Data Structure and Operations for Fuzzy Multisets

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Abstract. An overview of data structures and operations for fuzzy multisets is given. A simple linear list identified with an infinite dimensional vector is taken as an elementary data structure for fuzzy multisets. Fuzzy multiset operations are defined by corresponding vector operations. Two level sets of α-cut and ν-cut are defined and commutativity between them are described. Spaces of fuzzy multisets are also discussed.

1 Introduction

Multisets which have been considered by several authors (e.g., [6, 9, 1]) with application to database queries (e.g., [3]) are attracting more attention of researchers in relation to new computation paradigms [2].

Fuzzy multisets have also been studied [23, 5, 10, 19, 18, 11, 7]. This paper is not intended to provide a comprehensive overview, but focuses several features of fuzzy multisets with discussion of data structure for representing fuzzy multisets. A linear list or a simple vector is taken as the elementary data structure. This structure is not only for convenience but reveals fundamental characteristic of multisets as results of linear data processing. Thus a novel type of image of a function is discussed.

A characteristic in fuzzy multisets is their inherent infiniteness. We hence discuss infinite–dimensional spaces of fuzzy multisets.

We omit most of proofs of propositions, as they are not difficult and readers can try to prove by themselves. Moreover applications are not discussed here, they are studied in other papers [12, 13, 16].

2 Preliminaries

Assume throughout this paper that the universal set X is finite. We thus write $X = \{x_1, x_2, \ldots, x_n\}$ or $X = \{x, y, \ldots, z\}$. Scalars are denoted by $a, b, \ldots$

Fuzzy Sets

To show operations for fuzzy multisets as extensions of those for fuzzy sets, we first provide notations for fuzzy sets. Infix notations for max and min are denoted by $\vee$ and $\wedge$, respectively as in most literature [25].

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As is well–known, for fuzzy sets A and B of X:

1.
$$A \subseteq B \iff \mu_A(x) \le \mu_B(x), \quad \forall x \in X.$$

2.
$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x).$$

3.
$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x).$$

4. A weak α-cut is denoted by $[A]_\alpha$ here:
$$x \in [A]_\alpha \iff \mu_A(x) \ge \alpha.$$
Moreover, a strong α-cut is denoted by $]A[_\alpha$:
$$x \in\,]A[_\alpha \iff \mu_A(x) > \alpha.$$

5. The t-norms and conorms are used as generalizations of fuzzy set union and intersection. We omit details of t-norms [20, 4], and note only that they are respectively denoted by $t(a, b)$ and $s(a, b)$. When used as set operations, the operations are denoted by
$$\mu_{A\mathbf{T}B}(x) = t(\mu_A(x), \mu_B(x)), \quad \mu_{A\mathbf{S}B}(x) = s(\mu_A(x), \mu_B(x)).$$
A particular t-norm of algebraic product is denoted by $a \cdot b$ whereby
$$\mu_{A\cdot B}(x) = \mu_A(x) \cdot \mu_B(x),$$

Important properties of commutativity between an operation and an α-cut hold:

$$\begin{aligned}
[A \cup B]_\alpha &= [A]_\alpha \cup [B]_\alpha, \\
[A \cap B]_\alpha &= [A]_\alpha \cap [B]_\alpha, \\
]A \cup B[_\alpha &=]A[_\alpha \cup\,]B[_\alpha, \\
]A \cap B[_\alpha &=]A[_\alpha \cap\,]B[_\alpha.
\end{aligned}$$

We also have

$$\begin{aligned}
A \subseteq B &\iff [A]_\alpha \subseteq [B]_\alpha, \quad \forall \alpha \in (0, 1], \\
A \subseteq B &\iff\,]A[_\alpha \subseteq\,]B[_\alpha, \quad \forall \alpha \in [0, 1).
\end{aligned}$$

For a given function $f \colon X \to Y$, we moreover have

$$]f(A)[_\alpha = f(]A[_\alpha).$$

Multisets

A multiset M of X is characterized by the count function $C_M\colon X \to \boldsymbol{N}$, where $\boldsymbol{N} = \{0, 1, 2, \ldots\}$. Thus, $C_M(x)$ is the number of occurrences of the element $x \in X$.

The following is basic relations and operations for crisp multisets;

I. (inclusion):

$$M \subseteq N \Leftrightarrow C_M(x) \le C_N(x), \ \ \forall x \in X.$$

II. (equality):

$$M = N \Leftrightarrow C_M(x) = C_N(x), \ \ \forall x \in X.$$

III. (union):

$$C_{M\cup N}(x) = C_M(x) \vee C_N(x).$$

IV. (intersection):

$$C_{M\cap N}(x) = C_M(x) \wedge C_N(x).$$

V. (sum):

$$C_{M+N}(x) = C_M(x) + C_N(x).$$

VI. (Cartesian product): For $x \in X$ and $y \in Y$,

$$C_{M\times N}(x, y) = C_M(x) \cdot C_N(y).$$

VII. ($\boldsymbol{\nu}$-cut):

A ν-cut $M \mapsto [M]^\nu$ transforms a multiset into an ordinary set:

$$C_M(x) \ge \nu \Leftrightarrow x \in [M]^\nu$$
$$C_M(x) < \nu \Leftrightarrow x \notin [M]^\nu$$

It is reasonable to assume that the number $C_M(\cdot)$ should be finite.

Example 1. Let us consider a simple example of $X = \{a, b, c, d\}$, $C_M(a) = 2$, $C_M(b) = 3$, $C_M(c) = 1$, and $C_M(d) = 0$. We write $M = \{2/a, 3/b, 1/c\}$ by ignoring zero occurrence $0/d$. Alternatively, we can write $M = \{a, a, b, b, b, c\}$. notice also that we can exchange the order of elements in $\{\cdots\}$ as in ordinary sets

$$\{a, a, b, b, b, c\} = \{c, a, b, b, a, b\}. \qquad (1)$$

A Sorting Operation. A sorting operation is essential in processing multisets. Let us take the last example in which $\{c, a, b, b, a, b\}$ should be arranged into the sorted form of $\{a, a, b, b, b, c\}$ in order to check (1). This means that we actually handle sequences (c, a, b, b, a, b) and (a, a, a, b, b, c) as an abstract data structure instead of $\{a, a, b, b, b, c\}$ and $\{c, a, b, b, a, b\}$.

3 Fuzzy Multisets

Fuzzy multiset A of X (often called fuzzy bag) is characterized by the function $C_A(\cdot)$ of the same symbol, but the value $C_A(x)$ is a finite multiset in $I = [0, 1]$ [23]. In other words, given $x \in X$,

$$C_A(x) = \{\mu, \mu', \dots, \mu''\}, \qquad \mu, \mu', \dots, \mu'' \in I.$$

Assume $\mu, \mu', \dots, \mu''$ are nonzero for simplicity. We write

$$A = \{\{\mu, \mu', \dots, \mu''\}/x, \dots\}$$

or

$$A = \{(x, \mu), (x, \mu'), \dots, (x, \mu''), \dots\}.$$

As a data structure, we introduce an infinite–dimensional vector:

$$C_A(x) = (\mu, \mu', \dots, \mu'', 0, 0, \dots).$$

Collection of such vectors is denoted by $\mathcal{V}$:

$$\mathcal{V} = \{\, (\mu, \mu', \dots, \mu'', 0, 0, \dots) : \quad \mu, \mu', \dots, \mu'' \in I \,\}$$

A sorting operation to multisets in I is important in defining operations for fuzzy multisets. This operation denoted by S ($S \colon \mathcal{V} \to \mathcal{V}$) rearranges the sequence in $\mathcal{V}$ into the decreasing order:

$$S((\mu, \mu', \dots, \mu'', 0, 0, \dots)) = (\nu^1, \nu^2, \dots, \nu^p, 0, 0, 0)$$

where

$$\nu^1 \geq \nu^2 \geq \cdots \geq \nu^p > 0$$

and

$$\{\mu, \mu', \dots, \mu''\} = \{\nu^1, \nu^2, \dots, \nu^p\}.$$

Thus we can assume

$$C_A(x) = (\nu^1, \nu^2, \dots, \nu^p, 0, 0, 0) \tag{2}$$

Moreover we define the length of $C_A(x)$ by

$$L(C_A(x)) = p$$

The above sorted sequence for $C_A(x)$ is called the standard form for a fuzzy multiset, as many operations are defined in terms of the standard form.

Additional operations on $\mathcal{V}$ are necessary in order to define fuzzy multiset operations. Assume

$$k = (k_1, k_2, \dots, k_q, \dots, k_r, 0, 0, \dots),\ l = (l_1, l_2, \dots, l_q, \dots, l_s, 0, 0, \dots) \in \mathcal{V}$$

where k_i $(i = 1, \dots, r)$ and l_j $(j = 1, \dots, s)$ are nonzero.

Then we define

$$k \vee l = (\max\{k_1, l_1\}, \dots, \max\{k_q, l_q\}, \dots), \tag{3}$$

$$k \wedge l = (\min\{k_1, l_1\}, \dots, \min\{k_q, l_q\}, \dots), \tag{4}$$

$$k \cdot l = (k_1 \cdot l_1, \dots, k_q \cdot l_q, \dots), \tag{5}$$

$$k \mid l = (k_1, k_2, \dots, k_r, l_1, l_2, \dots, l_s, 0, 0, \dots) \tag{6}$$

Assume $k_1 \geq k_2 \geq \cdots \geq k_r$; let $\alpha \in (0,1]$ and $\nu \in \boldsymbol{N}$. Then we define

$$|k| = \sum_{i=1}^{r} k_i, \tag{7}$$

$$[[k]]_\alpha = j \quad (k_j \geq \alpha, k_{j+1} < \alpha), \tag{8}$$

$$]]k[[_\alpha = j' \quad (k_{j'} > \alpha, k_{j'+1} \leq \alpha), \tag{9}$$

$$[[k]]^\nu = k_\nu \tag{10}$$

Note that in the last equation (10), $k_\nu = 0$ if and only if $\nu > r$.

Moreover we define inequality of the two vectors:

$$k \leq l \iff k_i \leq l_i, \quad i = 1, 2, \dots \tag{11}$$

We now define fuzzy multiset operations.

1. **inclusion:**

$$A \subseteq B \iff S(C_A(x)) \leq S(C_B(x)), \quad \forall x \in X.$$

2. **equality:**

$$A = B \iff S(C_A(x)) = S(C_B(x)), \quad \forall x \in X.$$

3. **union:**

$$C_{A \cup B}(x) = S(C_A(x)) \vee S(C_B(x)).$$

4. **intersection:**

$$C_{A \cap B}(x) = S(C_A(x)) \wedge S(C_B(x)).$$

5. **sum:**

$$C_{A+B}(x) = S(S(C_A(x)) \mid S(C_B(x))).$$

6. **product:**

$$C_{A \cdot B}(x) = S(C_A(x)) \cdot S(C_B(x)).$$

7. **$\boldsymbol{\alpha}$-cuts:**

$$C_{[A]_\alpha}(x) = [[S(C_A(x))]]_\alpha,$$

$$C_{]A[_\alpha}(x) =]]S(C_A(x))[[_\alpha.$$

8. ***ν*-cut:**

$$C_{[A]^\nu}(x) = [[S(C_A(x))]]^\nu.$$

9. **Cardinality:**

$$|A| = \sum_{x \in X} |C_A(x)|.$$

We obviously have

$$\begin{aligned} L(C_{A\cup B}(x)) &= \max\{L(C_A(x)), L(C_B(x))\}, \\ L(C_{A\cap B}(x)) &= \min\{L(C_A(x)), L(C_B(x))\}, \\ L(C_{A+B}(x)) &= L(C_A(x)) + L(C_B(x)), \\ L(C_{A\cdot B}(x)) &= \min\{L(C_A(x)), L(C_B(x))\}. \end{aligned}$$

The reason why we define fuzzy multiset operations using the sorting operation S is shown in next propositions.

Proposition 1. An ordinary fuzzy set F and a crisp multiset M can be embedded into the collection of all fuzzy multisets. Let the embedding map be $\mathcal{I}$. Namely,

$$C_{\mathcal{I}(F)}(x) = (\mu_F(x), 0, 0, \dots),$$

and

$$C_{\mathcal{I}(M)}(x) = (1, 1, \dots, 1, 0, 0, \dots),$$

where the number of 1's in the right hand side is equal to $C_M(x)$. Assume F_1 and F_2 are ordinary fuzzy sets, and M_1 and M_2 are crisp multisets. We then have

$$\begin{aligned} \mathcal{I}(F_1 \cup F_2) &= \mathcal{I}(F_1) \cup \mathcal{I}(F_2), \\ \mathcal{I}(F_1 \cap F_2) &= \mathcal{I}(F_1) \cap \mathcal{I}(F_2), \\ \mathcal{I}(M_1 \cup M_2) &= \mathcal{I}(M_1) \cup \mathcal{I}(M_2), \\ \mathcal{I}(M_1 \cap M_2) &= \mathcal{I}(M_1) \cap \mathcal{I}(M_2), \\ \mathcal{I}(M_1 + M_2) &= \mathcal{I}(M_1) + \mathcal{I}(M_2), \\ [\mathcal{I}(F)]_\alpha &= \mathcal{I}([F]_\alpha), \\ ]\mathcal{I}(F)[_\alpha &= \mathcal{I}(]F[_\alpha), \\ [\mathcal{I}(M)]^\nu &= \mathcal{I}([M]^\nu). \end{aligned}$$

Proposition 2. For two fuzzy multisets A and B of X, the following relations hold.

$$\begin{aligned}
&A \subseteq B \iff [A]_\alpha \subseteq [B]_\alpha, \quad \forall \alpha \in (0,1], \\
&A \subseteq B \iff]A[_\alpha \subseteq]B[_\alpha, \quad \forall \alpha \in [0,1). \\
&[A \cup B]_\alpha = [A]_\alpha \cup [B]_\alpha, \\
&[A \cap B]_\alpha = [A]_\alpha \cap [B]_\alpha, \\
&[A + B]_\alpha = [A]_\alpha + [B]_\alpha, \\
&]A \cup B[_\alpha =]A[_\alpha \cup]B[_\alpha, \\
&]A \cap B[_\alpha =]A[_\alpha \cap]B[_\alpha, \\
&]A + B[_\alpha =]A[_\alpha +]B[_\alpha, \\
&[A \cup B]^\nu = [A]^\nu \cup [B]^\nu, \\
&[A \cap B]^\nu = [A]^\nu \cap [B]^\nu.
\end{aligned}$$

The proof is easy and omitted. Notice also that

$$[A + B]^\nu \neq [A]^\nu + [B]^\nu$$

in general.

Proposition 3. For fuzzy multisets A, B, and C of X, the commutative, associative, and distributive laws holds for operations $\cup$ and $\cap$. Namely, the collection of all fuzzy multisets of X forms a distributive lattice.

$$\begin{aligned}
A \cup B &= B \cup A, \\
A \cap B &= B \cap A, \\
A \cup (B \cup C) &= (A \cup B) \cup C, \\
A \cap (B \cap C) &= (A \cap B) \cap C, \\
(A \cap B) \cup C &= (A \cup C) \cap (B \cup C), \\
(A \cup B) \cap C &= (A \cap C) \cup (B \cap C).
\end{aligned}$$

The proof is omitted here (see Chapter 2 of [7]). Readers should note the properties in Proposition 2 is used in the proof.

Proposition 4. Let A be an arbitrary fuzzy multiset of X. Then,

$$\begin{aligned}
&[[A]_\alpha]^\nu = [[A]^\nu]_\alpha, \\
&[]A[_\alpha]^\nu =][A]^\nu[_\alpha.
\end{aligned}$$

holds. Namely, an α-cut and a ν-cut are commutative.

Example 2. Suppose $X = \{a, b, c, d\}$ and

$$\begin{aligned}
A &= \{\{0.3, 0.5\}/a, \{0.7\}/b, \{0.9\}/c\}, \\
B &= \{\{0.4, 0.4\}/a, \{0.2, 0.5\}/b\}.
\end{aligned}$$

We can represent A as

$$A = \{(a, 0.3), (a, 0.5), (b, 0.7), (c, 0.9)\}.$$

In the standard form,

$$A = \{(0.5, 0.3)/a, (0.7)/b, (0.9)/c\},$$
$$B = \{(0.4, 0.4)/a, (0.5, 0.2)/b\}.$$

where zero elements are ignored. We have

$$A \cup B = \{(0.5, 0.4)/a, (0.7, 0.2)/b, (0.9)/c\},$$
$$A \cap B = \{(0.4, 0.3)/a, (0.5)/b\},$$
$$A + B = \{(0.5, 0.4, 0.4, 0.3)/a, (0.7, 0.5, 0.2)/b, (0.9)/c\},$$
$$[A]_{0.3} = \{2/a, 1/b, 1/c\},$$
$$]A[_{0.3} = \{1/a, 1/b, 1/c\},$$
$$[A]^2 = \{0.3/a\}.$$

4 Images and α-Cuts

Let f be a mapping of X into Y. Let us consider two images:

$$f(A) = \bigcup_{x \in A} \{f(x)\} \tag{12}$$

and

$$f\langle A \rangle = \sum_{x \in A} \{f(x)\} \tag{13}$$

where A is a fuzzy multiset of X.

Example 3. Suppose

$$A = \{(a, 0.3), (a, 0.5), (b, 0.7), (c, 0.9)\}$$

and $f(a) = v$, $f(b) = f(c) = w$. Then,

$$f(A) = \{(v, 0.5), (w, 0.9)\},$$
$$f\langle A \rangle = \{(v, 0.3), (v, 0.5), (w, 0.7), (w, 0.9)\}.$$

When A is an ordinary fuzzy set, the former coincides with the ordinary extension principle. Moreover for an arbitrary fuzzy multiset A, $f(A)$ is an ordinary fuzzy set: there is no pair $(x, \nu), (x', \nu') \in f(A)$ such that $x = x'$. Therefore $f(A)$ is inappropriate as a mapping of fuzzy multisets.

For example, let Id be an identity mapping of X onto X: $Id(x) = x$. Suppose A is a fuzzy multiset. Then

$$Id(A) \neq A$$

in general. Instead, we have

$$Id(A) = [A]^1$$

using ν-cut with $\nu = 1$.

On the other hand, the latter can generate a fuzzy multiset from an ordinary fuzzy set. Notice in particular that the latter image uses a simple rewriting of symbols. We thus consider $f\langle A\rangle$ alone hereafter. Remark also that

$$Id\langle A\rangle = A.$$

We have the following.

Proposition 5. For every fuzzy multiset A and B of X,

$$f\langle [A]_\alpha\rangle = [f\langle A\rangle]_\alpha,$$
$$f\langle]A[_\alpha\rangle =]f\langle A\rangle[_\alpha,$$
$$f\langle A+B\rangle = f\langle A\rangle + f\langle B\rangle.$$

The proof is omitted.

5 Other Operations

We glimpse other operations for fuzzy multisets of which detailed discussion is omitted.

***t**-Norms*

Let

$$C_A(x) = (\nu_A^1, \ldots, \nu_A^p, \ldots),$$
$$C_B(x) = (\nu_B^1, \ldots, \nu_B^p, \ldots).$$

Then,

$$C_{A\mathbf{T}B}(x) = (t(\nu_A^1, \nu_B^1), \ldots, t(\nu_A^p, \nu_B^p), \ldots),$$
$$C_{A\mathbf{S}B}(x) = (s(\nu_A^1, \nu_B^1), \ldots, s(\nu_A^p, \nu_B^p), \ldots).$$

In particular,

$$C_{A\cdot B}(x) = (\nu_A^1 \cdot \nu_B^1, \ldots, \nu_A^p \cdot \nu_B^p, \ldots).$$

Cartesian Product

Let A and B respectively be fuzzy multiset of X and Y:

$$C_A(x) = (\nu_A^1, \nu_A^2, \ldots), \qquad C_B(y) = (\nu_B^1, \nu_B^2, \ldots),$$

for $x \in X$ and $y \in Y$. Then,

$$C_{A\times B}(x, y) = \{\min\{\nu_A^1, \nu_B^1\}, \min\{\nu_A^1, \nu_B^2\}, \ldots, \min\{\nu_A^i, \nu_B^j\}, \ldots\}.$$

Thus the multiset for $C_{A\times B}(x, y)$ has every combination of nonzero elements of $C_A(x)$ and $C_B(y)$. For crisp multisets M of X and N of Y, we have

$$\mathcal{I}(M \times N) = \mathcal{I}(M) \times \mathcal{I}(N).$$

Complement of Multiset

We have seen the collection of all fuzzy multisets forms a distributive lattice, the discussion of a complement of a crisp or fuzzy multiset has problems. A way to define a complement is to introduce a value of infinity $+\infty$ and assume $C_M(x)\colon X \to \boldsymbol{N} \cup \{+\infty\}$. It then is not difficult to see that a complement M^C can be defined:

$$C_{M^C}(x) = \begin{cases} 0 & (C_M(x) > 0), \\ +\infty & (C_M(x) = 0) \end{cases}$$

For a fuzzy multiset A, this suggests

$$C_{A^C}(x) = \emptyset \quad (C_A(x) \neq \emptyset),$$
$$C_{A^C}(x) = (+\infty, +\infty, \dots) \quad (C_A(x) = \emptyset).$$

The last definition includes an artificial sequence of $+\infty$ as all elements. Nevertheless, introduction of the infinite element is necessary from the viewpoint of intuitionistic logic, where a complement should be defined to satisfy the axiom of Heyting algebra [21].

For every fuzzy multiset A, the following is valid.

$$A^C \cap A = \emptyset,$$
$$A \subseteq (A^C)^C.$$

6 Spaces of Fuzzy Multisets

We have assumed $C_A(x)$ is finite at the beginning. However, extension to infinite fuzzy multisets is straightforward:

$$C_A(x) = (\nu^1, \dots, \nu^p, \dots)$$

in which we admit infinite nonzero elements. A reasonable assumption to this sequence is

$$\nu^j \to 0 \quad (j \to +\infty).$$

Since this assumption ensures α-cuts $[A]_\alpha$ and $]A[_\alpha$ are finite for all $\alpha \in (0, 1]$.

Metric spaces can be defined on the collections of fuzzy multisets of X. For example, let

$$C_A(x) = (\nu_A^1, \dots, \nu_A^p, \dots), \quad C_B(x) = (\nu_B^1, \dots, \nu_B^p, \dots),$$

Then we can define

$$d_1(A, B) = \sum_{x \in X} \sum_{j=1}^{\infty} |\nu_A^j - \nu_B^j|,$$

which is an ℓ_1 type metric. Moreover we can also define

$$d_2(A, B) = \sqrt{\sum_{x \in X} \sum_{j=1}^{\infty} |\nu_A^j - \nu_B^j|^2},$$

as the ℓ_2 type metric. Moreover a scalar product $< A, B >$ is introduced in the latter space:

$$< A, B > = |A \cdot B| = \sum_{x \in X} |C_{A \cdot B}(x)|$$

using the algebraic product. We have

$$d_2(A, B) = < A, A > + < B, B > -2 < A, B > .$$

It is not difficult to see the metric space with d_1 is a Banach space and that with $< A, B >$ is a Hilbert space [8]. Such metrics are useful in discussing fuzzy multiset model for data clustering [16].

7 Conclusion

Multisets and fuzzy multisets are based on the idea of simple sequential processing of linear lists, as seen in the discussion of $f\langle\cdot\rangle$ which is the sequential rewriting of symbols. Moreover the sorting operation is most important in fuzzy multiset operations.

Spaces in fuzzy multisets are used for clustering and information retrieval [16]. Recent methods in pattern recognition such as the support vector machines [22] can be used in fuzzy multiset space and application to document clustering, which will be shown by us in near future.

Although we have omitted relations between multisets and rough sets [17, 24]. Readers can see, e.g., [15] for the related discussion.

Since a fuzzy multiset represents multiple occurrence of an object with possibly different memberships, it is adequate for information retrieval model especially on the web. There are much room for further studies both in theory and applications.

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