Enlargement of Filtration and Additional Information in Pricing Models: Bayesian Approach

Dario GASBARRA[1], Esko VALKEILA[2], and Lioudmila VOSTRIKOVA[3]

[1] University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FIN-00014, Finland.
Dario.Gasbarra@rni.helsinki.fi
[2] Institute of Mathematics, P.O. Box 1100, FIN-02015 Helsinki University of Technology, Finland.
Esko.Valkeila@tkk.fi
[3] Departement de Mathématiques, Université d'Angers, France.
vostrik@univ-angers.fr

Summary. We show how the dynamical Bayesian approach can be used in the initial enlargement of filtrations theory. We use this approach to obtain new proofs and results for Lévy processes. We apply the Bayesian approach to some problems concerning asymmetric information in pricing models, including so-called weak information approach introduced by Baudoin, as well as some other approaches. We give also Bayesian interpretation of utility gain related to asymmetric information.

Key words: dynamical Bayesian modelling, enlargement of filtration, asymmetric information, Lévy processes

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1 Introduction

The initial enlargement of filtrations is an important topic in the theory of stochastic processes, and it was studied in the fundamental works of Jeulin [20], Jacod [18], Stricker and Yor [23] and Yor [24, 25] and others.

Recent interest to this question comes from pricing models in stochastic finance, where the enlargement of filtrations theory is an important tool in modelling of asymmetric information between different agents and the possible additional gain due to this information (see Amendinger et al. [1], Imkeller et al. [16] Baudoin [3, 4], Elliot and Jeanblanc [13] and others). For an approach based on anticipating calculus, see, e.g., [21].

The initial enlargement of filtration consists in the following.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and let X be a semimartingale with the $(P, \mathbf{F})$-triplet $T = (B, C, \nu)$ of predictable characteristics of (we refer to [19] and Section 2 for more details on semimartingales). Suppose that we are given a random variable ϑ on $(\Omega, \mathcal{F})$ such that $\sigma(\vartheta) \subsetneq \mathcal{F}_0$. Define now $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\vartheta)$; then $\Gamma = (\mathcal{G}_t)_{t\geq 0}$ is the initially enlarged filtration. The main problems studied are: is the $\mathbf{F}$-semimartingale X still a semimartingale with respect to the filtration Γ and if this is true, what is the new triplet $T^\vartheta = (B^\vartheta, C^\vartheta, \nu^\vartheta)$ with respect to (P, Γ)?

Surprising at the first glance [and very natural, in fact] the Bayesian approach proposed in the papers by Dzhaparidze et al. [9, 10] is closely related to the problem of enlargement of filtrations. In the Bayesian approach one of the basic concepts is the arithmetic mean measure. This means the following. Suppose that on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ we observe a semimartingale $X = (X_t)_{t\geq 0}$, and the law P^θ of X depends of a parameter $\theta \in \Theta$. Assume that θ is a value of some random variable ϑ, taking values in a measurable Polish space $(\Theta, \mathcal{A})$ where $\mathcal{A}$ is the Borel σ-algebra. Denote the law of the random variable ϑ by α. We suppose that for each $\theta \in \Theta$ the measure P^θ is absolutely continuous with respect to P and that the density process z^θ is measurable with respect to $\mathcal{F} \otimes \mathcal{A}$. Then we can introduce on the original space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ the arithmetic mean measure $\bar{P}^\alpha$: for $B \in \mathcal{F}$

$$\bar{P}^\alpha(B) := \int_\Theta P^\theta(B)\alpha(d\theta) = \int_\Theta \int_B z^\theta dP\alpha(d\theta).$$

One can interpret the measure $\bar{P}^\alpha$ also as a 'randomised experiment'. In [9, 10] it is shown how to compute the predictable characteristics of X with respect to the arithmetic mean measure $\bar{P}^\alpha$ given the characteristics T^θ of X with respect to P^θ.

The Bayesian approach to the initial enlargement of filtration goes as follows. Suppose for simplicity that the initial σ-algebra is trivial. Let X be a semimartingale with the $(P, \mathbf{F})$-triplet $T = (B, C, \nu)$. We suppose that we have, in addition, a random variable $\vartheta : (\Omega, \mathcal{F}) \to (\Theta, \mathcal{A})$ with values in a Polish space and the prior law α.

We consider next the product space $(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{A}, \mathbb{G}, \mathbb{P})$ with the filtration $\mathbb{G} = (\mathbb{G}_t)_{t\geq 0}$ defined by $\mathbb{G}_t = \mathcal{F}_t \otimes \mathcal{A}$ and $\mathbb{P}$ is the joint law of $(X(\omega), \vartheta(\omega))$. Let $t \in \mathbb{R}_+$ and α^t be the regular a posteriori distribution of the random variable ϑ given the information $\mathcal{F}_t$:

$$\alpha^t(\omega, \theta) := P(\vartheta \in d\theta | \mathcal{F}_t)(\omega).$$

Assume now that $\alpha^t \ll \alpha$. Then, according to the results of Jacod [18] the process $z^\theta = (z_t^\theta)_{t\geq 0}$ where

$$z_t^\theta(\omega) := \frac{d\alpha^t(\omega, \theta)}{d\alpha(\theta)},$$

is a $(P, \mathbf{F})$-martingale with $z_0^\theta = 1$. Define now a measure P^θ by

$$dP_t^\theta := z_t^\theta dP_t,$$

where the subscript means the restriction of the measure to the sub-σ-algebra $\mathcal{F}_t$. Then the process X is also a $(P^\theta, \mathbf{F})$-semimartingale. If we know the structure of the density martingale z^θ, then, using the Itô formula, we can write a semimartingale decomposition of it and the $(P^\theta, \mathbf{F})$-triplet $T^\theta = (B^\theta, C, \nu^\theta)$. Finally, if T^θ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable, one obtains the $(P, \varGamma)$-triplet of the semimartingale X by replacing in T^θ the fixed parameter θ by the random variable ϑ. This method is relatively simple and gives a unifying approach to various concrete models like diffusion processes, counting processes and Lévy processes. It can also be used outside the semimartingale world. Some applications will be given in the paper [12].

The paper contains two parts. The first one is devoted to the initial enlargement of filtration. We begin with reminding some basic facts on semimartingale characteristics and the Girsanov theorem. Then we apply the Bayesian approach to the initial enlargement. For somewhat related studies see [6, 14]. We continue by giving some examples of the initial enlargement with the final value. The Bayesian approach can be developed for the progressive enlargement of filtration as well. This will be done in a later work.

The second part is devoted to so-called weak information introduced in Baudoin [3, 4]. We show that the notion of weak information can be interpreted as changing the "true" prior α, the law of the random variable ϑ, to another prior distribution γ for the random variable ϑ. After this the whole analysis can be reduced to the computation of the $\bar{P}^\gamma$-characteristics of the semimartingale X.

Some preliminary results of the Bayesian approach were already obtained in [11]. We extend and generalize the results in various directions: in addition to several examples and new applications, we give a Bayesian interpretation of the so-called additional utility of an insider, or of a weak insider and, finally, the gain on false information.

2 Characteristics of a semimartingale

We shall work with a semimartingale X defined on a filtered space $(\varOmega, \mathcal{F}, \mathbf{F}, P)$. Recall some facts concerning the triplet T of a semimartingale X. Since the triplet T depends on the probability measure P and on the filtration, we keep track of the measures and filtrations in what follows. We assume that $\mathbf{F} := \mathbf{F}^X$ is the right-continuous version of the natural filtration of X (completed by P-null sets and that $\mathcal{F} = \mathcal{F}_\infty^X$.

Let μ be the jump measure of X, i.e.

$$\int_0^t \int_{|x|>\epsilon} x\mu(ds, dx) := \sum_{s \le t} \varDelta X_s \mathbf{1}_{\{|\varDelta X_s| > \epsilon\}}.$$

We use the standard notation from [19] and [15]: if $\mu := \mu^X$ is the jump measure of the semimartingale X, then $g * \mu$ means the integral with respect to the jump measure, $g * \nu$ denotes the integral with respect to the $(P, \mathbf{F})$-compensator ν of μ; later $g \cdot U$ is the stochastic integral with respect to a local martingale U or Riemann–Stieltjes integral with respect to a bounded variation process U.

Suppose that the semimartingale X has characteristics $T = (B, C, \nu)$ with respect to $(P, \mathbf{F})$. Recall that this means the following (see [19] for more details and unexplained terminology). Let $l : \mathbb{R} \to \mathbb{R}$ be a truncation function: $l(x) = x$ in the neighborhood of zero and l has a compact support. Then one can write the semimartingale X as

$$X = (X - X(l)) + X(l),$$

where $X(l)$ is a purely jump process, namely, the process with 'big' jumps defined as

$$X(l)_t := \sum_{s \le t} (\Delta X_s - l(\Delta X_s))$$

with $\Delta X_s = X_s - X_{s-}$.

Having bounded jumps, the process $\tilde{X} = (X - X(l))$ is a special semimartingale and allows the representation

$$\tilde{X}_t = X_0 + X_t^c + \int_0^t \int_{\mathbb{R} \setminus \{0\}} l(x)(\mu(ds, dx) - \nu(ds, dx)) + B_t(l),$$

where X^c is the continuous local martingale part of X, ν is the $(P, \mathbf{F})$ compensator of μ, $B_t(l)$ is the unique $(P, \mathbf{F})$-predictable locally integrable process such that the process $\tilde{X} - B(l)$ is a $(P, \mathbf{F})$-local martingale. Let C be the continuous process such that the process $(X^c)^2 - C$ is a $(P, \mathbf{F})$-local martingale. Having all this we have defined the *triplet of predictable characteristics* of a semimartingale X as $T = (B(l), C, \nu)$. Later we write B instead of $B(l)$.

Consider the class $\mathcal{G}$ of real bounded Borel functions on $\mathbb{R}$ vanishing in a neighborhood of 0. If η and $\tilde{\eta}$ are measures on $\mathbb{R}$ such that $\eta(|x| > \epsilon) < \infty$ and $\tilde{\eta}(|x| > \epsilon) < \infty$, and if for all $g \in \mathcal{G}$

$$\int_{\mathbb{R}} g(x)\eta(dx) = \int_{\mathbb{R}} g(x)\tilde{\eta}(dx)$$

then $\eta = \tilde{\eta}$.

Recall Theorem II.2.21 from [19, p.80]

Theorem 2.1. *A semimartingale X has the $(P, \mathbf{F})$-triplet $T = (B, C, \nu)$ if and only if*

- *The process $M(l) := X - X(l) - B - X_0$ is a local martingale.*

- *The process*

$$N(l) := M(l)^2 - C^2 - l^2 * \nu - \sum_{s \le \cdot} (\Delta B_s)^2$$

is a local martingale.
- *The process* $U(l) := g * (\mu - \nu)$ *is a local martingale whatever is* $g \in \mathcal{G}$.

Assume moreover that we have on $(\Omega, \mathcal{F}, \mathbf{F}, P)$ a family of probability measures P^θ with $\theta \in \Theta$ such that $P^\theta_t \ll P_t$ for all $t \in \mathbb{R}_+$.

Let $\theta \in \Theta$ be fixed. Then X is a $(P^\theta, \mathbf{F})$-semimartingale with a triplet $T^\theta = (B^\theta, C^\theta, \nu^\theta)$ where

$$\begin{aligned} B^\theta &= B + \beta^\theta \cdot C + (Y^\theta - 1) l * \nu, \\ C^\theta &= C, \\ \nu^\theta &= Y^\theta \cdot \nu, \end{aligned} \tag{2.1}$$

with certain $(P^\theta, \mathbf{F})$-predictable processes $\beta^\theta = (\beta^\theta_t)_{t \ge 0}$ and $Y^\theta = (Y^\theta_t)_{t \ge 0}$ such that for all $t \in \mathbb{R}^+$

$$((\beta^\theta)^2 \cdot C)_t + (|(Y^\theta - 1) l| * \nu)_t < \infty. \tag{2.2}$$

For more details see [19].

We denote by P^θ_t and P_t the restrictions of the corresponding measures on $\mathcal{F}_t$ and we define the density process $z^\theta = (z^\theta_t)_{t \ge 0}$ with

$$z^\theta_t = \frac{dP^\theta_t}{dP_t}.$$

We note that the density process is $(P, \mathbf{F})$-martingale with the property $\inf_{t \in [0,T]} z^\theta_t > 0$ P-a.s. for each $T > 0$, and we define the *stochastic logarithm* m^θ of z^θ by

$$dm^\theta := dz^\theta / z^\theta_-. \tag{2.3}$$

Then m^θ is a $(P, \mathbf{F})$-local martingale and z^θ is the *stochastic exponential* of m^θ:

$$z^\theta_t = \mathcal{E}(m^\theta)_t.$$

Assume now that X is a $(P, \mathbf{F})$-semimartingale with a triplet $T = (B, C, \nu)$ and that the natural filtration $\mathbf{F}$ of X has the *predictable representation property* : a local martingale M with respect to this filtration has the representation:

$$M = M_0 + H \cdot X^c + W * (\mu - \nu). \tag{2.4}$$

Here the predictable process H belongs to the space L^2_{loc} of locally square-integrable processes with respect to C and the function $W = W_t(\omega; x)$ belongs to $G_{loc}(\mu)$. For information on the space $G_{loc}(\mu)$ see [19, II.1.1,pp. 72-74]. On the predictable representation property one can consult [19, p.185].

By the predictable representation property we have that the local martingale m^θ from (2.3) has the following semimartingale representation

$$m^\theta = \beta^\theta \cdot X^c + \left(Y^\theta - 1 + \frac{\hat{Y}^\theta - \hat{1}}{1 - \hat{1}} \right) * (\mu - \nu), \tag{2.5}$$

where the processes β^θ and Y^θ are the same as in (2.1) and the "hat" processes are related to the jumps of the compensator ν, namely

$$\hat{1}_t(\omega) := \nu(\omega; \{t\} \times \mathbb{R}_0)$$

and

$$\hat{Y}_t^\theta(\omega) := \int_{\mathbb{R}_0} Y_t^\theta(\omega, x)\nu(\omega, \{t\}, dx).$$

So, to find the triplet T^θ we can read β^θ and Y^θ from (2.5) and use (2.1) .

3 Arithmetic mean measure

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with $\mathcal{F} = \mathcal{F}_\infty$. Suppose that we are given a parametric family of probability measures $(P^\theta)_{\theta \in \Theta}$ where θ belongs to a measurable Polish space $(\Theta, \mathcal{A})$.

We make the following assumption.

Assumption 3.1 *For each $\theta \in \Theta$ the probability P^θ is locally absolute continuous with respect to P .*

Then we can define density process: for each $\theta \in \Theta$ and $t \in \mathbb{R}_+$

$$z_t^\theta = \frac{dP_t^\theta}{dP_t}$$

where P_t^θ and P_t are the restrictions of P^θ and P on $\mathcal{F}_t$. We consider measurable with respect to θ versions of the density processes. Given a probability measure α on $(\Theta, \mathcal{A})$ and $t \in \mathbb{R}_+$ and $B \in \mathcal{F}_t$, we define *the arithmetic mean measure*:

$$\bar{P}_t^\alpha(B) := \int_\Theta P_t^\theta(B)\alpha(d\theta) = \int_{\Theta \times B} z_t^\theta P(d\omega)\alpha(d\theta), \qquad \bar{P}_t^\alpha.$$

Remark 1. In the case of the initial enlargement by a random variable ϑ such that $\alpha = \mathcal{L}(\vartheta | P)$, considered in Section 4, we have $\bar{P}^\alpha = P$. This follows from the fact that in this case P^θ is the regular conditional law of X given $\vartheta = \theta$.

We see that $\bar{P}_t^\alpha$ is absolutely continuous with respect to P_t but, in general, P_t^θ is not absolutely continuous with respect to $\bar{P}_t^\alpha$. For this reason we add another assumption.

Assumption 3.2 *For each $\theta \in \Theta$ the probability P^θ is locally absolute continuous with respect to $\bar{P}^\alpha$.*

Assume again that X is a $(P, \mathbf{F})$-semimartingale with triplet $T = (B, C, \nu)$ and having the representation property. Then X is a $(P^\theta, \mathbf{F})$-semimartingale with a triplet $T^\theta = (B^\theta, C^\theta, \nu^\theta)$ where B^θ , C^θ , ν^θ are given in (2.1).

The next theorem is a generalization of a result by Kolomiets.

Theorem 3.1. *Suppose that the assumptions 3.1 and 3.2 hold and X is a $(P, \mathbf{F})$-semimartingale with triplet $T = (B, C, \nu)$. Then X is also a $(\bar{P}^\alpha, \mathbf{F})$-semimartingale with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ defined by*

$$\begin{aligned} \bar{B} &= E_\alpha \bar{z}^\theta_- \cdot B^\theta = B + E_\alpha \bar{z}^\theta_- \beta^\theta \cdot C + E_\alpha \bar{z}^\theta_- (Y^\theta - 1) l * \nu, \\ \bar{C} &= C, \\ \bar{\nu} &= E_\alpha \bar{z}^\theta_- Y^\theta \cdot \nu, \end{aligned} \tag{3.1}$$

where $\bar{z}^\theta$ is the density of P^θ with respect to the arithmetic mean measure $\bar{P}^\alpha$.

For the proof see [8, Theorem 3.3].

To interchange the order of integration in (3.1) by using the Fubini theorem we introduce the following notation. For each $t \in \mathbb{R}_+$ we define a posteriori measure α^t. To do it for $B \in \mathcal{A}$ we put

$$\alpha^t(B) := \frac{\int_B z^\theta_t \alpha(d\theta)}{\int_\Theta z^\theta_t \alpha(d\theta)}.$$

Let us define $\alpha^{t-}(d\theta)$ in the following natural way: for $B \in \mathcal{A}$

$$\alpha^{t-}(B) := \frac{\int_B z^\theta_{t_-} \alpha(d\theta)}{\int_\Theta z^\theta_{t_-} \alpha(d\theta)}.$$

Assuming that β^θ_t and Y^θ_t are integrable with respect to α^{t-}, we put

$$\bar{\beta}_t = E_{\alpha^{t-}} \beta^\theta_t, \quad \bar{Y}_t = E_{\alpha^{t-}} Y^\theta_t. \tag{3.2}$$

Theorem 3.2. *Suppose that the assumptions 3.1 and 3.2 hold and for $t > 0$*

$$E_{\alpha^{t-}} |\beta^\theta_t| \cdot C_t + E_{\alpha^{t-}} |Y^\theta - 1| l * \nu_t < \infty. \tag{3.3}$$

Then X is a $(\bar{P}^\alpha, \mathbf{F})$-semimartingale with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ defined by

$$\begin{aligned} \bar{B} &= B + \bar{\beta} \cdot C + (\bar{Y} - 1) l * \nu \\ \bar{C} &= C, \\ \bar{\nu} &= \bar{Y} \cdot \nu \end{aligned} \tag{3.4}$$

where $\bar{\beta}$ and $\bar{Y}$ are given in (3.2).

Proof To prove our result we use the classical Fubini theorem. In order to do it, we show that $\bar{B}$ is the process of locally P-integrable variation. In fact, for all $t > 0$

$$\mathrm{Var}(\bar{B})_t \leq \mathrm{Var}(B)_t + E_\alpha \bar{z}^\theta_- |\beta^\theta| \cdot C_t + E_\alpha \bar{z}^\theta_- |Y^\theta - 1| l * \nu_t.$$

Using classical Fubini theorem for positive functions in last two integrals and integration with respect the measure α^{t-} we have: for all $t > 0$

$$\mathrm{Var}(\bar{B})_t \leq \mathrm{Var}(B)_t + E_{\alpha^-} |\beta^\theta| \cdot C_t + E_{\alpha^-} |Y^\theta - 1| l * \nu_t.$$

We define a localizing sequence as follows. Put

$$\tau_n = \inf\{t \geq 0 : \; E_{\alpha^{t-}} |\beta^\theta| \cdot C_t + E_{\alpha^{t-}} |Y^\theta - 1| l * \nu_t + \mathrm{Var}(B)_t > n\}. \quad (3.5)$$

and notice that τ_n is $\mathbf{F}$-stopping time. Moreover, since the jumps of considered processes are bounded by a constant, we can easily verify that

$$E_{\bar{P}^\alpha}[E_{\alpha^{t-}} |\beta^\theta| \cdot C_{\tau_n} + E_{\alpha^{t-}} |Y^\theta - 1| l * \nu_{\tau_n} + \mathrm{Var}(B)_{\tau_n}] < n + 3 \max_{x \in \mathbb{R}} l(x),$$

where l is the truncation function. Now, we notice that the sequence of $\mathbf{F}$-stopping times τ_n increases to infinity due to the condition (3.3). Then, we localize with τ_n and apply the classical Fubini theorem to (3.1) and we have (3.4). □

Remark 2. Theorem 3.2 is a special case of the stochastic Fubini theorem. Namely, we know that

$$z^\theta_t = \mathcal{E}(m^\theta)_t,$$

where

$$m^\theta = \beta^\theta \cdot X^c + \left(Y^\theta - 1 + \frac{\hat{Y}^\theta - \hat{1}}{1 - \hat{1}}\right)$$

Then by Theorem 3.2 we have the following variant of stochastic Fubini theorem

$$\bar{z}_t = \int_\Theta z^\theta_t \alpha(d\theta) = \mathcal{E}(\bar{m})_t$$

with

$$\bar{m} = \bar{\beta} \cdot X^c + \left(\bar{Y} - 1 + \frac{\hat{\bar{Y}} - \hat{1}}{1 - \hat{1}}\right).$$

Sometimes the verification of the condition (3.3) can be difficult and we can be interested to replace it by another condition expressed in terms of the density process. For instance, we can use the following assumption.

Assumption 3.3 *There exists a localizing sequence of* $\mathbf{F}$- *stopping times* τ_n *such that for every* $n \geq 1$

$$E \int_\Theta [z^\theta, z^\theta]^{1/2}_{\tau_n} \alpha(d\theta) < \infty$$

where E *is the expectation with respect to the initial measure* P.

Theorem 3.3. *Suppose that the assumptions 3.1, 3.2, 3.3 hold. Then X is a $(\bar{P}^\alpha, \mathbf{F})$-semimartingale with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ defined by (3.4).*

Proof In fact, we have only to show that the assumption 3.3 implies the local integrability of the variation of $\bar{B}$. Since B is locally integrable with respect to the arithmetic mean measure, which follows from the fact that the jumps of B are bounded by a constant, we have only to show that there exists a localizing sequence of stopping times s_n such that for each n

$$E_{\bar{P}^\alpha}\left(E_{\alpha^-}|\beta^\theta| \cdot C_{\tau_n} + E_{\alpha^-}|Y^\theta - 1| l * \nu_{\tau_n}\right) < \infty. \tag{3.6}$$

Let

$$\bar{Z}_t = \frac{d\bar{P}_t}{dP_t}.$$

We remark that

$$\bar{Z}_t = \int_\Theta z_t^\theta \alpha(d\theta).$$

Using the fact that $\bar{Z}$ is a positive $(P, \mathbf{F})$-martingale and the observation that we are dealing with the predictable positive processes, we obtain:

$$\begin{aligned}
&E_{\bar{P}^\alpha}\left(E_{\alpha^-}|\beta^\theta| \cdot C_{\tau_n} + E_{\alpha^-}|Y^\theta - 1| l * \nu_{\tau_n}\right)\\
&= E_P \bar{Z}_{\tau_n}\left(E_{\alpha^-}|\beta^\theta| \cdot C_{\tau_n} + E_{\alpha^-}|Y^\theta - 1| l * \nu_{\tau_n}\right)\\
&= \int_\Theta E_P\{z_-^\theta|\beta^\theta| \cdot C_{\tau_n} + z_-^\theta|Y^\theta - 1| l * \nu_{\tau_n}\}\alpha(d\theta)\\
&= \int_\Theta E_P\{z_-^\theta|\beta^\theta| \cdot C_{\tau_n} + z_-^\vartheta|Y^\theta - 1| l * \mu_{\tau_n}^X\}\alpha(d\theta)\\
&= \int_\Theta E_P\{\mathrm{Var}([z^\theta, X(l) - B])_{\tau_n}\}\alpha(d\theta)
\end{aligned}$$

Let

$$\tau_n' = \inf\{t \geq 0 : \sup_{0 \leq s \leq t} |X_s(l) - B_s| > n\}$$

and $s_n = \tau_n' \wedge \tau_n$. By the Fefferman inequality, (see [15, Theorem 10.17]) and the fact that $X(l) - B$ is $(P, \mathbf{F})$-local martingale we deduce that

$$E_P \mathrm{Var}([z^\theta, X(l) - B])_{s_n} \leq \| (X(l) - B)^{s_n} \|_{BMO} \, E_P[z^\theta, z^\theta]_{s_n}^{1/2}.$$

We remark that

$$\| (X(l) - B)^{s_n} \|_{BMO} \leq 2(n + 2 \max_x l(x))$$

where l is truncation function. So, after integration with respect to α, we obtain from assumption 3.3 that (3.6) holds, and, hence, $\bar{B}$ has locally integrable variation with respect to $\bar{P}^\alpha$. □

4 Initial enlargement

4.1 Triplet and initial enlargement

Let X be a semimartingale on a filtered space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the (right-continuous completed) natural filtration of X. Let $T = (B, C, \nu)$ be the $(P, \mathbf{F})$-triplet of X.

Suppose that we have also a random variable ϑ with values in measurable Polish space $(\Theta, \mathcal{A})$. Define the initially enlarged filtration $\Gamma = (\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\vartheta)).$$

Our problem is to find the semimartingale decomposition of X with respect to the enlarged filtration Γ.

Let α be the distribution of the random variable ϑ, i.e. $P(\vartheta \in d\theta) = \alpha(d\theta)$. Let for α^t be its regular conditional distribution with respect to the σ-algebra $\mathcal{F}_t$. Following Bayesian terminology we say that α is the *a priori distribution* and α^t is the *a posteriori distribution* of the random variable ϑ with respect to the information $\mathcal{F}_t$.

We make the following standing assumption.

Assumption 4.1 *The posterior distributions α^t and the prior distribution α satisfy: for each $t \in [0, T]$ we have P-a.s.*

$$\alpha^t \ll \alpha. \tag{4.1}$$

We make a stop to discuss the right-continuity of the filtration Γ: in Amendinger [2, Proposition 3.3] it is shown that under the assumption $\alpha^t \sim \alpha$ we have that $\mathcal{G}_t = F_t \vee \sigma(\vartheta)$. Inspecting the proof of this result in [2], one can see that, in fact, it is sufficient to assume only assumption 4.1. So, under assumption 4.1 we have $\mathcal{G}_t = F_t \vee \sigma(\vartheta)$.

We consider next the product space $(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{A}, \mathbb{G}, \mathbb{P})$ where the filtration $\mathbb{G} = (\mathbb{G}_t)_{t\geq 0}$ is defined by

$$\mathbb{G}_t = \bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{A}) \tag{4.2}$$

and $\mathbb{P}$ is the joint law of $(\omega, \vartheta(\omega))$. Again, under assumption 4.1 we can take $\mathbb{G}_t = F_t \otimes \mathcal{A}$.

Denote the optional and predictable σ-algebras on $(\Omega \times \mathbb{R}^+)$ with respect to $\mathbf{F}$ by $\mathcal{O}(\mathbf{F})$ and $\mathcal{P}(\mathbf{F})$. With the filtration $\mathbb{G}$ we have that

$$\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$$

and

$$\mathcal{O}(\mathbf{F}) \otimes \mathcal{A} \subset \mathcal{O}(\mathbb{G}).$$

The following result is due to Jacod [18, Lemme 1.8., p.18-19].

Lemma 4.1. *Under assumption 4.1 there exists a strictly positive* $\mathcal{O}(\mathbb{G})$-*measurable function* $(\omega, t, \theta) \mapsto z_t^\theta(\omega)$ *such that:*

1. *For each* $\theta \in \Theta$, z^θ *is a* $(P, \mathbf{F})$-*martingale.*
2. *For each* $t \in \mathbb{R}_+$, *the measure* $z_t^\theta \alpha(d\theta)$ *is a version of the regular conditional distribution* $\alpha^t(d\theta)$ *so that* $P_t \times \alpha$-*a.s.*

$$\frac{d\alpha^t}{d\alpha}(\theta) = z_t^\theta. \tag{4.3}$$

For each $\theta \in \Theta$ define also a measure P^θ:

$$dP_t^\theta := z_t^\theta dP_t. \tag{4.4}$$

The measure P^θ is absolutely continuous with respect to the P, and so X is a $(P^\theta, \mathbf{F})$-semimartingale with the $(P^\theta, \mathbf{F})$-triplet $T^\theta = (B^\theta, C, \nu^\theta)$.

Next we indicate how one can use the prior and posterior distributions to obtain the semimartingale decomposition of a $(P, \mathbf{F})$-semimartingale with respect to the filtration Γ.

1. We are given a semimartingale X with $(P, \mathbf{F})$-triplet $T = (B, C, \nu)$, where the natural filtration $\mathbf{F}$ has the representation property, random variable ϑ, prior $\alpha(d\theta) = P(\vartheta \in d\theta)$ and posterior $\alpha^t(d\theta) = P(\vartheta \in d\theta|\mathcal{F}_t)$.
2. Compute $\dfrac{d\alpha^t}{d\alpha}(\theta)$ with the Itô formula as $\mathcal{E}(m^\theta)$ and read β^θ and Y^θ from the representation (2.5), use (2.1) to obtain T^θ .
3. If T^θ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable, replace θ by ϑ in T^θ to obtain the triplet of X with respect to (P, Γ).

In the following theorem we give the link between the Girsanov theorem and enlargement of filtrations.

Theorem 4.1. *Assume that the process* X *is a* $(P, \mathbf{F})$-*semimartingale with triplet* $T = (B, C, \nu)$ *and we have the martingale representation property with respect to natural filtration* $\mathbf{F}$. *Let* ϑ *be a random variable such that the assumption (4.1) is satisfied. Suppose also that* $L^1(\Omega, \mathcal{F}, P)$ *is separable and the condition (3.3) holds.*

Then the following conditions are equivalent:

(a) X *is a* $(P^\theta, \mathbf{F})$-*semimartingale with triplet* $T^\theta = (B^\theta, C, \nu^\theta)$ *on the space* $(\Omega, \mathcal{F}, \mathbf{F}, P)$ *for* α-*almost all* θ *and the application* $T' : (\omega, t, \theta) \to T_t^\theta(\omega)$ *is* $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-*measurable.*

(b) X *is a* $(\mathbb{P}, \mathbb{G})$-*semimartingale with triplet* $T^{'} : (\omega, t, \theta) \to T_t^\theta(\omega)$ *on the product space* $(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{A}, \mathbb{G}, \mathbb{P})$ *where* $\mathbb{P}$ *is the joint law of* $(\omega, \vartheta(\omega))$.

(c) X *is a* (P, Γ)-*semimartingale on* $(\Omega, \mathcal{F}, P)$ *with triplet* $T^\vartheta = (B^\vartheta, C, \nu^\vartheta)$.

Remark 1. It should be noticed that separability condition will be used only in the direction: $c) \Rightarrow b) \Rightarrow a)$.

To prove the theorem we need some lemmas concerning the transformation of triplets, stopping times and martingales.

Lemma 4.2. *The function* $X : (\omega, t, \theta) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is* $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$*-measurable if and only if* $X^\vartheta : (\omega, t, \vartheta(\omega)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is* $\mathcal{P}(\Gamma)$*-measurable.*

Proof It is sufficient to establish the property on semi-algebras generating the corresponding σ-algebras. Let now $a, b, c \in \mathbb{R}, a < b, A \in \mathcal{F}_a, B \in \mathcal{A}$ and

$$X(\omega, t, \theta) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_A(\omega)\mathbf{1}_B(\theta). \tag{4.5}$$

Then X is an element of semi-algebra generating $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$ and

$$X^\vartheta(\omega, t, \vartheta(\omega)) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_A(\omega)\mathbf{1}_B(\vartheta(\omega)) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_{A\cap\vartheta^{-1}(B)}(\omega). \tag{4.6}$$

Since the set $A \cap \vartheta^{-1}(B)$ belongs to $\mathcal{F}_a \vee \sigma(\vartheta)$, it belongs also to $\mathcal{G}_a$, and the function X^ϑ defined by (4.6) is an element of $\mathcal{P}(\Gamma)$.

Inversely, let $a, b, c \in \mathbb{R}, a < b, C \in \mathcal{G}_{a-}$, then

$$X^\vartheta(\omega, t, \vartheta(\omega)) = c\mathbf{1}_{(a,b]}(t)\mathbf{1}_C(\omega) \tag{4.7}$$

is an element of semi-algebra generating $\mathcal{P}(\Gamma)$. Since $\mathcal{G}_{a-} = \bigvee_{s<a}(\mathcal{F}_s \vee \sigma(\vartheta))$ it suffices to consider elements of the generating algebra $\bigcup_{s<a}(\mathcal{F}_s \vee \sigma(\vartheta))$. In turn, if $C \in \bigcup_{s<a}(\mathcal{F}_s \vee \sigma(\vartheta))$, then there exists $s < a$ such that $C \in \mathcal{F}_s \vee \sigma(\vartheta)$. Next, the σ-algebra $\mathcal{F}_s \vee \sigma(\vartheta)$ is generated by the sets $A \cap \vartheta^{-1}(B)$ with $A \in \mathcal{F}_s$ and $B \in \mathcal{A}$. So, we have to consider only the elements X^ϑ of the form (4.7) with $C = A \cap \vartheta^{-1}(B)$. But the corresponding application X is (4.5) and it is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable. □

Lemma 4.3. *Let for each* $\theta \in \Theta$ *the process* $(X_t^\theta)_{t\geq 0}$ *be an* $\mathbf{F}$*-adapted càdlàg process. Let* $L > 0$ *and*

$$\tau_L^\theta = \inf\{s \geq 0 : \ X_s^\theta(\omega) > L\}. \tag{4.8}$$

If the application $X : (\omega, t, \theta) \to X_t^\theta$ *is* $\mathcal{O}(\mathbb{G})$*-measurable, then*

$$\tau_L^\vartheta = \inf\{s \geq 0 : \ X_s^{\vartheta(\omega)}(\omega) > L\}$$

is a Γ*-stopping time.*

Proof Let $t \in \mathbb{R}_+$. Then

$$\{(\omega, \theta) : \ \tau_L^\theta > t\} = \{(\omega, \theta) : \ \sup_{s\leq t} X_s^\theta \leq L\} \in \mathbb{G}_t$$

where $\mathbb{G}_t$ is defined by (4.2). It means that for all $u > t$

$$\{(\omega, \theta) : \tau_L^\theta > t\} \in \mathcal{F}_u \otimes \mathcal{A}.$$

Since $\mathcal{F}_u \otimes \mathcal{A}$ is generated by the semi-algebra of the sets $A \times B$ with $A \in \mathcal{F}_u$ and $B \in \mathcal{A}$, we can restrict ourselves to this special type of sets. But

$$\{\omega : (\omega, \vartheta(\omega)) \in A \times B\} \in \mathcal{F}_u \vee \sigma(\vartheta)$$

and, hence, for $u > t$

$$\{\omega : \tau_L^\vartheta > t\} \in \mathcal{F}_u \vee \sigma(\vartheta).$$

Then, τ_L^ϑ is a Γ-stopping time. □

Lemma 4.4. *Let $\theta \in \Theta$ and $(M_t^\theta)_{t\geq 0}$ be an $\mathbf{F}$-adapted càdlàg process. Let M be the application $(t, \omega, \theta) \to M_t^\theta(\omega)$. Suppose that $L^1(\Omega, \mathcal{F}, P)$ is separable. Then the following conditions are equivalent:*

a) M^θ is $(P^\theta, \mathbf{F})$-martingale for α-almost all θ and M is $\mathcal{O}(\mathbb{G})$-measurable process,
b) M is a $(\mathbb{P}, \mathbb{G})$-martingale,
c) M^ϑ is a (P, Γ)-martingale.

Proof We show that

$$a) \overset{(i)}{\Rightarrow} c) \overset{(ii)}{\Rightarrow} b) \overset{(iii)}{\Rightarrow} a).$$

(i): Let E be the expectation with respect to P and $\mathbf{E}$ be the expectation with respect to $\mathbb{P}$, the joint law of $(\omega, \vartheta(\omega))$. For each $s < t, A \in \mathcal{F}_s, B \in \mathcal{A}$

$$E(\mathbf{1}_A(\omega)\mathbf{1}_B(\vartheta(\omega))(M_t^\vartheta - M_s^\vartheta)) = \mathbf{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M_t^\theta - M_s^\theta)).$$

Let E_α be the expectation with respect to α and E_θ is the expectation with respect to P^θ. Then by the Fubini theorem and conditioning we obtain:

$$\mathbf{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M_t^\theta - M_s^\theta)) = E_\alpha[\mathbf{1}_B(\theta)E_\theta(\mathbf{1}_A(\omega)E_\theta(M_t^\theta - M_s^\theta|\mathcal{F}_s))] = 0$$

since M^θ is a martingale α-a.s. with respect to $(P^\theta, \mathbf{F})$. Hence, P-a.s.

$$E(M_t^\vartheta - M_s^\vartheta|\mathcal{F}_s \vee \sigma(\vartheta)) = 0.$$

Since M^ϑ is càdlàg, using corollary 2.4 of [22], p.59, we have:

$$E(M_t^\vartheta - M_s^\vartheta|\mathcal{G}_s) = \lim_{u\downarrow s} E(M_t^\vartheta - M_s^\vartheta|\mathcal{F}_u \vee \sigma(\vartheta)) = 0$$

which gives $c)$.

(ii): If M^ϑ is (P, Γ)-martingale, then for each $t \in \mathbb{Q}^+$ the random variable M_t^ϑ is $\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(\vartheta))$-measurable and it can be written in the form $M_t^\vartheta(\omega) = M(\omega, t, \vartheta(\omega))$ (P-a.s.) where M is measurable with respect to the filtration $\mathbb{G}_t = \bigcap_{s>t}(\mathcal{F}_t \otimes \mathcal{A})$. Taking a right-continuous version having left-hand limits we obtain the application $M : (\omega, t, \theta) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is $\mathcal{O}(\mathbb{G})$-measurable. For all $s < t$ and $A \in \mathcal{F}_s,\ B \in \mathcal{A}$ we have:

$$\mathbf{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M(\omega,t,\theta)-M(\omega,s,\theta))=E(\mathbf{1}_A(\omega)\mathbf{1}_B(\vartheta(\omega))(M_t^\vartheta-M_s^\vartheta))=0$$

which means that $\mathbb{P}$-a.s.

$$\mathbf{E}(M(\omega,t,\theta)-M(\omega,s,\theta)|\mathcal{F}_s\otimes\mathcal{A})=0$$

and we have b) in the same way as c) before, since M is càdlàg.

(iii): If we have b), then for each (ω,t,θ) we have $M_t^\theta=M(\omega,t,\theta)$. For $A\in\mathcal{F}_s$ and $B\in\mathcal{A}$ we obtain by the Fubini theorem

$$\begin{aligned}0&=\mathbf{E}(\mathbf{1}_A(\omega)\mathbf{1}_B(\theta)(M(\omega,t,\theta)-M(\omega,s,\theta)))\\&=E_\alpha(\mathbf{1}_B(\theta)E_\theta(\mathbf{1}_A(\omega)(M_t^\theta-M_s^\theta))).\end{aligned}$$

Hence, for each $s<t$ and α-a.s.

$$E_\theta(\mathbf{1}_A(M_t^\theta-M_s^\theta))=0.$$

The measurability problem which may occur here is that α-a.s. set can depend on A and s. Since $L^1(\Omega,\mathcal{F},P)$ is separable, we obtain that α-a.s. for all s and all $\mathcal{F}_s$-measurable bounded functions g_s

$$E_\theta(g_s(M_t^\theta-M_s^\theta))=0$$

and, hence,

$$E_\theta(M_t^\theta-M_s^\theta|\mathcal{F}_s)=0$$

which gives a).

□

Proof We show that a), b), c) are equivalent. With the notation of Theorem 2.1, the processes $M^\theta(l)$, $N^\theta(l)$ and $U^\theta(l)$ are $(P,\mathbf{F})$-local martingales. Since the semimartingale $\tilde{X}$ has bounded jumps, all these local martingales are also locally bounded, i.e. for each θ there exists a localizing sequence τ_L^θ such that the stopped processes are bounded. By Lemma 4.3 the replacing θ by ϑ in stopping times gives $\tau_L^\vartheta(\omega)$ which is a (P,Γ)-stopping time. Moreover, the application $\tau_L:(\omega,t,\theta)\to\tau_L^\theta$ is a $(\mathbb{P},\mathbb{G})$-stopping time.

Next, by Lemma 4.2 the replacing of θ by ϑ in T^θ which supposed to be $\mathcal{P}(\mathbf{F})\otimes\mathcal{A}$-measurable, gives T^ϑ which is $\mathcal{P}(\Gamma)$-measurable. Moreover, the application $T':(\omega,t,\theta)\to T^\theta$ is $\mathcal{P}(\mathbb{G})$-measurable.

Finally, the claim follows from Lemma 4.4 which guaranties the conservation of martingale properties in the case of replacing θ by the variable ϑ and in the case of replacing of the initial space by the product space. □

In the considered case where P^θ is the conditional law of semimartingale X given $\vartheta=\theta$, one can rewrite the assumption 3.3 in terms of the so-called decoupling measure Q as in [14]. Let us suppose that the density process $z=(z^\theta)_{\theta\in\Theta}$ is $\mathcal{O}(\mathbf{F})\otimes\mathcal{A}$-measurable. Then we can replace θ by ϑ to obtain z^ϑ. We denote by P_t and Q_t the restrictions of the measures P and Q to $\mathcal{G}_t$

where $\Gamma = (\mathcal{G}_t)_{t\geq 0}$ is the filtration enlarged by the initial value ϑ. If $z_t^\vartheta > 0$ P-a.s. for all $t > 0$, we can define Q by

$$dQ_t = (z_t^\vartheta)^{-1} dP_t.$$

The decoupling measure has the following property: (Q, Γ)- triplet of X is the same as the $(P, \mathbf{F})$- triplet of X and $\mathcal{L}(\vartheta|Q) = \mathcal{L}(\vartheta|P)$. We can also use an another definition of a decoupling measure Q, namely, as the solution of the following martingale problem, if it exists and unique: the (Q, Γ)-triplet of X is the same as the $(P, \mathbf{F})$-triplet of X and $\mathcal{L}(\vartheta|Q) = \mathcal{L}(\vartheta|P)$.

Remark 2. If $z_t^\vartheta > 0$ P-a.s. for all $t > 0$, the assumption 3.3 is equivalent to the assumption:

$$E_Q[z^\vartheta, z^\vartheta]^{1/2}_{\tau_n} < \infty \tag{4.9}$$

for some localizing sequence of $\mathbf{F}$-stopping times τ_n. We note that $[z^\vartheta, z^\vartheta]^{1/2}$ is (Q, Γ)-locally integrable (see [19, Corollary I.4.55]). Here we require the existence of a localizing sequence of $\mathbf{F}$-stopping times.

Theorem 4.2. *Under the settings of Theorem 4.1, assume that a) and (4.9) hold. Then X is a (P, Γ)-semimartingale with the triplet $T^\vartheta = (B^\vartheta, C, \nu^\vartheta)$.*

Proof Using the proof of Theorem 4.1 we note that it remains to prove that B^ϑ is of locally integrable variation with respect to P. Since B^ϑ is obtained from B^θ by replacing θ by ϑ, we have:

$$\mathrm{Var}(B^\vartheta)_t \leq \mathrm{Var}(B)_t + |\beta^\vartheta| \cdot C_t + |Y^\vartheta - 1| l * \nu_t.$$

Since B is locally integrable with respect to P, the question of local integrability of B^ϑ is reduced to the existence of a localizing sequence of $\mathbf{F}$-stopping times τ_n such that for each n

$$E_P\left(|\beta^\vartheta| \cdot C_\tau + |Y^\vartheta - 1| l * \nu_{\tau_n}\right) < \infty. \tag{4.10}$$

We have:

$$\begin{aligned}
&E_P\left(|\beta^\vartheta| \cdot C_{\tau_n} + |Y^\vartheta - 1| l * \nu_{\tau_n}\right) \\
&= E_Q\{z_\tau^\vartheta\left(|\beta^\vartheta| \cdot C_{\tau_n} + |Y^\vartheta - 1| l * \nu_{\tau_n}\right)\} \\
&= E_Q\{z_-^\vartheta |\beta^\vartheta| \cdot C_{\tau_n} + z_-^\vartheta |Y^\vartheta - 1| l * \nu_{\tau_n}\} \\
&= E_Q\{z_-^\vartheta |\beta^\vartheta| \cdot C_{\tau_n} + z_-^\vartheta |Y^\vartheta - 1| l * \mu^X_{\tau_n}\} \\
&= E_Q \mathrm{Var}([z^\vartheta, X(l) - B])_{\tau_n}.
\end{aligned}$$

By the Fefferman inequality, (see [15, Theorem 10.17]) and the fact that $X(l) - B$ is both (Q, Γ)- and $(P, \mathbf{F})$-local martingale we deduce that

$$E_Q \mathrm{Var}([z^\vartheta, X(l) - B])_{\tau_n} \leq \| (X(l) - B)^{\tau_n} \|_{BMO} \, E_Q[z^\vartheta, z^\vartheta]^{1/2}_{\tau_n}.$$

From Proposition 2.38 in [17] it follows easily that the $(P, \mathbf{F})$-local martingale $(X(l) - B)$ is $(P, \mathbf{F})$-locally in BMO since it has bounded jumps, and by assumption (4.9) there is a localizing sequence of $\mathbf{F}$-stopping times τ_n tending to infinity which makes the last expression finite. Hence, the inequality (4.10) holds and B^ϑ has locally integrable variation with respect to P. □

Remark 3. Assumption (4.9) can be expressed in term of information. More precisely,

$$E_Q[z^\vartheta, z^\vartheta]_\tau^{1/2} \leq C(1 + E_Q z_\tau^\vartheta \log z_\tau^\vartheta).$$

The boundedness of this information was used in [10] to verify the stochastic Fubini theorem.

4.2 Initial enlargement and Gaussian martingales

Let us first consider a classical example of the initial enlargement of filtration. Here X is a continuous Gaussian martingale with respect to the measure P starting from zero and such that there exists $\lim_{t\to\infty} X_t = X_\infty$.

Let $\vartheta = X_\infty$. We denote by $\langle X \rangle$ the predictable quadratic variation of X and we put $\langle X \rangle_{t,\infty} := \langle X \rangle_\infty - \langle X \rangle_t$.

The prior distribution $\alpha(d\theta) := P(\vartheta \in d\theta)$ is a $\mathcal{N}(0, \langle X \rangle_\infty)$ and the posterior distribution α^t of ϑ given $\mathcal{F}_t$ is $\mathcal{N}(X_t, \langle X \rangle_{t,\infty})$.

Assume $\langle X \rangle_{t,\infty} > 0$ for all $t \in \mathbb{R}_+$, then α^t is equivalent to α, so the assumption (4.1) is valid.

From the Itô formula with the function $f(x, y) = x^2/y$ applied to the first term in exponential we have:

$$\begin{aligned}
\frac{d\alpha^t}{d\alpha}(\theta) &= \frac{\sqrt{\langle X \rangle_\infty}}{\sqrt{\langle X \rangle_{t,\infty}}} \exp\left\{ -\frac{(\theta - X_t)^2}{2\langle X \rangle_{t,\infty}} + \frac{\theta^2}{2\langle X \rangle_\infty} \right\} \\
&= \exp\left\{ \int_0^t \beta_s^\theta dX_s - \frac{1}{2} \int_0^t \left(\beta_s^\theta\right)^2 d\langle X \rangle_s \right\},
\end{aligned}$$

where

$$\beta_s^\theta := \frac{\theta - X_s}{\langle X \rangle_{s,\infty}}.$$

Since β^θ is a predictable process for each $\theta \in \Theta$, continuous in θ uniformly in $t \in [0, T]$ for each $T > 0$, the application $(\omega, t, \theta) \to \beta_t^\theta$ is $\mathcal{P}(\mathbf{F}) \otimes \mathcal{A}$-measurable. By Theorem 4.1 we can now conclude that the process

$$X_t - \int_0^t \frac{X_\infty - X_s}{\langle X \rangle_{s,\infty}} d\langle X \rangle_s$$

is a (P, Γ)-Gaussian martingale with the bracket $\langle X \rangle$.

We give some special cases of the above results.

- Let Y be a Brownian motion and put $X_t = \int_0^t a_s dY_s$, where a is deterministic square-integrable function on $\mathbb{R}_+$. If $a_s := I_{(0,T]}(s)$, then we have: $\vartheta = Y_T$, $\langle X\rangle_{t,\infty} = T - t$ for $t \le T$ and $\beta_s^\theta = \dfrac{\theta - Y_s}{T-s}$; this implies the classical representation of the Brownian bridge
$$Y_t = \int_0^t \frac{Y_T - Y_s}{T-s} ds + Y_t^\Gamma,$$
where Y^Γ is a Brownian motion with respect to Γ.
- In the previous case take $a = I_{(0,T+\eta]}$. We obtain the case of *final value distorted by a small noise* example from [1].
- Assume that Y is a fractional Brownian motion and let $X_t := E[Y_T|F_t^Y]$ be the prediction martingale. This example and related will be studied in detail in [12].

4.3 Initial enlargement in the Poisson filtration

Assume that X is a Poisson process with intensity 1 on $(\Omega, \mathcal{F}, \mathbf{F}, P)$ stopped in time T and let $\vartheta = X_T$. Here the prior distribution α is $Poisson(T)$ and the posterior distribution

$$\alpha^t(\theta) = \begin{cases} e^{T-t}\frac{(T-t)^{\theta - X_t}}{(\theta - X_t)!} & \text{if } \theta \ge X_t, \\ 0 & \text{if } \theta < X_t. \end{cases} \tag{4.11}$$

Next, for all $t \in [0, T[$ we have $\alpha^t \ll \alpha$ and

$$\frac{d\alpha^t}{d\alpha}(\theta) = e^{-t}\frac{(T-t)^{\theta - X_t}}{T^\theta} I_{\{\theta \ge X_t\}} \frac{\theta!}{(\theta - X_t)!}.$$

We put $Y_s^\theta := \dfrac{\theta - X_{s-}}{T-s}$ and note that Y^θ is a predictable process such that $0 \le Y_s^\theta < \infty$ for all $s \in [0,T]$ – this follows from the fact that $\Delta X_T = 0$ $\mathbb{P}$-a.s. Since

$$\frac{d\alpha^t}{d\alpha}(\theta) = \exp\left\{\int_0^t (Y_s^\theta - 1) ds\right\} \prod_{s \le t} \left(Y_s^\theta\right)^{\Delta X_s},$$

we obtain that with respect to the filtration Γ the standard Poisson process has the semimartingale representation:

$$X_t = n_t + \int_0^t \frac{X_T - X_{s-}}{T-s} ds, \qquad t < T$$

where $n = (n_t)_{t \ge 0}$ is a (P, Γ)-martingale.

4.4 Lévy processes: initial enlargement with the final value

Let X be a Lévy process. Then for each $\lambda \in \mathbb{R}$ the characteristic function of X_t is

$$Ee^{i\lambda X_t} = e^{-t\psi(\lambda)}$$

where ψ is characteristic exponent given by

$$\psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x I_{\{|x|<1\}}\right)\pi(dx)$$

with π a measure on $\mathbb{R}$ verifying $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$. The $(P, \mathbf{F})$-triplet of X is $T = (aI, \sigma^2 I, Leb \otimes \pi)$, where $I_t = t$.

We consider again stopped in T process and we take $\vartheta := X_T$. The process X is a time-homogeneous Markov process with independent increments and hence

$$\alpha^t(d\theta) = P(X_T \in d\theta | X_t) = P(X_{T-t} + x \in d\theta)|_{x=X_t}.$$

To be able to continue we assume that the law of the random variable X_s has a density $f(s, y)$ with respect to fixed dominating measure η, i.e. for $B \in \mathcal{B}(\mathbb{R})$

$$P(X_s \in B) = \int_B f(s, y)\eta(dy).$$

Moreover, we assume that $f \in C_b^{1,2}(\mathbb{R}^+ \times U)$ where U is an open set belonging to $\mathbb{R}$.

Since $\alpha^t \prec\prec \alpha$ for $t \in [0, T[$, we can write that η-a.s.

$$\frac{d\alpha^t}{d\alpha}(\theta) = \frac{f(T-t, \theta - X_t)}{f(T, \theta)}. \qquad (4.12)$$

Use the Itô formula to obtain that

$$\begin{aligned} f(T-t, \theta - X_t) = f(T, \theta) &- \int_0^t \frac{\partial f}{\partial s}(T-s, \theta - X_{s-})ds \\ &- \int_0^t \frac{\partial f}{\partial x}(T-s, \theta - X_{s-})dX_s \\ &+ \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 f}{\partial x^2}(T-s, \theta - X_{s-})ds \\ &+ \sum_{s \le t}\left(\Delta f(T-s, \theta - X_s) + \frac{\partial f}{\partial x}(T-s, \theta - X_{s-})\Delta X_s\right). \end{aligned} \qquad (4.13)$$

We know that the expression in (4.12) is a $(P, \mathbf{F})$-martingale for each θ. So, we can identify the continuous martingale part on the right-hand side of (4.13) and then the continuous martingale part of (4.12) as

$$-\int_0^t \frac{\frac{\partial f}{\partial x}(T-s, \theta - X_{s-})}{f(T, \theta)} dX_s^c. \qquad (4.14)$$

Recall that $z_t^\theta = \dfrac{d\alpha^t}{d\alpha}(\theta)$. According to the Girsanov theorem the term β^θ in the equation (2.1) is obtained as (for more details on this kind of computations see [19, Lemma III.3.31])

$$\begin{aligned}\beta_t^\theta &= \frac{d\langle z^\theta, X^c\rangle_t}{z_{t-}^\theta d\langle X^c, X^c\rangle_t} = \frac{-\frac{\partial f}{\partial x}(T-t, \theta - X_{t-})}{f(T-t, \theta - X_{t-})} \\ &= -\frac{\partial}{\partial x}\log f(T-t, x)|_{x=\theta - X_{t-}}. \end{aligned} \tag{4.15}$$

Consider next the pure jump martingale in (4.12): we have that

$$\Delta f(T-t, \theta - X_t) = f(T-t, \theta - X_t) - f(T-t, \theta - X_{t-})$$

and so

$$\frac{\Delta z_t^\theta}{z_{t-}^\theta} = \frac{f(T-t, \theta - X_t)}{f(T-t, \theta - X_{t-})} - 1,$$

from this we obtain (for more details, see [19, p. 175]) that the P^θ compensator ν^θ of μ^X is

$$\nu^\theta(dt, du) = \frac{f(T-t, \theta - (X_{t-} + u))}{f(T-t, \theta - X_{t-})}\pi(du)dt. \tag{4.16}$$

Moreover, since the expression on the right-hand side of (4.12) is a martingale, the function $f(t,u)$ satisfies the following integro-differential equation, which might be called Kolmogorov backward integro-differential equation:

$$\begin{aligned}\frac{\partial f}{\partial t}(T-t, \theta - x) &= \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(T-t, \theta - x) - a\frac{\partial f}{\partial x}(T-t, \theta - x) \\ &\quad + \int_{\mathbb{R}} \Big(f\left(T-t, \theta - (x+y)\right) \\ &\quad - f\left(T-t, \theta - x\right) + \frac{\partial f}{\partial x}\left(T-t, \theta - x\right)y\Big)\pi(dy)\end{aligned} \tag{4.17}$$

with the boundary condition $f(T, \theta - x) = \delta_{\{0\}}(\theta - x)$.

Example: Brownian motion

We look again the Brownian case, as in Subsection 4.2, but now using the Lévy processes approach. Since the triplet of X is $T = (0, \sigma^2 I, 0)$, the equation (4.17) is reduced to:

$$\frac{\partial f}{\partial t}(T-t, \theta - x) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(T-t, \theta - x)$$

with boundary condition $f(T, \theta - x) = \delta_{\{0\}}(\theta - x)$.

It is well-known that the solution is

$$f(T-t,\theta-x)=\frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{-\frac{(\theta-x)^2}{2(T-t)}\right\}$$

and so $\beta^\theta=\dfrac{\theta-X_s}{T-s}$ and a new drift is $B_t^\theta=\displaystyle\int_0^t\frac{\theta-X_s}{T-s}ds$.

Example: Gamma process

Let X be a Gamma process. This means that the $(P,\mathbf{F})$-triplet of X is $T=(\frac{a}{b}t,0,\frac{a}{u}e^{-bu}dudt)$. We know also that the density $f(t,x)=P(X_t\in dx)$ is $f(t,x)=\frac{b^{at}}{\Gamma(at)}x^{at-1}e^{-bx}$ with some parameters $a,b>0$ (see [5, p.73]). In particular, we have that $X_t-\frac{a}{b}t$ is a $(P,\mathbf{F})$-martingale.

Put again $\vartheta=X_T$ and we have from (4.16) that the (P^θ,F) compensator is

$$\nu^\theta(dx,dt)=\left(1-\frac{x}{\theta-X_{t-}}\right)^{a(T-t)-1}\frac{a}{x}dxdt.$$

Hence, $(P^\theta,\mathbf{F})$-drift of the process X is

$$\int_0^t\int_0^{\theta-X_{t-}}x\left(1-\frac{x}{\theta-X_{s-}}\right)^{a(T-s)-1}\frac{a}{x}dxdt=\int_0^t\frac{\theta-X_{s-}}{T-s}ds,$$

and this means that the process $X_t-\frac{a}{b}t-\int_0^t\frac{\theta-X_{s-}}{T-s}ds$ is a $(P^\theta,\mathbf{F})$-martingale.

Example: Poisson process

We look again at the Poisson case, as in subsection 4.3. We indicate briefly how one can use the approach described in 4.4, where we know only the triplet of the process X. So, let X be a Poisson process with intensity λ.

Put again $\vartheta=X_T$. Put $p(t,k):=P(X_t=k)$ and we assume that for $k\geq 0$ the functions $p(\cdot,k)\in C^1(\mathbb{R}_+)$.

We know (see (4.12)) that

$$\frac{d\alpha^t}{d\alpha}(\theta)=\frac{p(T-t,\theta-X_t)}{p(T,\theta)}.$$

We start with the trivial identity, which is the analog of the Itô formula here:

$$\begin{aligned}&p(T-t,\theta-X_t)=\\&p(T,\theta)-\int_0^t p_t(T-s,\theta-X_{s-})ds+\sum_{s\leq t}\Delta p(T-s,\theta-X_s).\end{aligned}\tag{4.18}$$

Using the fact that $\Delta X_t\in\{0,1\}$, we have the identity

$$\Delta p(T-s,\theta-X_s) = (p(T-s,\theta-(X_{s-}+1)) - p(T-s,\theta-X_{s-}))\Delta X_s;$$

since the right-hand side of (4.18) is a $(P,\mathbf{F})$-martingale, we obtain that the functions $p(t,k)$ satisfy the following system of differential equations:

$$p_t(T-s,k) = \lambda(p(T-s,k) - p(T-s,k+1)) \tag{4.19}$$

and, hence,

$$p(T-s,k) = e^{-\lambda(T-s)}\frac{(\lambda(T-s))^k}{k!}$$

is the solution of (4.19) with the boundary condition $p(T,\theta-x) = \delta_{\{0\}}(\theta-x)$. It remains to note that

$$p(T-s,k) - p(T-s,k+1) = p(T-s,k)\left(\frac{k+1}{\lambda(T-s)} - 1\right) \tag{4.20}$$

and we can conclude that with respect to the measure P^θ the process X has intensity $\frac{\theta-X_{s-}}{T-s}$. This means that the process $X_t - \int_0^t \frac{\theta-X_{s-}}{T-s}ds$ is a $(P^\theta,\mathbf{F})$-martingale.

5 Weak information

In this and in the next sections we discuss briefly some other applications of the Bayesian viewpoint related with the enlargement and arithmetic mean measure.

5.1 Weak insider information

The notion of weak information in mathematical finance was introduced by Baudoin [3, 4]. Before we discuss briefly this notion, recall our basic setup. We have a semimartingale X on a filtered space $(\Omega,\mathcal{F},\mathbf{F},P)$ with the right-continuous version of natural filtration $\mathbf{F} = (\mathcal{F}_t^X)_{t\geq 0}$ completed by the P-null sets of $\mathcal{F}$, and $\mathcal{F} = \mathcal{F}_\infty^X$. We assume the predictable representation property for $\mathbf{F}^X$ and we denote by $T = (B,C,\nu)$ the $(P,\mathbf{F})$-triplet of X.

Let ϑ be a F_T-measurable random variable with the values in a measurable Polish space $(\Theta,\mathcal{A})$. Let $\alpha := \mathcal{L}(\vartheta|P)$, $\alpha^t(d\theta) := P(\vartheta\in d\theta|\mathcal{F}_t)$, assume that we have (4.1), and define z_t^θ by (4.3) and finally put $dP_t^\theta = z_t^\theta dP_t$. Recall that in this case the arithmetic mean measure is

$$\bar{P}_t^\alpha(B) := \int_\Theta P_t^\theta(B)\alpha(d\theta) = P(B).$$

In particular, the $(P,\mathbf{F})$-triplet of the semimartingale X does not change under the arithmetic mean measure $\bar{P}^\alpha$ (see Remark 1).

Consider three types of agents in the pricing model, where the stock price is given by the semimartingale X: ordinary agents, strong insiders and weak insiders. We do not want to go in too detailed description of the pricing model, but we define these three types by giving the information and the (historical) probability of the agent.

- *ordinary agents* For the ordinary agent the information is given by $\mathbf{F}$, the probability is P and he uses the triplet $T = (B, C, \nu)$ to build his strategy.
- *strong insiders* For the strong insider the information is given in the pair (X, ϑ), and we can model this by an initial enlargement of the filtration. By using Theorem 4.1 we see that one possibility to model strong insider is to change the probability P to P^θ, and the strong insider works with the filtration $\mathbf{F}$ and the new triplet T^θ.

We describe the notion of *weak insider* in more detail. Let γ be the probability distribution on $(\Theta, \mathcal{A})$. Following [3, p. 112] we assume that $\gamma \ll \alpha$. Then it is easy to see that $\bar{P}^\gamma \ll \bar{P}^\alpha = P$, where

$$\bar{P}_t^\gamma(B) = \int_{\Theta \times B} z_t^\theta \gamma(d\theta) dP,$$

and the measure $\bar{P}^\gamma$ is the arithmetic mean measure with respect to the prior distribution γ; in [3] the corresponding measure on $(\Omega, \mathcal{F}, \mathbf{F})$ is called the *minimal probability associated with the conditioning* (T, ϑ, γ).

Hence, we can model the weak insiders as follows:

- *weak insiders* For the weak insider the information is given by the filtration $\mathbf{F}$, but he changes the probability measure P to the measure $\bar{P}^\gamma$ and he works with the triplet $\bar{T}^\gamma = (\bar{B}^\gamma, C, \bar{\nu}^\gamma)$.

Assume that we have

$$\gamma^t \ll \gamma$$

and we have assumption 3.3 with respect to the measure $P \otimes \gamma$.

We can now use Theorem 3.1 to compute the new triplet with respect to the measure $\bar{P}^\gamma$ and we obtain:

$$\begin{aligned} \bar{B}^\gamma &= B + \bar{\beta}^\gamma \cdot C + (\bar{Y}^\gamma - 1) l * \nu, \\ \bar{C}^\gamma &= C, \\ \bar{\nu}^\gamma &= \bar{Y}^\gamma \cdot \nu, \end{aligned} \tag{5.1}$$

where the predictable local characteristics $\bar{\beta}^\gamma$ and $\bar{Y}^\gamma$ are given by

$$\bar{\beta}_t^\gamma = E_{\gamma^{t-}} \beta_t^\theta, \quad \bar{Y}_t^\gamma = E_{\gamma^{t-}} Y_t^\theta \tag{5.2}$$

with γ^t and γ^{t-} be the a posteriori distributions under γ. Recall that γ^t is defined by :

$$\gamma^t(A) := \frac{\int_A z_t^\theta \gamma(d\theta)}{\int_\Theta z_t^\theta \gamma(d\theta)}, \qquad A \in \mathcal{A},$$

and γ^{t-} is given by the same formula with replacing z_t^θ by z_{t-}^θ.

Define now $\bar{m}^\gamma$ as

$$\bar{m}^\gamma = \bar{\beta}^\gamma \cdot X^c + \left(\bar{Y}^\gamma - 1 + \frac{\hat{\bar{Y}}^\gamma - \hat{1}}{1 - \hat{1}}\right) * (\mu - \nu),$$

then we have that

$$\frac{d\bar{P}_t^\gamma}{dP_t} = \mathcal{E}(\bar{m}^\gamma)_t.$$

By definition of $\bar{P}_t^\gamma$ and γ^t we have also that

$$\frac{d\gamma^t}{d\gamma}(\theta) = \frac{dP_t^\theta}{d\bar{P}_t^\gamma} = \frac{dP_t^\theta}{dP_t}\frac{dP_t}{d\bar{P}_t^\gamma} = z_t^\theta \frac{1}{\mathcal{E}(\bar{m}^\gamma)_t}.$$

In comparison with $\dfrac{d\alpha^t}{d\alpha}(\theta)$ which is equal to z_t^θ ($P_t \times \alpha$ -a.s.), it means that

$$\frac{d\gamma^t}{d\gamma}(\theta) = \frac{d\alpha^t}{d\alpha}(\theta)\frac{1}{\mathcal{E}(\bar{m}^\gamma)_t}.$$

Example: Brownian motion

Let X be a Brownian motion stopped in T and suppose that the Brownian filtration $\mathbf{F}$ is enlarged by $\vartheta = X_T$. In this example $T = (0, I, 0)$ and

$$\beta^\theta = \frac{\theta - X_t}{T - t}.$$

Consider the example of *final value distorted with a noise*. We suppose that the weak insider knows in advance the value y of random variable $Y = X_T + \epsilon$, where ϵ is independent of X_T and has $\mathcal{N}(0, \eta^2)$ as law. The prior of the insider with weak information is $\gamma = P(X_T|Y)$, which by the normal correlation theorem is $\mathcal{N}(m, \sigma^2)$ with $\sigma^2 = (T^{-1} + \eta^{-2})^{-1}$ and $m = Y\sigma^2/\eta^2$.

For $t < T$ the posterior distribution is $\gamma^t := P(X_T|Y, X_t)$, which by the normal correlation theorem is $\mathcal{N}(m_t, \sigma_t^2)$ with $\sigma_t^2 = ((T-t)^{-1} + \eta^{-2})^{-1}$ and $m_t = (Y\eta^{-2} + X_t(T-t)^{-1})\sigma_t^2$.

According to previous results on triplets the drift of X under the insider measure is given by

$$\bar{B}_t^\gamma = \int_0^t \frac{E_{\gamma^s}\vartheta - X_s}{T - s} ds. \tag{5.3}$$

Since

$$E_{\gamma^s}\vartheta = \frac{Y(T-s) + X_s\eta^{-2}}{T - s + \eta^{-2}},$$

we have after simplifications that

$$\bar{B}_t^\gamma = \int_0^t \frac{Y - X_s}{T - s + \eta^2} ds.$$

Remark 1. One can analyze the increasing information along the same lines. By this we mean that the insider obtains dynamically more and more precise information about the random variable ϑ. A model of this type is the following: in addition to the price process X the insider observes the process Y, where

$$Y_t = \vartheta + \epsilon_t,$$

where ϵ is a semimartingale, independent of the random variable ϑ such that $\epsilon_t \to 0$ P-a.s. as $t \to T$. This kind of models are analyzed in [7].

6 Additional expected logarithmic utility of an insider

6.1 Introduction

We consider the pricing model with two assets, the stock (risky asset) and the bond (riskless asset). We assume as in [1] that the process X has the dynamics

$$dX_t = \mu_t d\langle M\rangle_t + dM_t \tag{6.1}$$

where μ is a predictable process and M is a continuous Gaussian martingale with deterministic bracket $\langle M\rangle$. We assume that the interest rate r is equal to zero, so the bond price $B_t = 1$ for all t.

We assume that the stock price S has the dynamics

$$dS_t = S_t dX_t.$$

For the investment strategy π we have the portfolio dynamics

$$dV_t^\pi = \pi_t V_t^\pi dX_t.$$

Then it can be shown that with respect to the logarithmic utility, the average optimal strategy π^o of an ordinary investor is $\pi^o := \mu$. Note that here the average optimal strategy is computed with respect to the measure P.

6.2 Additional expected utility of strong insiders

Now consider a strong insider who knows the final value of the stock. We assume that it is the same as the insider knows the final value of the martingale M_T. Put again $\vartheta = M_T$.

Then he can model the dynamics of X as

$$dX_t = (\mu_t + \beta_t^\theta) d\langle M\rangle_t + dM_t^\theta. \tag{6.2}$$

Here M^θ is a continuous Γ-martingale with

$$M_t^\theta = M_t - \int_0^t \beta_s^\theta d\langle M\rangle_s$$

and

$$\beta_t^\theta = \frac{\theta - M_t}{\langle M\rangle_{t,T}}$$

where $\langle M\rangle_{t,T} = \langle M\rangle_T - \langle M\rangle_t$. Again the optimal expected investment strategy of an insider agent for the logarithmic utility is $\pi^i = \mu + \beta^\theta$. Note that the expectation is computed with respect to the measure $\mathbb{P}$ which is the joint law of $(M, \vartheta(\omega))$. The log-value of the optimal strategy for an ordinary investor is

$$\log V_t^{\pi^o} = \log V_0 + \int_0^t \mu_s dM_s + \frac{1}{2}\int_0^t \mu_s^2 d\langle M\rangle_s. \tag{6.3}$$

Similarly, the log-value of the optimal strategy for the insider investor is

$$\log V_t^{\pi^i} = \log V_0 + \int_0^t (\mu_s + \beta_s^\theta) dM_s^\theta + \frac{1}{2}\int_0^t (\beta_s^\theta + \mu_s)^2 d\langle M\rangle_s. \tag{6.4}$$

To calculate the expectation $\mathbf{E}$ we need the following lemma.

Lemma 6.1. *Let $u^\theta = (u_t^\theta)_{t\geq 0}$ be a positive $\mathbf{F}$-adapted càdlàg process for each $\theta \in \Theta$. Suppose that the application $u : (\omega, t, \theta) \to u_t^\theta(\omega)$ is $\mathcal{O}(\mathbb{G})$-measurable with $\mathbb{G}$ defined by (4.2). Then*

$$\mathbf{E}\int_0^t u_s^\theta d\langle M\rangle_s = E\int_0^t \bar{u}_s^\alpha d\langle M\rangle_s \tag{6.5}$$

where and $\bar{u}_s^\alpha$ is the posterior mean of u_s^θ, i.e.

$$\bar{u}_s^\alpha = E_{\alpha^{s-}} u_s^\theta$$

Proof Recall first the following fact. Assume that $y = (y_t)_{t\geq 0}$ is a positive uniformly integrable $(P, \mathbf{F})$-martingale and D is a predictable increasing process with $D_0 = 0$. Then by [15, Theorem 5.16, p. 144 and Remark 5.3, p. 137]

$$Ey_t D_t = E\int_0^t ({}^pY)_s dD_s = \mathbf{E}\int_0^t Y_{s-} dD_s. \tag{6.6}$$

Since z^θ is the conditional density of the law of X given $\vartheta = \theta$ with respect to P, we have using (6.6) and the ordinary Fubini theorem that

$$\begin{aligned}
\mathbf{E}\int_0^t u_s^\theta d\langle M\rangle_s &= E\left(\int_\Theta z_t^\theta \int_0^t u_s^\theta d\langle M\rangle_s d\alpha\right) = \int_\Theta E\left(z_t^\theta \int_0^t u_s^\theta d\langle M\rangle_s\right) d\alpha \\
&= \int_\Theta E\int_0^t z_{s-}^\theta u_s^\theta d\langle M\rangle_s d\alpha = E\int_0^t \left(\int_\Theta z_{s-}^\theta u_s^\theta d\alpha\right) d\langle M\rangle_s \\
&= E\int_0^t \bar{u}_s^\alpha d\langle M\rangle_s.
\end{aligned}$$

This proves (6.5). □

Let us now compute the expected utility from the insider point of view. This means that we take the expectation of (6.4) with respect to the insider measure $\mathbb{P}$ which is the joint law of (ω, ϑ). In the computation we use the fact that the martingale M has a drift $\int_0^{\cdot} \beta_s^\theta d\langle M\rangle_s$ with respect to the insider measure. We obtain:

$$\begin{aligned}
\mathbf{E}(\log V_t^{\pi^i} - \log V_t^{\pi^o}) &= \frac{1}{2}\mathbf{E}\int_0^t (\mu_s + \beta_s^\theta)^2 d\langle M\rangle_s \\
&\quad -\frac{1}{2}\mathbf{E}\int_0^t \mu_s^2 d\langle M\rangle_s - \mathbf{E}\int_0^t \mu_s dM_s \\
&= \frac{1}{2}\mathbf{E}\int_0^t (\beta_s^\theta)^2 d\langle M\rangle_s \\
&= \frac{1}{2}E\int_0^t \bar{v}_s^\alpha(\beta) d\langle M\rangle_s
\end{aligned}$$

where $\bar{v}_s^\alpha(\beta)$ is the posterior variance of the process β_s^θ. Next we give the Bayesian interpretation of this result. Note first that the Kullback–Leibler information in the prior with respect to posterior is

$$I(\alpha|\alpha^\tau) := E_{\alpha^\tau} \log \frac{d\alpha^\tau}{d\alpha}(\theta).$$

In our case we have:

$$\mathbf{E}(\log V_t^{\pi^i} - \log V_t^{\pi^o}) = EI(\alpha|\alpha^t).$$

For more information on this kind of computations we refer to [10].

We compute next the difference of the expected gain from the ordinary agent point of view. This has the interpretation that an ordinary agent has excess to the insider information, but he thinks that this is false. We model this by the measure $P \otimes \alpha$ — this means that the ordinary agent does not change his triplet. So the expected utility gain has to be calculated using the measure $P \otimes \alpha$. With a similar computation to the previous one we obtain that

$$E_{P\otimes\alpha}(\log V_t^{\pi^o} - \log V_t^{\pi^i}) = \frac{1}{2}E_{P\otimes\alpha}\int_0^t (\beta_s^\theta)^2 d\langle M\rangle_s.$$

The Kullback–Leibler information in the posterior α^τ with respect to the prior α is define by

$$I(\alpha^\tau|\alpha) := E_\alpha \log \frac{d\alpha}{d\alpha^\tau}.$$

For our model we can conclude that

$$E_{P\otimes\alpha}(\log V_t^{\pi^o} - \log V_t^{\pi^i}) = EI(\alpha^t|\alpha).$$

Note that the differences of the expected gains are in both cases positive — this reflects the fact the the investors act optimally according to their own model.

6.3 Additional expected logarithmic utility of weak insider

Assume that γ and α are two different equivalent priors for the parameter ϑ; we can define the arithmetic mean measures $\bar{P}^{\gamma}$ and $\bar{P}^{\alpha}$; we can compute the $(\mathbf{F}, \bar{P}^{\gamma})$ and $(\mathbf{F}, \bar{P}^{\alpha})$-triplets of the semimartingale X by (3.1). Note that here we do not assume that α is the marginal law of the parameter ϑ.

Denote the optimal strategies based on the weak information for the prior γ and α by $\pi^{w,\gamma}$ and $\pi^{w,\alpha}$, respectively.

Then, with a familiar computation

$$E_{\bar{P}^{\gamma}}\left(\log V_t^{w,\gamma} - \log V_t^{w,\alpha}\right) = \frac{1}{2} E_{\bar{P}^{\gamma}}\left(\int_0^t (\overline{\beta}_s^{\gamma} - \overline{\beta}_s^{\alpha})^2 d\langle M\rangle_s\right) \tag{6.7}$$

where

$$\bar{\beta}_s^{\gamma} = E_{\gamma^{s-}}\beta_s^{\theta}, \qquad \bar{\beta}_s^{\alpha} = E_{\alpha^{s-}}\beta_s^{\theta}.$$

Note that the right-hand side of (6.7) is nothing else but thye Kullback–Leibler information of $\bar{P}^{\alpha}$ in $\bar{P}^{\gamma}$ and, hence,

$$E_{\bar{P}^{\gamma}}\left(\log V_t^{w,\gamma} - \log V_t^{w,\alpha}\right) = I(\bar{P}^{\alpha}|\bar{P}^{\gamma})_t.$$

Note that

$$\begin{aligned}
&0 \le I(\bar{P}^{\alpha}|\bar{P}^{\gamma})_t = E_{\bar{P}^{\gamma}} \log \frac{d\bar{P}_t^{\gamma}}{d\bar{P}_t^{\alpha}} = \\
&= \int_{\Theta}\int_{\Omega}\left\{\log \frac{dP_t^{\theta}}{d\bar{P}_t^{\alpha}} - \log \frac{dP_t^{\theta}}{d\bar{P}_t^{\gamma}}\right\} P_t^{\theta}(d\omega)\gamma(d\theta) \\
&= E_{\gamma}\left\{I(P_t^{\theta}|\bar{P}_t^{\alpha}) - I(P_t^{\theta}|\bar{P}_t^{\gamma})\right\} = E_{\bar{P}_t^{\gamma}}\left\{I(\alpha|\alpha^t) - I(\gamma|\gamma^t)\right\}
\end{aligned}$$

In particular, this means that

$$E_{\bar{P}_t^{\gamma}} I(\gamma|\gamma^t) = \inf_{\alpha} E_{\bar{P}_t^{\gamma}} I(\alpha|\alpha^t)$$

where the infimum is taken over all measures α which are equivalent to γ. The interpretation is that if one believes in his own prior γ, he expects to get less information from the data than any other person using the same model with a “wrong” prior.

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