First Steps Towards Computably-Infinite Information Systems

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Abstract. In order to characterize the metric of exact subsets of infinite information systems, [51] studied the asymptotic behaviour of ω–chains of graded indiscernibility relations. The SFP object underlying the universe of exact sets presented in [2] provides a concrete example of an infinite graded information system. By controlling the asymptotic behaviour of ω–Sequences of Finite Projections, the theory of graded chains of indiscernibility relations articulates the fine structure of SFP objects, providing a metric over exact sets.

Keywords: domain theory, effective domain theory, exact sets, Frege sets, graded indiscernibility, infinite information systems, Myhill-Sheperdson Theorem, modal logic, non–standard analysis, numeration theory, rough sets, SFP objects.

1 Infinite Information Systems

Though distinct fields of computer science, Rough Set Theory (RST) [44] and Domain Theory (DT) [58] both study the approximation of concepts in infinite information systems. The former studies the topological approximation of subsets by upper and lower bounding subsets, while the latter studies the order–theoretic approximation of functions under an information ordering. However their techniques diverge, these fields are intricately related when attention is focused on infinite information systems arising in connection with the inverse limits of Sequences of Finite Projections (SFP) [47].

Given a set of points U, RST studies the field of subsets of U under upper and lower approximation operations, typically those of the quasi–discrete topology induced by an equivalence relation E on U [44]. In the finite case, the field of exact subsets forms a compact metric Boolean algebra. In order to extend this characterization to the exact subsets of infinite information systems [51] studied the asymptotic behaviour of approximations by descending chains of equivalence relations $\{E_i\}$ over U such that $E_{i+1} \subseteq E_i$, called "graded chains of indiscernibility relations". The limit E_∞ of this chain is an equivalence relation called *the indiscernibility relation graded by* $\{E_i\}$. The collection of all E_i-granules forms a base for a "graded" topology in [51]. We shall show that this graded topology is identical to the Pawlak topology induced by E_∞.

J.F. Peters et al. (Eds.): Transactions on Rough Sets II, LNCS 3135, pp. 151–188, 2004.

By integrating the "distance" between two elements of U as the stage at which they are discerned in the chain $\{E_i\}$, [51] obtains a metric distance $-$ which we call a *graded metric* $-$ between subsets of U which are closed in the graded topology. As Pawlak's topology is quasi–discrete, we conclude that the "closed" subsets are precisely the exact subsets of U whence the graded metric obtains over the exact subsets of U. As an example of an infinite information system based upon graded indiscernibility, we use the theory of SFP objects to construct a universe of exact sets by placing an approximation space (U, E) in a type–lowering retraction with 2^U. Accordingly, this universe of exact sets is accompanied by a graded metric.

By dualizing Polkowski's construction, the concept of an *inversely graded* indiscernibility relation and metric is obtained. A. Robinson's [54] ultra product construction of non–standard real numbers provides a viable example of this. This conclusion is consistent with the analogy developed in [4] that the universe of exact sets is to standard set theory as Robinson's non–standard analysis is to standard analysis. We hope that this development suggests a foundational role for the study of infinite information systems in rough set theory and analysis.

2 Rough Set Theory

Let $U \neq \emptyset$ and $E \subseteq U \times U$ be an equivalence relation. In rough set theory [44], E is typically interpreted as a relation of *indiscernibility* with respect to some prior family of concepts (subsets of U). The pair (U, E) is called an *approximation space*. For each point $u \in U$, let $[u]_E$ denote the equivalence class of u under E. E-classes $[u]_E$ are called (E-) *granules* or *elementary* subsets. Let $A \subseteq U$;

$$Int(A) =_{df} \{x \in U \mid (\forall y \in U)[xE\, y \rightarrow y \in A]\} = \cup\{[u]_E \mid [u]_E \subseteq X\},$$
$$Cl(A) =_{df} \{x \in U \mid (\exists y \in U)[xE\, y \wedge y \in A]\} = \cup\{[u]_E \mid [u]_E \cap X \neq \emptyset\},$$

are called the *lower* and *upper approximations* of A, respectively. $Cl(A)$ is a relational closure under E of the subset A. E-closed subsets of U are called (E-) *complete (or exact)*. It is natural to regard the exact subsets of U as the "parts" of U and elementary subsets as the "atomic parts" of U [1]. $\mathcal{C}(E)$ denotes the family of E–exact subsets of U and, for $A \subseteq U$, $\mathcal{C}(A)$ the family of E–exact subsets of A. The family,

$$\mathcal{B}_E = \{[x]_E : x \in U\},$$

is the partition of U induced by the equivalence relation E and it forms a base for a quasi–discrete topology Π_E over U. It is clear that the open sets in this topology are precisely the exact sets.

[1] RST marks the re–emergence in the Warsaw School of Mathematics of mereological concepts such as "part" and "whole" that were originally introduced into set theory by Leśniewski [36]; see e.g., rough mereology [48–50]. For an example of the mereological conception of set in the recent philosophical literature $-$ motivated independently of RST $-$ see [37].

Given an approximation space (U, E) and $A, B \subseteq U$, [44] defines A and B to be *roughly equal*, symbolically, $A \sim_E B$, iff $Int(A) = Int(B)$ and $Cl(A) = Cl(B)$. Hence, $A \sim_E B$ iff A and B are equivalent with respect to the partition topology $(U, \mathcal{C}(E))$ induced by (U, E). Rough equality classes $[A]_{\sim_E}$ are called *topological rough sets* [63]. In the context of (U, E), $[A]_{\sim_E}$ may be identified with the interval of subsets lying between $Int(A)$ and $Cl(A)$, in which A is approximated from "below" by its interior and from "above" by its closure.

3 Proximal Frege Structures

The comprehension principle of naive set theory is based upon Frege's [21, 22] ill–fated [23–25] idea that for every concept X there is an object, $\widehat{X}$, called *the extension* of X, which comprises precisely the objects falling under X. $\widehat{X}$ is, intuitively, the set of objects u such that u falls under X, i.e., $\{u : X(u)\}$. Frege thus posited the existence of a "type–lowering correspondence", terminology established in [5] for functions $f : 2^U \rightarrow U$, holding between the concept universe 2^U and the universe of objects U. Frege attempted to govern his introduction of extensions by adopting the principle that $\widehat{X}$ and $\widehat{Y}$ are strictly identical just in case precisely the same objects fall under the concepts X and Y. That this "Basic Law" contradicts Cantor's Theorem[2] in requiring f to be injective and leads to Russell's paradox regarding "the class of all classes which do not belong to themselves" [23, 55, 25], is now well appreciated. [15] initiated the proof theoretic study of Frege's extension function in consistent theories extending second order logic. Models of these theories are obtained by placing a domain of discourse U in a type–lowering correspondence with its power set 2^U and are called "Frege structures" after terminology established in [1]. The rediscovery of Frege's Theorem[3] [43, 64, 9] in type–lowering extensions of second order logic re–focused interest on Frege structures. [3, 2, 6] presented Frege structures as *retraction pairs* of type–lowering and raising maps, i.e., pairs (f, g) of functions $f : 2^U \rightarrow U$, $g : U \rightarrow 2^U$ such that $f(g(u)) = u$. f retracts 2^U onto U (determining sethood) and g is the adjoining section (determining elementhood). The Frege structure (U, f, g) is a model of abstract set theory in which Cantor's "consistent multiplicities" [13, p. 443.] can be identified with precisely those subsets of U which are elements of the section $g[U]$ of the retraction, whence the fundamental question [6] arises: which subsets are *these*?

[2] According to Cantor's Theorem [12], the power set 2^U is of strictly larger cardinality than U.

[3] In the *Grundlagen der Arithmetik* [21], Frege sketched a derivation of Peano's postulates for the arithmetic of natural numbers from Cantor's Principle of equality for the cardinal numbers [11], that sets have the same cardinal number just in case they are equipollent. Called "Hume's Principle" (HP) by Boolos [9], this principle is presented in *Grundlagen* §63 as the statement that, for any concepts F and G: the number of Fs = the number of Gs iff F is equipollent with G. The derivation of the infinity of the natural number series in second order logic + HP − a theory Boolos calls "Frege's Arithmetic" − is now called "Frege's Theorem" [9].

One concrete answer is provided along the lines of Rough Set Theory. Accordingly, one may augment the retraction pair (f, g) with an equivalence relation E on U and characterize the section as comprised of precisely the E–closed subsets of U [2]. Comprehension is governed by the method of upper and lower approximations in the resulting universe (U, E, f, g) of *abstract* sets, called [2] a "proximal" Frege structure. The leading idea is that logical space has an atomic structure and this granularity is the form of set theoretic comprehension.

Let (U, E) be an approximation space and $\ulcorner\cdot\urcorner : 2^U \to U$, $\llcorner\cdot\lrcorner : U \to 2^U$ be functions. Assume further,

1. $\ulcorner\cdot\urcorner$ is a retraction, i.e., $\ulcorner\llcorner u\lrcorner\urcorner = u$ (i.e., $\ulcorner\cdot\urcorner \circ \llcorner\cdot\lrcorner = 1_U$), thus $\llcorner\cdot\lrcorner$ is the adjoining section.
2. The operator $\llcorner\cdot\lrcorner \circ \ulcorner\cdot\urcorner$ is a closure (under E). This is that for every $X \subseteq U$, $\llcorner\ulcorner X\urcorner\lrcorner$ is E–closed and
$$X \subseteq \llcorner\ulcorner X\urcorner\lrcorner.$$
So,
$$\llcorner\ulcorner X\urcorner\lrcorner = Cl(X).$$
3. The E–closed subsets of U are precisely the $X \subseteq U$ for which $\llcorner\ulcorner X\urcorner\lrcorner = X$. They are fixed–points of the operator $\llcorner\cdot\lrcorner \circ \ulcorner\cdot\urcorner$.

Then $(U, E, \ulcorner\cdot\urcorner, \llcorner\cdot\lrcorner)$ is called a *proximal Frege structure (PFS)*. Elements of U are called *Frege sets*. The indiscernibility relation E of a PFS is usually denoted $\equiv$. The fundamental idea behind this rather curious mathematical structure is to use the approximation space [44] to tame the reflexive universe U of sets,

$$U \lhd 2^U,$$

where "$U \lhd 2^U$" indicates that $\ulcorner\cdot\urcorner$ retracts the power set of U onto U.

It follows that for every $u \in U$,

$$\llcorner u\lrcorner = \llcorner\ulcorner\llcorner u\lrcorner\urcorner\lrcorner,$$

and hence, $\llcorner u\lrcorner$ is $\equiv$–exact. Suppose X is $\equiv$–closed, then $\llcorner\ulcorner X\urcorner\lrcorner = X$. Thus,

$$X \in \llcorner U\lrcorner = Image(\llcorner\cdot\lrcorner).$$

So, the set of $\equiv$–closed subsets of U is precisely the image of $\llcorner\cdot\lrcorner$. In algebraic terms it is the kernel of the retraction mapping.

Further, we have the isomorphism $\mathcal{C}(\equiv) \approx U$ given by,

$$i : \mathcal{C}(\equiv) \to U : X \mapsto \ulcorner X\urcorner,\ j : U \to \mathcal{C}(\equiv) : u \mapsto \llcorner u\lrcorner.$$

All of this may be summarized by the equation,

$$\mathcal{C}(\equiv) \approx U \lhd 2^U,$$

where $\mathcal{C}(\equiv)$ is the family of $\equiv$-closed subsets of U.

Let $\mathcal{F} = (U, \equiv, \ulcorner\cdot\urcorner, \llcorner\cdot\lrcorner)$ be a proximal Frege structure. Writing "$u_1 \in_{\mathcal{F}} u_2$" for "$u_1 \in \llcorner u_2 \lrcorner$", U interpreted as a universe of abstract (or type–free)sets; $\llcorner\cdot\lrcorner$ supports the relation of set membership holding between type–free sets (elements of U). Note that as $\llcorner u_2 \lrcorner$ is $\equiv$–closed, whenever $u_1 \equiv v$,

$$u_1 \in \llcorner u_2 \lrcorner \leftrightarrow v \in \llcorner u_2 \lrcorner.$$

$\mathcal{F}$ thus validates the Principle of Naive Comprehension (PNC),

$$u \in_{\mathcal{F}} \ulcorner X \urcorner \Leftrightarrow X(u),$$

for exact ($\equiv$–closed) subsets X of U. When X fails to be elementary, the equivalence also fails and is replaced by a pair of approximating conditionals,

$$(1)\ u \in_{\mathcal{F}} \ulcorner Int(X) \urcorner \Rightarrow X(u);$$
$$(2)\ X(u) \Rightarrow u \in_{\mathcal{F}} \ulcorner Cl(X) \urcorner.$$

Note here we now use applicative grammar "$X(u)$" ("u falls under X) to indicate that u is an element of $X \subseteq U$. Let "$\{x : X(x)\}$" denote $\ulcorner X \urcorner$. While "$\{u \in U \mid X(u)\}$" denotes a *subset* of U, the expression "$\{x : X(x)\}$ denotes an *element* of U. We distinguish the indiscernibility class $[u]_{\equiv}$ of a Frege set u from Frege set $\ulcorner [u]_{\equiv} \urcorner$ that represents $[u]_{\equiv}$; the latter is denoted $\overline{\{u\}}$, and is called the *proximal singleton* of u. Let $x, y \in U$. Then, both

$$(a)\ (\forall u)(u \in_{\mathcal{F}} x \leftrightarrow u \in_{\mathcal{F}} y) \leftrightarrow x = y;$$
$$(b)\ (\forall u)(x \in_{\mathcal{F}} u \leftrightarrow y \in_{\mathcal{F}} u) \leftrightarrow x \equiv y.$$

Theorem 1. ***(Cocchiarella 1972)*** *There are $x, y \in U$ such that $x \equiv y$ but $x \neq y$.*

Proof. As assuming the failure of the theorem (i.e., assuming the identity of indiscernibles), [15, 16] derives Russell's contradiction directly, the theorem follows.

Cocchiarella's theorem ensures that whenever we have a PFS $\mathcal{F} = (U, \equiv, \ulcorner\cdot\urcorner, \llcorner\cdot\lrcorner)$, the indiscernibility relation $\equiv$ is coarser than the strict identity.

Since elements of U represent elementary subsets of U, the complete Boolean algebra,

$$(\mathcal{C}(\equiv), \cup, \cap, -, \emptyset, U),$$

is isomorphic to $(U, \cup_U, \cap_U, -_U, \ulcorner\emptyset\urcorner, \ulcorner U \urcorner)$ under the restriction $\ulcorner\cdot\urcorner \restriction \mathcal{C}(\equiv)$ of the type–lowering retraction to elementary subsets of U. Here, $\cup_U, \cap_U, -_U$ denote the definitions of union, intersection and complementation natural to type–free sets, e.g.,

$$u_1 \cup_U u_2 =_{df} \ulcorner \llcorner u_1 \lrcorner \cup \llcorner u_1 \lrcorner \urcorner$$

etc., (in most contexts the subscripted "U" is suppressed). We define "$u_1 \subseteq u_2$" to be "$\llcorner u_1 \lrcorner \subseteq \llcorner u_2 \lrcorner$", i.e., *inclusion* of Frege sets is the partial ordering naturally associated with the Boolean algebra of Frege sets.

3.1 Counterparts and Inner and Outer Penumbra

Since unions of $\mathcal{F}$-sets are exact,

$$\{x \in U \mid (\exists\, g \in U)[f \equiv g \wedge x \in_{\mathcal{F}} g]\},$$

is an exact subset of U. Let $a \in U$; define the *outer penumbra* of a, symbolically, $\Diamond a$, to be the $\mathcal{F}$-set $\cup[a]_{\equiv}$; dually, define the *inner penumbra*, $\Box a$, to be the $\mathcal{F}$-set $\cap[a]_{\equiv}$. These operations, called the *penumbral modalities*, are interpreted using David Lewis' counterpart semantics for modal logic [37]. Your "counterpart" (in a given world) is a person more qualitatively similar to you than any other object (in that world). You are necessarily (possibly) P iff all (some) of your counterparts are P. Similarly, if a and b are indiscernible $\mathcal{F}$-sets, a is more similar to b than any *other* (i.e., discernible!) $\mathcal{F}$-set. Thus we call a $\mathcal{F}$-set b a *counterpart* of a $\mathcal{F}$-set a whenever $a \equiv b$. Then $\Box a$ ($\Diamond a$) represents the set of $\mathcal{F}$–sets that belong to all (some) of a's counterparts. In this sense, we can say that a $\mathcal{F}$-set x *necessarily* (*possibly*) belongs to a just in case x belongs to $\Box a$ ($\Diamond a$).

Define $a\langle x\rangle =_{df} x \in_{\mathcal{F}} \Diamond a$; thus,

$$a\langle x\rangle \Leftrightarrow x \in_{\mathcal{F}} \Diamond a \Leftrightarrow x \in \llcorner \Diamond a \lrcorner.$$

This is the *outer membership relation*, also written $x \in_{\Diamond} a$. Thus, for all $a, x \in U$,

$$x \in_{\mathcal{F}} a \rightarrow x \in_{\Diamond} a.$$

It is equally clear that the converse of this does not hold. Dually, define $a[x] =_{df} x \in_{\mathcal{F}} \Box a$; thus,

$$a[x] \Leftrightarrow x \in_{\mathcal{F}} \Box a \Leftrightarrow x \in \llcorner \Box a \lrcorner.$$

This is the inner *membership relation*, also written $x \in_{\Box} a$.

The principle that disjoint sets are discernible is a triviality of classical set theory. However, it is independent of the axioms for PFS given thus far. *The Discernibility of the Disjoint* (DoD) [2, 3] is the principle that disjoint $\mathcal{F}$-sets be discernible.

We say that a PFS $\mathcal{F}$ is a *plenum* iff it satisfies the following condition of plenitude: For all $a, b \in U$, $\Box a \equiv a \equiv \Diamond a$, whence $\Box a \equiv \Diamond a$. Further, suppose $a \subseteq b$ and $a \equiv b$. Then, for all $c \in U$, $a \subseteq c \subseteq b \Rightarrow c \equiv b$. Plenitude is a consistent extension of the notion of a PFS which tells us that $\mathcal{F}$ inclusion is well–behaved with respect to indiscernibility. It is a nontrivial generalization, via "blurring", of a truism of classical set theory that if a is a subset of b and a is indiscernible from b, then any c which is in between a and b as a subset is indiscernible from b. If $\mathcal{F}$ is a plenum, then,

Lemma 1. *For all $a \in U$ and $X \subseteq U$ such that $x \equiv a$ $(x \in X)$, we have $\cup X$, $\cap X \equiv a$.*

Proof. $\Box a \subseteq \cup X, \cap X \subseteq \Diamond a$, so the desired result follows from Plenitude by the fact that $\Box a \equiv \Diamond a$.

Corollary 1. $([a]_\equiv, \cap_U, \cup_U, \Box a, \Diamond a)$ *is a complete lattice with least (greatest) element* $\Box a$ $(\Diamond a)$.

The canonical PFS M_{max} to whose domain theoretic construction we now turn is a concrete example of a plenum which satisfies DoD.

4 The Canonical PFS M_{max}

4.1 Basic Domain Theory

Cpo's and Continuous Functions. We assume the standard terminology and notation for partially ordered sets (posets). When a poset X has the smallest element, we denote it by $\bot_X$. We may drop the subscript X and write just $\bot$, as long as doing so does not cause confusion.

A poset X with smallest element such that every directed subset S of it has a least upper bound $\bigvee S$ (called a *directed limit of* S) in X is called a *complete partial order (cpo).* A singleton clearly is a cpo and we call it a *trivial cpo.* Let A and B be posets. A function $f : A \to B$ is *monotonic* (or *monotone*) if it is order preserving. Moreover, it is *continuous* if for each directed subset S of A,

$$f(\bigvee S) = \bigvee f(S) = \bigvee\{f(x) \mid x \in S\}.$$

It is clear that all continuous functions are monotonic. So, continuous functions are precisely those which preserve directed limits.

The collection of cpo's and that of monotonic functions form a category which we denote by $\mathcal{MCPO}$, abbreviated by $\mathcal{M}$. The collection of cpo's and that of continuous functions form a category which is denoted by $\mathcal{CCPO}$, abbreviated $\mathcal{C}$. $\mathcal{CCPO}$ is a full subcategory of $\mathcal{MCPO}$.

Retraction and Embedding. For cpo's A and B, we call a pair $(s : A \to B,\ r : B \to A)$ of monotonic (continuous) functions an *m(c)-retraction pair* (from B *to* A) iff

$$r \cdot s = 1_A;$$

s is called the *m(c)-section* and r is called the *m(c)-retraction.* If there is an m–retraction pair from A to B, we say A is an *m–retract of* B, in symbols,

$$A \lhd_{\mathcal{M}} B.$$

The concept of *c–retract*,

$$A \lhd_{\mathcal{C}} B,$$

can be defined similarly. If the retraction pair is a pair of morphisms in a subcategory $\mathcal{S}$ of $\mathcal{MCPO}$, we may write

$$A \lhd_{\mathcal{S}} B.$$

For cpo's A and B, let a pair $(f : A \to B, g : B \to A)$ of monotonic (continuous) functions be called an *m(c)–embedding pair (from A to B)* if

$$f \cdot g \leq 1_B, \quad g \cdot f = 1_A.$$

f is called the *m(c)–embedding* and g is called the *m(c)–projection*. Sometimes, embedding pairs are called *m(c)–projection pairs*. It can easily be verified that an m(c)–embedding uniquely determines the corresponding projection and vice versa. So, we may write f^R for g when (f, g) is an $m(c)$–embedding pair. If there is an $m(c)$–embedding $f : A \to B$, we write

$$A \curvearrowright B.$$

If the embedding pair is a pair of morphisms in a subcategory $\mathcal{S}$ of $\mathcal{MCPO}$, we may write

$$A \curvearrowright_{\mathcal{S}} B.$$

If it does not cause confusion, we will drop the subscript $m(c)$ in the foregoing.

It is clear that if $(f : A \to B, g : B \to A)$ is an m–embedding pair then (g, f) is a retraction pair. If $(f : A \to B, g : B \to A)$ is an m–embedding pair then

$$f(\bot_A) = \bot_B \text{ and } g(\bot_B) = \bot_A .$$

It also is easy to verify that if $(f : A \to B, g : B \to A)$ is a monotonic isomorphism pair, then both f and g are continuous.

The collection of cpo's and that of $m(c)$–projection pairs form a category that is denoted by $\mathcal{M}(\mathcal{C})$-$\mathcal{PCPO}$. $\mathcal{M}(\mathcal{C})$-$\mathcal{PCPO}$ is a non–full subcategory of $\mathcal{M}(\mathcal{C})\mathcal{CPO}$.

Function Spaces of Cpo's. Suppose A and B are cpo's. For every monotonic functions $f, g : A \to B$, we define a relation $f \leq g$ by

$$f \leq g \text{ iff } f(a) \leq g(a).$$

This relation clearly is a partial order over the set $[A \to B]_{\mathcal{M}}$ of all monotone functions from A to B. The restriction of this relation to the continuous functions is a partial order over the set $[A \to B]_{\mathcal{C}}$ of all continuous functions from A to B. For a poset A and a cpo B, $[A \to B]_{\mathcal{M}}$ and $[A \to B]_{\mathcal{C}}$ are cpo's.

Assume $(f_i : A_i \to B_i, f_i^R : B_i \to A_i),\ i = 1, 2$ are $m(c)$–projection pairs. They induce an $m(c)$–projection pair,

$$([f_1 \to f_2] : [A_1 \to A_2] \to [B_1 \to B_2], \quad [f_1^R \to f_2^R] : [B_1 \to B_2] \to [A_1 \to A_2]),$$

given by,

$$[f_1 \to f_2](w) = f_2 \cdot w \cdot f_1^R$$
$$[f_1^R \to f_2^R](w) = f_2^R \cdot w \cdot f_1.$$

Suppose A and B are finite cpo's; then all monotonic functions from A to B are continuous and

$$[A \to B]_{\mathcal{M}} = [A \to B]_{\mathcal{C}}.$$

The Inverse Limit Construction. By a *sequence of c–embeddings of cpo's*, we mean the following diagram:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots \qquad \text{(Diagram D)}$$

where $f_n, n \in \omega$ are c–embeddings from A_n to A_{n+1}. The *inverse limit* of this sequence is a cpo,

$$\{x \in \Pi_{i\in\omega} A_i \mid x_i = f_i^R(x_{i+1})\},$$

with the point–wise ordering. We denote this cpo by A_∞. For each $n \in \omega$, we clearly have $\bot_{A_n} = f_n^R(\bot_{A_{n+1}})$ and $\bot_{A_\infty}$ is given as

$$\bot_{A_\infty} = \langle \bot_{A_0}, \bot_{A_1}, ...\rangle.$$

For each $n \in \omega$, there is a c–embedding pair $(f_{n\infty}, f_{n\infty}^R)$ from A_n to A_∞ such that

$$f_{n\infty}(x)(i) = \begin{cases} f_i^R \cdot f_{i+1}^R \cdot \ldots \cdot f_n^R(x) & \text{if } i < n \\ x & \text{if } i = n \\ f_n \cdot \ldots \cdot f_i(x) & \text{if } i > n, \end{cases}$$

$$f_{n\infty}^R(x) = x_n.$$

For each directed subset X of A_∞, the least upper bound of X is

$$\bigvee X = \bigvee_{n\in\omega} f_{n\infty}(\bigvee f_{n\infty}^R(X)).$$

The following naturally expected lemma states that $f_{n\infty}^R(x)$ is the n–th approximation of $x \in A_\infty$.

Proposition 1. *$\{f_{n\infty} \cdot f_{n\infty}^R(x) \mid n \in \omega\}$ is a chain in A_∞. Moreover,*

$$x = \bigvee\{f_{n\infty} \cdot f_{n\infty}^R(x) \mid n \in \omega\}.$$

We may write $\tilde{x}_n$ to denote $f_{n\infty} \cdot f_{n\infty}^R(x) = f_{n\infty}(x_n)$. Hence,

$$x = \bigvee\{\tilde{x}_n \mid n \in \omega\} = \bigvee_{n\in\omega} \tilde{x}_n.$$

Each element b of $\cup\{f_{n\infty}(A_n) \mid n \in \omega\}$ is *"compact"* since for every directed subset S of A_∞, if $b \leq \bigvee S$ then $b \leq s$ for some $s \in S$. Moreover, for every element s of A_∞, there is a directed set S_s of compact elements of A_∞ such that $s = \bigvee S_s$. Hence A_∞ is an *"algebraic cpo"*.

Maximal Elements of Cpo's Every chain is a directed set, whence it follows from (an equivalent to) the Axiom of Choice that every cpo has at least one maximal element. Indeed:

Lemma 2. *Let D be a cpo. Any element of D extends into a maximal element of D.*

Proof. Let $d \in D$ and let $U_d = \{x \in D \mid d \leq x\}$. Every chain in U_d is clearly a chain in D. Hence it has a least upper bound l in D. Clearly $l \in U_d$. Suppose there is an upper bound u of this chain in U_d such that

$$u \leq l.$$

Then, since $u \in D$, we have $u = d$. Hence, l is a least upper bound of this chain in U_d. By (an equivalent to) The Axiom of Choice, U_d has a maximal element m with respect to the induced partial ordering. Assume there is an element x of D such that

$$x \geq m.$$

Then,

$$x \geq m \geq d$$

and $x \in U_d$. Hence, $x = m$. Therefore this m is maximal with respect to the partial ordering on D.

Lemma 3. *Let A and B be cpo's such that $A \lhd_{\mathcal{M}} B$. Then, for each maximal element a of A, there is a maximal element b of B such that*

$$a = r(b),$$

where r is the m–retraction map from B to A.

Proof. Since $r : B \rightarrow A$ is surjective, there is $x \in B$ such that $a = r(x)$. Let $m \in B$ be a maximal extension of x. Then, due to the maximality of a and monotonicity of r, we have

$$r(m) = r(x) = a.$$

Lemma 4. *(The Isolation Lemma) Suppose A and B are nontrivial cpo's and consider the cpo $[A \rightarrow B]_{\mathcal{M}}$. For every maximal element f of $[A \rightarrow B]_{\mathcal{M}}$ and for all maximal elements a of A, $f(a)$ is maximal in B. In particular,*

$$f(a) \neq \perp .$$

Proof. Suppose for some $a \in A_{max}$, $f(a)$ is not maximal in B. Define $f' : A \rightarrow B$ by

$$f'(x) = \begin{cases} b & \text{if} \quad x = a \\ f(x) & \text{else,} \end{cases} \qquad (x \in A),$$

where b is an element of B with $b \gneq f(a)$. Since a is maximal, f' is monotone and hence $f' \in [A \rightarrow B]_{\mathcal{M}}$. Clearly $f' \gneq f$ and this is a contradiction.

This result states that maximal elements of cpo's are isolated with respect to monotonic functional application.

4.2 The Theory of SFP Objects

SFP objects were first studied by Plotkin [47] and Smyth [59] in order to build mathematical models of nondeterministic computation. Notwithstanding the original motivation, one finds that these objects are very important from the foundational point of view. The construction of SFP objects suggests that they are the smallest transfinite structures which bridge the finite world with the infinite world. The intellect's leap from finite to infinite goes through this transfinite transition between finitude and full scale actual infinity.

Consider a sequence of c–embeddings of cpo's,

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots \qquad \text{(Diagram D)}$$

where $f_n, n \in \omega$ are c–embeddings from A_n to A_{n+1}. Let A_∞ be the inverse limit of this sequence, i.e.,

$$A_\infty = \{x \in \Pi_{i\in\omega} A_i \mid x_i = f_i^R(x_{i+1})\}$$

with the point–wise ordering. In case, in the diagram D, all A_i, $i \in \omega$ are finite cpo's, we call the inverse limit A_∞ an *SFP object*. For any SFP object A_∞, $\cup\{f_{n\infty}(A_n) \mid n \in \omega\}$ is a countable set. Hence, A_∞ is an *"ω-algebraic cpo"*. We denote the category of SFP objects and monotonic (continuous) functions by $\mathcal{M}(\mathcal{C})\mathcal{SFP}$.

We consider the following sequences of c–embeddings of finite cpo's,

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots$$

$$B_0 \xrightarrow{g_0} B_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} B_n \xrightarrow{g_n} B_{n+1} \xrightarrow{g_{n+1}} \cdots$$

Now consider the following sequence of c–embeddings of finite cpo's,

$$C_0 \xrightarrow{h_0} C_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{n-1}} C_n \xrightarrow{h_n} C_{n+1} \xrightarrow{h_{n+1}} \cdots$$

where $C_n = [A_n \to B_n]_\mathcal{M}$ and

$$h_n(w) = [f_n \to g_n](w) = g_n \cdot w \cdot f_n^R$$
$$h_n^R(w) = [f_n^R \to g_n^R](w) = g_n^R \cdot w \cdot f_n.$$

It is clear that $C_n, n \in \omega$ are finite cpo's (in fact, they are sets of monotone functions from A_n to B_n), and hence C_∞ also is an SFP object.

Proposition 2. *Let A_∞, B_∞, and C_∞ be as above. We have*

$$C_\infty \approx_{\mathcal{CCPO}} [A_\infty \to B_\infty]_\mathcal{C}.$$

Since treating this isomorphism explicitly is notationally cumbersome, we identify $[A_\infty \to B_\infty]_\mathcal{C}$ and C_∞. *To within this isomorphism,* each continuous function $u \in [A_\infty \to B_\infty]_{\mathcal{C}}$ can be considered as the limit of the chain $\{\tilde{u}_n\}_{n\in\omega}$, where

$$\tilde{u}_n = h_{n\infty} \cdot h^R_{n\infty}(u).$$

More formally,

$$u = \bigvee_{i\in\omega} \tilde{u}_i.$$

Furthermore, for each $n \geqslant 1$, we have $u_n = h^R_{n\infty}(u) = g^R_{n\infty} \cdot u \cdot f_{n\infty}$ and $\tilde{u}_n = h_{n\infty}(u_n)$.

Now suppose u is a non–continuous monotonic function from A_∞ to B_∞. Then, $g^R_{n\infty} \cdot u \cdot f_{n\infty}$ is a monotonic function from A_n to B_n; hence it still is an element of C_n. Note that $u \notin [A_\infty \to B_\infty]_{\mathcal{C}}$ whence u_n is not defined. However, for each $n \in \omega$

$$\begin{aligned}
&(h_{(n+1)\infty}(g^R_{(n+1)\infty} \cdot u \cdot f_{(n+1)\infty}))(x)\\
&\quad = g_{(n+1)\infty} \cdot g^R_{(n+1)\infty} \cdot u \cdot f_{(n+1)\infty} \cdot f^R_{(n+1)\infty}(x)\\
&\quad \geq g_{(n+1)\infty} \cdot g^R_{(n+1)\infty} \cdot u \cdot f_{n\infty} \cdot f^R_{n\infty}(x)\\
&\quad \geq g_{n\infty} \cdot g^R_{n\infty} \cdot u \cdot f_{n\infty} \cdot f^R_{n\infty}(x)\\
&\quad = (h_{n\infty}(g^R_{n\infty} \cdot u \cdot f_{n\infty}))(x)
\end{aligned}$$

where $x \in A_\infty$. Hence,

$$\{h_{n\infty}(g^R_{n\infty} \cdot u \cdot f_{n\infty}) \mid n \in \omega\}$$

still is a chain in C_∞.

Theorem 2. *(Maximum Continuous Approximation Theorem) Let A_∞ and B_∞ be SFP objects. Suppose u is a monotonic function from A_∞ to B_∞. Then,*

$$\bigvee\{h_{n\infty}(g^R_{n\infty} \cdot u \cdot f_{n\infty}) \mid n \in \omega\} \leq u.$$

Furthermore, u is continuous iff equality holds. Moreover, u has a largest continuous function,

$$\bigvee\{h_{n\infty}(g^R_{n\infty} \cdot u \cdot f_{n\infty}) \mid n \in \omega\},$$

that approximates u.

This theorem indicates that, over SFP objects, monotonic functions and continuous functions are very closely related. This theme is developed in the forthcoming section on hyper continuous functions of SFP objects.

Maximal Elements of SFP Objects.

Theorem 3. *Consider an SFP object A_∞ that is the inverse limit of the following sequence of c–embeddings of finite cpo's,*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n,} A_{n+1} \xrightarrow{f_{n+1}} \cdots .$$

Then, every compact element of A_∞ extends into a maximal element.

Proof. Let e be a compact element of A_∞. Then, for some $k \in \omega$ and $e_k \in A_k$,

$$e = f_{k\infty}(e_k).$$

Let m_k be a maximal extension of e_k in A_k. For each $j \geqslant k$, let m_{j+1} be a maximal element of A_{j+1} such that

$$m_j = f_j(m_{j+1}).$$

Now, consider the following infinite tuple m,

$$m = < f_0^R \cdot ... \cdot f_{k-1}^R(m_k), ..., f_{k-1}^R(m_k), m_k, m_{k+1}, m_{k+2}, ... > .$$

It is clear that $m \in A_\infty$. Furthermore, $e \leq m$. Now assume $x \geq m$ in A_∞. Then, for all $j \geqslant k$,

$$f_{j\infty}^R(x) \geq f_{j\infty}^R(m) = m_j.$$

By the maximality of m_j, we have,

$$f_{j\infty}^R(x) = m_j.$$

Also for all $j < k$, we have,

$$f_{j\infty}^R(x) \geq f_{j\infty}^R(m) = f_j^R \cdot ... \cdot f_{k-1}^R(m_k),$$

and by the maximality of $f_0^R \cdot ... \cdot f_{k-1}^R(m_k)$, we have,

$$f_{j\infty}^R(x) = f_j^R \cdot ... \cdot f_{k-1}^R(m_k).$$

Therefore $x = m$. Hence, m is maximal in A_∞.

Remark 1. Note that this extension result does not use the Axiom of Choice. This indicates the intrinsic difference between the concept of actual infinity, e.g., infinite cpo's, and the transcendental infinity of SFP objects. The former requires Choice to reach maximal elements while the latter does not. This seems to be a counterexample to Cantor's fundamental postulate that the nature of infinite objects is a uniform generalization of that of finite ones.

SFP Solutions to Recursive Domain Equations. As a standard result, there is a non–trivial SFP object D_∞ such that

$$D_\infty \approx_{\mathcal{CSFP}} [D_\infty \to E]_{\mathcal{C}},$$

where E is an SFP object. We present an outline of the construction of such D_∞. Define finite cpo's recursively as follows,

$$D_0 = \bigcirc,$$

where $\bigcirc$ is a trivial cpo with sole element $\bot_\bigcirc$, and,

$$D_{n+1} = [D_n \to E]_{\mathcal{M}}.$$

Define $u_0 : D_0 \to D_1$ and $u_0^R : D_1 \to D_0$ as

$$u_0(\bot_{\bigcirc}) = \bot_{D_1} \text{ and } u_0^R(x) = \bot_{\bigcirc} .$$

Define $u_{n+1} : D_n \to D_{n+1}$ and $u_{n+1}^R : D_{n+1} \to D_n$ as

$$u_{n+1}(w) = [u_n \to 1_E](w) = 1_E \cdot w \cdot u_n^R = w \cdot u_n^R$$
$$u_{n+1}^R(w) = [u_n^R \to 1_E^R](w) = 1_E^R \cdot w \cdot u_n = w \cdot u_n .$$

It is clear that (u_0, u_0^R) is a c-projection pair. Also $(1_E, 1_E^R) = (1_E, 1_E)$ is a c-projection pair. Therefore, (u_n, u_n^R) is a c-projection pair for all $n \in \omega$. Since D_n are all finite cpo's, D_∞ is an SFP object.

Clearly, for each $x, y \in D_\infty$ and for each $n \in \omega$, we have

$$x_{n+1}(y_n) \in E_n = E.$$

Also, it can readily be seen that $x_{n+1}(y_n)$, $n \in \omega$ forms a chain in E. Let

$$\Gamma : D_\infty \to [D_\infty \to E]_{\mathcal{C}}$$

be such that

$$\Gamma(x)(y) = \bigvee_{n \in \omega} x_{n+1}(y_n).$$

Then, Γ is a continuous function. Now, define

$$\Omega : [D_\infty \to E]_{\mathcal{C}} \to D_\infty$$

by

$$\Omega(v) = \bigvee_{n \in \omega} f_{n\infty}(v_{(n)}).$$

Then, Ω is a continuous function. Moreover,

$$\Gamma \cdot \Omega = 1_{[D_\infty \to E]_{\mathcal{C}}} \text{ and } \Omega \cdot \Gamma = 1_{D_\infty}.$$

4.3 Hyper-continuous Functions

The theory of maximal continuous approximations gives rise to a curious subclass of monotonic functions. Suppose A and B are SFP objects. A subspace $\lessdot A \to B \gtrdot \subseteq [A \to B]_{\mathcal{M}}$ is said to be *normal* iff for all $m \in [A \to B]_{\mathcal{M}}, c_m \in \lessdot A \to B \gtrdot$ where c_m is the maximal continuous approximation of m.

Trivially, $[A \to B]_{\mathcal{M}}$ is normal. Suppose m is an element of $[A \to B]_{\mathcal{C}}$. Then $c_m = m \in [A \to B]_{\mathcal{C}}$. Thus $[A \to B]_{\mathcal{C}}$ also is normal. Clearly, $[A \to B]_{\mathcal{C}}$ is the smallest normal space and $[A \to B]_{\mathcal{M}}$ is the largest normal space. Moreover, any subspace $\mathcal{F}$ such that

$$[A \to B]_{\mathcal{C}} \subseteq \mathcal{F} \subseteq [A \to B]_{\mathcal{M}}$$

is a normal space. *In conclusion, the normal spaces are precisely subspaces in between* $[A \to B]_{\mathcal{C}}$ *and* $[A \to B]_{\mathcal{M}}$*, inclusive.*

Suppose $\lessdot A \to B \gtrdot$ is normal and X is a cpo. A monotonic function $f : \lessdot A \to B \gtrdot \to X$ is said to be *hyper continuous* if for every $m \in \lessdot A \to B \gtrdot, f(m) = f(c_m)$. In words, *hyper continuous functions are those monotone functions which can not distinguish* m *from* c_m. We denote the set of all hyper continuous functions from $\lessdot A \to B \gtrdot$ to X by $\prec \lessdot A \to B \gtrdot \to X \succ$.

In the following, we study some technical but elementary properties of hyper continuous functions.

Lemma 5. *Let* $\lessdot A \to B \gtrdot$ *be normal and* X *be a cpo, then* $\prec \lessdot A \to B \gtrdot \to X \succ$ *is a cpo.*

Proof. Let $\circledast : \lessdot A \to B \gtrdot \to X$ be such that $\circledast(f) = \perp$ for all $f \in \lessdot A \to B \gtrdot$. Suppose $\circledast(m) = x$. Clearly $x = \perp$, and hence $\circledast(c_m) = x$. Hence $\circledast$ is the smallest hyper continuous function from $\lessdot A \to B \gtrdot$ to X. Suppose D is a directed subset of $\prec \lessdot A \to B \gtrdot \to X \succ$. Clearly it is a directed subset of $[\lessdot A \to B \gtrdot \to X]_{\mathcal{M}}$. Let δ be the least upper bound of D in $[\lessdot A \to B \gtrdot \to X]_{\mathcal{M}}$. It suffice to show that δ is hyper continuous. Suppose $m \in \lessdot A \to B \gtrdot$. For each $d \in D$, we have $d(m) = d(c_m)$. Hence,

$$\delta(m) = \bigvee_{d \in D} d(m) = \bigvee_{d \in D} d(c_m) = \delta(c_m).$$

We have established that δ is hyper continuous.

Lemma 6. *Suppose* $f : \lessdot A \to B \gtrdot \to X$ *is hyper continuous. Moreover, assume* f *is 1-1. Then* $\lessdot A \to B \gtrdot = [A \to B]_{\mathcal{C}}$.

Proof. Let $m \in \lessdot A \to B \gtrdot$. We have $f(m) = f(c_m)$. Since f is 1-1, $m = c_m$. Hence $\lessdot A \to B \gtrdot = [A \to B]_{\mathcal{C}}$.

Corollary 2. *Let* $\lessdot A \to B \gtrdot \neq [A \to B]_{\mathcal{C}}$*. Moreover, assume* $f : \lessdot A \to B \gtrdot \to X$ *is hyper continuous. Then,* f *is not injective.*

This Corollary establishes the following important result:

Theorem 4. *There are continuous functions which are not hyper continuous.*

Proof. Suppose $\lessdot A \to B \gtrdot$ is normal and $\lessdot A \to B \gtrdot \neq [A \to B]_{\mathcal{C}}$. Moreover $\lessdot A \to B \gtrdot$ is a cpo. Consider the identity function $id_{\lessdot A \to B \gtrdot}$ over $\lessdot A \to B \gtrdot$. It is injective and clearly continuous. Also $\lessdot A \to B \gtrdot$ is a cpo. Hence f is not hyper continuous.

Despite this general result, under a special circumstance, continuous functions become hyper continuous. Indeed,

Lemma 7. *We have*

$$\prec [A \to B]_{\mathcal{C}} \to X \succ = [[A \to B]_{\mathcal{C}} \to X]_{\mathcal{M}}.$$

Proof. For all $m \in [A \to B]_{\mathcal{C}}$, we have $m = c_m$. Hence all monotonic functions from $[A \to B]_{\mathcal{C}}$ to X are hyper continuous.

This also implies that whenever the normal space $\lessdot A \to B \gtrdot$ is an SFP object then all continuous functions from $\lessdot A \to B \gtrdot$ to a cpo X are hyper continuous. Furthermore, this lemma implies the following important result,

Corollary 3. *There are hyper continuous functions which are not continuous.*

Proof. Let $f : [A \to B]_{\mathcal{C}} \to X$ be a monotone function which is not continuous. This is hyper continuous but not continuous.

Corollary 4. *Suppose A, B and X are SFP objects; Then,*

$$\prec [A \to B]_{\mathcal{C}} \to X \succ \curvearrowleft_{\mathcal{M}} [[A \to B]_{\mathcal{C}} \to X]_{\mathcal{C}}.$$

Proof. By the maximal continuous approximation theorem, we have

$$[[A \to B]_{\mathcal{C}} \to X]_{\mathcal{C}} \curvearrowright_{\mathcal{M}} [[A \to B]_{\mathcal{C}} \to X]_{\mathcal{M}}.$$

In what follows, we present a general method of forcing monotone functions into hyper continuous functions and their relations.

Theorem 5. *Assume $\lessdot A \to B \gtrdot$ is normal, X is a cpo and f is a monotone function from $\lessdot A \to B \gtrdot$ to X. Then, there is a hyper continuous function $\bar{f} : \lessdot A \to B \gtrdot \to X$ such that $\bar{f} \leq f$. If f is hyper continuous then $f = \bar{f}$.*

Proof. Define $\bar{f} : \lessdot A \to B \gtrdot \to X$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \text{ is continuous} \\ f(c_x) & \text{if } x \text{ is monotone.} \end{cases}$$

This is to say $\bar{f}(x) = f(c_x)$. Suppose $x, y \in \lessdot A \to B \gtrdot$ and $x \leq y$. Let $\tilde{x}_i = h_{i\infty}(g^R_{i\infty} \cdot x \cdot f_{i\infty})$ and $\tilde{y}_i = h_{i\infty}(g^R_{i\infty} \cdot y \cdot f_{i\infty})$. Then,

$$c_x = \bigvee_{i \in \omega} \tilde{x}_i, \quad c_y = \bigvee_{i \in \omega} \tilde{y}_i$$

and, $\tilde{x}_i \leq \tilde{y}_i$ for all $i \in \omega$. Hence, $c_x \leq c_y$ and so

$$\bar{f}(x) = f(c_x) \leq f(c_y) = \bar{f}(y).$$

We have shown that $\bar{f}$ is monotone. Moreover,

$$\bar{f}(x) = f(c_x) = f(c_{c_x}) = \bar{f}(c_x).$$

Hence $\bar{f}$ is hyper continuous. Clearly $\bar{f} \leq f$. If f is hyper continuous, then

$$\bar{f}(x) = f(c_x) = f(x).$$

Corollary 5. *Assume $\lessdot A \to B \gtrdot$ is normal and X is a cpo. We have*

$$[\lessdot A \to B \gtrdot\ \to X]_{\mathcal{M}} \curvearrowleft_{\mathcal{MCPO}} \prec \lessdot A \to B \gtrdot\ \to X \succ .$$

Proof. Define

$$i : [\lessdot A \to B \gtrdot\ \to X]_{\mathcal{M}} \to \prec \lessdot A \to B \gtrdot\ \to X \succ$$

by

$$i(f) = \bar{f}.$$

Let $j : \prec \lessdot A \to B \gtrdot\ \to X \succ \to [\lessdot A \to B \gtrdot\ \to X]_{\mathcal{M}}$ be inclusion. Clearly both of them are monotone. We have $j \cdot i(f) = \bar{f} \leq f$. and $i \cdot j(f) = i(f) = f$.

On the other hand, from the maximal continuous approximation theorem it follows that when X is an SFP object and $\lessdot A \to B \gtrdot\ = [A \to B]_{\mathcal{C}}$, for every hyper continuous function $f :\ \lessdot A \to B \gtrdot\ \to X$, there is a continuous function $c_f : \bar{f} :\ \lessdot A \to B \gtrdot\ \to X$ such that $c_f \leq f$, and such forcing constitutes the following embedding,

$$\begin{aligned}[\lessdot A \to B \gtrdot\ \to X]_{\mathcal{C}} \curvearrowright_{\mathcal{MCPO}} \prec \lessdot A \to B \gtrdot\ \to X \succ \\ = [\lessdot A \to B \gtrdot\ \to X]_{\mathcal{M}}.\end{aligned}$$

For normal spaces $\lessdot A_i \to B_i \gtrdot$, $i = 1, 2, ...n$, and a cpo X; we can define the concept of *hyper continuous n–ary functions*,

$$f :\ \lessdot A_1 \to B_1 \gtrdot\ \times \cdots \times\ \lessdot A_n \to B_n \gtrdot\ \to\ T$$

as monotone functions such that

$$f(x_1, ..., x_n) = f(c_{x_1}, ..., c_{x_n}).$$

Similar embedding results as for binary and multi–ary cases in general hold.

Suppose $\lessdot A \to B \gtrdot$ and $\prec \lessdot A \to B \gtrdot\ \to X \succ$ are normal. Define a function $Ap : \prec \lessdot A \to B \gtrdot\ \to X \succ \times \lessdot A \to B \gtrdot\ \to X$ as

$$Ap(f, d) = c_f(c_d).$$

Lemma 8. *Ap is hyper continuous.*

Proof. We have

$$Ap(f, d) = c_f(c_d) = c_{c_f}(c_{c_d}) = Ap(c_f, c_d).$$

Hence Ap is hyper continuous. This can be proved alternatively as follows: The function ap given by

$$ap(f, d) = f(d)$$

is continuous. Hence $Ap = \overline{ap}$ is hyper continuous.

4.4 Balancing a Hyper-continuous Function Space with Its Domain

We have seen that by restricting the members of a function space to continuous functions, we can balance the size of the space of functions with the size of its domain in the category $\mathcal{CSFP}$ of continuous functions over SFP objects. In what follows, we show that this balancing of size can be achieved in the category $\mathcal{MCPO}$ of monotone functions over cpo's.

Suppose D_∞ is a solution to

$$D \approx_{\mathcal{CSFP}} [D \to E]_{\mathcal{C}}$$

in $\mathcal{CSFP}$. Hence we can identify D_∞ with $[D_\infty \to E]_{\mathcal{C}}$. Precisely speaking, we ought to be explicit about the isomorphism pair for $D_\infty \approx_{\mathcal{CSFP}} [D_\infty \to E]_{\mathcal{C}}$. However, to avoid notational complexity, we identify $d \in D_\infty$ with the isomorphic image of d, unless doing so causes confusion. Let $M = \prec D_\infty \to E \succ$; then $M = [D_\infty \to E]_{\mathcal{M}}$. Hence,

$$D_\infty \approx_{\mathcal{CCPO}} [D_\infty \to E]_{\mathcal{C}} \curvearrowright_{\mathcal{M}} [D_\infty \to E]_{\mathcal{M}} = M,$$

and we have

$$[D_\infty \to E]_{\mathcal{C}} \curvearrowright_{\mathcal{M}} M.$$

Furthermore,

$$D_\infty \curvearrowright_{\mathcal{M}} M.$$

To within the identification of D_∞ with $[D_\infty \to E]_{\mathcal{C}}$, this embedding is the inclusion map ic and the adjoint projection is the monotone map

$$c : m \mapsto c_m.$$

Notice that, in general, M is a cpo but is not an SFP object.

Theorem 6. $M \approx_{\mathcal{MCPO}} \prec M \to E \succ$.

Proof. Let

$$(\beta : [D_\infty \to E]_{\mathcal{C}} \to M, \alpha : M \to [D_\infty \to E]_{\mathcal{C}})$$

be the embedding pair for

$$[D_\infty \to E]_{\mathcal{C}} \curvearrowright_{\mathcal{M}} M.$$

So, $\alpha(m) = c_m$ and β is the inclusion $ic : u \mapsto u$. Define

$$\Phi : M \to \prec M \to E \succ \text{ and } \Psi : \prec M \to E \succ \to M$$

by

$$\Phi(a) = a \cdot \alpha \quad \Psi(b) = b \restriction [D_\infty \to E]_{\mathcal{C}}.$$

If for $m \in M$ and compact $x \in E$, $(\Phi(a))(m) \geq x$, then

$$(\Phi(a))(c_m) = a(c_{c_m}) = a(c_m) = (\Phi(a))(m) \geq x.$$

Hence $\Phi(a)$ is hyper continuous and is indeed in $\prec M \rightarrow E \succ$. Moreover, for $a \geq a'$, and $m \in M$, we have,

$$\Phi(a)(m) = a(c_m) \geq a'(c_m) = \Phi(a')(m).$$

Hence Φ is monotone. For all $d \in D_\infty$, we have,

$$(\Psi(b))(d) = b(d),$$

and, since $b \in \prec M \rightarrow E \succ$ is monotone, $\Psi(b)$ is monotone. So, $\Psi(b)$ indeed is in M. Moreover, if $b \geq b'$ and then

$$(\Psi(b))(d) = b(d) \geq b'(d) = (\Psi(b'))(d).$$

Hence Ψ is monotone. We have seen that Φ and Ψ are both well–defined. Now we verify that they constitute an isomorphism pair. For each $b \in \prec M \rightarrow E \succ$ and $m \in M$, we have,

$$(\Phi \cdot \Psi(b))(m) = (\Psi(b) \cdot \alpha)(m) = b(c_m) = b(m).$$

So, $\Phi \cdot \Psi = 1_{\prec M \rightarrow E \succ}$. Moreover, for each $a \in M$ and $d \in D_\infty$, we have.

$$(\Psi \cdot \Phi(a))(d) = \Psi(a \cdot \alpha)(d) = a \cdot \alpha(d) = a(c_d) = a(d).$$

Therefore, $\Psi \cdot \Phi = 1_M$.

The following immediate consequence of this isomorphism holds.

Lemma 9. *For all $m \in M$, we have,*

$$\forall x \in D_\infty.[m(x) \leq m'(x)] \text{ iff } \forall u \in M.[\Phi(m)(u) \leq \Phi(m')(u)].$$

We may write $m \prec u \succ$ for $\Phi(m)(u)$. To be specific, we will denote the partial ordering on M by $\leq$ and that on $\prec M \rightarrow E \succ$ by $\preceq$. Therefore this lemma states

$$m \leq m' \text{ iff } \Phi(m) \preceq \Phi(m').$$

Moreover, we have,

Theorem 7. *For all $d \in D_\infty$ and $m \in M$, $m \prec d \succ = m(c_d)$. For each $m \in M$, let $apply_m : M \rightarrow E$ be*

$$apply_m(x) = m \prec x \succ .$$

It is clear that $apply_m$ is hyper continuous. However, $apply : M \times M \rightarrow E$ defined as

$$apply((m, x)) = m \prec x \succ$$

is monotone but not hyper continuous.

4.5 The Construction of M_{max}

The construction of the canonical PFS M_{max} may now be given in three stages. First, using the theory of SFP objects, we construct a continuum reflexive structure D_∞ satisfying $D_\infty \approx_{\mathcal{CSFP}} [D_\infty \to T]_{\mathcal{C}}$, where T is the domain of three–valued truth,

$$\begin{array}{ccc} \underline{true} & & \underline{false} \\ \nwarrow & & \nearrow \\ & \perp & \end{array}$$

under the information ordering $\leq_k$ and $[D_\infty \to T]_{\mathcal{C}}$ is the space of all continuous functions from D_∞ to T under the information ordering associated with nested partial characteristic functions. Here, the bottom value $\perp$ represents a truth–value gap. The truth value domain $(T, \leq_k)$ is clearly an SFP object. In truth theory, this structure $(T, \leq_k)$ is used to construct term models of partial logic; the third truth value $\perp$ is successfully used as the semantic value of logically paradoxical sentences [28, 35, 20].

The reason for adopting $(T, \leq_k)$ as our domain of truth values is two– fold. First, it is an SFP structure and hence we can use it as a fixed parameter in recursive structural equations. Second, we can use $\perp$ to capture the behavior of logical paradoxes in set theory. There is another partial ordering associated with the underlying set T. This partial ordering, called the *truth ordering* and denoted by $\leq_t$, is described by the following diagram,

$$\begin{array}{c} \underline{true} \\ \uparrow \\ \perp \\ \uparrow \\ \underline{false}. \end{array}$$

Following Kleene [33], the strong three–valued Boolean operators $\overline{\wedge}, \overline{\vee}$, and $\overline{\neg}$ are defined on T by,

$\overline{\wedge}$: the greatest lower bound operation with respect to $\leq_t$
$\overline{\vee}$: the least upper bound operation with respect to $\leq_t$
$\overline{\neg}$: $(\underline{true}) = \underline{false}$
$\overline{\neg}$: $(\underline{false}) = \underline{true}$
$\overline{\neg}$: $(\perp) = \perp$.

[7] introduces a complete set of truth functions for Kleene's strong 3–valued logic; here it suffices to observe that strong material conditional $\overline{\Rightarrow}$ and biconditional $\overline{\Leftrightarrow}$ are monotone operations over $(T, \leq_k)$ under their classical definitions in terms of strong negation and disjunction (or conjunction). The quantifiers $\overline{\forall}, \overline{\exists}$ are introduced in terms of arbitrary meet $\bigwedge$ and join $\bigvee$ on $(T, \leq_t)$. These connectives are monotone functions over $(T, \leq_k)$ but fail to be continuous due to their infinitary nature: D_∞ is not closed under $\bigwedge$ and $\bigvee$. This hampers D_∞ as

a first order model for naive set theory[4], as does the connected failure of strict identity to be even monotone as a binary function $D_\infty \times D_\infty \to T$.

However, we have seen that each monotone function $f : D_\infty \to T$ is maximally approximated by a unique continuous function c_f in $[D_\infty \to T]_\mathcal{C}$, whence c_f in D_∞ under representation. This gives rise to the concept of hyper continuity.

Next, we have seen that the space M of monotone functions from D_∞ to T is a solution for the reflexive equation $M \approx_\mathcal{M} \prec M \to T \succ$, where $\prec M \to T \succ$ is the space of all hyper continuous functions from M to T. Throughout the rest of this section, we consider a solution $M = [D_\infty \to T]_\mathcal{M}$ for $W \approx_\mathcal{M} \prec W \to T \succ$ as discussed in the previous section. Note that a monotone function $f : M \to T$ is hyper continuous just in case

$$c_x = c_y \;\Rightarrow\; f(x) = f(y) \qquad (x, y \in M),$$

i.e., over M, the equivalence relation of sharing a common maximal continuous approximation is a congruence for all hyper continuous functions. The isomorphism pair,

$$(\Phi : M \to \prec M \to T \succ,\ \Psi : \prec M \to T \succ \to M),$$

yields application $\cdot \prec \cdot \succ : M \times M \to T$ defined by

$$b(a) = \Phi(b)(a) \qquad (a, b \in M).$$

Any hyper continuous map $f : M \to T$ is represented by a unique object $\Psi(f)$ in M such that

$$\Psi(f)(a) = f(a).$$

Writing "$x \bar{\in} y$" for "$y \prec x \succ$ ", we pass to a universe $(M, \bar{\in})$ of partial (non–bivalent) sets–in–extension similar to the set theories of Gilmore [28] and Scott [57]. Let $x \equiv_M y =_{df} (\forall m \in M)(m \bar{\in} x \overline{\leftrightarrows} m \bar{\in} y)$. $(M, \equiv_M)$ is thus an equivalence relation and a congruence for all hyper continuous functions from M to T. Although $x \equiv_M y$ is hyper continuous in both x and y, the hyper continuity of indiscernibility will be established only in the context of the identity theory of *maximal* elements of M, to which we now pass.

Maximal Elements of *M*. Since $(M, \leq)$ is a complete partial order, M is closed under least upper bounds of $\leq$–chains. Hence, there are $\leq$–maximal elements of M by (an equivalent to) the Axiom of Choice. Let M_{max} be the set of maximal elements of M.

Theorem 8. *Let $a, b \in M_{max}$. Then $a \prec b \succ \neq \bot$.*

[4] Historically, this failure of quantificational continuity surfaced in the functional setting of Church's quantified λ-calculus in the form of Rossers's Paradox [34], a functional version of Russell's Paradox.

Proof. Suppose $a \prec b \succ = a(c_b) = \perp$. Let $X = \{x \in D_\infty \mid c_b \leq x\}$. To show that X is a directed subset of D_∞, suppose that $d_1, d_2 \in X$ lack an upper bound in $(D_\infty, \leq)$. Then there is $d_3 \in D_{max}$ such that $d_1(d_3) = \underline{true}$ and $d_2(d_3) = \underline{false}$. Then, since $c_b \leq d_1, d_2$, we have $c_b(d_3) = \perp$. Since b is maximal in M and d_3 is maximal in D_∞, $b(d_3) \neq \perp$ by The Isolation Lemma. So we assume, without loss of generality, that $b(d_3) = \underline{true}$; the case $b(d_3) = \underline{false}$ is treated in a manner similar to the following: Since $d_1(d_3) = \underline{true}$ by the continuity of d_1 there is a compact approximation e to d_3 in D_∞, i.e., $e \leq d_3$, such that $d_1(e) = \underline{true}$. Define $f : D_\infty \to T$ by

$$f(d) = \begin{cases} \underline{true} & \text{if} \quad e \leq d \\ c_b(d) & \text{else,} \end{cases} \qquad (d \in D_\infty).$$

To show that f is a function (i.e, it is single–valued), suppose there is a $d \in D_\infty$ such that $e \leq d$ and $f(d) = c_b(d) = \underline{false}$. Since $c_b \leq d_1$, by monotonicity $d_1(d) = \underline{false}$. But as $d_1(e) = \underline{true}$ and $e \leq d_1$, by monotonicity $d_1(d) = \underline{true}$, a contradiction. Thus, f is a function extending c_b, i.e., $c_b < f$. Indeed f is monotone by construction. Further, we claim f is continuous: let $\{y_\alpha\}$ be a directed subset of D_∞ and $y = \lim\{y_\alpha\}$; we need to show there is an ordinal α with $f(y) = f(y_\alpha)$. If $e \leq y$, then by the compactness of e, there is an ordinal $\beta \leq \alpha$ and $y_\beta \in \{y_\alpha\}$ such that $e \leq y_\beta$. Then,

$$f(y_\beta) = f(e) = f(y) = \underline{true}$$

with the first and second identity by the montonicity of f, and the last by the definition of f, whence y_β is the desired y_α. Otherwise, $f(y) = c_b(y)$ and the desired y_α is given by the continuity of c_b. This contradicts the maximality of c_b among continuous approximation to b. Thus, X is a directed subset of D_∞.

(Note that, as D_∞ is a cpo, X has a least upper bound $d^* \in D_\infty$, whence $d^* \in D_{max}$ since X is upward closed under $\leq$. By the Isolation Lemma, $a \prec d^* \succ \neq \perp$, since a is maximal in M and d^* is maximal in D_{max}.)

Define $g : D_\infty \to T$ by

$$g(d) = \begin{cases} \underline{true} & \text{if} \quad c_b \leq d \\ a(d) & \text{else,} \end{cases} \qquad (d \in D_\infty).$$

Then g is a function as X is directed. f is monotone by construction. Further, $a < g$, contrary to a's maximality in $(M, \leq)$. Thus $a \prec b \succ \neq \perp$, as required.

We say $a \in M$ is *max–defined* iff for all maximal elements m of M, $a \prec m \succ \neq \perp$. By the previous Lemma, all elements of M_{max} are max–defined.

Lemma 10. *If $f : M \to T$ is hyper continuous, then $\Psi(f) \in M$ is max–defined iff for all $x \in M_{max}$, $f(x) \neq \perp$.*

Proof. Suppose $\Psi(f) \in M$ is max–defined. Then, for all $x \in M_{max}$,

$$f(x) = \Phi \cdot \Psi(f)(x) = \Psi(f) \prec x \succ \neq \perp .$$

Conversely, assume for all $x \in M_{max}$, $f(x) \neq \perp$. We then have

$$\Psi(f) \prec x \succ = \Phi \cdot \Psi(f)(x) = f(x) \neq \perp .$$

Hence $\Psi(f)$ is max–defined.

Theorem 9. *(The Semi–lattice Lemma) Suppose $a \in M$ is max–defined. Then the collection,*

$$\{b \in M \mid a \leq b\},$$

is an upper semi–lattice.

Corollary 6. *Every element of M which is max–defined has a unique maximal extension $m_a \in M$.*

Proof. Since M is a cpo, any $a \in M$ which is max–defined has a maximal extension. Let a' and a'' be maximal extensions of a. Then, by The Semi–Lattice Lemma, $a' \vee a''$ exists in M; whence $a' = a''$.

If a is max–defined, then the unique maximal extension m_a of a agrees with a on maximal elements. So, *intuitively, we can consider m_a the completion of a with respect to non–maximal arguments.*

There is a trivial but important consequence to this Corollary.

Theorem 10. *Suppose $m \in M$ is max–defined and $\mathbf{m}$ is the unique maximal extension of m. Then, for all $x \in M_{max}$*

$$\mathbf{m} \prec x \succ = m \prec x \succ = \Phi(m)(x).$$

Theorem 11. *Suppose $a', a'' \in M$ are max–defined and they agree on maximal elements of M. Then, they have the same maximal extension.*

Corollary 7. *Suppose $a, a' \in M$ are maximal and they agree on maximal elements of M. Then, $a = a'$.*

Max–indiscernibility If $a, a' \in M$ are such that for all $m \in M_{max}$,

$$m \prec a \succ = m \prec a' \succ,$$

then we say a and a' are *max–indiscernible*, in symbols $a \equiv_{max} b$. Obviously if $a = b$ then a and b are max–indiscernible. Moreover, $a, a' \in M$ are max–indiscernible iff for all maximal elements f of $\prec M \to T \succ$, $f(a) = f(b)$. It is easy to verify that $\equiv_{max}$ is an equivalence relation on M.

Theorem 12. *Suppose $a, a' \in M_{max}$ agree on maximal elements of M. Then, they are max–indiscernible.*

Proof. Immediate from Corollary 7.

The max–indiscernibility relation restricted to maximal elements of M can be represented by the following hyper continuous relation $\overline{\equiv}_{max} : M \times M \to T$,

$$a \overline{\equiv}_{max} b = \overline{\forall} m \in M_{max}.[m \prec a \succ \ \overline{\Leftrightarrow}\ m \prec b \succ].$$

In fact, we have,

Lemma 11. *Let $a, b \in M_{max}$. Then, a and b are max–indiscernible iff $a \overline{\equiv}_{max} b = \underline{true}$. Moreover, a, b are* not *max–indiscernible iff $a \overline{\equiv}_{max} b = \underline{false}$.*

Moreover,

Lemma 12. *For all $a \in M$, $a \overline{\equiv}_{max} c_a = \underline{true}$.*

Proof. For all $m \in M_{max}$, due to the hyper continuity of $\Phi(m)$, we have

$$m \prec a \succ = \Phi(m)(a) = m(c_a) = \Phi(m)(c_a) = m \prec c_a \succ .$$

Due to the nature of $\overline{\forall}$ and $\overline{\Leftrightarrow}$, we have $a \overline{\equiv}_{max} b \neq \perp$ iff for all $m \in M_{max}$, $m \prec a \succ \neq \perp$ and $m \prec b \succ \neq \perp$. The following lemma immediately follows,

Lemma 13. *Let $a, b \in M$. If $a \overline{\equiv}_{max} b = \perp$ then, one of a or b is not maximal.*

Proof. Suppose $a \overline{\equiv}_{max} b = \perp$. By the observation above, for some $m \in M_{max}$,

$$m \prec a \succ \neq \perp \quad \text{or} \quad m \prec b \succ \neq \perp .$$

Hence, either a or b is not maximal.

Remark 2. The converse of this result does not hold. Indeed, by Lemma 12, for all $a \in M_{max} - D_\infty, a \overline{\equiv}_{max} c_a \neq \perp$, but c_a is not maximal.

Theorem 13. *Suppose $m \in M$ is max–defined and* **m** *is the unique maximal extension of m. Then $\mathbf{m} \equiv_{max} m$.*

Max–Coextensiveness If $a \in M$ and $b \in M$ agree on all elements m of M_{max}, in the sense that

$$a \prec m \succ = b \prec m \succ,$$

then we say a and b are *max–coextensive*, in symbols $a \sim_{max} b$. This is equivalent to saying that, for all maximal elements x of D_∞,

$$a(x) = b(x).$$

It can readily be verified that $\sim_{max}$ is an equivalence relation on M. This relation restricted to max–defined elements of M has the following monotone (but not hyper continuous) representation,

$$a \overline{\sim}_{max} b = \overline{\forall} m \in M_{max}.[a \prec m \succ \ \overline{\Leftrightarrow}\ b \prec m \succ].$$

Indeed, we have,

Lemma 14. *Suppose a and b are max–defined. Then, a and b are max–coextensive iff* $a \overline{\sim}_{max}\, b = \underline{true}$. *Moreover, a and b fail to be max–coextensive iff* $a \overline{\sim}_{max}\, b = \underline{false}$.

Due to the nature of $\overline{\forall}$ and $\overline{\Leftrightarrow}$, we have $a \overline{\sim}_{max}\, b \neq \perp$ iff for all $m \in M_{max}, a \prec m \succ \neq \perp$ and $b \prec m \succ \neq \perp$. This establishes the following lemma,

Lemma 15. *Suppose* $a, b \in M$. *Either a or b is not max–defined iff*

$$a \overline{\sim}_{max}\, b = \perp .$$

We call $\equiv_{max}$ and $\sim_{max}$ restricted to M_{max} the *indiscernibility* and the *co–extensiveness* relations over M_{max}, respectively.

Theorem 14. *Let* $a, b \in M_{max}$. *Then* $a \sim_{max} b$ *iff* $a = b$.

Theorem 15. *Let* $a, b \in M_{max}$. *Then* $a \equiv_{max} b$ *iff* $c_a = c_b$.

The indiscernibility relation $\equiv_{max}$ is thus coarser than co–extensiveness $\sim_{max}$, which is strict identity $=$ over M_{max} and the restriction of a monotone function from $M \times M$ to T to maximal elements of M.

$(M_{max}, \equiv_{max})$. $(M_{max}, \equiv_{max})$ is a "reduct" of $(M, \equiv_M)$ in the sense that it preserves the discriminative capacities of M : if a and $b \in M_{max}$ fail to be discerned by elements of M_{max}, then they fail to be discerned by elements of M,

$$a \ \equiv_{max} \ b \Leftrightarrow a \equiv_M b \qquad (a, b \in M),$$

a corollary of the theorem that, under $\leq$, every element a of M which is max–defined has a unique $\leq$ maximal extension m_a in M_{max} which agrees with a on all elements of M_{max} i.e., $a \sim_{max} m_a$.

A subset X of M_{max} is *exact* iff X is closed under $\equiv_{max}$. Throughout this section "X" ranges over exact subsets. We now establish that:

$$\mathcal{C}(\equiv_{max}) \approx M_{max} \lhd 2^{M_{max}}.$$

Let $\{M_{max} \to 2\}$ be the family of all characteristic functions f_X of exact subsets X of M_{max}. For each exact $X \subseteq M_{max}$, let $g(X) : M \to T$ be the hyper continuous function extending f_X by assigning the function value $\perp$ to all non maximal arguments in M,

$$g(X)(m) = \begin{cases} f(m) \text{ if } & m \in M_{max} \\ \perp & \text{else,} \end{cases} \qquad (m \in M).$$

Then $f_X \leq g(X)$. Since f is max–defined, $\Psi(g(X))$ has a unique maximal extension $\mathbf{\Psi(g(X))} \in M$.

Define $l : \{M_{max} \to 2\} \to M_{max}$ by

$$l(f_X) = \mathbf{\Psi(g(X))}.$$

Adjoint–wise, define $r : M_{max} \to \{M_{max} \to 2\}$ by

$$r(m) = \Phi(m) \restriction M_{max} \qquad (m \in M_{max}).$$

Then $l(r(m)) = m$ $(m \in M_{max})$, whence (l, r) is an isomorphism pair of M_{max} and $\{M_{max} \to 2\}$. Let $\mathfrak{G} =_{df} (M_{max}, \equiv_{max}, \ulcorner\cdot\urcorner, \llcorner\cdot\lrcorner)$, where

$$\ulcorner.\urcorner : 2^{M_{max}} \to M_{max}, \qquad \llcorner\cdot\lrcorner : M_{max} \to 2^{M_{max}}$$

are defined by $\ulcorner Y \urcorner = l(Cl(Y))$ $(Y \in 2^{M_{max}})$, and $\llcorner m \lrcorner =$ the subset X of M_{max} whose characteristic function is $r(m)$ (i.e., such that $f_X = r(m)$) $(m \in M_{max})$.

Then $\mathfrak{G}$ is a PFS. The cardinality of M_{max} is that of $2^{\mathbb{R}}$ (hypercontinuum) but merely that of 2^{ω} (continuum) up to $\equiv_{max}$.

See [2] for a proof that $\mathfrak{G}$ is a plenum satisfying DoD.

4.6 Graded Indiscernibility in M_{max}

The Maximum Ccontinuous Approximation Theorem asserts that every $D_{n+1} = [D_n \to T]_{\mathcal{M}}$ induces the following indiscernibility E_{n+1} on M_{max}. Let $x, y \in M_{max}$ and let c_x, c_y be maximum continuous approximations of x, y respectively. Let d_x, d_y the projection of c_x, c_y to D_{n+1}. Then $E_{n+1} \subseteq M_{max} \times M_{max}$ is the equivalence relation defined by

$$xE_{n+1}y \Leftrightarrow d_x = d_y \ (x, y \in M_{max}).$$

Then $E_{n+1} \subseteq E_n$; $\{E_n\}$ is a descending chain of equivalence relations on M_{max} approximating $\equiv_{max}$. This is an interesting example of an "infinitary information system" in the sense of [51], to which we now turn.

5 Graded Indiscernibility and Metric

Polkowski [51] induced a graded indiscernibility relation over a universe U from an infinite sequence of attributes a_i via the conversion of an attribute a_i to an equivalence relation IND_a such that

$$IND_{a_i}(x, y) \Leftrightarrow a_i(x) = a_i(y).$$

Such sequences of attributes are called "infinite information systems".

[51] considers a descending chain of equivalence relations $\{E_i\}$ over U such that $E_{i+1} \subseteq E_i$. We call the equivalence relation $E_\infty = \bigcap_{i \in \omega} E_i$ an *indiscernibility relation graded by* $\{E_i\}$. By $[x]_i$ we mean the equivalence class $\{y : xE_iy\}$. It is clear that

$$[x]_{n+1} \subseteq [x]_n.$$

A subset $T \subseteq U$ is *n–complete (or n–exact)* if

$$T = \bigcup \{[x]_n : x \in T\}.$$

Otherwise, T is said to be *n–rough*. For each n, the family,

$$\mathcal{B}_n = \{[x]_n : x \in U\},$$

is the partition of U induced by the equivalence relation E_n and it forms a base for a Polkowski-Pawlak topology Π_n over U. It is clear that open sets in this topology are precisely n–exact sets. Given a subset $T \subseteq U$, it is easy to verify that T is n–exact iff

$$Cl_n(T) = Int_n(T),$$

where $CL_n(T)$ and $INT_n(T)$ are closure and the interior in Π_n of T, respectively.

We have

$$[x]_n = \left[\bigcup\{[z]_{n+1} : z \in [x]_n - [x]_{n+1}\}\right] \cup [x]_{n+1}.$$

So, all (basic) open sets in Π_n are open in Π_{n+1}. That is Π_{n+1} *is finer than* Π_n. Similarly, all (basic) open sets in Π_n are open in Π_∞ and Π_∞ is finer than Π_n.

Having established that Π_∞ is an upper bound of the chain $\Pi_n \subseteq \Pi_{n+1}$, it is natural to "expect" that

$$\Pi_\infty = \bigcup_{n\in\omega} \Pi_n,$$

which we will examine in what follows.

Let

$$\mathcal{B}_{\mathcal{A}} = \bigcup_n \mathcal{B}_n = \{[x]_n : n \in \omega, x \in U\}.$$

It can readily be shown that $\mathcal{B}_{\mathcal{A}}$ forms a base for a topology $\Pi_{\mathcal{A}}$ over U. It is clear that all (basic) open sets of Π_n are open in $\Pi_{\mathcal{A}}$. So, $\Pi_{\mathcal{A}}$ is finer than Π_n.

It is clear that

$$\Pi_{\mathcal{A}} \supseteq \bigcup_{n\in\omega} \Pi_n$$

in lieu of the definition of $\mathcal{B}$. Conversely assume X is open in $\Pi_{\mathcal{A}}$. Then, for some $u \subseteq \omega$ and $w \subseteq U$,

$$X = \cup\mathcal{X},$$

where $\mathcal{X} = \{[x]_n : n \in u, x \in \omega\}$. For each n, let

$$\mathcal{X}_n = \{[x]_n : x \in w\}.$$

Then

$$\bigcup \mathcal{X}_n \in \Pi_n,$$

and

$$X = \bigcup_{n\in\omega}\bigcup \mathcal{X}_n \in \bigcup_{n\in\omega} \Pi_n.$$

We have shown that

$$\Pi_{\mathcal{A}} = \bigcup_{n\in\omega} \Pi_n.$$

So, $\Pi_{\mathcal{A}}$ is the least upper bound of the chain $\Pi_n \subseteq \Pi_{n+1}$, $n \in \omega$. It now follows that

$$\Pi_{\mathcal{A}} \subseteq \Pi_\infty.$$

This can be verified by the following direct argument too: As we have

$$[x]_n = \left[\bigcup\{[z]_\infty : z \in [x]_n - [x]_\infty\}\right] \cup [x]_\infty,$$

$[x]_n$ is open in Π_∞. Conversely,

$$[x]_\infty = \bigcap_{n\in\omega} [x]_n.$$

We know that $(\Pi_n, \subseteq)$ is a complete lattice, as $(\Pi_n, \cup, \cap, \neg, \varnothing, U)$ is a complete Boolean algebra, and hence,

$$\left(\bigcup_n \Pi_n, \subseteq\right)$$

is a complete lattice with the top element U and the bottom element $\varnothing$. Therefore, $\bigcap_{n\in\omega}[x]_n \in \bigcup_n \Pi_n$. We have shown that

$$\Pi_{\mathcal{A}} \supseteq \Pi_\infty.$$

In summary,

Proposition 3. $\Pi_{\mathcal{A}} = \bigcup_{n\in\omega} \Pi_n = \Pi_\infty$.

This allows us to introduce the following definition: $T \subseteq U$ is $\Pi_{\mathcal{A}}$*–exact* iff it is Π_∞-exact, i.e.,

$$T = \cup\{[x]_\infty : x \in T\}.$$

It immediately follows that T is $\Pi_{\mathcal{A}}$-exact iff

$$T = Cl_{\Pi_{\mathcal{A}}}(T) = Int_{\Pi_{\mathcal{A}}}(T),$$

where $Cl_{\Pi_{\mathcal{A}}}(T)$ and $Int_{\Pi_{\mathcal{A}}}(T)$ are the closure and the interior of T with respect to the topology $\Pi_{\mathcal{A}}$. A subset of U which is not $\Pi_{\mathcal{A}}$-exact is called $\Pi_{\mathcal{A}}$*–rough.*

For each $n \in \omega$, we define a function $d_n : U \times U \to \mathbf{R}^+$, where $\mathbf{R}^+$ is the collection of nonnegative real numbers, as follows,

$$d_n(x, y) = \begin{cases} 1 \text{ if } [x]_n \neq [y]_n \\ 0 \text{ otherwise.} \end{cases}$$

Then,

$$d_n(x, x) = 0, \quad d_n(x, y) = d_n(y, x).$$

We now define a function $d : U \times U \to \mathbf{R}^+$ as

$$d(x, y) = \sum_n 10^{-n} \cdot d_n(x, y),$$

where the convergence of the right side of the equation can readily be verified. It immediately follows that d is a pseudo–metric on U. [So is d_n.] This pseudo–metric d contains the information on when x and y became indiscernible in the chain E_i. In particular,

Lemma 16. *$d(x, y) = 0$ whenever $xE_\infty y$.*

Remember that as $(\Pi_\mathcal{A}, U, \varnothing, \cup, \cap, \neg)$ is a complete Boolean algebra, all open sets in $\Pi_\mathcal{A}$ are closed and vice versa. Now, according to Hausdorff's account, the following defines a metric on $\Pi_\mathcal{A}$:

$$d_H(K, L) = \max\{\max_{x\in K} dist(x, L), \max_{x\in L} dist(x, K)\},$$

where

$$dist(x, L) = \min\{d(x, z) : z \in L\}.$$

It is well–known that every $\Pi_\mathcal{A}$-rough set X can be characterized by a pair of $\Pi_\mathcal{A}$-exact sets (Q, T) where

$$Q = Cl_{\Pi_\mathcal{A}}(X), \quad T = U - Int_{\Pi_\mathcal{A}}(X).$$

Given two $\Pi_\mathcal{A}$-rough sets X_1, X_2, we can define the distance metric D by

$$D(X_1, X_2) = \max\{d_H(Q_1, Q_2), d_H(T_1, T_2)\}.$$

If X is $\Pi_\mathcal{A}$-exact, then,

$$Q = Cl_{\Pi_\mathcal{A}}(X) = X, \quad T = U - Int_{\Pi_\mathcal{A}}(X) = U - X.$$

It can readily be verified that

$$d_H(Q_1, Q_2) = d_H(X_1, X_2) = d_H(T_1, T_2).$$

Therefore, $D(X_1, X_2) = d_H(X_1, X_2)$.

Let $d_H \times d_H : \Pi_\mathcal{A} \times \Pi_\mathcal{A} \to \mathbf{R}^+$ be such that

$$d_H \times d_H((K, K'), (L, L')) = d_H(K, L) + d_H(K', L').$$

Then,

$$\begin{aligned} d_H \times d_H((K, K'), (M, M')) &= d_H(K, M) + d_H(K', M') \\ &\leq d_H(K, L) + d_H(L, M) + d_H(K', L') + d_H(L', M') \\ &= d_H \times d_H((K, K'), (L, L')) \\ &\quad + d_H \times d_H((L, L'), (M, M')). \end{aligned}$$

We have shown that $d_H \times d_H$ is a metric on $\Pi_\mathcal{A} \times \Pi_\mathcal{A}$.

Proposition 4. *$\cup, \cap : \Pi_\mathcal{A} \times \Pi_\mathcal{A} \to \Pi_\mathcal{A}$ are continuous maps with respect to $d_H \times d_H$ and d_H.*

Proposition 5. *$\neg : \Pi_\mathcal{A} \to \Pi_\mathcal{A}$ is continuous with respect to d_H and d_H.*

Let $D \times D : 2^U \times 2^U \to \mathbf{R}^+$ be such that

$$D \times D(X, Y) = D(X) + D(Y).$$

Then, it is a metric on $2^U \times 2^U$. Furthermore,

Proposition 6. *$\cup, \cap : 2^U \times 2^U \to 2^U$ are continuous maps with respect to $D \times D$ and D.*

Proposition 7. *$\neg : 2^U \to 2^U$ is continuous with respect to D and D.*

6 Metric and Measure on PFS

In the sequel, the technique of inducing metrics over the subsets of infinitary information systems is applied to induce a metric on the universe of a graded PFS. There is an natural way of inducing a graded indiscernibility relation in the context of a PFS. The above construction of the canonical PFS M_{max} induces a graded indiscernibility relation over the universe of Frege sets. This and the metric it induces on the universe of M_{max} can be captured by the following elaboration of the definition of a PFS.

Let $\mathcal{F} = (U, E, \ulcorner\cdot\urcorner, \llcorner\cdot\lrcorner)$ be a PFS and assume that E is the graded indiscernibility relation over U induced by a descending chain E_n, $n \in \omega$. One now has a metric d_H over $\Pi_{\mathcal{A}}$. The retraction mapping $\ulcorner\cdot\urcorner$ maps $\Pi_{\mathcal{A}}$–exact sets onto U and hence induces a metric d_h over U such that

$$d_h(\ulcorner Q\urcorner, \ulcorner P\urcorner) = d_H(Q, P).$$

Naturally, 2^U is equipped with the metric D. As observed earlier, d_H is the restriction of D to $\Pi_{\mathcal{A}}$–exact sets. Then $\mathcal{F}$ is *graded by* $E_n, n \in \omega$ and called a *graded PFS*.

Note that:

1. $\ulcorner\cdot\urcorner$ is continuous with respect to D and d_h,
2. $\llcorner\cdot\lrcorner$ is continuous with respect to d_h and D.

Assume $\mathcal{F}$ is graded by E_n, $n \in \omega$. It follows that

Proposition 8. *If $X \equiv Y$, then $d_H(\llcorner X\lrcorner, \llcorner Y\lrcorner) = 0$.*

Given an element X of 2^U, we say it is an *infinitesimal* if

$$d_H(X, \varnothing) = 0.$$

It immediately follows that for any $X, Y \in 2^U$,

$$d_H(X - Y, \varnothing) = 0 \Leftrightarrow d_H(X, Y) = 0.$$

We define a norm on every element X of 2^U as follows,

$$\|X\| = d_H(X, \varnothing).$$

It can readily be shown that $\|X\|$ defines a measure on 2^U.

7 Inversely Graded Indiscernibility and Metric

Given a set U, consider an ascending chain of equivalence relations $\{E_i\}$ over U such that $E_i \subseteq E_{i+1}$. We call the equivalence relation $E_\infty = \bigcup_{i \in \omega} E_i$ an *indiscernibility relation inversely graded by* $\{E_i\}$. By $[x]_i$ we mean the equivalence class $\{y : xE_iy\}$. It is clear that

$$[x]_n \subseteq [x]_{n+1}.$$

We have

$$[x]_{n+1} = \left[\bigcup\{[z]_n : z \in [x]_{n+1} - [x]_n\}\right] \cup [x]_n.$$

So, all (basic) open sets in Π_{n+1} are open in Π_n. This is to say

$$\Pi_{n+1} \subseteq \Pi_n.$$

Similarly, all (basic) open sets in Π_∞ are open in Π_n, i.e.,

$$\Pi_\infty \subseteq \Pi_n$$

for all $n \in \omega$.

Note that as the Pawlak topology is quasi–discrete, the collection of the *cogranules*, i.e.,

$$\mathcal{C}_n = \{U - [x]_n : x \in U\},$$

is a *cobase* base for a topology on U. Let

$$\mathcal{C}_\mathcal{R} = \bigcap_{n\in\omega} \mathcal{C}_n.$$

It can readily be shown that $\mathcal{C}_\mathcal{R}$ forms a cobase for a topology $\Pi_\mathcal{R}$ over U.

As the Pawlak topology is quasi–discrete and is a complete Boolean algebra, in lieu of duality, we have

Proposition 9. $\Pi_\mathcal{R} = \bigcap_{n\in\omega} \Pi_n = \Pi_\infty$.

This allows us to introduce the following definition: $T \subseteq U$ is $\Pi_\mathcal{R}$*–exact* iff it is Π_∞–exact, i.e.,

$$T = \cup\{[x]_\infty : x \in T\}.$$

It immediately follows that T is $\Pi_\mathcal{R}$–exact iff

$$T = Cl_{\Pi_\mathcal{R}}(T) = Int_{\Pi_\mathcal{R}}(T),$$

where $Cl_{\Pi_\mathcal{R}}(T)$ and $Int_{\Pi_\mathcal{R}}(T)$ are the closure and the interior of T with respect to the topology $\Pi_\mathcal{R}$. A subset of U which is not $\Pi_\mathcal{R}$-exact is called $\Pi_\mathcal{R}$*–rough.*

For each $n \in \omega$, we define a function $d_n : U \times U \to \mathbf{R}^+$, where $\mathbf{R}^+$ is the collection of nonnegative real numbers, as follows:

$$d_n(x.y) = \begin{cases} 0 \text{ if } [x]_n = [y]_n \\ 1 \text{ otherwise.} \end{cases}$$

Then

$$d_n(x,x) = 0, \quad d_n(x,y) = d_n(y,x).$$

Now define a function $d : U \times U \to \mathbf{R}^+$ as

$$d(x,y) = \sum_n 10^{-n} \cdot d_n(x,y),$$

where the convergence of the right side of the equation can readily be verified. It immediately follows that d is a pseudo-metric on U. (So is d_n.) This pseudo-metric d contains the information determining when x and y became discernible in the chain E_i. In particular,

Lemma 17. *$d(x,y) = 0.\overline{1} = \frac{1}{9}$ whenever $\neg(xE_\infty y)$.*

Recall that, since $(\Pi_{\mathcal{R}}, U, \varnothing, \cup, \cap, \neg)$ is a complete Boolean algebra, all open sets in $\Pi_{\mathcal{R}}$ are closed and vice versa. According to Hausdorff's account, the following defines a metric on $\Pi_{\mathcal{R}}$,

$$d_H(K,L) = \max\{\max_{x\in K} dist(x,L), \max_{x\in L} dist(x,K)\}$$

where

$$dist(x,L) = \min\{d(x,z) : z \in L\}.$$

It is well–known that every $\Pi_{\mathcal{R}}$–rough set X can be characterized by a pair of $\Pi_{\mathcal{R}}$–exact sets (Q,T) where

$$Q = Cl_{\Pi_{\mathcal{R}}}(X), \quad T = U - Int_{\Pi_{\mathcal{R}}}(X).$$

Given two $\Pi_{\mathcal{R}}$-rough sets X_1, X_2, define the distance metric D by

$$D(X_1,X_2) = \max\{d_H(Q_1,Q_2), d_H(T_1,T_2)\}.$$

If X is $\Pi_{\mathcal{R}}$–exact then,

$$Q = Cl_{\Pi_{\mathcal{R}}}(X) = X, \quad T = U - Int_{\Pi_{\mathcal{R}}}(X) = U - X.$$

It can readily be verified that

$$d_H(Q_1,Q_2) = d_H(X_1,X_2) = d_H(T_1,T_2).$$

Therefore, $D(X_1,X_2) = d_H(X_1,X_2)$.

Let $d_H \times d_H : \Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}} \to \mathbf{R}^+$ be such that

$$d_H \times d_H((K,K'),(L,L')) = d_H(K,L) + d_H(K',L').$$

Then,

$$\begin{aligned} d_H \times d_H((K,K'),(M,M')) &= d_H(K,M) + d_H(K',M') \\ &\leq d_H(K,L) + d_H(L,M) + d_H(K',L') + d_H(L',M') \\ &= d_H \times d_H((K,K'),(L,L')) \\ &\quad + d_H \times d_H((L,L'),(M,M')). \end{aligned}$$

We have shown that $d_H \times d_H$ is a metric on $\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}$.

Proposition 10. *$\cup, \cap : \Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}} \to \Pi_{\mathcal{R}}$ are continuous maps with respect to $d_H \times d_H$ and d_H.*

Proposition 11. *$\neg : \Pi_{\mathcal{R}} \to \Pi_{\mathcal{R}}$ is continuous with respect to d_H and d_H.*

Let $D \times D : 2^U \times 2^U \to \mathbf{R}^+$ be such that

$$D \times D(X,Y) = D(X) + D(Y).$$

Then, $D \times D$ is a metric on $2^U \times 2^U$. Furthermore,

Proposition 12. *$\cup, \cap : 2^U \times 2^U \to 2^U$ are continuous maps with respect to $D \times D$ and D.*

Proposition 13. *$\neg : 2^U \to 2^U$ is continuous with respect to D and D.*

8 An Example of Inversely Graded Indiscernibility

Let $\mathcal{L}_{\mathcal{R}}$ be the first order language of real numbers. Let $\mathcal{R}$ be the standard model of $\mathcal{L}_{\mathcal{R}}$ and let R be the universe of $\mathcal{R}$. $X \subseteq R$ is said to be *definable* iff there is a formula $A(x)$ of $\mathcal{L}_{\mathcal{R}}$ such that

$$X = \{x \in \mathcal{R} \mid \mathcal{R} \models A(x)\}.$$

Let $X \subseteq \mathbf{M}$ and $\mathcal{D}(X)$ be the family of all definable subsets of X. $\mathcal{F} \subseteq \mathcal{D}(X)$ is a *d-filter* on X iff

$$\begin{array}{l} X \in \mathcal{F} \\ A, B \in \mathcal{F} \Rightarrow (A \cap B) \in \mathcal{F} \\ A \in \mathcal{F}, A \subseteq B \in \mathcal{D}(X) \Rightarrow B \in \mathcal{F} \\ \emptyset \notin \mathcal{F}. \end{array}$$

It is clear that if $\mathcal{F}$ is a d-filter on X then,

$$A \in \mathcal{F} \Rightarrow (X - A) \notin \mathcal{F}.$$

Let $\mathcal{F}$ be a d-filter on X. If

$$A \in 2^X \Rightarrow A \in \mathcal{F} \vee (X - A) \in \mathcal{F}$$

then $\mathcal{F}$ is said to be an *ultra d–filter on* $\mathcal{D}(X)$.

Let $\mathcal{F} \subseteq \mathcal{D}(X)$; $\mathcal{F}$ is *d–consistent* iff the closure of $\mathcal{F}$ under finite intersections does not contain $\emptyset$ as an element. Let

$$UC(\mathcal{F}) =_{df} \{Y \in \mathcal{D}(X) \mid (\exists F \in \mathcal{F})(F \subseteq Y)\}.$$

Lemma 18. *Let $\mathcal{F}$ be a d–filter on X. Let $Y \subseteq X$. Suppose $\{Y\} \cup \mathcal{F}$ is not d–consistent. Then $\{X - Y\} \cup \mathcal{F}$ is d–consistent.*

Since $\mathcal{D}(X)$ is countably infinite, it may be enumerated as $\mathcal{D}(X)_i : i < \omega$.

Lemma 19. *(***Lindenbaum***) Every d–filter $\mathcal{F}$ has an ultra d–filter extension.*

Proof. Define an infinite sequence,

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_k \subseteq \ldots \subseteq \Gamma_\omega,$$

of families of definable subsets of X by

$$\begin{array}{lll} \Gamma_0 = & \mathcal{F}, & \\ \Gamma_{k+1} = & UC(\Gamma_k \cup \{\mathcal{D}(X)_{k+1}\}) & \text{if this } \Gamma_k \cup \{\mathcal{D}(X)_{k+1}\} \text{ is d-consistent} \\ & \Gamma_k & \text{otherwise,} \end{array}$$

and

$$\Gamma_\infty = \cup\{\Gamma_k \mid k < \omega\}.$$

Then, it follows that Γ_ω is a maximal d–filter which contains the d–filter $\mathcal{F}$. Furthermore, $\Gamma_k,\ k < \omega$ are all filters. Let $\{S_i \mid i \in \mathcal{D}(X)\}$ be a $\mathcal{D}(X)$-family of sets. On the product,

$$\Pi_{i\in I} S_i = \{g : \mathcal{D}(X) \to \cup_{i\in I} S_i \mid \text{for all } i \in \omega,\ g(i) \in S_i\},$$

we define the following equivalence relation,

$$g_1 \equiv_{\Gamma_k} g_2 \Leftrightarrow \{i \in \mathcal{D}(X) \mid g_1(i) = g_2(i)\} \in \Gamma_k,\ k \in \omega.$$

It is clear that $\equiv_{\Gamma_k},\ k \in \omega$ is a descending ω–sequence of equivalence relations and,

$$g_1 \equiv_{\Gamma_\infty} g_2 \Leftrightarrow \{i \in \mathcal{D}(X) \mid g_1(i) = g_2(i)\} \in \Gamma_\infty,$$

is the limit of it.

Then each cofinite set is definable (even in first order Peano arithmetic). Therefore the collection of cofinite sets $\mathcal{F}$ is a d–filter. In lieu of Lindenbaum's Lemma, there is an ω–chain,

$$\mathcal{F} = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_k \subseteq \ldots \subseteq \Gamma_\infty,$$

of filters approximating an ultra–filter Γ_∞. Define an ultra–power $\mathcal{R}^*$ of $\mathcal{R}$ by

$$\mathcal{R}^* = \mathcal{R}^\omega/\mathcal{F}.$$

Γ_∞ yields an indiscernibility relation $\equiv_{\Gamma_\infty}$on $\mathcal{R}^*$ inversely graded by $\equiv_{\Gamma_k},\ k \in \omega$. This in turn induces a metric on $\mathcal{R}^*$ as discussed in the previous section.

9 Metrically Open Subsets of a Graded Plenum

We conclude our study of exact subsets of infinite information systems with an open problem in the theory of the graded plenum M_{max}.

Let $(U, E, \ulcorner\cdot\urcorner, \llcorner\cdot\lrcorner)$ be a plenum graded by $E_n,\ n \in \omega$. The metric $d(x, y) : U \times U \to R$ induced by the graded indiscernibility relation on U gives rise to the metric space (U, d). The collection of all d–open balls forms a basis for a topology (U, τ_d) called the *metric topology* of (U, d).

By effectivizing the SFP construction of D_∞ as in effective domain theory [30], we can single out the computable elements of D_∞ and thence define "computable" elements of M_{max} as those whose maximal continuous approximation are computable. As effective DT indicates, these computable elements are now represented by Gődel numbers (programs) and we can define "computable exact subset of M_{max}" (equivalently, "computable element of M_{max}") by effectively enumerating these programs. The recursion theorist A.I. Mal'cev [40] proposed two different versions of the notion of a "computable program set" in the context of numeration theory, the general theory of Gődel numberings [17, 18]. The one of interest to RST deals with "index sets" − sets of natural numbers which are closed under the relation of program equivalence − and "extensional" program transformations – effective number theoretic functions which preserve program equivalence. Accordingly, this extensional recursion theory identifies the computable program sets with the r.e. index sets, i.e., r.e. exact sets.

Given an acceptable indexing of r.e. subsets of natural numbers, Myhill–Sheperdson's [41] result in recursion theory states that every extensional program transformation is an effective enumeration operator. (As a corollary, we have the result by Rice–Shapiro [53] that every r.e. index set is upward closed under subset inclusion). This and Thomas Streicher [61], and Yu Ershov's [19] result in effective DT established that extentional computability entails continuity.

It thus appears natural to conjecture that all computable elements of M_{max} are metric open.

As RST passes from finite (definitely computable) to infinite information systems it may be desirable to consider the first steps, towards computably infinite information systems.

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