Remarks on Risk Neutral and Risk Sensitive Portfolio Optimization

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Summary. In this note it is shown that risk neutral optimal portfolio strategy is nearly optimal for risk sensitive portfolio cost functional with negative risk factor that is close to zero.

Key words: risk sensitive control, discrete-time Markov processes, splitting, Poisson equation, Bellman equation

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1 Introduction

We consider a market with m risky assets. Denote by $S_i(t)$ the price of the i-th asset at time t. We shall assume that the prices of assets depend on k economic factors $x_i(n)$, $i = 1, \ldots, k$, with values changing at discrete times $n = 0, 1, \ldots$, so that for $t \in [n, n+1)$ the prices satisfy the equation

$$\frac{dS_i(t)}{S_i(t)} = a_i(x(n))dt + \sum_{j=1}^{k+m} \sigma_{ij}(x(n))dw_j(t), \tag{1.1}$$

where $(w(t) = (w_1(t), w_2(t), \ldots, w_{k+m}(t))$ is a $(k+m)$-dimensional Brownian motion defined on a given probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$. The economic factors $x(n) = (x_1(n), \ldots, x_k(n))$ evolve according to the equation

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$$\begin{aligned} x_i(n+1) &= x_i(n) + b_i(x(n)) + \sum_{j=1}^{k+m} d_{ij}(x(n))[w_j(n+1) - w_j(n)] \\ &= g(x(n), W(n)), \end{aligned} \tag{1.2}$$

where $W(n) := (w_1(n+1) - w_1(n), \ldots, w_{k+m}(n+1) - w_{k+m}(n))$.

We assume that a, b are bounded and continuous vector functions, and σ, d are bounded and continuous matrix functions of suitable dimensions. Additionally we shall assume that the matrix dd^T (^{T} stands for transpose) is nondegenerate. Notice that equation (1.2) corresponds to the discretization of a diffusion process. The set of factors may include dividend yields, price-earning ratios, short term interest rates, the rate of inflation see e.g. [1]. The dynamics of such factors is usually modelled using diffusions, frequently linear as in the case when a is a function of a spot interest rate governed by the Vasicek process (see [1]). Our assumptions concerning boundedness of the vector functions a and b may be relaxed allowing linear growth. However in this case we need more complicated assumptions to obtain analogs of Lemmas 3.2, 3.3 and Corollary 3.1 which are important in the proof of Proposition 3.1.

Assume that starting with an initial capital $V(0)$ we invest in the given assets. Let $h_i(n)$ be the part of the wealth process located in the i-th asset at time n, which is assumed to be nonnegative. The choice of $h_i(n)$ depends on our observation of the asset prices and economic factors up to time n. Denoting by $V(n)$ the wealth process at time n and by $h(n) = (h_1(n), \ldots, h_m(n))$ our investment strategy at time n, we have that $h(n) \in U = \{(h_1, \ldots, h_m),\ h_i \geq 0,\ \sum_{i=1}^m h_i = 1\}$ and

$$\frac{V(n+1)}{V(n)} = \sum_{i=1}^{m} h_i(n) \xi_i(x(n), W(n)), \tag{1.3}$$

where

$$\xi_i(x(n), W(n)) = \exp\left(a_i(x(n)) - \frac{1}{2} \sum_{j=1}^{k+m} \sigma_{ij}^2(x(n)) + \sum_{j=1}^{k+m} \sigma_{ij}(x(n))[w_j(n+1) - w_j(n)] \right).$$

We are interested in the following investment problems:
maximize the *risk neutral cost functional*

$$J_x^0(\{h(n)\}) = \liminf_{n \to \infty} \frac{1}{n} E_x \left[\ln V(n)\right] \tag{1.4}$$

and maximize the *risk sensitive cost functional*

$$J_x^\gamma(\{h(n)\}) = \frac{1}{\gamma} \limsup_{n \to \infty} \frac{1}{n} \ln E_x \left[V(n)^\gamma\right] \tag{1.5}$$

with $\gamma < 0$. Using (1.3) we can rewrite the cost functional (1.4) as

$$J_x^0(\{h(n)\}) = \liminf_{n\to\infty} \frac{1}{n} E_x \left[\sum_{t=0}^{n-1} \ln \left(\sum_{i=1}^m h_i(t)\xi_i(x(t), W(t)) \right) \right]$$
$$= \liminf_{n\to\infty} \frac{1}{n} E_x \left[\sum_{t=0}^{n-1} c(x(t), h(t)) \right], \tag{1.6}$$

with $c(x,h) = E\left\{\ln\left(\sum_{i=1}^m h_i\xi_i(x, W(0))\right)\right\}$. It is clear that risk neutral cost functional J^0 depends on the uncontrolled Markov process $(x(n))$ and we practically maximize the cost function c itself. Consequently an optimal control is of the form $(\hat{u}(x(n)))$, where $\sup_h c(x,h) = c(x, \hat{u}(x))$ and the Borel measurable function $\hat{u} : R^k \mapsto U$ exists by continuity of c for fixed $x \in R^k$. This control does not depend on asset prices and is a time independent function of current values of the factors x only. The Bellman equation corresponding to the risk neutral control problem is of the form

$$w(x) + \lambda = \sup_h \left[c(x,h) + Pw(x)\right] \tag{1.7}$$

where $Pf(x) := E_x\{f(x(1))\}$ for $f \in b\mathcal{B}(R^k)$ - the space of bounded Borel measurable functions on R^k, is a transition operator corresponding to $(x(n))$. In Section 2 we shall show that there are solutions w and λ to the equation (1.7) and λ is the optimal value of the cost functional J^0.
Letting

$$\zeta_n^{h,\gamma}(\omega) := \prod_{t=0}^{n-1} \exp\left(\gamma \ln\left(\sum_{i=1}^m h_i(t)\xi_i(x(t), W(t))\right)\right)$$
$$\left(E\left[\exp\left(\gamma \ln\left(\sum_{i=1}^m h_i(t)\xi_i(x(t), W(t))\right)\right) |\mathcal{F}_{t-1}\right]\right)^{-1}$$

consider a probability measure $P^{h,\gamma}$ defined by its restrictions $P_{|n}^{h,\gamma}$ to the first n times, given by the formula

$$P_{|n}^{h,\gamma}(d\omega) = \zeta_n^{h,\gamma}(\omega) P_{|n}(d\omega).$$

Then the cost functional (1.5) can be rewritten as

$$J_x^\gamma(\{h(n)\}) = \frac{1}{\gamma} \limsup_{n\to\infty} \frac{1}{n} \ln E_x \left[\exp\left(\gamma \sum_{t=0}^{n-1} \ln\left(\sum_{i=1}^m h_i(t)\xi_i(x(t), W(t))\right)\right)\right]$$
$$= \frac{1}{\gamma} \limsup_{n\to\infty} \frac{1}{n} \ln E_x^{h,\gamma} \left[\exp\left(\sum_{t=0}^{n-1} c_\gamma(x(t), h(t))\right)\right], \tag{1.8}$$

with

$$c_\gamma(x,h) := \ln\left(E\left[\left(\sum_{i=1}^{m} h_i \xi_i(x, W(0))\right)^\gamma\right]\right). \tag{1.9}$$

The risk sensitive Bellman equation corresponding to the cost functional J^γ is of the form

$$e^{w_\gamma(x)} = \inf_h \left[e^{(c_\gamma(x,h)-\lambda_\gamma)} \int_E e^{w_\gamma(y)} P^{h,\gamma}(x, dy)\right]. \tag{1.10}$$

where for $f \in b\mathcal{B}(R^k)$

$$P^{h,\gamma} f(x) = E\left[\left(\sum_{i=1}^{m} h_i \xi_i(x, W(0))\right)^\gamma \exp\left(-c_\gamma(x,h)\right) f\left(g(x, W(0))\right)\right], \tag{1.11}$$

with g as in (1.2) and where $\frac{1}{\gamma}\lambda_\gamma$ is the optimal value of the cost functional (1.8). Notice that under measure $P^{h,\gamma}$ the process $(x(n))$ is still Markov but with controlled transition operator $P^{h,\gamma}(x, dy)$. Following [6] we shall show that

$$\frac{1}{\gamma}\lambda_\gamma \to \lambda \tag{1.12}$$

whenever $\gamma \uparrow 0$.

In what follows we distinguish two special classes of controls (h_n): *Markov controls* $\mathcal{U}_M = \{(h(n)) : \ h(n) = u_n(x(n))\}$, where $u_n : R^k \mapsto U$ is a sequence of Borel functions, and *stationary controls* $\mathcal{U}_s = \{(h_n) : \ h(n) = u(x(n))\}$, where $u : R^k \mapsto U$ is a Borel function. We shall denote by $\mathcal{B}(R^k)$ the set of Borel subsets of R^k and by $\mathcal{P}(R^k)$ the set of probability measures on R^k.

The study of risk sensitive portfolio optimization has been originated in [1] and then continued in a number of papers, in particular, in [16]. Risk sensitive cost functional was studied in papers [13], [6], [7], [3], [4], [12], [2], [8] and references therein. In this paper using techniques based on the splitting of Markov processes (see [15]) we study the Poisson equation for additive cost functional, the solution of which is also a solution to the risk neutral Bellman equation. We then consider the problem of risk sensitive portfolio optimization with risk factor close to 0. We generalize the result of [16], where uniform ergodicity of factors was required and using [8] we show the existence of the solution to the Bellman equation for small risk in a more general ergodic case. The proof that a nearly optimal continuous risk neutral control function is also nearly optimal for risk sensitive cost functional with risk factor close to 0 is based on a modification of the arguments in [6] using some results from the theory of large deviations.

2 Risk neutral Bellman equation

By the nondegeneracy of the matrix dd^T there exists a compact set $C \subset R^k$, for which we can find a closed ball in R^k, $\beta > 0$ and $\nu \in \mathcal{P}(R^k)$ such that

$\nu(C) = 1$ and $\forall_{A \in \mathcal{B}(R^k)}$

$$\inf_{x \in C} P(x, A) \geq \beta \nu(A). \tag{2.1}$$

We fix a compact set C, $\beta > 0$ and $\nu \in \mathcal{P}(R^k)$ satisfying the above minorization property. Additionally assume that the set C is ergodic, i.e.

$$\forall_{x \in R^k} \; E_x \{\tau_C\} < \infty \quad \text{and} \quad \sup_{x \in C} E_x \{\tau_C\} < \infty,$$

where $\tau_C = \inf \{i > 0 : \; x_i \in C\}$.
Consider a splitting of the Markov process $(x(n))$ (see [15]).
Let $\hat{R^k} = \left\{ C \times \{0\} \cup C \times \{1\} \cup (R^k \setminus C) \times \{0\} \right\}$ and $\hat{x}(n) = (x^1(n), x^2(n))$ be a Markov process defined on $\hat{R^k}$ such that

(i) when $(x^1(n), x^2(n)) \in C \times \{0\}$, $x^1(n)$ moves to y accordingly to $(1-\beta)^{-1}(P(x^1(n), dy) - \beta\nu(dy))$ and whenever $y \in C$, $x^2(n)$ is changed into $x^2(n+1) = \beta_{n+1}$, where β_n is i.i.d.

$$P\{\beta_n = 0\} = 1 - \beta, \quad P\{\beta_n = 1\} = \beta,$$

(ii) when $(x^1(n), x^2(n)) \in C \times \{1\}$, $x^1(n)$ moves to y accordingly to ν and $x^2(n+1) = \beta_{n+1}$,
(iii) when $(x^1(n), x^2(n)) \in R^k \setminus C \times \{0\}$, $x^1(n)$ moves to y accordingly to $P(x^1(n), dy)$ and whenever $y \in C$, $x^2(n)$ is changed into $x^2(n+1) = \beta_{n+1}$.

Let $C_0 = C \times \{0\}$, $C_1 = C \times \{1\}$.
Following [8] and [15] we have

Proposition 2.1. *For $n = 1, 2 \dots$ we have P-a.e.*

$$P\left(\hat{x}(n) \in C_0 | \hat{x}(n) \in C_0 \cup C_1, \hat{x}(n-1), \dots, \hat{x}(0)\right) = 1 - \beta. \tag{2.2}$$

The process $(\hat{x}(n) = (x^1(n), x^2(n)))$ is Markov with transition operator $\hat{P}(\hat{x}(n), dy)$ defined by (i)-(iii). Its first coordinate $(x^1(n))$ is also a Markov process with transition operator $P(x^1(n), dy)$. Furthermore, for any bounded Borel measurable function $f : (R^k)^{n+1} \mapsto R$ we have

$$E_x \{f(x(1), x(2), \dots, x(n))\} = \hat{E}_{\delta_x^*} \left\{ f(x^1(1), x^1(2), \dots, x^1(n)) \right\} \tag{2.3}$$

where $\delta_x^* = \delta_{(x,0)}$ for $x \in R^k \setminus C$ and $\delta_x^* = (1-\beta)\delta_{(x,0)} + \beta\delta_{(x,1)}$ for $x \in C$ and $\hat{E}_\mu$ stands for conditional law of Markov process $(\hat{x}(n))$ with initial law $\mu \in \mathcal{P}(\hat{R^k})$.

Proof. Since the Markov property of $(x^1(n))$ is fundamental in this paper we recall this proof from [8] leaving the proof of other statements to the reader. For $A \in R^k$ we have

$$
\begin{aligned}
&P\left(x^1(n+1) \in A | x^1(n), x^1(n-1), \ldots, x^1(0)\right) \\
&= P\left(x^1(n+1) \in A | x^1(n), x^2(n) = 0, x^1(n-1), \ldots, x^1(0)\right) \\
&P\left(x^2(n) = 0 | x^1(n), x^1(n-1), \ldots, x^1(0)\right) \\
&+ P\left(x^1(n+1) \in A | x^1(n), x^2(n) = 1, x^1(n-1), \ldots, x^1(0)\right) \\
&P\left(x^2(n) = 1 | x^1(n), x^1(n-1), \ldots, x^1(0)\right).
\end{aligned}
$$

In the case when $x^1(n) \in C$, the right-hand side of the last equation is equal to

$$
\frac{P^{a_n}(x^1(n), A) - \beta\nu(A)}{1-\beta}(1-\beta) + \beta\nu(A) = P^{a_n}(x^1(n), A).
$$

For $x^1(n) \notin C$, it is equal to $P^{a_n}(x^1(n), A)$, which completes the proof of the Markov property of $(x^1(n))$.

□

By the assumption on C and the construction of the split Markov process we immediately have

Corollary 2.1. *$\hat{E}_x\left[\tau_{C_1}\right] < \infty$ for $x \in \hat{R^k}$ and $\sup_{x \in C_1} \hat{E}_x\left[\tau_{C_1}\right] < \infty$.*

Lemma 2.1. *Given $h(n) \in \mathcal{U}_M$ there is a unique $\lambda(\{h(n)\})$ such that for $x \in C_1$*

$$
\hat{E}_x\left[\sum_{t=1}^{\tau_{C_1}}\left(c(x^1(t), h(t)) - \lambda(\{h(n)\})\right)\right] = 0. \tag{2.4}
$$

Proof. Notice that for $x \in C_1$ the mapping

$$
D : \lambda \mapsto \hat{E}_x\left[\sum_{t=1}^{\tau_{C_1}}\left(c(x^1(t), h(t)) - \lambda\right)\right]
$$

is continuous and strictly decreasing. Since the values of this mapping for $\|c\|$ and $-\|c\|$ are, respectively, nonpositive and nonnegative, there is a unique λ for which the mapping attains 0.

□

For Borel measurable $u : R^k \mapsto U$ let

$$
\hat{w}^u(x) = \hat{E}_x\left[\sum_{t=0}^{\tau_{C_1}}\left(c(x^1(t), u(x^1(t))) - \lambda(u)\right)\right], \tag{2.5}
$$

where we use the notation $\lambda(u) = \lambda(\{u(x(n))\})$.

Lemma 2.2. *Function $\hat{w}^u$ defined in (2.5) is the unique (up to an additive constant) solution to the additive Poisson equation (APE) for the split Markov process $(\hat{x}(n))$:*

$$
\hat{w}^u(x) = c(x^1, u(x^1)) - \lambda(u) + \int_{\hat{R^k}} \hat{w}^u(y)\hat{P}(x, dy). \tag{2.6}
$$

Furthermore, if $\hat{w}$ and λ satisfy the equation

$$\hat{w}(x) = c(x^1, u(x^1)) - \lambda + \int_{\hat{R}^k} \hat{w}(y)\hat{P}(x, dy) \tag{2.7}$$

then $\lambda = \lambda(u)$ (defined in Lemma 2.1) and $\hat{w}$ differs from $\hat{w}^u$ by an additive constant.

Proof. In fact, we have using (2.4)

$$\begin{aligned}
\hat{E}_x\left[w(\hat{x}(1))\right] &= \hat{E}_x\left[\chi_{\hat{x}(1)\in C_1}\hat{E}_{x(1)}\left[\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t), u(x^1(t))) - \lambda(u)\big)\right]\right] \\
&\quad + \hat{E}_x\left[\chi_{\hat{x}(1)\notin C_1}\hat{E}_{x(1)}\left[\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t), u(x^1(t))) - \lambda(u)\big)\right]\right] \\
&= \hat{E}_x\left[\chi_{\hat{x}(1)\in C_1}\big(c(x^1(1), u(x^1(1))) - \lambda(u)\big)\right] \\
&\quad + \hat{E}_x\left[\chi_{\hat{x}(1)\notin C_1}\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t), u(x^1(t))) - \lambda(u)\big)\right] \\
&= \hat{E}_x\left[\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t), u(x^1(t))) - \lambda(u)\big)\right] - \big(c(x^1, u(x^1)) - \lambda(u)\big)
\end{aligned}$$

from which (2.6) follows. If $\hat{w}^u$ is a solution to (2.6) then by iteration we obtain that

$$\hat{w}^u(x) = \hat{E}_x\left[\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t), u(x^1(t))) - \lambda(u)\big) + \hat{E}_{\hat{x}_{\tau_{C_1}}}\left[\hat{w}^u(\hat{x}(1))\right]\right], \tag{2.8}$$

where by the construction of the split Markov process

$$\hat{E}_{x_{\tau_{C_1}}}\left[\hat{w}^u(\hat{x}(1))\right] = (1-\beta)\int_{R^k}\hat{w}^u(z, 0)\nu(dz) + \beta\int_{R^k}\hat{w}^u(z, 1)\nu(dz).$$

Consequently, $\hat{w}^u$ differs from $\hat{w}^u$ defined in (2.5) only by an additive constant. Similarly, if $\hat{w}$ and λ are solutions to (2.7) then $\hat{w}$ differs from

$$\tilde{w}(x) = \hat{E}_x\left[\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t), u(x^1(t))) - \lambda\big)\right]$$

by an additive constant $\hat{E}_z\left\{\hat{w}(\hat{x}(1))\right\}$ with $z \in C_1$. Since $\tilde{w}$ itself is a solution to (2.7) we have that $\hat{E}_z\left\{\tilde{w}(\hat{x}(1))\right\} = 0$ for $z \in C_1$. Therefore, for $z \in C_1$

$$0 = \hat{E}_z\left[\tilde{w}(\hat{x}(1))\right] = \hat{E}_z\left[\chi_{\hat{R^k}\setminus C_1}(\hat{x}(1))\sum_{t=1}^{\tau_{C_1}}\big(c(x^1(t),u(x^1(t)))-\lambda\big)\right.$$
$$\left. +\chi_{C_1}(\hat{x}(1))\hat{E}_{\hat{x}(1)}\left[\sum_{t=0}^{\tau_{C_1}}\big(c(x^1(t),u(x^1(t)))-\lambda\big)\right]\right]$$
$$= \hat{E}_z\left[\sum_{t=1}^{\tau_{C_1}}\big(c(x^1(t),u(x^1(t)))-\lambda\big)\right]$$

and by Lemma 2.1 we have $\lambda = \lambda(u)$ which completes the proof.

□

Corollary 2.2. *Given a solution* $\hat{w}^u : \hat{R^k} \mapsto R$ *to the APE* (2.6) *we have that* w^u *defined by*

$$w^u(x) := \hat{w}^u(x,0) + 1_C(x)\beta\left[\hat{w}^u(x,1) - \hat{w}^u(x,0)\right] \tag{2.9}$$

is a solution to the APE for the original Markov process $(x(n))$

$$w^u(x) = c(x,u(x)) - \lambda(u) + \int_{R^k} w^u(y)P(x,dy). \tag{2.10}$$

Furthermore if w^u *is a solution to* (2.10) *then* $\hat{w}^u$ *defined by*

$$\hat{w}^u(x^1,x^2) = c(x^1,u(x^1)) - \lambda(u) + \hat{E}_{x^1,x^2}\left[w^u(x^1(1))\right] \tag{2.11}$$

is a solution to (2.6).

Proof. By (2.2) we have

$$\begin{aligned}\hat{E}_x\left[\hat{w}^u(\hat{x}(1))\right] &= \hat{E}_x\left[\hat{E}_x\left[\hat{w}^u(\hat{x}(1))|x^1(1)\right]\right]\\ &= \hat{E}_x\left[\chi_C(x^1(1))\left[(1-\beta)\hat{w}^u(x^1(1),0) + \beta\hat{w}^u(x^1(1),1)\right]\right.\\ &\quad\left. + \chi_{E\setminus C}(x^1(1))\hat{w}^u(x^1(1),0)\right]\\ &= \hat{E}_x\left[w^u(x^1(1))\right].\end{aligned} \tag{2.12}$$

Therefore by (2.6) we obtain that w^u defined in (2.9) is a solution to (2.10). Assume now that w^u is a solution to (2.10). Then by (2.3)

$$\hat{E}_{\delta^*_x}\left[w^u(x^1(1))\right] = E_x\left[w^u(x(1))\right]$$

and for $\hat{w}^u$ given in (2.11) we obtain (2.9). From (2.9) we obtain (2.12) which in turn by (2.11) shows that $\hat{w}^u$ is a solution to (2.6).

□

Remark 2.1. The APE has been a subject of intensive studies in [14] (together with the so called multiplicative Poisson equation). The results given above show that the use of splitting techniques provides an explicit form for the solutions to this equation.

The value of $\lambda(u)$ has another important characterization. Namely, we have

Proposition 2.2. *For Borel measurable $u : R^k \to U$ the value $\lambda(u)$ defined in Lemma 2.1 is equal to*

$$\lambda(u) = \lim_{n\to\infty} \frac{1}{n} E_x \left[\sum_{t=0}^{n-1} c(x(t), u(x(t))) \right] \tag{2.13}$$

Proof. Let $\lambda > \lambda(u)$. For $z \in C_1$ we have

$$\hat{E}_z \left[\sum_{t=1}^{\tau_{C_1}} \left(c(x^1(t), u(x^1(t))) - \lambda \right) \right] < 0$$

and consequently for $N \geq N_0$

$$\hat{E}_z \left[\sum_{t=1}^{\tau_{C_1} \wedge N} \left(c(x^1(t), u(x^1(t))) - \lambda \right) \right] \leq 0. \tag{2.14}$$

Let

$$w_N^u(x) = \hat{E}_x \left[\sum_{t=0}^{\sigma_{C_1} \wedge N-1} \left(c(x^1(t), u(x^1(t))) - \lambda \right) \right] \tag{2.15}$$

with $\sigma_{C_1} = \inf \{t \geq 0 : \ \hat{x}(t) \in C_1\}$.
For $x \notin C_1$

$$\begin{aligned} w_{N+1}^u(x) &= \hat{E}_x \Bigg[c(x^1(0), u(x^1(0))) - \lambda \\ &\quad + \hat{E}_{\hat{x}(1)} \left[\sum_{t=0}^{\sigma_{C_1} \wedge N-1} \left(c(x^1(t), u(x^1(t))) - \lambda \right) \right] \Bigg] \\ &= \hat{E}_x \left[c(x^1(0), u(x^1(0))) - \lambda + w_N^u(\hat{x}(1)) \right] \end{aligned} \tag{2.16}$$

and for $x \in C_1$ by (2.14) we have

$$\begin{aligned} w_{N+1}^u(x) &= c(x^1(0), u(x^1(0))) - \lambda \\ &\geq \hat{E}_x \Bigg[c(x^1(0), u(x^1(0))) - \lambda \\ &\quad + \hat{E}_{\hat{x}(1)} \left[\sum_{t=0}^{\sigma_{C_1} \wedge N-1} \left(c(x^1(t), u(x^1(t))) - \lambda \right) \right] \Bigg] \\ &= \hat{E}_x \left[c(x^1(0), u(x^1(0))) - \lambda + w_N^u(\hat{x}(1)) \right]. \end{aligned} \tag{2.17}$$

Consequently,

$$w_{N+1}^u(x) \geq \hat{E}_x\left[c(x^1(0), u(x^1(0))) - \lambda + w_N^u(\hat{x}(1))\right] \tag{2.18}$$

and by iteration for $N \geq N_0$

$$\begin{aligned} w_{N+k}^u(x) &\geq \hat{E}_x\left[\sum_{t=0}^{k-1}\big(c(x^1(t), u(x^1(t))) - \lambda\big) + w_N^u(\hat{x}(k))\right] \\ &\geq \hat{E}_x\left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t))) - \lambda - \|c\|N\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{k}\hat{E}_x\left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t)))\right] \\ &\leq \frac{1}{k}\|c\|N + \frac{1}{k}\sup_N \hat{E}_x\left[\sum_{t=1}^{\sigma_{C_1}\wedge N-1}\big(c(x^1(t), u(x^1(t))) - \lambda(u)\big)\right] + \lambda \end{aligned}$$

and, consequently,

$$\limsup_{k\to\infty}\frac{1}{k}\hat{E}_x\left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t)))\right] \leq \lambda.$$

With λ decreasing to $\lambda(u)$, we obtain

$$\limsup_{k\to\infty}\frac{1}{k}\hat{E}_x\left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t)))\right] \leq \lambda(u). \tag{2.19}$$

Assume now that $\lambda < \lambda(u)$. For $z \in C_1$ we have

$$\hat{E}_z\left[\sum_{t=1}^{\tau_{C_1}}\big(\gamma c(x^1(t), u(x^1(t))) - \lambda\big)\right] > 0$$

and, consequently, for $N \geq N_0$

$$\hat{E}_z\left[\sum_{t=1}^{\tau_{C_1}\wedge N}\big(c(x^1(t), u(x^1(t))) - \lambda\big)\right] \geq 0. \tag{2.20}$$

Therefore, for w_N^u defined in (2.15), similarly to (2.16)-(2.17), we have

$$w_{N+1}^u(x) \leq \hat{E}_x\left[c(x^1(0), u(x^1(0))) - \lambda + w_N^u(\hat{x}(1))\right] \tag{2.21}$$

and by iteration for $N \geq N_0$

$$\begin{aligned} w^u_{N+k}(x) &\leq \hat{E}_x \left[\sum_{t=0}^{k-1} \big(c(x^1(t), u(x^1(t))) - \lambda \big) + w^u_N(\hat{x}(k)) \right] \\ &\leq \hat{E}_x \left[\sum_{t=0}^{k-1} \big(c(x^1(t), u(x^1(t))) - \lambda \big) + \|c\| N \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{k} \hat{E}_x \left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t))) \right] \\ &\geq -\frac{1}{k} \|c\| N + \frac{1}{k} \inf_N \hat{E}_x \left[\sum_{t=1}^{\sigma_{C_1} \wedge N-1} \big(c(x^1(t), u(x^1(t))) - \lambda(u) \big) \right] + \lambda \end{aligned}$$

and

$$\liminf_{k\to\infty} \frac{1}{k} \hat{E}_x \left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t))) \right] \geq \lambda$$

and, finally,

$$\liminf_{k\to\infty} \frac{1}{k} \hat{E}_x \left[\sum_{t=0}^{k-1} c(x^1(t), u(x^1(t))) \right] \geq \lambda(u) \tag{2.22}$$

which together with (2.19) completes the proof.

□

We summarize the results of this section in the following

Theorem 2.1. *There exists a unique (up to an additive constant) function $w : R^k \mapsto R$ and a unique constant λ which are solutions to the Bellman equation* (1.7)*. Furthermore, λ is the optimal value of the cost functional J^0.*

Proof. Notice that for $\hat{u}$ optimal we find w and λ as a solution to the APE

$$w(x) = c(x, \hat{u}(x)) - \lambda + \int_{R^k} w(y) P(x, dy),$$

which exist by Lemmas 2.1, 2.2 and Corollary 2.2. By Proposition 1.17, λ is an optimal value of the cost functional J^0. Uniqueness up to an additive constant of w follows from uniqueness of the solutions to APE for the split Markov process (Lemma 2.2) and Corollary 2.2.

□

3 Risk sensitive asymptotics

In what follows we shall assume that $\gamma \in (-1, 0)$. The following estimation will be useful in this section

Lemma 3.1. *We have*

$$e^{\gamma\|a\|} \le E\left[\left(\sum_{i=1}^m h_i\xi_i(x,W(0))\right)^\gamma\right] \le e^{|\gamma|\|a\|+\frac{1}{2}\gamma^2\|\sigma^2\|}. \tag{3.1}$$

Proof. Since $r(z)=z^\gamma$ is convex, by the Jensen inequality we have

$$E\left[\left(\sum_{i=1}^m h_i\xi_i(x,W(0))\right)^\gamma\right] \le \sum_{i=1}^m h_i E\left[(\xi_i(x,W(0)))^\gamma\right].$$

Using the Hölder inequality twice we have

$$E\left[\left(\sum_{i=1}^m h_i\xi_i(x,W(0))\right)^\gamma\right] \ge \frac{1}{E\left[\left(\sum_{i=1}^m h_i\xi_i(x,W(0))\right)^{-\gamma}\right]} \ge \frac{1}{\left(\sum_{i=1}^m h_i E\left[\left(\sum_{i=1}^m \xi_i(x,W(0))\right)\right]\right)^{-\gamma}}.$$

Then using standard estimations for ξ_i we easily obtain (3.1).

□

Immediately from Lemma 3.1 we have

Corollary 3.1.

$$\limsup_{\gamma\to 0}\sup_{x\in R^k}\sup_{h\in U}\left|E\left[\left(\sum_{i=1}^m h_i\xi_i(x,W(0))\right)^\gamma\right]-1\right|=0 \tag{3.2}$$

and

$$\lim_{\gamma\to 0}\sup_{x\in R^k}\sup_{h\in U}|c_\gamma(x,h)|=0. \tag{3.3}$$

We furthermore have

Lemma 3.2.

$$\lim_{\gamma\to 0}\frac{1}{\gamma}c_\gamma(x,h)=c(x,h) \tag{3.4}$$

and the limit is increasing and uniform in x and h from compact subsets.

Proof. By the Hölder inequality $\frac{1}{\gamma}c_\gamma(x,h)$ is increasing in γ. Using l'Hôpital's rule for $\gamma\to 0$ we identify the limit as $c(x,h)$. Since the functions $c(x,h)$ and $c_\gamma(x,h)$ are continuous, by Dini's theorem the convergence is uniform on compact sets.

□

Lemma 3.3. *We have that*

$$\sup_{A\in\mathcal{B}(R^k)}\sup_{x\in R^k}\sup_{h\in U}\left|\frac{P^{h,\gamma}(x,A)}{P(x,A)}-1\right|\to 0 \tag{3.5}$$

as $\gamma\to 0$.

Proof. Notice that by the Hölder inequality we have

$$P^{h,\gamma}(x,A) \leq e^{-c_\gamma(x,h)} e^{\frac{1}{2}c_{2\gamma}(x,h)} \sqrt{P(x,A)} \tag{3.6}$$

and

$$P(x,A) \leq e^{\frac{1}{2}c_\gamma(x,h)} e^{-\frac{1}{2}\gamma\|a\|} \sqrt{P^{h,\gamma}(x,A)} \tag{3.7}$$

from which (3.5) easily follows.

□

In what follows we shall assume that for some $\gamma < 0$ we have

$$E_x\left[e^{|\gamma|\tau_C}\right] < \infty \tag{3.8}$$

for $x \in R^k$ and

$$\sup_{x\in C} E_x\left[e^{|\gamma|\tau_C}\right] < \infty. \tag{3.9}$$

where C is the same compact set as in Section 2.
We recall the following fundamental result from [8].

Theorem 3.1. *For $\gamma < 0$ sufficiently close to 0 there exists λ^γ and a continuous function $w_\gamma : R^k \mapsto R$ such that the Bellman equation* (1.10) *is satisfied. Moreover $\frac{1}{\gamma}\lambda^\gamma$ is an optimal value of the cost functional J_x^γ and the control $\hat{u}(x_n)$, where $\hat{u}$ is a Borel measurable function for which the infimum in the right hand side of* (1.10) *is attained, is an optimal control within the class of all controls from $\mathcal{U}_s$.*
Furthermore, if for an admissible control (h_n) we have that

$$\limsup_{t\to\infty} E_x^{(h_n)}\left[\left(E_{x_t}^{h_t}\left[e^{w_\gamma(x_1)}\right]\right)^\alpha\right] < \infty$$

for every $\alpha > 1$, then $\frac{1}{\gamma}\lambda^\gamma \leq J_x^\gamma((h_n))$.

Notice now that by the Hölder inequality the value of the functional J^γ is increasing in $\gamma < 0$ and, by the Jensen inequality, is dominated by the value of J^0. Consequently, the same holds for the optimal values of the cost functionals, i.e.

$$\frac{1}{\gamma}\lambda_\gamma \leq \lambda. \tag{3.10}$$

Furthermore, there is a sequence u_n of continuous functions from R^k to U such that $c(x, u_n(x))$ converges uniformly in x from compact subsets to $\sup_{h\in U} c(x,h)$. By Lemma 2.1 and Theorem 2.1 we immediately have that $\lambda((u_n)) \to \lambda$ as $n \to \infty$. This means that for any $\varepsilon > 0$ there is an ε-optimal continuous control function u_ε. We are going to show that for each $\varepsilon > 0$

$$J^\gamma(u_\varepsilon(x(n))) \to J^0(u_\varepsilon(x(n))) \tag{3.11}$$

as $\gamma \to 0$. Since the proof will be based, following Section 5 of [6], upon large deviation estimates, we need the following assumption:

(A) there is a continuous function $f_0 : R^k \mapsto [1,\infty)$ such that for each positive integer n the set $K_n := \left\{x \in R^k : \frac{f_0(x)}{Pf_0(x)} \le n\right\}$ is compact.

Remark 3.1. By direct calculation one can show that for a large class of ergodic processes $(x(n))$ function $f_0(x) = e^{c\|x\|^2}$ satisfies (A) for small c. To be more precise, assume for simplicity that $k = 1$ and $|x+b(x)| \le \beta|x|$ for a sufficiently large x with $0 < \beta < 1$. Then for $0 < c < \frac{1-\beta^2}{2dd^T}$ assumption (A) holds.

Proposition 3.1. *Under* (A) *for continuous control function* $u : R^k \mapsto U$ *we have*

$$J^\gamma(u(x(n))) \to J^0(u(x(n))) \tag{3.12}$$

as $\gamma \to 0$.

Proof. Under (A) using Lemma 3.3 we see that the set

$$K_n^{u,\gamma} := \left\{x \in R^k : \frac{f_0(x)}{P^{u,\gamma} f_0(x)} \le n\right\}$$

is compact for each n. Therefore, by Theorem 4.4 of [10] we have an upper large deviation estimate for empirical distributions of Markov process with transition operator $P^{u(x),\gamma}(x,\cdot)$. Using the theorem in Section 3 of [11] we also have a lower large deviation estimate. Consequently, we have a large deviation principle corresponding to the rate function

$$I^{u,\gamma}(\nu) := \sup_{h\in H} \int_{R^k} \ln \frac{h(x)}{P^{u(x),\gamma}h(x)} \nu(dx), \tag{3.13}$$

where H is the set of all bounded functions $h : R^k \mapsto R$ such that $\frac{1}{h(x)}$ is also bounded and $\nu \in \mathcal{P}(R^k)$. Therefore, by Varadhan's theorem (Theorem 2.1.1 of [5]) we have

$$\begin{aligned} &\frac{1}{\gamma} \lim_{n\to\infty} \frac{1}{n} \ln E_x^{h,\gamma}\left[\exp\left(\sum_{t=0}^{n-1} c_\gamma(x(t),h(t))\right)\right] \\ &= \inf_{\nu\in\mathcal{P}(R^k)} \left(\int_{R^k} \frac{1}{\gamma} c_\gamma(z,u(z))\nu(dz) - \frac{1}{\gamma} I^{u,\gamma}(\nu)\right). \end{aligned} \tag{3.14}$$

There is a sequence of measures ν_{γ_i} with $\gamma_i \to 0$ as $i \to \infty$ such that

$$\begin{aligned} &\int_{R^k} \frac{1}{\gamma_i} c_{\gamma_i}(z,u(z))\nu_{g_i}(dz) - \frac{1}{\gamma_i} I^{u,\gamma_i}(\nu_{\gamma_i}) \\ &\le \inf_{\nu\in\mathcal{P}(R^k)} \left(\int_{R^k} \frac{1}{\gamma_i} c_{\gamma_i}(z,u(z))\nu(dz) - \frac{1}{\gamma_i} I^{u,\gamma_i}(\nu)\right) + \frac{1}{i}. \end{aligned} \tag{3.15}$$

Since from (3.1)

$$\frac{1}{\gamma}\lim_{n\to\infty}\frac{1}{n}\ln E_x^{h,\gamma}\left[\exp\left(\sum_{t=0}^{n-1}c_\gamma(x(t),h(t))\right)\right]\le\|a\| \tag{3.16}$$

we have that $I^{u,\gamma_i}(\nu_{\gamma_i})\to 0$. We shall show that the sequence (ν_{γ_i}) is tight. Applying Fatou's lemma to the sequence $\{f_0\wedge N\}$ with $N\to\infty$ we obtain that

$$\int_{R^k}\ln\frac{f_0(x)}{P^{u(x),\gamma}f_0(x)}\nu_{\gamma_i}(dx)\le I^{u,\gamma_i}(\nu_{\gamma_i}). \tag{3.17}$$

By (3.5) for $\varepsilon>0$ there is γ_0 such that for $\gamma\ge\gamma_0$

$$(1-\varepsilon)Pf_0(x)\le P^{u(x),\gamma}f_0(x)\le(1+\varepsilon)Pf_0(x). \tag{3.18}$$

Therefore, by (3.17)

$$\int_{R^k}\ln\frac{f_0(x)}{Pf_0(x)}\nu_{\gamma_i}(dx)\le I^{u,\gamma_i}(\nu_{\gamma_i})+\ln(1+\varepsilon) \tag{3.19}$$

for $i>i_0$. Let $\rho_n:=\inf_{x\in K_n}\ln\frac{f_0(x)}{Pf_0(x)}$. Then

$$\rho_n\nu_{\gamma_i}(K_n)+\ln n\nu_{\gamma_i}(K_n^c)\le I^{u,\gamma_i}(\nu_{\gamma_i})+\ln(1+\varepsilon) \tag{3.20}$$

where $K_n^c:=R^k\setminus K_n$. Consequently,

$$\ln n\nu_{\gamma_i}(K_n^c)\le\frac{I^{u,\gamma_i}(\nu_{\gamma_i})+\ln(1+\varepsilon)-\rho_n}{\ln n-\rho_n} \tag{3.21}$$

and since $\ln n\ge 1+\rho_n$ for sufficiently large n, we have the tightness of the measures ν_{γ_i}. By the Prohorov theorem there exists a subsequence of ν_{γ_i}, for simplicity still denoted by ν_{γ_i}, and a probability measure $\bar\nu$ such that $\nu_{\gamma_i}\to\bar\nu$ as $i\to\infty$. Since by (3.5) $I^{u,\gamma}(\nu)$ converges uniformly to $I^u(\nu):=\sup_{h\in H}\int_{R^k}\ln\frac{h(x)}{P^{u(x)}h(x)}\nu(dx)$ as $\gamma\to 0$ and I^u is a nonnegative lower semicontinuous function, we have that $I^u(\bar\nu)=0$. By Lemma 2.5 of [9] the measure $\bar\nu$ is invariant for the transition operator $P(x,\cdot)$. Therefore, by Lemma 3.2

$$\lim_{i\to\infty}\frac{1}{\gamma_i}\lim_{n\to\infty}\frac{1}{n}\ln E_x^{h,\gamma_i}\left[\exp\left(\sum_{t=0}^{n-1}c_{\gamma_i}(x(t),h(t))\right)\right]$$
$$\ge\lim_{i\to\infty}\int_{R^k}\frac{1}{\gamma_i}c_{\gamma_i}(z,u(z))\nu_{\gamma_i}=\int_{R^k}c(z,u(z))\bar\nu(dz)=J^0(u(x(n)) \tag{3.22}$$

and using the fact that the cost functional J^γ is increasing in γ we obtain (3.12), which completes the proof.

□

We are now in position to summarize the results of this section.

Theorem 3.2. *Under* (A) *a continuous ε-optimal control function u_ε for J^0 is also a 2ε-optimal control function for J^γ provided $0 > \gamma > \gamma_0$. Consequently convergence* (1.12) *holds.*

Remark 3.2. One can expect that at least a subsequence of $\frac{1}{\gamma} w_\gamma(x)$ converges to $w(x)$ uniformly on compact subsets, as $\gamma \to 0$, where w is a solution to the risk neutral Bellman equation (1.7). Unfortunately, the authors were not able to show this.

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