the market evaluation of the manufacturing arm of the company. This makes plausible the assumption that the agent may borrow funds until this level.

The set of admissible strategies, denoted by $\mathcal{A}(y)$, depends on the initial endowment $y := (x, k)$.

We shall assume that $\mathcal{A}(y) \neq \emptyset$, i.e. at least one admissible strategy u does exist. Obviously, this is always the case when R and Q are non-negative, since $u = (0, 0, 0)$ belongs to $\mathcal{A}(y)$.

Recall that $\tilde{P} = Z_T P$ where

$$Z_t = \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right\},$$

with $\theta_s := -\Sigma_s^{-1} b_s$. Under $\tilde{P}$

$$\tilde{W}_t := W_t - \int_0^t \theta_s ds$$

is a Wiener process. Due to the boundedness of θ the random variable Z_T is square integrable. Thus, the random variable ζ in (2.2) belongs to $L_1(\tilde{P})$. In particular, the conditional expectation in (2.7) is well-defined. Moreover, for an admissible strategy, we have

$$\int_0^T R(s, K_s)ds + Q(K_T) \in L^1(\tilde{P}).$$

Remark. The completeness of the financial market, i.e. the uniqueness of the martingale measure, is essential for our further development: we rely on the martingale representation theorem. The latter does not hold for more general models of incomplete market (which may constitute one of possible directions of future studies) where the natural extension of the admissibility condition (2.7) involves the supremum of expectations over the set of all martingale measures.

3 Existence and Structure of the Optimal Control

Take an arbitrary admissible control. Under the measure $\tilde{P}$ the dynamics of the phase variable (2.6) can be rewritten as follows:

$$X_t = x + \int_0^t (R(s, K_s) - \mathbf{1}I_s - c_s)ds + \int_0^t \pi_s d\tilde{M}_s, \tag{3.1}$$

where $\tilde{M}$ is a (square integrable) martingale with respect to $\tilde{P}$. Notice that $X \geq 0$ while the ordinary integral above is less or equal to $\zeta \in L^1(\tilde{P})$, see the assumption (2.2). Thus, with respect to $\tilde{P}$, the stochastic integral, being a local martingale dominating an integrable random variable, namely, $-(x+\zeta)$, is a supermartingale.

Substituting the expression (3.1) into (2.7), we obtain the formula

$$V_t = x + \tilde{E}\left[\int_0^T R(s, K_s)ds + Q(K_T)|\mathcal{F}_t\right] - \int_0^t (\mathbf{1}I_s + c_s)ds + \int_0^t \pi_s d\tilde{M}_s.$$

The definition of admissibility implies, in particular, that $\tilde{E}V_T \geq 0$. Due to the supermartingale property, the expectation of the stochastic integral with respect to $\tilde{P}$ is negative and we infer the inequality

$$\tilde{E}\int_0^T c_s ds \leq x - H(I) \tag{3.2}$$

where

$$H(I) := \tilde{E}\left[\int_0^T (\mathbf{1}I_s - R(s, K_s))ds - Q(K_T)\right]. \tag{3.3}$$

Let us denote by $\mathcal{C}(y)$ the set of pairs of production and investment processes (I, c) for which (3.2) holds.

The next lemma is established in the same way as in the classical consumption–investment model, see, e.g., the textbook [12].

Lemma 3.1. *For any given $(I, c) \in \mathcal{C}(y)$ there exists a portfolio process π such that $(\pi, I, c) \in \mathcal{A}(y)$.*

Proof. Let $(I, c) \in \mathcal{C}(y)$. Noticing that $H(I)$ is finite, we consider the non-negative process V with

$$V_t := \tilde{E}\left[\int_0^T (\mathbf{1}I_s + c_s)ds|\mathcal{F}_t\right] - \int_0^t (\mathbf{1}I_s + c_s)ds$$
$$+x - \tilde{E}\left[\int_0^T (\mathbf{1}I_s + c_s - R(s, K_s))ds - Q(K_T)\right].$$

It can be written in the form

$$V_t = x + \tilde{E}\left[\int_0^T R(s, K_s)ds + Q(K_T)|\mathcal{F}_t\right] - \int_0^t (\mathbf{1}I_s + c_s)ds + M_t^V - M_0^V,$$

where

$$M_t^V := \tilde{E}\left[\int_0^T (\mathbf{1}I_s + c_s - R(s, K_s))ds - Q(K_T)|\mathcal{F}_t\right].$$

By the martingale representation theorem

$$M_t^V - M_0^V = \int_0^t \pi_s d\tilde{M}_s$$

and we infer easily from (2.7) and (3.1) that the triplet $(\pi, I, c) \in \mathcal{A}(y)$. □

The conclusion following from this lemma is very important: solving the original problem with a seemingly complicated "pointwise" constraint (2.7) is reduced to the solving of a much simpler problem with a single "traditional" inequality constraint given by a convex functional, with a consequent search for the corresponding investment strategy. Moreover, it is easily seen that the search for the optimal production and optimal consumption also can be done in a separate consecutive way. Indeed, since the utility function is increasing, for a given production strategy I with $H(I) \leq x$ (such a strategy exists as there is an admissible strategy u), the corresponding maximal value of the functional is attended on a consumption strategy for which (3.2) holds with the equality. The maximal possible value will correspond to I^o on which $H(I)$ attains minimum. The existence of the optimal I^o as well as the solution of the consumption problem satisfying (3.2) follows from the Komlós theorem - we recall the arguments in Proposition 1 of the next section dealing with the optimal production strategy. Summarizing, we arrive to the following

Theorem 3.1. *In the solution $(\pi^o, I^o, c^o) \in \mathcal{A}(y)$ of the consumption-investment problem with production possibilities the optimal investment I^o in manufacturing arm is the minimizer for the problem with the functional (3.3) and the dynamics (2.1). The optimal consumption process $c^o \geq 0$ is the* *solution of the maximization problem (2.5) under the constraint (3.2). The optimal portfolio strategy* π^o *is the unique square-integrable predictable process satisfying the identity*

$$M_t^{V^o} = M_0^{V^o} + \int_0^t \pi_s^o d\tilde{M}_s$$

with

$$M_t^{V^o} := \tilde{E}\left[\int_0^T (\mathbf{1}I_s^o + c_s^o - R(s, K_s^o))ds - Q(K_T^o)|\mathcal{F}_t\right].$$

4 Optimal Production Investment

Let us consider separately the optimal control problem

$$H(I) := \tilde{E}\left[\int_0^T (\mathbf{1}I_s - R(s, K_s))ds - Q(K_T)\right] \to \min \tag{4.1}$$

over the convex set $\mathcal{I}$ of all Γ-valued predictable processes I and where K is given by (2.1)[3]. This problem belongs to the well-studied class of convex problems for which one can use duality methods.

Proposition 1. *The minimization problem (2.1), (4.1) has a solution.*

Proof. Now standard (and fast) way to prove the existence in the convex optimal control problems is the reference to the Komlós theorem. The latter claims that for any L^1-bounded sequence of random variables ξ_n there exist a random variable $\xi \in L^1$ and a subsequence ξ_{n_k} converging to ξ a.s. in the Cesaro sense.

Let $H^o = \inf_{I\in\mathcal{I}} H(I)$ and let $H(I^n) \to H^o$ for some $I^n \in \mathcal{I}$. Due to the boundedness of Γ we can apply the Komlós theorem to I^n considering these processes as random variables on the space $(\Omega\times[0,T], \mathcal{P}, d\tilde{P}dt)$, where $\mathcal{P}$ is the predictable σ-algebra. Renumbering, we may assume without loss of generality that the original sequence converges $d\tilde{P}dt$-a.e. to some I in Cesaro sense. This means simply that the controls $\bar{I}^n := n^{-1}\sum_{j=1}^n I^j$ converge (a.e.) to I^o which is, clearly, an element of $\mathcal{I}$. Let us denote by $\bar{K}_n$ and K^o the corresponding capital accumulation processes. The solution of (2.1) can be written explicitly via the (stochastic) Cauchy formula. The latter implies that, outside a null-set, the sequence $\bar{K}_t^n(\omega)$ converges to $K_t^o(\omega)$ whatever is $t \in [0, T]$. Moreover, the sequence $\sup_t \bar{K}_t^n(\omega)$ is bounded (by a constant depending on ω). Recalling the hypothesis $R(s, K) \le f(s)(1 + lK)$, we deduce from here, using the Fatou lemma for the integral and the continuity of R and Q in K, that

[3] Economically, this form suggests the minimization of losses, i.e. the manufacturing, presumably, is non-rentable; in more optimistic situation one could consider the problem $-H(I) \to \max$, the maximization of profits.

$$\int_0^T (\mathbf{1}I_s^o - R(s,K_s^o))ds - Q(K_T^o) \le \liminf \left[\int_0^T (\mathbf{1}I_s^o - R(s,\bar K_s^n))ds - Q(\bar K_T^n)\right].$$

Taking the $\tilde P$-expectation with of the both side of this inequality and applying again the Fatou lemma, this time with respect to $\tilde P$ (justified because the random variable ζ in (2.2) belongs to $L^1(\tilde P)$) we obtain:

$$H(I^o) \le \liminf H(\bar I^n) \le \liminf n^{-1}\sum_{j=1}^n H(I^j) = H^o.$$

Thus, $H(I^o) = H^o$, i.e. I^o is the optimal control. □

We shall assume from now on that $R(t,K)$ and $Q(K)$ have derivatives in the variable K. The particular structure of the problem (2.1), (4.1) (linear dynamics and convex functional) implies that the necessary condition of optimality given the Bismut stochastic maximum principle, see [2], [3], is also a sufficient one. For the considered case the arguments are easy and the proof can be done in a few lines. For the reader's convenience we give them instead sending him to a general theory presented in [20].

Isolating the $\tilde P$-martingale term and using the abbreviation $\mu_t := \lambda_t - \sigma_t\theta_t$, we rewrite the dynamics of manufacturing capital in vector notations as

$$dK_t = (I_t - \operatorname{diag} K_t\mu_t)dt + \operatorname{diag} K_t\,\sigma_t d\tilde W_t, \qquad K_0 = k, \tag{4.2}$$

and introduce the Hamiltonian

$$\mathcal{H}(t,K,I,p,h) := \langle p, I - \operatorname{diag} K\,\mu_t\rangle + \langle h, \operatorname{diag} K\,\sigma_t\rangle + R(t,K) - \langle \mathbf{1}, I\rangle,$$

where $p \in \mathbf{R}^m$ while h and $\operatorname{diag} K\,\sigma_t$ are $m\times d$-matrices interpreted as elements of $\mathbf{R}^{md}$. Exceptionally, we use here the notation $\langle .,.\rangle$ for scalar products following the traditional and easy to memorize form which was suggested by Bismut. Note that the second term can be written as $\operatorname{tr} h(\operatorname{diag} K\,\sigma_t)^*$, where * denotes the transpose and tr the trace.

The maximum principle claims that the pair (I^o, K^o) satisfying the equation

$$dK_t^o = (I_t^o - \operatorname{diag} K_t^o\mu_t)dt + \operatorname{diag} K_t^o\sigma_t d\tilde W_t, \qquad K_0^o = k, \tag{4.3}$$

is optimal for the problem (4.1), (4.2) if there exist a continuous predictable processes p with square integrable sup norm and a process $h \in L^2(\Omega\times[0,T],\mathcal{P},d\tilde P dt)$ solving the m-dimensional backward stochastic differential equation (BSDE)

$$dp_t = -\nabla\mathcal{H}(t,K_t^o,I_t^o,p_t,h_t)dt + h_t d\tilde W_t, \qquad p_T = \nabla Q(K_T^o), \tag{4.4}$$

where ∇ is the gradient in the variable K, specifically,

$$dp_t = (\operatorname{diag}\mu_t\,p_t - \nabla R(t,K_t^o) - \widehat h_t)dt + h_t d\tilde W_t, \qquad p_T = \nabla Q(K_T^o), \tag{4.5}$$

where $\widehat{h}_t^i = \sum_j h_t^{ij} \sigma_t^{ij}$ and the following relation holds:

$$\mathcal{H}(t, K_t^o, I_t^o, p_t, h_t) = \max_{I \in \Gamma} \mathcal{H}(t, K_t^o, I, p_t, h_t) \qquad d\tilde{P}dt\text{-a.e.} \tag{4.6}$$

For brevity we shall call any quadruplet of processes I^o, K^o, p, and h satisfying the above relations and the integrability assumption a *Bismut quadruplet*.

Knowing that the processes p and h satisfying (4.5) exist, there is almost nothing to prove. Indeed, let I be an arbitrary Γ-valued predictable process. Using (4.3) and (4.5) we get by the Ito formula that

$$\begin{aligned} d(p_t K_t) &= (p_t \mathrm{diag}\, \mu_t\, K_t - \nabla R(t, K_t^o) K_t - \mathrm{tr}\, h(\mathrm{diag}\, K\, \sigma_t)^*)dt \\ &\quad + p_t(I_t - \mathrm{diag}\, K_t \mu_t)dt + \mathrm{tr}\, h(\mathrm{diag}\, K\, \sigma_t)^* dt + dN_t \\ &= (p_t I_t - \nabla R(t, K_t^o) K_t)dt + dN_t \end{aligned}$$

where N is a square integrable martingale with respect to $\tilde{P}$.

Writing this in the integral form and observing that the expectation of stochastic integral vanishes we arrive to the formula

$$\tilde{E} \int_0^T p_t I_t dt = \tilde{E} \nabla Q(K_T^o) K_T - p_0 k + \tilde{E} \int_0^T \nabla R(t, K_t^o) K_t dt.$$

This formula holds, in particular, for I^o and K^o. Taking the difference of the identities for the optimal and an arbitrary and using the concavity of R and Q, we obtain easily that

$$\tilde{E} \int_0^T p_t (I_t^o - I_t) dt \le \tilde{E} \int_0^T (R(t, K_t^o) - R(t, K_t)) dt + \tilde{E}(Q(K_T^o) - Q(K_T)). \tag{4.7}$$

But the maximum principle (4.6) implies

$$\int_0^T \mathbf{1}(I_t^o - I_t) dt \le \int_0^T p_t (I_t^o - I_t) dt \qquad \tilde{P}\text{-a.s.} \tag{4.8}$$

and we deduce from these two inequalities that $H(I^o) \le H(I)$.

Due to the simplicity of our problem we can see easily that the stochastic maximum principle is the necessary condition: the optimal pair is the component of a Bismut quadruplet. Indeed, starting from the optimal pair (I^o, K^o) we can define p and h satisfying (4.5). The optimality of (I^o, K^o) implies that in (4.7) and (4.8) we have equalities. But the fulfillment of (4.8) for any $I = (I_t)$ is equivalent to (4.6).

Summarizing, we have the following.

Proposition 2. *A pair (I^o, K^o) satisfying (4.3) is an optimal solution of the problem (3.3), (4.2) if and only if it can be complimented to a Bismut quadruplet.*

In the case where $\sigma = 0$ and, therefore, h appears only in the diffusion term, the linear backward equation is especially simple and can be "solved" easily. Indeed, the m-dimensional random variable

$$\xi := \int_0^T e_s^{-\lambda} \nabla R(s, K_s^o) ds + e_T^{-\lambda} \nabla Q(K_T^o)$$

with

$$e_t^\lambda := \operatorname{diag} \left\{ e^{\int_0^t \lambda_s^1 ds}, ..., e^{\int_0^t \lambda_s^m ds} \right\}$$

is a square integrable functional of the Wiener process. By the martingale representation theorem

$$\tilde{E}(\xi|\mathcal{F}_t) = \tilde{E}\xi + \int_0^t \varphi_s d\tilde{M}_s$$

for some matrix-valued process $\varphi \in L^2(\Omega \times [0,T], \mathcal{P}, d\tilde{P}dt)$ of an appropriate dimension. It is easy to see that $h_t := e_t^\lambda \varphi_t$ and

$$p_t := e_t^\lambda \tilde{E}\xi - e_t^\lambda \int_0^t e_s^{-\lambda} \nabla R(s, K_s^o) ds + e_t^\lambda \int_0^t \varphi_s d\tilde{M}_s$$

is the solution of the backward stochastic equation (4.5).

In the case $d = 1$ we can get an "explicit" solution of the BSDE for arbitrary σ by making at first the equivalent change of the probability measure, removing the term $\widehat{h}$ from the drift (under this measure the process with $d\tilde{W}_t' := d\tilde{W}_t + \sigma_t dt$ Wiener). In general case we use just a reference to an existence theorem for the solution of a linear BSDE. An appropriate result can be found, e.g., in [6].

However, though attractive, the stochastic maximum principle is not very helpful in getting the optimal solution. In the case when $\sigma = 0$ and the coefficients are deterministic, it is "degenerated" to the ordinary Pontryagin maximum principle (of a deterministic problem). The latter is a powerful tool of the optimal control theory which allows to analyze the structure of the optimal control. We do this by considering examples.

5 Special Cases

5.1 Deterministic Dynamics: Examples.

The separation result has an important consequence for the case of the model where the values of the production assets may only depreciate (i.e. $\sigma = 0$) and the parameters λ^i are deterministic. The problem becomes deterministic:

$$H(K) := \int_0^T (\mathbf{1} I_t - R(t, K_t)) dt - Q(K_T) \to \min, \tag{5.1}$$

$$\dot{K}_t^i = I_t^i - \lambda_t^i K_t^i, \qquad K_0^i = k^i, \tag{5.2}$$

where $I = (I_t)$ is a Borel function taking values in $\Gamma \subset \mathbf{R}_+^m$.

The necessary and sufficient condition of optimality is the classical Pontriagin maximum principle. More specifically, a pair (I^o, K^o) is optimal for the problem (5.1), (5.2) if and only if it is a part of the "Pontryagin triplet" (I^o, K^o, p) satisfying the following relations:

$$\dot{K}_t^o = I_t^o - \operatorname{diag} \lambda_t K_t^o, \qquad K_0^o = k, \tag{5.3}$$

$$\dot{p}_t = p_t \operatorname{diag} \lambda_t - \nabla R(t, K_t^o), \qquad p_T = \nabla Q(K_T^o), \tag{5.4}$$

$$(p_t - \mathbf{1}) I_t^o = \max_{I \in \Gamma} (p_t - \mathbf{1}) I_t \quad a.e. \tag{5.5}$$

Due to the number of parameters involved, the complete analysis of this system seems to be rather complicated. We restrict ourselves to the scalar problem with constant coefficients and $\Gamma = [0, a]$ and provide several examples where the solution can be obtained explicitly. For $m = 1$ we have:

$$\dot{K}_t^o = I_t^o - \lambda K_t^o, \qquad K_0^o = k, \tag{5.6}$$

$$\dot{p}_t = \lambda p_t - R'(K_t^o), \qquad p_T = Q'(K_T^o), \tag{5.7}$$

$$(p_t - 1) I_t^o = \max_{I \in \Gamma} (p_t - 1) I_t \quad a.e. \tag{5.8}$$

Case study: scalar homogeneous model with $Q = \text{const}$ (such a situation may arise in practice) and $R(K) = (\kappa/\gamma) K^\gamma$, $\kappa > 0$, $\gamma \in]0, 1[$.

Due to the continuity, near the right extremity T of the time interval the dual variable p is close to the value $p_T = 0$; more precisely, it decreases to zero because the equation (5.7) implies that the derivative $\dot{p}_T = -\kappa (K_T^o)^{\gamma - 1} < 0$. Now put $T_1 := \sup\{t \geq 0 : \; p_t \geq 1\}$ (with the convention that $T_1 = 0$ if the set is empty). The maximum relation ensures that $I_t^o = 0$ on $]T_1, T]$. If $T_1 = 0$, the phase trajectory is the decreasing exponential $K_t^o = k e^{-\lambda t}$ while the trajectory of the dual variable is

$$p_t = e^{\lambda t} \int_t^T e^{-\lambda s} R'(K_s^o) ds = k^{\gamma - 1} \frac{\kappa}{\lambda \gamma} e^{\lambda t} (e^{-\lambda \gamma t} - e^{-\lambda \gamma T}).$$

To be compatible with the maximum principle the right-hand side should be less or equal to unity on the whole interval $[0, T]$ and this requirement is met when the initial endowment $k \geq k^c$ where the threshold is given by

$$k^c = \sup_{t \leq T} \left[\frac{\kappa}{\lambda \gamma} e^{\lambda t} (e^{-\lambda \gamma t} - e^{-\lambda \gamma T}) \right]^{\frac{1}{1-\gamma}}.$$

Thus, for large k the control $I_t^o = 0$. We shall have, for large initial endowments in production assets, the similar structure of the optimal control also for the model where $Q'(K) \to 0$ as $K \to \infty$.

Qualitatively, this result means that in the case of small marginal liquidation value the investor having high level of initial manufacturing facilities is not motivated in their further development.

The situation seems to be rather different for $k < k^c$. Then necessarily I^o is not equal to zero on a certain non-null subset of $[0, T_1]$. Let us show that for some range of parameters, $I^o_t = aI_{[0,T_1]}$.

So, suppose that on $[0, T_1]$ the control $I^o_t = a$ and, therefore, on this interval the state dynamics is given by the formula

$$K^o_t = ke^{-\lambda t} + \frac{a}{\lambda}(1 - e^{-\lambda t}) = \frac{a}{\lambda} + \left(k - \frac{a}{\lambda}\right) e^{-\lambda t}. \tag{5.9}$$

First, we consider the simplest particular case where $k = a/\lambda$. Then $K^o_t = k$ on $[0, T_1[$ (the maximal level of investments keeps the production capacity constant) and, according to (5.7), $\dot{p}_{T_1} = \lambda - \kappa k^{\gamma-1}$. For $t \in [T_1, T]$ we have the formula $K^o_t = ke^{\lambda T_1} e^{-\lambda t}$ and, hence, on this interval

$$p_t = k^{\gamma-1} e^{\lambda(\gamma-1)T_1} \frac{\kappa}{\lambda\gamma} e^{\lambda t}(e^{-\lambda\gamma t} - e^{-\lambda\gamma T}).$$

Note that the point $T_1 \in]0, T[$ can be defined from the equation $p_{T_1} = 1$ which solution does exist for $k < k^c$. On the interval $[0, T_1]$ the function p solving the differential equation

$$\dot{p}_t = \lambda p_t - \kappa k^{\gamma-1}, \qquad p_{T_1} = 1,$$

and hence given by the formula

$$p_t = \frac{\kappa}{\lambda} k^{\gamma-1} + \left(1 - \frac{\kappa}{\lambda} k^{\gamma-1}\right) e^{-\lambda(T_1 - t)}$$

should be larger or equal to unity. If also $k < (\kappa/\lambda)^{\frac{1}{1-\gamma}}$, the value of derivative $\dot{p}_{T_1} < 0$. Taking into account that the trajectory cannot cross the unit level upwards with negative value of derivative (always equal to $\lambda - \kappa k^{\gamma-1}$), we conclude that the control $aI_{[0,T_1]}$ is optimal for such values of the initial endowment k.

If $k > a/\lambda$, the trajectory supposed to be optimal decreases on $[0, T_1]$ from its initial value k. For $k < (\lambda/\kappa)^{\frac{1}{1-\gamma}}$, we have $\dot{p}_{T_1} < 0$, i.e. the dual variable cross the unit level at T_1 and cannot do this before.

If $k < a/\lambda$, the candidate for the optimal trajectory on $[0, T_1]$ increases from k to a certain value which is less than a/λ. At least, in the case of the small ratio a/λ (i.e., when $\lambda < \kappa(a/\lambda)^{\gamma-1}$), we can conclude again that $p_t > 1$ on $[0, T_1[$ and, therefore, $I^o_t = aI_{[0,T_1]}$ is the optimal control.

In short, for initial endowments k less than a certain critical value k_c (in some case, with appropriate restrictions on other parameters), the optimal strategy is of the bang-bang form and requires at the beginning of the planning interval intensive investments in the production assets.

However, in the range $]k_c, k^c[$ the structure of the optimal control may be more involved and even not of the bang-bang type.

5.2 Deterministic Dynamics: Turnpike Behavior

To investigate the general structure of the optimal control in the problem (5.1), (5.2), we exclude the control variable from the functional using the expressions $I_t^i = \dot{K}_t^i + \lambda_t^i$ given by (5.2). After simple transformations we arrive to the problem with the functional depending only of the phase variable:

$$\int_0^T \Phi(t, K_t)dt + S(K_T) \to \min, \tag{5.10}$$

$$\dot{K}_t^i = I_t^i - \lambda_t^i K_t^i, \qquad K_0^i = k^i, \tag{5.11}$$

where the functions $\Phi(t, K) := \lambda_t K - R(t, K)$ and $S(K) := \mathbf{1}K - Q(K) - \mathbf{1}k$ are convex in K.

It is well-known that, under minor assumptions, the optimal trajectory in models of such type exhibits, on a large time interval, a turnpike behavior: it coincides, except initial and final periods, with the function $\widehat{K}$ where $\widehat{K}_t$ is the minimizer of the function $\Phi(t, .)$, i.e. the root of the equation $\nabla\Phi(t, K) = 0$.

To be specific, we consider again the one-dimensional time-homogeneous model assuming also that $k < a/\lambda$, $\Phi'(a/\lambda) > 0$, $\Phi'(0) = -\infty$. Then any trajectory K evolves in the interval $[0, a/\lambda]$; it increases if $I = a$ and decreases if $I = 0$.

Now the dual variable $\psi = p - 1$ solves the equation

$$\dot{\psi}_t = \lambda\psi_t + \Phi'(K_t^o), \qquad \psi_T = -S'(K_T^o). \tag{5.12}$$

and the maximum principle says that $I_t^o = 0$ if $\psi_t < 0$, and $I_t^o = a$ if $\psi_t > 0$. It is convenient to introduce an auxiliary function $q_t := e^{-\lambda t}\psi_t$ having the same sign as ψ_t; its derivative $\dot{q}_t = e^{-\lambda t}\Phi'(K_t^o)$.

Let $t_1 := \inf\{t : \ q_t = 0\}$, $t_2 := \sup\{t : \ q_t = 0\}$. Notice that if $[t_1, t_2]$ is not a singleton, then on this interval $q = 0$. Indeed, suppose that there is a subinterval $]t', t''[$ where $q < 0$ but $q_{t'} = q_{t''} = 0$. Since on this subinterval the control $I^o = 0$, the trajectory K^o is decreasing, the trajectory $\Phi'(K^o)$ is also decreasing and so is $-\dot{q}$. This is impossible and, therefore, q cannot deviate from zero downwards. Similarly, if $q > 0$ on $]t', t''[$ and q vanishes at the extremities, then on this interval $I^o = a$, the trajectory K^o increases as well as $\Phi'(K^o)$. Thus,

$$\dot{\psi}_{t'} = \Phi'(K_{t'}^o) < \Phi'(K_{t''}^o) = \dot{\psi}_{t''}$$

in contradiction with the inequalities $\dot{\psi}_{t'} \geq 0$, $\dot{\psi}_{t''} \leq 0$.

The equation (5.12) necessitates that $\Phi'(K^o) = 0$ on $[t_1, t_2]$, i.e. $K^o = \widehat{K}$ where $\widehat{K}$ is the minimizer of Φ; the optimal control is $I^o = \widehat{K}\lambda$. The left extremity coincides with zero if and only if $k = \widehat{K}$. If $t_1 > 0$, there are two possible cases: 1) on $[0, t_1[$ the dual variable ψ is strictly negative, $I^o = 0$ and the trajectory K^o decreases from k to the value $\widehat{K}$; 2) on $[0, t_1[$ the dual variable ψ is strictly positive, $I^o = a$ and the trajectory K^o increases from

k to the value $\widehat{K}$. In both cases the interval $[0,t_1]$ does not depend on the terminal part of the functional and $t_1 < T$ for sufficiently large T.

The case $t_2 = T$ is exceptional. This means that $0 = \psi_T = -S'(\widehat{K})$, i.e., $\widehat{K}$ minimizes also the function S. Otherwise, the interval $[t_2, T]$ is not a singleton. The optimal control on this interval depends on the sign of $S'(\widehat{K})$. Suppose, e.g., that $S'(\widehat{K}) > 0$. Let $I^o = 0$. Then ψ is strictly negative, the trajectory K^o decreases from the value $\widehat{K}$, $\Phi'(K^o) < 0$ and, therefore, $\dot{\psi} = \lambda\psi + \Phi'(K^o) < 0$, i.e., the trajectory ψ decreases from zero. Since $-S'$ is a decreasing function, the transversality condition $\psi_T = -S'(K^o_T)$ will be met for a certain (uniquely defined) value of t_2 (of course, the time horizon should be large enough).

The above arguments show that, for a long time interval, the optimal investments in the manufacturing consist in keeping the production on a specific "turnpike" level which depends only of the technology used and not of the initial capital and the liquidation value. This level should be attained in the fastest way at the beginning of the planning period. At the end of the period, the investment policy is to leave the turnpike quickly to profit from the selling of the manufacturing arm.

5.3 Remark on the HJB equation

The case where the fluctuations of the price of production assets are assumed (i.e. σ is not zero) can be studied by methods of dynamic programming. The problem of interest can be imbedded in the family of stochastic control problems parameterized by initial date t and the initial endowment x (we prefer x to k here for notational convenience). The HJB equation is as follows:

$$V_t + \inf_{I\in[0,a]} \left[\frac{1}{2}\sigma^2 x^2 V_{xx} + (I - \mu x)V_x + (I - R(x))\right] = 0$$

with the terminal condition $V(T,x) = -Q(x)$. The number H^o we are interested in is $V(0,k)$. The above equation can be rewritten in the form

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} - \mu x V_x + aI_{\{V_x<-1\}} - R(x) = 0.$$

One can prove that the Bellman function V of the problem is a viscosity solutions of this equation which is unique in an appropriate class but a detailed discussion is beyond the scope of the present paper.

5.4 Piecewise-linear utility function

As we just see, in some cases the production problem may admit an explicit solution otherwise the value H^o can be find numerically. An attractive feature of the considered setting is that the investing problem is well-studied and also admits cases with explicit solutions. The most famous one is the problem with $U(c) = \rho/c^\rho$ found by Merton.

We discuss here an example where the utility function is linear up to a saturation point, i.e.

$$U(c) = cI_{\{c\le C\}} + CI_{\{c>C\}}.$$

Thus, the optimal control problem is read now:

$$J(c) := E\int_0^T e^{-\beta t}U(c_t)dt \to \max$$

over all non-negative predictable processes c such that

$$E\int_0^T Z_t c_t dt \le x - H(I^o).$$

Clearly, in our search for the optimum we can consider the subset of controls for which the constraint is satisfied with an equality.

The solution can be found easily using the Lagrange multiplier method removing the above constraint. Arguing formally, we write the unconstrained problem

$$E\int_0^T [e^{-\beta t}U(c_t) - \theta Z_t c_t]dt \to \max$$

where the multiplier $\theta \ge 0$. Its solution is any non-negative predictable process $c = (c_t)$ maximizing pointwise the integrand. Of course, the solution depends of the unknown Lagrange multiplier θ. Let

$$c_t^*(\theta) := CI_{\{\theta Z_t > e^{-\beta t}\}}.$$

Define on $\mathbf{R}_+$ the function

$$f(\theta) := E\int_0^T Z_t c_t^*(\theta)dt = C\int_0^T \tilde{P}(e^{\beta t}Z_t < 1/\theta)dt$$

which is continuous and decreasing from $f(0) = CT$ to $f(\infty) = 0$.

Let us show that the optimal consumption process is $c^o := c^*(\theta^*)$ where θ^* is defined as the solution of the equation $f(\theta^*) = x - H(I^o)$ and this solution we assume existing (otherwise the problem is trivial with the optimal solution $c_t^o = C$). Indeed, let $c = (c_t)$ be an arbitrary consumption process satisfying the constraint with the equality. Then

$$J(c^o) - J(c) = E\int_0^T [e^{-\beta t}U(c^o) - \theta^* Z_t c_t^o - e^{-\beta t}U(c_t) + \theta^* Z_t c_t]dt$$

and we get the result because the right-hand side is non-negative due to the choice of c^o as the maximizer of the unconstrained problem with the multiplier θ^*.

Acknowledgment

The authors are grateful to Andrei Dmitruk to whom they are indebted for the arguments on the turnpike behavior used in Subsection 4.2. His expertise in the Pontriagin maximum principle is greatly appreciated. We expressed also our thanks to the anonymous referee for helpful remarks.

References

1. Aubin, J.-B.: *Optima and Equilibria. An Introduction to Nonlinear Analysis.* Berlin Heidelberg New York: Springer 1993
2. Bismut J.-M. Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44**, 384–404 (1973)
3. Bismut J.-M. An introductory approach to duality in optimal stochastic control. *SIAM Review* **20**, 1, 62–78 (1978)
4. Cox J.C., Huang C.: Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econ. Theory* **49**, 33–83 (1989)
5. Cuoco D.: Optimal consumption and equilibrium prices with portfolio contraints and stochastic income. *J. Econ. Theory* **72**, 33–73 (1997)
6. El Karoui N., Peng S., Quenez M.-C.: Backward stochastic differential equation in finance. *Math. Finance*, **7**, 1, 1–71 (1997)
7. Harrison M., Pliska S.: Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications **11**, 215–260 (1981)
8. Hirayama T., Kijima M.: A generalized consumption/investment decision problem with production possibilities. Working paper (1991)
9. Karatzas I.: Optimization problems in the theory of continuous trading. SIAM J. Control and Optimization **27**, 1221–1259 (1989)
10. Karatzas I., Lehoczky J.P, Shreve S.E.: Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. trading. SIAM J. Control and Optimization **25**, 1557–1586 (1987)
11. Karatzas I., Lehoczky J.P, Sethi S.P., Shreve S.E.: Explicit solution of a general consumption/investment problem. trading. Math. Oper. Res. **11**, 261–294 (1986)
12. Karatzas I., Shreve S.E.: Brownian Motion and Stochastic Calculus. Berlin Heidelberg New York: Springer 1988
13. Karatzas I., Shreve S.E.: Methods of Mathematical Finance. Berlin Heidelberg New York: Springer 1998
14. Kort P.M.: The influence of a stochastic environement on the firm’s optimal dynamic investment policy. Optimal Control Theory and Economic Analysis **3**, 247–257. Ed.: G. Feichtinger. Amsterdam: North-Holland 1971
15. Merton R.C.: Lifetime portfolio selection under uncertainty: the continuous-time case. Rev. Econ. Stat. **51**, 247–257 (1969)
16. Merton R.C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **3**, 373–413 (1971)
17. Pliska S.: A stochastic calculus model of continuous trading: optimal portfolio. Math. Oper. Res. **11**, 371–382 (1986)
18. Rockafellar R.T.: Convex Analysis. Princeton: Princeton University Press 1970
19. Zame W.R.: Competetive equilibria in production economies with an infinite dimensional space. Econometrica **55**, 1075–1108 (1987)
20. Yong J., Zhou X.Y. Stochastic Control. Hamiltonian Systems and HJB Equations. Berlin Heidelberg New York: Springer 1999