2

Geometric Fragmentation Statistics

The statistical issues governing the fragmentation of a body are not well in hand even to this day. Perhaps foremost in the list of objectives is the prediction of the distribution in the size of fragments resulting from a fragmentation event. One intriguing approach to this problem has simply been to investigate the statistically most random way of partitioning a given topology into a number of discrete entities. This approach to statistical fragmentation has been commonly identified as geometric fragmentation.

As noted in the introduction, Mott was initially led in pursuit of a theoretical description of the distribution in fragments from a fragmenting cylinder event by then recent work of Lineau (1936). Fragmenting munitions data available to Mott at the time appeared consistent with the exponential expression resulting from Lineau's one-dimensional model. Thus, his early efforts focused on extending the same geometric statistics approach to the topology of a naturally fragmenting cylinder.

In the present section we pursue some of the approaches to obtaining representations of fragment size distributions using the methods of geometric fragmentation statistics. In particular the well-known Mott distribution will be developed and examined.

2.1 Lineau Distribution

Fundamental to geometric fragmentation are the theoretical efforts of Lineau (1936). He considered the elementary problem of an extended body such as a glass rod or a stretching wire subjected to forces resulting in the multiple fracturing of that body. If any point on the body is as likely as another to fracture the problem is statistically well posed. The problem is modeled as that of an infinite one-dimensional body, or line, in which breaks are introduced with equal probability at any point on that line, as is illustrated in Fig. 2.1.

Thus, as stated, the random geometric fragmentation of a one-dimensional body appears decidedly unambiguous. An analytic solution requires only a

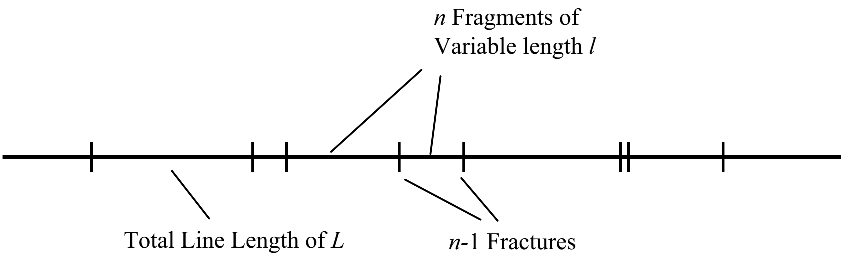

Fig. 2.1. Line of total length L broken at random into fragments of variable length l by $n-1$ fractures

proper probabilistic description of the random breaks, and the lengths of the segments delineated by these breaks. We shall show later that even this prescription for the statistical fragmentation of a one-dimensional body is arguable. At this point, however, we proceed with the solution leading to the one-dimensional Lineau fragment size distribution.

Consider a line of length L in which breaks on the line are introduced at random [Grady, 1990]. Since we are initially interested in partitioning the line into a large number of fragments (the average length is very small compared to the total length L) the finite length of the line is not of consequence and can effectively be considered infinite (Fig. 2.1). The average spacing between breaks λ or equivalently the frequency of breaks per unit length $h_o = 1/\lambda$ characterizes the statistical distribution. The random distribution of points on a line is described by Poisson statistics.

If an arbitrary length l of the line is examined then the probability of finding n points (fractures) within the length l is given by,

$$P\left(n,l\right)=\frac{\left(l/\lambda\right)^{n}e^{-l/\lambda}}{n!}\ . \tag{2.1}$$

The most probable distribution in fragment lengths is determined by observing that the probability of finding no fractures within the length l is,

$$P\left(0,l\right)=e^{-l/\lambda}\ , \tag{2.2}$$

while the probability of finding one fracture within the subsequent length increment dl is

$$P\left(1,dl\right)=\left(1/\lambda\right)dl\ . \tag{2.3}$$

The probability of occurrence of fragments of length l within a tolerance of increment dl is then,

$$f\left(l\right)dl=P\left(0,l\right)P\left(1,dl\right)=\left(1/\lambda\right)e^{-l/\lambda}dl\ , \tag{2.4}$$

where,

$$f\left(l\right)=\left(1/\lambda\right)e^{-l/\lambda}\ , \tag{2.5}$$

is the fragment length probability density distribution while the integral of $f(l)$,

$$F(l) = 1 - e^{-l/\lambda} , \tag{2.6}$$

is the cumulative fragment distribution.

If a distribution of N_o fragments satisfies the present statistical premises, then the analytic expression,

$$N(l) = N_o e^{-l/\lambda} , \tag{2.7}$$

characterizes the cumulative number distribution of fragments larger than length l. Assigning a density per unit length of the one-dimensional body it is readily shown that the cumulative mass fraction of fragments is given by,

$$M(l) = 1 - (1 + l/\lambda) e^{-l/\lambda} . \tag{2.8}$$

The latter is commonly a more tractable experimental description.

Equation (2.7) can be written in the differential form,

$$\frac{dN}{N} = -\frac{1}{\lambda} dl , \tag{2.9}$$

providing a useful form for generalizing to fragmentation events in which the distribution is biased toward specific fragment sizes. This is accomplished through a dependence of the distribution length scale $\lambda = \lambda(l)$ on the fragment size.

2.1.1 Binomial Distribution

When the number of breaks within the body length L is few then the fragment size probability distribution will depend on the body length. Here probabilistic aspects of the problem are governed by the binomial probability function,

$$P_{j,k}(p) = \frac{k!}{j!(k-j)!} p^j (1-p)^{k-j} , \tag{2.10}$$

when, $P_{j,k}(p)$ is the probability of j successes in k attempts while p is the probability of a single success.

Consider then a one-dimensional body of length L in which $n-1$ randomly distributed breaks partition the body into n fragments.

Consider further a region of length, $l < L$ within the domain of L. The probability of a single fracture occurring within the region is the ratio $p = l/L$. Thus, from the binomial probability function, the probability that none of the $n-1$ fractures occurs within the region l is just,

$$\begin{aligned} P_{0,n-1}(l/L) &= \frac{(n-1)!}{0!(n-1)!} \left(\frac{l}{L}\right)^0 \left(1 - \frac{l}{L}\right)^{n-1} , \\ &= (1 - l/L)^{n-1} . \end{aligned} \tag{2.11}$$

Given that the $n-1$ fractures are outside of the region l the probability that a single fracture occurs within the interval dl is the ratio $p = dl/\,(L-l)$. Again, from (2.10).

$$P_{1,n-1}\left(\frac{dl}{L-1}\right) = \frac{(n-1)!}{1!\,(n-2)!}\left(\frac{dl}{L-l}\right)^{1}\left(1-\frac{dl}{L-l}\right)^{n-2} , \qquad (2.12)$$
$$\cong \frac{n-1}{L}\left(\frac{dl}{1-l/L}\right) .$$

The probability of finding a fragment of length l within an interval dl is then the product of (2.11) and (2.12), or

$$f\,(l)\,dl = \frac{n-1}{L}\left(1-\frac{l}{L}\right)^{n-2} dl , \qquad (2.13)$$

where $f\,(l)$ is the fragment length probability density distribution. The cumulative probability distribution is then,

$$F\,(l) = 1-(1-l/L)^{n-1} . \qquad (2.14)$$

With the probability density function from (2.13) the expected value for the fragment length is found to be $\lambda = L/n$. Equation (2.14) can then be written,

$$F\,(l) = 1-e^{-(1-L/\lambda)ln(1-l/L)} , \qquad (2.15)$$

which, in the limit $\lambda \ll L$ and $l \ll L$ yields the cumulative fragment probability distribution for a Poisson process on an infinite line in (2.6).

Probability density curves for number of fragments equal to $n = 2, 3, 4$, and 5 are illustrated in Fig. 2.2, along with the Poisson approximation to the $n = 5$ fragments case.

2.2 Mott-Linfoot Fragment Distribution

Mott and Linfoot (1943) referenced the earlier work of Lineau (1936) and furthered his random geometric fragmentation ideas in pursuit of a sensible fragment size distribution relation for the description of fragmenting munitions. Their acceptance of the Lineau approach was bolstered by fragmenting munitions data available to them at the time which were found to plot reasonably linear in a log number versus cube root of the fragment mass representation. Since $m^{1/3}$ is proportional to a length measure of the fragment they reasoned that the same random variable considered in the Lineau one-dimensional development applied in the multidimensional fragmentation event. Further, in examining fragments from the available data, they observed that a substantial portion retained inner and outer surfaces of the original munitions case. This suggested that the fragmentation of a plate or areal region in which event

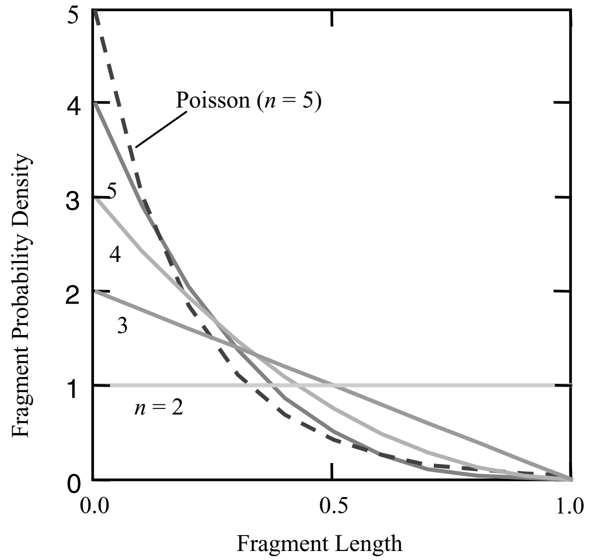

Fig. 2.2. Illustrates fragment probability distributions for fragmentation of unit length body into $n = 2$, 3, 4 and 5 fragments. *Dashed line* shows the Poisson distribution approximation to the $n = 5$ fragments case

the appropriate length scale would be proportional to $m^{1/2}$. Thus, a plot of log number versus $m^{1/2}$ should, by the reasoning given, provide a better fit to the fragment distribution data. In notation consistent with the development of the Lineau distribution in the preceding section, the fragment cumulative probability distribution proposed by Mott and Linfoot (1943) would be,

$$F(m) = 1 - e^{-(m/\mu)^{1/2}} , \tag{2.16}$$

where the characteristic mass μ is the distribution scale parameter. The corresponding probability density distribution is then,

$$f(m) = \frac{1}{2\mu}\left(\frac{m}{\mu}\right)^{-1/2} e^{-(m/\mu)^{1/2}} . \tag{2.17}$$

This distribution in various forms has been successfully used by numerous researchers over the past six decades to organize and compare vast amounts of exploding munitions fragmentation data. Mott expended considerable subsequent effort in a quest to justify the functional form assumed in (2.16) and (2.17).

2.2.1 Random Lines Fragmentation

In these initial efforts to justify their distribution Mott and Linfoot (1943) pursued a very reasonable geometric model. They considered the statistical partitioning of a surface by the random disposition of vertical and horizontal lines. The spacing of lines in the two orientations was assumed to be independently governed by the Lineau distribution. Thus,

$$f_x\left(x\right)=\frac{1}{x_o}e^{-x/x_o}\ , \tag{2.18}$$

and

$$f_y\left(y\right)=\frac{1}{y_o}e^{-y/y_o}\ , \tag{2.19}$$

where the average spacing or frequency of lines in the vertical and horizontal direction was allowed to differ. It is not difficult to see that this geometric model might sensibly replicate the statistical behavior of an exploding munition. The random lines correlate with observed longitudinal and transverse fractures, while the ratio x_o/y_o simulate the elongated nature or aspect ratio of exploding munitions fragments as illustrated in Fig. 2.3a.

We will subsequently show, as did Mott, that this distribution does not correspond well with the distribution in (2.16) and (2.17) (the Mott distribution) arrived at intuitively by Mott and Linfoot. In a later section, however, it will be shown that this geometric algorithm, when effectively generalized,

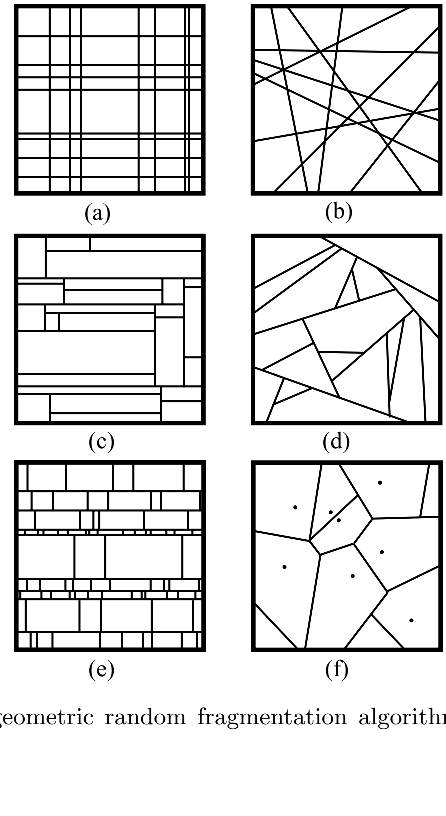

Fig. 2.3. Various geometric random fragmentation algorithms explored by Mott and others

quite nicely approximates the statistical representation of the biaxial fragmentation of expanding shells.

The probability density distribution over fragment length and width is then provided by a juxtaposition of (2.18) and (2.19),

$$f\left(x,y\right)=\frac{1}{x_o y_o}e^{-x/x_o-y/y_o} \ . \tag{2.20}$$

Mott and Linfoot (1943) then proceeded to solve for the distribution in fragment size through the following approach: Let $z=\sqrt{xy}$, where xy is the fragment area, provide a measure of the fragment size. The cumulative distribution for fragments of size larger than z is then provided by the integral expression,

$$1-F\left(z\right)=\iint\limits_{xy>z^2}\frac{1}{x_o y_o}e^{-x/x_o-y/y_o}dxdy \ . \tag{2.21}$$

The double integral over area is written,

$$\frac{1}{x_o y_o}\int\limits_0^\infty e^{-x/x_o}\left[\int\limits_{z^2/x}^\infty e^{-y/y_o}dy\right]dx \ , \tag{2.22}$$

which readily reduces to,

$$\frac{1}{x_o}\int\limits_0^\infty e^{-\frac{1}{x_o}\left(x+\frac{x_o}{y_o}\frac{z^2}{x}\right)}dx \ . \tag{2.23}$$

With the change of variable,

$$x=z\sqrt{\frac{x_o}{y_o}}\eta \ , \tag{2.24}$$

the integral becomes,

$$\frac{z}{\sqrt{x_o y_o}}\int\limits_0^\infty e^{-\frac{z}{\sqrt{x_o y_o}}\left(\eta+\frac{1}{\eta}\right)}d\eta \ . \tag{2.25}$$

Introducing the characteristic length $z_o=\sqrt{x_o y_o}$ and making the further change of variable $\eta=e^\theta$ yields,

$$\frac{z}{z_o}\int\limits_{-\infty}^\infty e^{-\frac{z}{z_o}\left(e^\theta+e^{-\theta}\right)}e^\theta d\theta \ , \tag{2.26}$$

which in turn transforms to the integral of the hyperbolic function,

$$2\frac{z}{z_o}\int_0^\infty e^{-2\frac{z}{z_o}\cosh\theta}\cosh\theta d\theta \ . \tag{2.27}$$

The solution of the integral was recognized by Mott and Linfoot as a modified Bessel function. Integral solutions for modified Bessel functions of integer order (Abramowitz and Stegun, 1954) provides,

$$K_n\left(u\right) = \int_0^\infty e^{-u\cosh\theta}\cosh n\theta d\theta \ , \tag{2.28}$$

for the modified Bessel function of order n. Thus, we arrive at the cumulative probability distribution over fragment size z,

$$F\left(z\right) = 1 - 2\frac{z}{z_o}K_1\left(2z/z_o\right) \ . \tag{2.29}$$

The probability density distribution follows directly from $dF\left(z\right)/dz = f\left(z\right)$ and the modified Bessel function relation (Abramowitz and Stegun, 1954),

$$\frac{d}{du}\left(uK_1\left(u\right)\right) = -uK_o\left(u\right) \ , \tag{2.30}$$

or,

$$f\left(z\right) = 4\frac{z}{z_o^2}K_o\left(2z/z_o\right) \ . \tag{2.31}$$

An alternative solution method is instructive. Again, start with (2.20) for the probability density distribution over fragment length and width. A transformation to a probability distribution $g\left(a,r\right)$ over the fragment area,

$$a = xy \ , \tag{2.32}$$

and the fragment aspect ratio,

$$r = x/y \ , \tag{2.33}$$

is sought.

The differential invariant,

$$f\left(x,y\right)dxdy = g\left(a,r\right)dadr \ ,$$

leads to

$$dxdy = \left|\frac{\partial\left(x,y\right)}{\partial\left(a,r\right)}\right| dadr \ ,$$

for the differential element through the transformation Jacobian (Buck, 1965). The transformed probability density function is then,

$$g\left(a,r\right) = f\left(x\left(a,r\right),y\left(a,r\right)\right)\left|\frac{\partial\left(x,y\right)}{\partial\left(a,r\right)}\right| \ . \tag{2.34}$$

Calculating the Jacobian through (2.32) and (2.33),

$$\left|\frac{\partial(x,y)}{\partial(a,r)}\right| = \frac{1}{2}r^{-1} , \tag{2.35}$$

yields,

$$g(a,r) = \frac{1}{2x_o y_o}\frac{1}{r}e^{-\left(\frac{1}{x_o}\sqrt{ar}+\frac{1}{y_o}\sqrt{a/r}\right)} , \tag{2.36}$$

for the probability density distribution in fragment area and aspect ratio.

To obtain the probability density distribution over area $h(a)$, irrespective of aspect ratio, integrate over all r,

$$h(a) = \frac{1}{2x_o y_o}\int_0^\infty \frac{1}{r}e^{-\left(\frac{1}{x_o}\sqrt{ar}+\frac{1}{y_o}\sqrt{a/r}\right)}dr . \tag{2.37}$$

Changing the integration variable through $r = (x_o/y_o)e^{2\eta}$ gives,

$$h(a) = \frac{2}{a_o}\int_0^\infty e^{-2\sqrt{\frac{a}{a_o}}\cosh\eta}d\eta , \tag{2.38}$$

where $a_o = x_o y_o$. The general integral relation for the modified Bessel function of (2.28) yields,

$$h(a) = \frac{2}{a_o}K_o\left(2\sqrt{a/a_o}\right) . \tag{2.39}$$

The present distribution function over fragment area is equivalent to that of Mott and Linfoot in (2.31) if the transformation $a = z^2$ is performed.

The cumulative distribution over fragment area $H(a)$ is readily obtained through the integration of (2.39) and the relation $K_1'(u) = -K_o(u)$ (Abramowitz and Stegun, 1954),

$$H(a) = 1 - 2\sqrt{a/a_o}K_1\left(2\sqrt{a/a_o}\right) . \tag{2.40}$$

The density distribution in (2.36) can be pursued further to provide the probability density function over aspect ratio $k(r)$ irrespective of fragment size. The integral over fragment area,

$$k(r) = \frac{1}{2x_o y_o}\int_0^\infty \frac{1}{r}e^{-\left(\frac{\sqrt{r}}{x_o}+\frac{1}{y_o\sqrt{r}}\right)\sqrt{a}}da , \tag{2.41}$$

through the transformation,

$$\xi = \left(\frac{\sqrt{r}}{x_o}+\frac{1}{(y_o\sqrt{r})}\right)\sqrt{a} ,$$

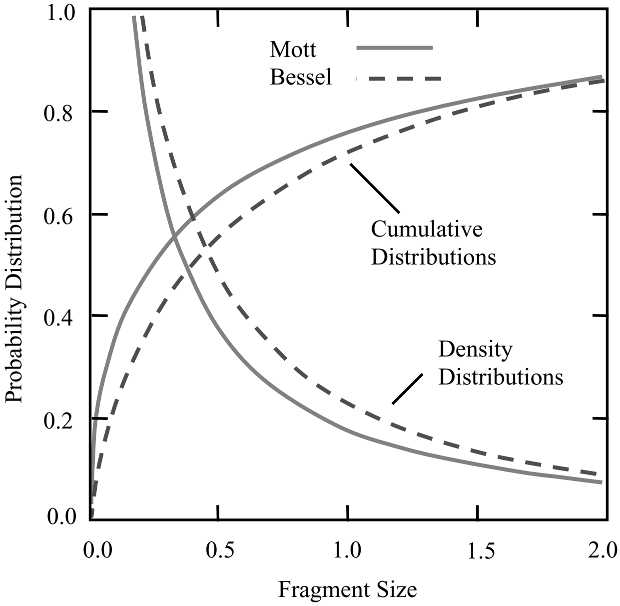

Fig. 2.4. A comparison of the Mott distribution and the Bessel fragment size (area) distribution resulting from the random Lineau placement of vertical and horizontal lines on the surface

yields,

$$k(r) = \frac{1}{r_o} \frac{1}{(1 + r/r_o)^2} \int_0^\infty \xi e^{-\xi} d\xi \ , \tag{2.42}$$

where $r_o = x_o/y_o$. Thus,

$$k(r) = \frac{1}{r_o} \frac{1}{(1 + r/r_o)^2} \ , \tag{2.43}$$

provides the probability density distribution over fragment aspect ratio.

The probability distribution (both density and cumulative) for the random orthogonal lines geometric fragmentation problem is compared with the Mott distribution in Fig. 2.4 with both distributions normalized to unity. This latter distribution is discussed further in a subsequent chapter. The comparison reveals differences, as was noted by Mott and Linfoot, which failed to provide the justification sought by them. A calculated distribution variance (second moment about the mean) of five for the Mott distribution (2.17) significantly exceeds a variance of three calculated for the Bessel distribution in (2.39). The Mott distribution would predict a larger number of both small and large fragments.

Mott and Linfoot then pursued the geometric fragmentation problem in which orientation as well as placement of lines on the area was a random variable as illustrated in Fig. 2.3b. They were unable, however, to determine the size distribution for this fragmentation algorithm except in the small fragment limit, which did agree with the proposed dependence of fragment number proportional to the square root of fragment area. Computer solutions of this geometric fragmentation algorithm [Grady and Kipp, 1985] suggest reasonable agreement with the Mott distribution over the full range of fragment sizes.

By considering the geometric problem of randomly oriented vertical and horizontal lines, and then the extension to randomly oriented lines as shown in Fig. 2.3b, Mott and Linfoot tacitly observe that the generated fragment size distribution would probably depend on the algorithm chosen to randomly partition the area. This algorithm dependence of geometric fragmentation methods will be pursued later.

2.2.2 Cylindrical Segmentation Fragmentation

First, however, it is of interest to outline the final geometric fragmentation algorithm pursued by Mott before this line of study was dropped by him. If the fragment size distribution generated in a random geometric fragmentation process is dependent on the fragmentation algorithm, as is becoming apparent, then an algorithm which most closely replicates the event of interest might be expected to better approximate the statistical features sought. Indeed, the present algorithm reasonably approximates the longitudinal fractures and subsequent circumferential breakup observed in a munition fragmentation event. This proximity to the problem of concern most likely guided Mott in selecting this final geometric fragmentation process for study.

This final algorithm is illustrated in Fig. 2.3e. The method consists of first inscribing randomly positioned horizontal lines, and then segmenting each horizontal strip with randomly positioned vertical lines where the average spacing within any strip is proportional to the width of that strip. In pursuing the size distribution solution to this problem, Mott also changed the functional form governing the random placement of horizontal lines and vertical line segments. We will here, however, proceed one step at a time and assume that the Lineau distribution governs the placement of the line and line segments as in the preceding exercise.

It is found in carrying through the solution for the fragment size distribution for this geometric algorithm, with a Lineau distribution of vertical lines and horizontal line segments, that the analysis is not tractable and that the resulting distribution does not converge. Mott must have also observed this difficulty and the observation may have motivated his selection of a distribution function different than the Lineau form. It is also possible that this selection was not arbitrary, but was motivated by physical ideas emerging from the more physically-based theories he was starting to pursue.

In any case, in pursuing the solution to this alternative geometric fragmentation algorithm posed by Mott, it becomes apparent that the probability density distribution for fragments over the areal region cannot simply be obtained by a juxtaposition of the two linear distributions as was done in the previous analysis. It will initially be necessary to work with number distributions because of difficulties in normalizing the probability distribution. Note first that given a probability density distribution in fragment lengths $f(x)$, the total length dL of fragments of length x within increment dx is just,

$$dL = xdN = N_o x f(x)\, dx \ , \tag{2.44}$$

or

$$L = N_o \int_0^\infty x f(x)\, dx = N_o \langle x \rangle \ , \tag{2.45}$$

where $\langle x \rangle$ is the expected value of x and N_o is the total fragment number. For the Lineau distribution as written in (2.18) the expected value is just $\langle x \rangle = L/N_o = x_o$.

In the geometric fragmentation algorithm illustrated in Fig. 2.3e the region is assumed to be of equal height and width L. Consider one strip of width y. The number of segments (fragments) within this one strip of length x, within increment dx, is just,

$$dN_x = N_{xo} \frac{1}{x_o} e^{-x/x_o} dx = \frac{L}{x_o^2} e^{-x/x_o} dx \ , \tag{2.46}$$

where the Lineau distribution in fragment lengths is assumed. Correspondingly, the number of strips of width, y within increment dy is

$$dN_y = \frac{L}{y_o^2} e^{-y/y_o} dy \ . \tag{2.47}$$

Thus, the number of fragments of length x and width y per unit area (setting $L^2 = 1$) is just the product,

$$dN = dN_x dN_y = \frac{1}{x_o^2 y_o^2} e^{-x/x_o - y/y_o} dx dy \ . \tag{2.48}$$

At this point Mott supplemented the geometric fragmentation algorithm with the assumption that within a strip of width y the average fragment length was proportional to y or,

$$x_o = py \ . \tag{2.49}$$

Mott suggested that the constant p was approximately 5 based on munitions fragments that he had the opportunity to inspect. The present assumption was clearly motivated by observation of fragments from a cylindrical munition fragmentation event in which the nature of the breakup leads to an abundance of elongated fragments.

The resulting fragment number distribution is accordingly,

$$dN = \frac{1}{p^2 y_o^2 y^2} e^{-y/y_o - x/py} dx dy \ . \tag{2.50}$$

We again introduce the change of variables,

$$a = xy, \quad r = x/y \ , \tag{2.51}$$

with Jacobian,

$$\left| \frac{\partial (x, y)}{\partial (a, r)} \right| = 1/2r \ , \tag{2.52}$$

leading to the number distribution,

$$dN = \frac{1}{2p^2 y_o^2} \frac{1}{a} e^{-\sqrt{\frac{a}{r y_o^2}} - r/p} da \ dr \ , \tag{2.53}$$

over fragment area a and aspect ratio r. The number distribution over fragment area is then the integral,

$$\frac{dN}{da} = n(a) = \frac{1}{2p^2 y_o^2} \frac{1}{a} \int_0^\infty e^{-\sqrt{\frac{a}{r y_o^2}} - r/p} dr \ , \tag{2.54}$$

or, with the variable change $r = p\eta$,

$$n(a) = \frac{1}{2p y_o^2} \frac{1}{a} \int_0^\infty e^{-\sqrt{\frac{a}{p y_o^2}} \frac{1}{\sqrt{\eta}} - \eta} d\eta \ . \tag{2.55}$$

Unfortunately, the integral within (2.55) is not finite. Cursory examination of the distribution shows an unbounded number density distribution as both area a and aspect ratio r become small. Thus, this very natural statistical fragmentation geometry, when combined with the Lineau (Poisson) placement of the fractures, leads to an ill-defined fragment distribution. Although not explicitly discussed in his reports Mott must have tread this path and encountered the same difficulty. Undaunted, he proposed a novel solution resulting in an analytically regular fragment size distribution for the geometric fragmentation problem addressed above. Some further background is necessary, however, to fully appreciate the approach he pursued. Mott's treatment of this final geometric fragmentation algorithm will be revisited in Sect. 2.5.

2.3 Poisson Fragment Distribution and Statistical Heterogeneity

Mott and Linfoot proposed a representation for the statistical distribution of fragment sizes resulting from a munitions fragmentation event which was independent of any specific fragmentation process, geometric or otherwise. Namely, that a measure of the fragment size (proportional to the square root of the fragment area) was distributed over fragment number according to the Poisson process put forth by Lineau (1936). They then pursued analytic methods to justify the assumed fragment distribution, an approach that served Mott well throughout the course of his theoretical career. As illustrated in the previous section, the statistical geometric methods were not fully successful in validating the proposed distribution.

Others have pursued alternative statistical assumptions regarding the distribution of fragment sizes and have similarly undertaken efforts to justify their assumptions. Here we consider the approach of Grady and Kipp (1985) as it parallels the fragmentation assumptions and geometric statistics justification attempts of Mott and Linfoot (1943). The comparison more starkly brings out features and weaknesses of the approach.

2.3.1 Grady–Kipp Postulate

Within the intervening years since the seminal study of Mott, considerable opportunity has risen to test the distribution of Mott and Linfoot. Although the linear dependence of the logarithm of fragment number against the square root of fragment mass proposed by Mott and Linfoot has in many comparisons been quite satisfactory, there have also been examples of obvious discrepancy. For example, munitions fragment data have been obtained which plot linear in log number versus cube root of fragment mass. This dependence has of course been suggested to apply to thick-walled munitions in which the preponderance of fragments are of size smaller than the wall thickness, as is tacit in the initial development of Mott and Linfoot. Other disparities between the Mott-Linfoot distribution and munitions fragmentation data have also been observed.

Grady and Kipp (1985) have offered an alternative development and explanation for the distributions in fragment sizes observed in munitions fragmentation. They first suggest that if such fragmentation can be represented by mechanism-independent statistical descriptions that perhaps fragment mass, as opposed to fragment size (either $m^{1/2}$ or $m^{1/3}$ in the Mott-Linfoot development), is the more appropriate random variable. They then propose that the mass of the fragment is distributed over fragment number according to a Poisson (or binomial if the fragment number is small) process, which parallels the development of Lineau in Sect. 2.1.

Thus, if the fragment mass is viewed as a random scalar variable, then the random fragmentation of the mass is analogous to the one-dimensional Lineau problem. Fragmentation is determined by breaks distributed randomly over the scalar measure of mass. The breaks determine a Poisson variate and lead to a cumulative fragment probability distribution,

$$F(m) = 1 - e^{-m/\mu} \, , \tag{2.56}$$

and density distribution,

$$f(m) = \frac{1}{\mu} e^{-m/\mu} \, . \tag{2.57}$$

In contrast to the Mott distribution, the present distribution keeps the same linear exponential functional form for both area and volume fragmentation.

2.3.2 Sequential Segmentation

Grady and Kipp (1985) also pursued justification of their fragment distribution relations through geometric fragmentation methods. With the availability of computational resources they were not restricted to geometries with analytic solutions. The algorithms pursued by them are illustrated in Figs. 2.3c and 2.3d. The method is as follows: A point was selected at random on the unit area. Then a random vertical or horizontal direction in Fig. 2.3c or random arbitrary direction in Fig. 2.3d was determined and a line drawn through the point and terminated at the area boundary. A second point was randomly selected and a random line again drawn bisecting the sub area within which the point fell. This process was sequentially repeated until the desired intensity of fragmentation was achieved.

It was found that distributions from both the horizontal and vertical lines, and randomly oriented lines, sequential segmentation geometric processes converged to the linear-exponential distributions in (2.56) and (2.57) with sufficient numbers of fragments for the geometric fragmentation of an area. With some reflection, it is recognized that this geometric algorithm is replicating the Poisson partitioning of a scalar area or volume. Thus, the agreement is expected.

The linear exponential (Poisson) density and cumulative distribution is shown in Fig. 2.5 and differs markedly from the Mott distribution. The much broader Mott distribution has a variance a factor of five larger than the Poisson distribution.

2.3.3 Statistical Heterogeneity

Considering the substantial difference between the exponential distribution and the Mott distribution in Fig. 2.5, and the historic success of the latter in describing munitions fragment distribution data, one may question how the exponential distribution can be offered as a viable representation. Grady and Kipp (1985) provide the following argument in support of the exponential distribution.

In the statistical fragmentation problems considered up to this point, statistical homogeneity over the fragmented region was tacitly assumed. Namely, the average fragment size did not vary from point to point within the region of consideration. In application, uniform or homogeneous fragmentation is usually not achieved. Normally, due to complexity of the device geometry and dynamic loading, the intensity of fracture will vary throughout the body and, correspondingly, the average fragment size will also be a function of position. A uniformly expanding ring or the uniform expansion of a spherical shell are unique experimental geometries in which nearly homogeneous fragmentation

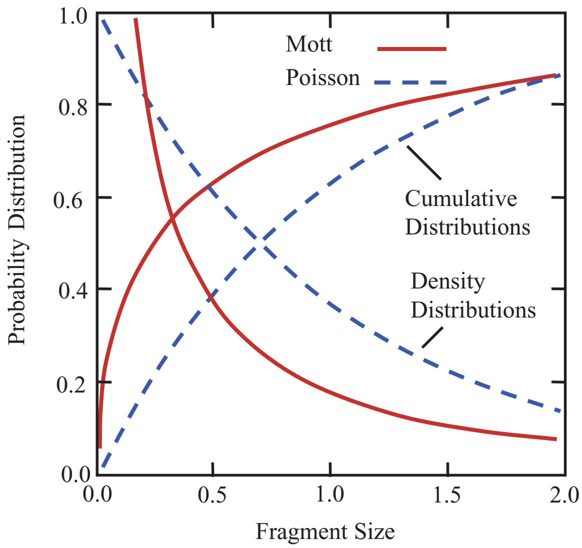

Fig. 2.5. A comparison of the Mott distribution and the Exponential, or Poisson, fragment size (area) distribution resulting from the random segmentation of the surface

is achieved. Most experimental geometries will lead to statistically inhomogeneous fragmentation. This concept was considered by Lineau (1936), but was not pursued.

Additionally, there is some evidence that fracture mechanisms may markedly differ for different parts of the fragment size distribution. This was briefly suggested by Mott for the fragmentation of exploding cylinders and has been pursued in more detail by Odintsov (1992). This possibility will be considered further later in this section.

The linear exponential distribution based on a Poisson process over a scalar mass region,

$$f(m) = \frac{1}{\mu} e^{-m/\mu} , \tag{2.58}$$

proposed by Grady and Kipp (1985), assumed statistical homogeneity with average mass μ constant over the region of interest. A second distribution with a different average fragment size could be described equally well with a distribution of the form of (2.58). A mixing of the two distributions would not be characterized by a linear exponential distribution. The distribution would, rather, be represented by the bi-linear form,

$$f(m) = \frac{g_1}{\mu_1} e^{-m/\mu_1} + \frac{g_2}{\mu_2} e^{-m/\mu_2} , \tag{2.59}$$

where g_1 and g_2 are the number fractions of the respective homogeneous distributions, while μ_1 and μ_2 are the corresponding average fragment masses.

More generally, any statistically inhomogeneous distribution could be approximated with a Poisson mixture [Puri and Goldie, 1979],

$$f(m) = \sum_{1}^{n} \frac{g_i}{\mu_i} e^{-m/\mu_i} \ . \tag{2.60}$$

It can be shown that any Poisson mixture representation of a fragment distribution will have a larger variance than a statistically homogeneous linear exponential representation of that same distribution. It is instructive to compare, for example, the bi-linear distribution from (2.59) with the Mott distribution. Normalizing the Mott distribution to the average fragment mass $x = m/\mu$,

$$f(x) = \frac{1}{\sqrt{2x}} e^{-\sqrt{2x}} \ , \tag{2.61}$$

and similarly the bi-linear distribution with $x = m/\mu$ and $\mu = g_1\mu_1 + g_2\mu_2$,

$$f(x) = \frac{g_1}{\alpha_1} e^{-x/\alpha_1} + \frac{g_2}{\alpha_2} e^{-x/\alpha_2} \ , \tag{2.62}$$

where $\alpha_1 = \mu_1/\mu$ and $\alpha_2 = \mu_2/\mu$.

Constrain the integral of the distribution and the first moment to unity in (2.62),

$$g_1 + g_2 = 1 \ , \tag{2.63}$$

$$\alpha_1 g_1 + \alpha_2 g_2 = 1 \ . \tag{2.64}$$

The second and third distribution moments (equivalently the distribution variance and skewness) can also be equated to the corresponding moments for the Mott distribution yielding,

$$2\left(g_1\alpha_1^2 + g_2\alpha_2^2\right) = 6 \ , \tag{2.65}$$

$$6\left(g_1\alpha_1^3 + g_2\alpha_2^3\right) = 90 \ , \tag{2.66}$$

uniquely constraining the four constants in the bi-linear distribution. A comparison of the Mott and bi-linear distributions (actually complementary cumulative distributions) with identical distribution moments is shown in Fig. 2.6. Although significant visual differences are observed, the bi-linear distribution does start to capture the important features of the Mott distribution. The distributions are compared in a semi-logarithmic representation in which a single exponential (Poisson) distribution plots linear. Better agreement can, of course, be achieved as more terms are included in the Poisson mixture representation.

A mixture of Weibull distributions (referred to as a hyper Weibull distribution) has been proposed by Odintsov (1992) of the form

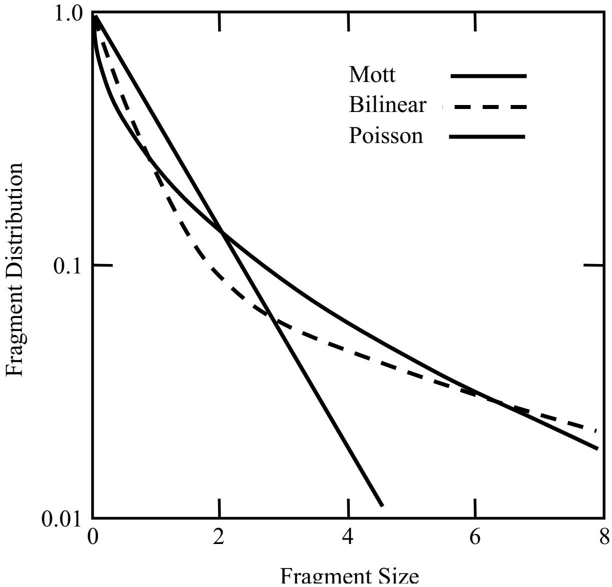

Fig. 2.6. Comparison of the Mott and Poisson fragment size complementary cumulative distribution with a bilinear (Poisson mixture) distribution approximation

$$f(m) = \sum_{1}^{n} g_i \frac{n_i}{\mu_i} \left(\frac{m}{\mu_i} \right)^{n_i - 1} e^{-(m/\mu_i)^{n_i}} , \tag{2.67}$$

with mean fragment size,

$$\mu = \sum_{1}^{n} g_i \mu_i \Gamma(1 + 1/n_i) . \tag{2.68}$$

This distribution, of course, reduces to the Poisson (hyper exponential) mixture provided in (2.60) when the n_i for each distribution component is set to unity. The latter specialized mixture was pursued in some detail be Odintsov.

The issue emphasized in the present section, however, is the statistically inhomogeneous character of experimental fragment distributions. Attempts to represent such distributions with analytic forms developed from homogeneous statistical fragmentation models will be at best approximate. Also, the theoretical logical inconsistencies are not fully satisfying. The introduction of Poisson mixtures to describe statistically inhomogeneous distributions is inherently reasonable. Acceptance of a Poisson (linear exponential) distribution as the homogeneous basis function as has been proposed [Grady and Kipp, 1985; Odintsov, 1992], has not been fully justified, and is open to criticism as later developments will illustrate.

2.3.4 Multimodal Distributions

The power of the mixture distribution representation is illustrated in the description of multimodal fragment distributions as emphasized by Odintsov (1992). Commonly the mass spectra of fragments over size is desired and is obtained from the probability distribution through,

$$dM = mdN = mN_o f(m)dm \; . \tag{2.69}$$

Let $\varphi(m) = dM/dm$ so that,

$$\varphi(m) = \frac{1}{\mu} m f(m) \; , \tag{2.70}$$

for the distribution of the mass of the fragment over the fragment size (or weight) m and with $\mu = 1/N_o$.

For a bilinear Poisson mixture normalized to a fragment size scale of $\mu = 1$ as in (2.62), the mass distribution in (2.70) becomes,

$$\varphi(x) = x\left(\frac{g_1}{\alpha_1} e^{-x/\alpha_1} + \frac{g_2}{\alpha_2} e^{-x/\alpha_2}\right) \; . \tag{2.71}$$

Two distributions are plotted from (2.71) and are shown in Fig. 2.7 for different values of the distribution parameters. In both distributions the number ratio is the same at $g_1/g_2 = 1$. When the size ratio is not large ($\alpha_1/\alpha_2 = 4$

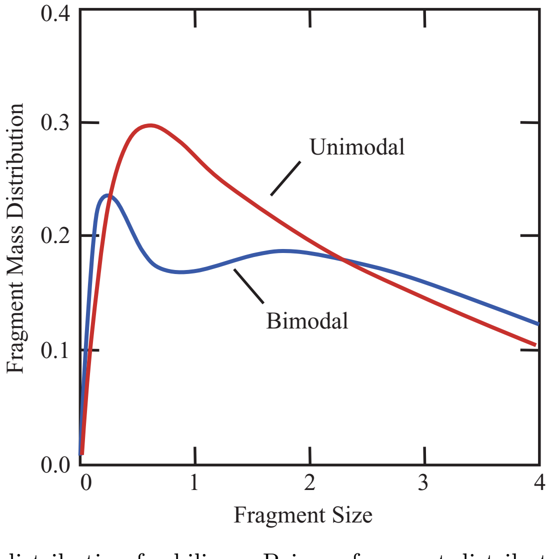

Fig. 2.7. Mass distribution for bilinear Poisson fragment distribution mixtures for selected distribution parameters illustrating both unimodal and bimodel character

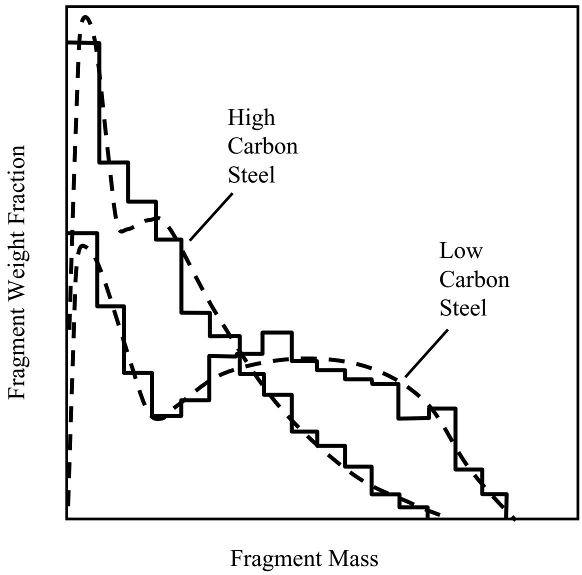

Fig. 2.8. Fragment mass distributions for explosive fragmentation of low carbon and high carbon content steel cylinders and associated bilinear exponential distribution fit in uncalibrated units [Odintsov, 1992]

in Fig. 2.7) the distribution is unimodal. As the size ratio increases (or correspondingly decreases), however, a mode separation is observed in the distribution yielding a distinct bimodal distribution ($\alpha_1/\alpha_2 = 10$ in Fig. 2.7).

Odintsov (1992) has reported detailed fragment distribution properties from explosion-induced natural fragmentation experiments on low-carbon and high-carbon steel cylinders. Histogram distributions for one low-carbon and one high-carbon steel tests are plotted in Fig. 2.8 in uncalibrated units. Bilinear curve fits to the data by Odintsov are also shown. The distribution for the low-carbon steel is distinctly bimodal where as that of high-carbon steel is nearly unimodal.

Odintsov attributes the bimodal character of the distributions to two distinct populations of fragments with possibly distinct fracture mechanisms. The first population is composed of the larger fragments created by through-the-thickness fractures, which retain sections of both the inner and outer surfaces of the original cylinder. The second population is composed of the smaller angular shards created by fracture intersections, near either the inner surface (shear fracture dominated) or the outer surface (tensile fracture dominated). These failure modes were also noted by Mott.

The more brittle high-carbon steel is dominated by fragments from the second population and consequently is nearly unimodal in character. The more ductile low-carbon steel has sensible contributions from both fragment populations leading to the observed bimodal nature of the distribution.

2.4 Voronoi-Dirichlet Fragment Distribution

The present discussions of random geometric fragmentation would be remiss without consideration of the Voronoi-Dirichlet construction [*e.g.*, Boots and Murdoch, 1983]. This method for the random partitioning of space has received by far the lion's share of attention in a much broader spectrum of literature. The resulting distributions have been proposed for such applications as the distribution of galactic matter throughout the universe [Kiang, 1966] and the formation of geologic columnar structures such as the Giant's Causeway in Northern Ireland [Weaire and Rivier, 1984] to name but a few.

The construction algorithm in two dimensions is illustrated in Fig. 2.3f. As in the Grady-Kipp construction, the method begins with a random (statistically homogeneous) distribution of points on the surface (or within the volume if three-dimensional space is considered). Space is then randomly partitioned by construction of perpendicular bisecting lines (or surfaces) as illustrated. On a regular (periodic) lattice of points the same process creates the Wigner-Seitz cells used, for example, in the construction of Brillouin zones in solid state physics [Kittel, 1971]. The space is also randomly partitioned through the reciprocal, or dual, Delauney construction [Watson, 1981] created through the joining, with lines (or surfaces), the points in each Voronoi-Dirichlet cell.

Analytic relations for the fragment size distributions resulting from the Voronoi-Dirichlet construction have not been directly determined. A computational determination of the resulting fragment size distributions has been widely pursued, however [e.g., Crain, 1978], and an analytic expression which successfully reproduce the computational distributions has been arrived at by intuitive means [Kiang, 1966].

Both the analytic distributions and the process of developing them are of interest to the present pursuit of statistical fracture through geometric means. First Kiang (1966) considered the one-dimensional Voronoi-Dirichlet construction, where points are distributed at random on a line (a Poisson process), and then the degenerate perpendicular bisector (the midpoint) of each point pair is determined. Thus, the Voronoi-Dirichlet distribution on a line is the dual of the Lineau distribution considered earlier (or the degenerate Delauney distribution). Whereas, in the Lineau distribution random points on the line were considered as breaks or fractures, in the present Voronoi-Dirichlet distribution these same random points constitute in some sense the centroid of fragments with fractures occurring at the bisector points.

2.4.1 One-Dimensional Voronoi–Dirichlet Distribution

The fragment size distribution for the one-dimensional Voronoi-Dirichlet distribution can be determined directly as follows. The probability of finding a length l between a Poisson point pair is given by the Lineau distribution,

$$f(l)dl = \frac{1}{\lambda}e^{-l/\lambda} \, . \tag{2.72}$$

The probability of finding a point pair of length l_1 adjacent to a point pair of length l_2 is then the product,

$$f(l_1)f(l_2)dl_1dl_2 = \frac{1}{\lambda^2}e^{-(l_1+l_2)/\lambda}dl_1dl_2 \ . \tag{2.73}$$

Implementing the transformation,

$$L = (l_1 + l_2)/2 \ , \tag{2.74}$$

$$\xi = (l_1 - l_2)/2 \ , \tag{2.75}$$

leads to the distribution,

$$f(L) = \frac{1}{\lambda^2}\int_{-L}^{L} e^{-2L/\lambda}d\xi \ , \tag{2.76}$$

where L is the length between midpoints of the point pairs. Integration provide the Voronoi-Dirichlet distribution of fragments on a line,

$$f(L) = \frac{2}{\lambda}\left(\frac{2L}{\lambda}\right)e^{-2L/\lambda} \ . \tag{2.77}$$

Comparison of the one-dimensional Voronoi distribution (2.77) and the Lineau, or Poisson, distribution is provided in Fig. 2.9.

2.4.2 Two and Three Dimensional Analytic Distributions

The distribution in (2.77) is a gamma function of order $n = 2$. Kiang (1966) offered without proof that symmetrically higher order gamma functions would provide analytic fragment distributions for Voronoi-Dirichlet partitioning of an area or a volume. Following Kiang we will write the general expression for the fragment distribution over mass,

$$f(m) = \frac{1}{\mu}\frac{n}{\Gamma(n)}\left(\frac{nm}{\mu}\right)^{n-1} e^{-nm/\mu} \ , \tag{2.78}$$

where $n = 2, 4$ or 6 for a line, surface or volume fragmentation, respectively.

Computational distributions from Voronoi-Dirichlet constructions on an area performed by Kiang (1966) were in acceptable agreement with (2.78) for $n = 4$. A degree of controversy was generated by Kiang's proposal among subsequent authors as to the adequacy of (2.78); both for and against. Apparently the construction of computer algorithms to generate Voronoi-Dirichlet fragment distributions is not a trivial exercise. In any case, for the present geometric fragmentation investigations, (2.78) is an adequate analytic representation of Voronoi-Dirichlet distributions in line and area fragmentation.

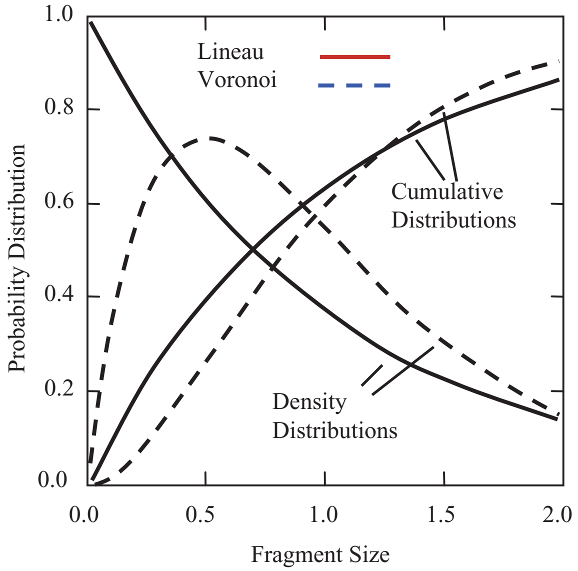

Fig. 2.9. A comparison of the one-dimensional Voronoi distribution and the Lineau, or Poisson, fragment length distribution resulting from the random segmentation of the line according to the respective algorithms

Fragment size (area) distributions resulting from both the Voronoi algorithm and the sequential segmentation algorithm (Poisson distribution) are compared with the Mott distribution from (2.17) in Fig. 2.10. The three density distributions are normalized to unit expected value. The comparisons reveal the stark differences resulting from differing randomization algorithms and differ markedly from the proposed distribution of Mott.

2.5 Mott Cylinder Segmentation Algorithm

Mott undertook one final attempt at justifying through geometric methods the proposed $m^{1/2}$ distribution. The approach was explored earlier in Sect. 2.2. From the elongated and sliver-shaped fragments recovered from exploding munitions tests Mott surmised that fracture in an end-detonated metal shell with cylindrical symmetry would occur through longitudinal running cracks with occasional crack branching and crack intersection resulting in the observed fragments. Thus he proposed the statistical algorithm illustrated in Fig. 2.3e and analyzed earlier with the Lineau distribution of the lines and line segments. To randomly distribute the longitudinal and transverse line features on the plane he made the interesting selection of the one-dimensional Voronoi-Dirichlet distribution discussed in the previous section rather than the Lineau distribution used in his earlier geometric pursuits. He may have made this

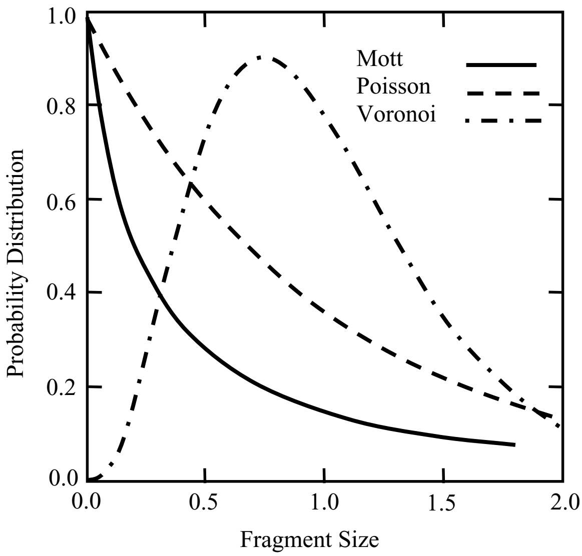

Fig. 2.10. Comparison of the Mott, Poisson and Voronoi distributions for the random fragmentation of an area

choice for one of two reasons. It is possible that he carried through the analysis using the Lineau distribution, as was attempted earlier in this chapter, and found (as was shown in the earlier section) that a solution could not be obtained. Alternatively, ideas to emerge in his later work, and to be discussed in the next chapter, may have influenced this selection: namely, that the physics of fracture interaction precludes the close proximity of parallel fracture, and thus limits the number of smaller fragments. The Voronoi-Dirichlet distribution is observed to better provide a statistical constraint limiting the number of the close parallel fractures and hence the number of smaller fragments.

Following the methods outlined previously, but using the Voronoi-Dirichlet distribution from (2.77) to determine the random placement of longitudinal lines and transverse line segments, the following size distribution over fragment area is obtained,

$$f(a) = \frac{2}{a_o}\frac{1}{\sqrt{4a/a_o}}\int_0^\infty \left(\xi\sqrt{4a/a_o}-1\right)\left(1+1/\xi^2\right)e^{-\xi\sqrt{4a/a_o}-1/\xi^2}d\xi\ . \quad (2.79)$$

This relation corresponds to the distribution provided by Mott and differs only in the distribution variable $\lambda = \sqrt{a/a_o}$ used by him where $a_o = py_o^2$. The corresponding cumulative distribution is then,

$$F(a) = \sqrt{4a/a_o}\int_0^\infty (1+1/\xi^2)e^{-\xi\sqrt{4a/a_o}-1/\xi^2}d\xi\ . \quad (2.80)$$

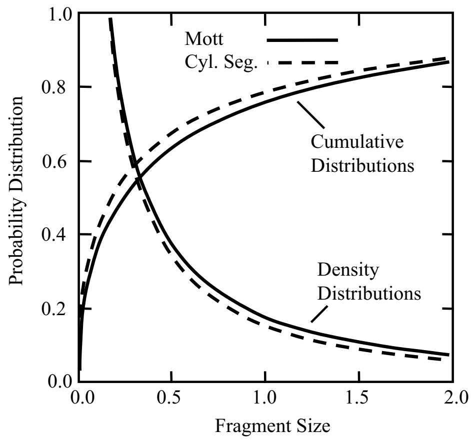

Fig. 2.11. A comparison of the cumulative and density fragment size distributions from the Mott distribution and the geometric cylinder segmentation distribution

The plot in Fig. 2.11 compares the Mott distribution with the cylindrical segmentation algorithm proposed by Mott using the Voronoi algorithm for randomly distributing the partitioning lines and line segments. This figure corresponds to Fig. 2.4 in which the Mott distribution is compared with the random vertical and horizontal lines algorithm. A final comparison is shown in Fig. 2.12 in which the density distributions resulting from both algorithms considered by Mott are compared with the Mott distribution over a wider

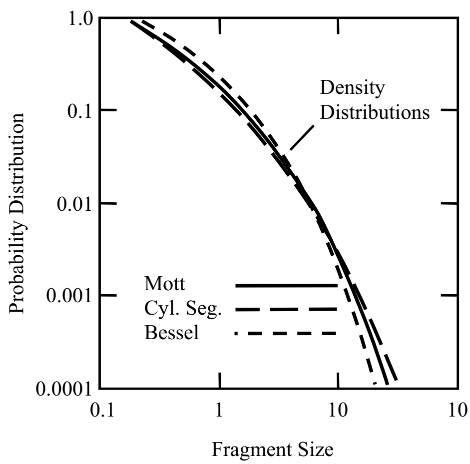

Fig. 2.12. A comparison of the probability density fragment distributions from the Mott distribution, the geometric cylinder segmentation distribution, and the geometric random horizontal and vertical lines (Bessel) distribution. Approximately 95% of the fragment area (mass) is included in the range of the plotted distributions

spectrum of fragment sizes. It is interesting that the distributions from the two algorithms tend to straddle the Mott-Linfoot proposed distribution, each with respectively larger and smaller variance. It is unlikely that anything else can be said.

At this point a degree of healthy suspicion as to the applicability of random geometric fragmentation algorithms to actual physical fragmentation phenomena should be embraced. Later, it will be shown that some utility of these methods can be made use of in modeling the statistical fragmentation phenomena, but they should be employed only with a sensible understanding of the underlying physics.

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