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A Theory of the Fragmentation of Shells and Bombs

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Summary. In a recent report on this subject[1], a tentative theory was put forward to account for the sizes of the fragments obtained from steel projectiles. In a further note[2], the theory was compared with the observed fragmentation of service shells. In this report an attempt is made to extend and to improve the theory, as far as is possible without a satisfactory theory of rupture in metals, which does not exist at present.

Before discussing the theory of fragmentation in Part II of this paper we shall give in Part I a summary of the information available about the velocities, weights and shapes of fragments and the mechanisms by which the explosive transfers its energy to them. We shall confine ourselves as far as possible to cylindrical projectiles of uniform diameter, both internal and external; shells with conical cavities are obviously less suitable for the deduction of theoretical conclusions. The rocket head is particularly suitable from this point of view, as is also the German 88 mm shell, and a special British 3.7″ shell recently fragmented by C.S.A.R., Millersford.

PART I

3.1 Expansion of the Casing

It is well known that steel casings expand considerably before rupture; this can be seen most clearly by examining the larger fragments which contain part of the inner and outer surfaces; the case has become thinner by an amount which varies very little from one fragment to another[3]. The present author has examined fragments from the following projectiles which have a uniform case

[1] A Theory of Fragmentation, by N.F. Mott and E.H. Linfoot, D.S.R. Extra-Rural Report A.C. 3348

[2] A.O.R.G. Memo. No. 24. “Fragmentation of H.E. Shells; a theoretical formula for the distribution of weights of fragments”

[3] Report R.C. 282 from Dept. of Metallurgy, University of Sheffield.

thickness: A German 88 mm shell, a special British 3.7″ shell with cylindrical cavity, and three rocket shells fragmented in the Safety in Mines Research Station, Buxton. The filling was TNT in each case; the results are as follows:

Table 3.1.

Type of Shell	Carbon content of shell %	External diameter (mm)	Thickness of Casing (mm)	Thickness of Fragment (mm)	Extension %
German AA 88 mm	0.7	88	15	11.8	27
British A.A. 3.7″ (cylindrical cavity)	0.4–0.5	94	16.5	12.8	30
Service A.A. rocket head	0.4–0.5	85	6.75	4.5	50
Thick cased rocket head	0.4–0.5	85	12.8	9	42
Thick cased rocket head	0.15	85	12.8	8.5	50

Further evidence is available from photographic records of the explosions of model bombs obtained at the Safety in Mines Research Station, Buxton[4]). According to these, model bombs 2″ in dia. with mild steel casings filled with tetryl expanded by the following amounts before breaking up:

Thickness of case (inches)	Expansion (%)
0.125	67
0.30	100

The result obtained that the thicker cased bomb expands further may however be due to end effects; it is not confirmed by the two rocket heads in Table 3.1.

3.2 Fragment Velocities

A theoretical treatment of the expansion of the casing of a long cylindrical cased charge of TNT has been given by G.I. Taylor[5]. Apart from the unknown end effect at the base of the shell, his results should be applicable to the nose-fuzed projectiles considered here.

According to Taylor the velocity of the casing can be expressed by the following formula:

$$V = V_o \, d_2 \big/ \sqrt{d_1^2 - d_2^2} \qquad (3.1)$$

[4] Report R.C. 236 from the Safety in Mines Research Station.
[5] Report to M. of H. S. No. R.C. 193.

where d_1, d_2 are the external and internal diameters of the casing before expansion, and V_o is given for different degrees of expansion in Table 3.2 Actual velocities calculated for certain shells are also given:

Table 3.2.

	Velocities in ft/sec				
% Expansion	11	30	67	124	200
V_o	2000	2400	2700	3000	3100
V (88 mm shell)	1750	2100	2400	2700	2800
V (3″ U.P.)	2750	3300	3700	4100	4250

These figures neglect the work done in deforming the case; assuming a constant[6] resistance to elongation T_o (poundals/sq.ft) and a density ρ for the steel, a short calculation gives for the reduction in velocity due to this cause

$$\delta V = \frac{T_o}{\rho V} \log(1 + \varepsilon) \tag{3.2}$$

Assuming T_o to be 30 tons/sq.in., we obtain the following values:

Table 3.3.

% Expansion	11	30	67	124	200
V (88 mm shell)	1700	2000	2300	2500	2550

The work done against the plastic forces does not decrease the fragment velocity appreciably, except perhaps for projectiles of very low charge-weight ratio (A.P. shells). The work done in rupturing the case is probably quite negligible.

It cannot be assumed that the fragments are projected from the shell with the velocity of the casing at the moment of break-up; the following observations show this:

(1) According to (unpublished) results obtained at Buxton, model bombs of similar dimensions made of steel and cast iron give fragments of about the same velocity. The cast iron gives very fine fragmentation and probably breaks up without plastic expansion.
(2) By grooving the charge, controlled fragments can be obtained of a desired size from U.P. casings. These fragments do not show thinning, but have

[6] In steels the resistance is, of course, not constant, but increases somewhat as the metal hardens.

the original thickness of the case. The case must therefore have broken before expansion. Nevertheless the velocity of the fragments is appreciably the same as for the normal shell without a grooved charge (unpublished results with model bomb).

Both these results show that the explosive must continue to exert pressure on the fragments after break-up, and up to about 20 or 30% expansion the pressure cannot depend much on whether the case has broken or not.

Evidence about fragment velocities is contradictory; at Buxton all fragments from a given model bomb are found to have approximately the same speed, except for a few very small ones of high velocity, probably acquired from the expanding gases after break-up; at Millersford, on the other hand, whilst most of the fragments from shells of the 88 mm or 3.7″ type have fragments with speeds in the range 2000−2500 ft/sec., there are a considerable number with much lower speeds down to 1000 ft/sec., and thus with speeds less than the calculated velocity of the casing before breakup. The origin of these is unexplained.

Photographic measurements of the velocity with which the casing of a model bomb expands have been made at Buxton; surprisingly enough, the velocity of the case comes out in one case to be *greater* than that of the fragments[7].

In view of these contradictory results we shall take theoretical values for the velocities of the casing, calculated as in Table 3.3; these agree at any rate as regards order of magnitude with observed fragment velocities.

3.3 Types of Fragmentation Observed

The cross sections of the large fragments from a cylindrical shell are usually of one or other of the types shown in Fig. 3.1; on the outside of the case (along AB) the rupture is brittle, with shear rupture from B to C. Types 1 and 4 are the commonest, with small pieces of triangular cross section frequently shearing off (as in type 5 in Fig. 3.1).

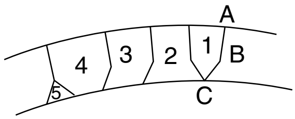

Fig. 3.1.

In some casings the rupture is by shear only, fragments of the types shown in Fig. 3.2 being observed. This has been observed both for mild steel and

[7] cf. Reference [4]; values given on pp. 2 and 5 for a model bomb with 0.018″ casing.

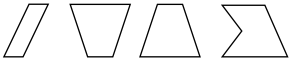

Fig. 3.2.

carbon steel casings. In the theories of part II we have limited ourselves to rupture which is at least partly brittle.

Fragments are commonly five to ten times as long as they are wide.

3.4 Weights of Fragments

The most usual classification is by weighing. The present writer has pointed out[8] that for many shells and bombs the weight distribution satisfies the following formula; the number of fragments with weights between m and $m+dm$ is equal to

$$Ce^{-M/M_A}dM, \qquad M = m^{1/2} \tag{3.3}$$

where C, M_A are constants. Since C depends on the total weight of the casing, the fineness of the fragmentation is given by a single parameter M_A. Apart from any theoretical significance of formula (3.3), it provides a convenient practical method of comparing the fragmentation of different projectiles.

Using ${M_A}^2$ as a measure of the mean fragment weight, the following factors appear to affect it in the following ways:

1. *Type of Steel*: there is little evidence that the tensile strength or yield point affects the fragmentation, but the carbon content certainly does. Thus two similar projectiles, the German 88 mm and the British 5.7″ shell give the following values of M_A:

	Diameter (mm)	Thickness (mm)	Steel, carbon %	M_A (ounce)$^{1/2}$
88 mm shell	88	15	0.7	0.19
3.7″ shell	94	16.5	0.4–5	0.36

   Also 3.25″ rocket heads of carbon (0.4%) and mild steels, thickness 0.5 gave the following values of M_A:

Carbon	0.30 (ounce)$^{1/2}$
Mild steel	0.33

2. *Calibre of shell*: for given charge-weight a big shell undoubtedly gives bigger fragments. For example, values of M_A for a large and for a small shell of similar capacities are (U.B. Proc. 21099 and 21051)

[8] Reference [1].

		M_A
95 mm shell Amatol 50/50		0.26
5.5″ (80 lb. shell) (140 mm Amatol 50/50		0.46

3. *Charge-weight ratio*: this affects both the thickness of the casing and its velocity at the moment of break-up. That the velocity at the moment of break-up has a profound influence on the fragmentation is shown by two facts:
   (a) That a 250 lb. bomb fragmented in water gives only about a quarter as many fragments as when exploded in air[9].
   (b) The well known gross fragmentation of that part of an H.E. shell with direct acting fuze which is in contact with the ground at the moment of explosion; large pieces can be picked from the crater.

   Apart from its influence on the velocity, a thin casing will of course give thinner fragments than a thick one. Whether it affects the other dimensions will be discussed below.

   Values of M_A for two otherwise similar rocket heads with thicknesses 0.265″ and 0.5″ are [2]

Thickness	0.263″	0.5″
M_A	0.134	0.255 (ounce)$^{1/2}$

The velocity of expansion could be altered at will without affecting the size or thickness of the casing by putting a lead covering round the outside of the shell. Experiments to determine the effect of this or the fragmentation would be of great interest. The pressure distribution within the case would also be altered (cf. Sect. 3.11).

3.5 Dimensions of Fragments

The primary process in fragmentation must be splitting parallel to the axis of the shell, with subsequent rupture at the ends, and production of secondary fragments of type 5 in Fig. 3.1. Assuming that cracking (e.g. along BC in Fig. 3.3) precedes shear rupture (e.g. along CD), the first task of any theory of fragmentation must be to account for the distance AB in Fig. 3.3 between the edges of the average fragment. The observed distributions of the breadth AB are shown in Figs. I and II at the end of this paper[10]. It is of course true

[9] Compilation of data on Trials on Explosive Effects of Aircraft Bombs. R.D. Woolwich, 1938

[10] In this report Mott included hand drawn sketches within the text identified as Arabic numbered figures as well as graphs appended at the end of the text identified as Roman numbered figures. The four graphs are identified as Figs. I, II, III and IV in this transcription. This identification agrees with the original with the exception of the present Fig. IV. Further author's notes will attempt to clarify this apparent miss-numbering in the original.

that the length AB often varies considerably along the length of a fragment, and a visual estimate of the mean breadth is subject to error; nevertheless the general shape of the curves is significant. We plot against fragment breadth not the total number of fragments, but the total length of all fragments (placed end to end) in each category.

The following points will be noted:

(a) The rather sharp cut-off for large breadths.
(b) The much narrower fragments obtained with the German 88 mm shell (0.7% carbon steel) than with the British 3.7″ shell or thick cased rocket head, (0.45% steel but similar diameter and casing thickness).
(c) The narrower fragments obtained with the thin cased (high capacity) rocket head than with the thick cased projectiles of similar steel.

The lengths of fragments from the German 88 mm shell are shown in Fig. III; the curve does not show the same cut-off at high values. In Fig. IV we show the length distribution for fragments of different breadths; there is obviously a rough correlation, broad fragments being longer[11]. The average length of fragments in different categories is given in Table 3.4.

Table 3.4.

	Lengths in mm							
Breadth (mm)	2–3	4	5	6	7	8	9	10
Thick-cased U.P. (carbon steel)		39	39	56	44	37	50	36
Thick-cased U.P. (mild steel)		34	35	33	36	47	58	54
Service U.P.			27	30	28	29		
German 88 mm	5.5	10	14.8		21.7			

Evidence for correlation between breadth and length is not marked except for the German shell. For the British shells a ratio of length to breadth of the order 5 seems to be normal, for the German shell a somewhat smaller value.

3.6 Weight Distribution of Fragments

The formula (3.3) was derived by the author[1] on the assumption of some sort of random break-up; Figs. 3.1 to 3.3 show however that neither the break-up parallel or perpendicular to the axis can be considered random as would be

[11] Although Mott refers to Figs. III and IV discussion in this paragraph is clearly covered by the data in Fig. III.

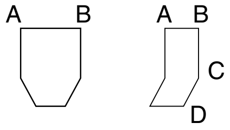

Fig. 3.3.

the case if the breadths were distributed according to the law: number of fragments with breadths between a and $a + da$ is preportional to $\exp(-a/a_o)da$. It therefore seems worth while to attempt a derivation of (3.3) from different assumptions.

Let us assume:

(a) that the casing is broken into strips and that the number of strips with breadths between x and $x + dx$ is

$$Cx \ \exp(-x/x_o) \ dx \tag{3.4}$$

This does not represent the facts exactly, but gives a nearer approximation than the random fracture.

(b) that each strip is broken up according to the same law, and that the average length of fragment is proportional to the thickness x of the strip. Thus from a strip of length l the number of fragments of length between y and $y + dy$ is

$$ye^{-y/px} \ l \, dy/(px)^3 \tag{3.5}$$

where p is a factor (of the order 5).

Then the number of fragments of area greater than a^2 is

$$\frac{Cl}{p^3} \int\!\!\int_{xy\,>a^2} \frac{y}{x^2} \left[\exp\left(-\frac{x}{x_o} - \frac{y}{px}\right)\right] dx\, dy$$

This reduces to

$$\text{const}\, \lambda \int_0^\infty \left(1 + \frac{1}{z^2}\right) \ \exp\left(-\lambda z - \frac{1}{z^2}\right) \ dz, \quad \lambda = a/x_o \ p^{1/2}$$

and thus the number of fragments with area such that $a(= \sqrt{\text{area}})$ lies between a and $a + da$ is

$$\text{const}\ f(\lambda) d\lambda$$

where

$$f(\lambda) = \int_0^\infty \left\{\left(1 + \frac{1}{z^2}\right) - \lambda\left(z + \frac{1}{z}\right)\right\} \exp\left(-\lambda z - \frac{1}{z^2}\right) dz \tag{3.6}$$

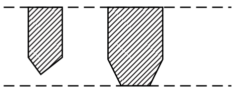

Fig. 3.4.

This function is plotted logarithmically in Fig. V over all values of λ from 0 to 10, i.e. over a range of $\log_{10} f$ equal to 3, which is about the range over which the fragment distribution is usually plotted. It will be seen that the deviation from a straight line is not very large[12].

Weight distributions of actual fragments are likely to deviate from this theoretical curve for the following reason : the narrower fragments frequently break as shown in Fig. 3.4, thus having a smaller depth than they should. Moreover the removal of the triangular pieces from the base of the smaller fragments will obviously make a greater proportional difference to their weight. This will result in a shift of the whole upper part of the curve in Fig. V somewhat to the left. On the other hand, on reaching the weight categories of the small triangular fragments, a large number of new fragments appear which are not included in the analysis given above. Thus the curve should appear as the dotted curve in Fig. V, which is very similar to those observed.

PART II

THEORY OF THE MEAN FRAGMENT SIZE

3.7 Dependence on Velocity

We consider that the fragmentation will be determined by the properties of the casing at the moment of break-up, and will not depend, for instance, on the pressures to which the case has been subjected during the expansion. The factors that may be of importance are thus

(a) Properties of the steel at the moment of rupture − for example the true ultimate tensile strength rather than the yield point.
(b) The rate of increase of plastic strain; this is equal to V/r, where V is the velocity of the case and r its radius.
(c) The thickness of the casing.
(d) The pressure of the explosive at the moment of break-up; according to Taylor's calculations this is from 60–25 tons/sq. in. for casings that break up after a 25 to 50% expansion; this is much less than the initial pressure, which is of the order 1000 tons/sq. in.

[12] A Fig. V does not appear in the original graphs however discussions in this paragraph clearly refer to the upper plot in the present Fig. IV.

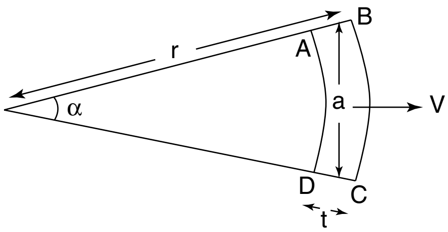

Fig. 3.5.

The theory which we shall develop suggests that (c) and (d) are of minor importance in determining the breadths and lengths of fragments. As in the author's previous report, we take the point of view that it is the kinetic energy of the case which tears it to pieces; the fragmentation would be almost the same if the expanding explosive could be miraculously removed just before the case broke up, leaving it to fly into pieces under its own momentum.

In the author's previous report[1] the following derivation of the fragment breadth was given. Suppose that ABCD in Fig. 3.5 is the cross section of a fragment which has just broken along BA, CD. The fragment is still in a state of plastic flow, the rate of increase of plastic strain being V/r. The kinetic energy of this flow of metal is

$$\frac{1}{2}\rho t\, V^2 \int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} r\theta^2 d\theta = \frac{1}{24} V^2\, t\rho a^3/r^2$$

It was argued that if this were greater than the energy Wt required to rupture the metal, the fragment would split in half. Thus the value

$$a = \left[\frac{24r^2W}{\rho V^2}\right]^{1/3} \tag{3.7}$$

would give an upper limit to the possible breadth of a fragment.

Agreement with observation, i.e. values of a of the order 1 cm, was obtained with values of W given by the notched bar impact test for a brittle steel, i.e. 40 ft/lbs. per sq. inch.[13] Since W occurs only as $W^{1/3}$, the values obtained are not very sensitive to W.

[13] Measurements were made at the N.P.L. of the Izod value of test pieces cut from a 3.7″ H. E. shell casing which had been extended 20% in the direction originally circumferential to the shell, to represent the state of the steel at the moment of rupture; values obtained for specimens with the usual 10 × 8 mm section at the notch were, for the energy absorbed to fracture

5.0 5.9 5.0 ft. lbs.

This gives 45 ft. lbs/sq. inch. (Ref. Eng. Dept/OYY/RE/B. 104 A, 5.3.43).

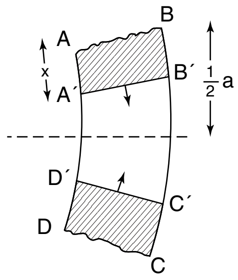

Fig. 3.6.

Equation (3.7) will certainly give a lower limit to the maximum fragment breadth, if W is the true fracture energy under the conditions existing in an H.E. shell. It is doubtful however if this bears any relation to the energy expended in the notched bar test, most of which is probably due to plastic deformation of the metal in the neighbourhood of the notch until the formation of a true crack of atomic width at its apex, leading to brittle rupture. The actual work necessary to separate two planes of atoms in a metal is of course much less, of the order 10^{-3} ft. lbs/sq. inch.

We shall therefore attempt a theory of fragmentation based on the assumption that the energy of fracture is negligible. In addition we shall make the following assumption: fracture can start at any one of a number of places on the surface or in the body of the casing, and once started will rapidly spread across it. During the initial stages of the expansion, it is very unlikely (or even impossible) that a crack will start anywhere; as the expansion increases the chance of a crack forming in any part of the case increases. We introduce a function $f(s)\,ds\,dx$, which gives the chance that a crack will form on a length dx of the circumference of the casing as the strain increases from s to $s + ds$. We may take $f(s)$ to be zero up to a certain value of s (the rupture point), or we may assume a very rapid increase of $f(s)$ in the neighbourhood of the rupture point. We shall find that the form of $f(s)$ determines the mean fragment size.

As before we consider a fragment that has just broken along the lines AB, CD (Fig. 3.6), and ask whether it is likely to break again. As soon as a fracture has formed along AB, for instance, the metal in the neighbourhood of AB will stop flowing. A boundary A′B′ between the part of the metal which is still in plastic flow and the metal which has stopped flowing will move downwards with a velocity that can be calculated. It will soon reach the boundary C′D′ moving upwards from the lower crack; when this has happened, no further crack can form. For a fragment of average width, therefore, the chance of a new crack forming before A′B′ and C′D′ meet each other must be small. This chance can be calculated by comparing the function $f(s)$, giving the rate of formation of cracks, with the time available before the surfaces join.

The velocity with which the surface A′B′ moves can be determined as follows, if we assume that this is small compared with the velocity of sound in steel: Let a be the breadth of the fragment, x the breadth of the part that has stopped flowing, and T_o the stress required to cause plastic flow. The velocity upwards of all material above A′B′ is then

$$\frac{V}{r}\left(\frac{1}{2}a - x\right)$$

Therefore the equation of motion of the block ABB′A′ is

$$T_o = -\rho x \frac{d}{dt}\left\{\left(\frac{1}{2}a - x\right)\frac{V}{r}\right\}$$

which gives

$$T_o = \frac{\rho V}{r} x \frac{dx}{dt} \tag{3.8}$$

Thus

$$\frac{1}{2}x^2/t = rT_o/\rho V \ , \tag{3.9}$$

and the time which the fragment takes to stop expanding is

$$a^2\rho V/8\,T_o r \ ,$$

which is of order 10^{-6} secs. if a ~7 mm The increase in the strain s of the material during this time is of the order 10^{-2}.

From (3.9) we find

$$\dot{x} = T_o r/\rho V x \sim 6 \times 10^4/x \text{ cm/sec}$$

so the velocity, except for very thin fragments, is considerably less than that of sound in steel (5×10^5 cm/sec.).

We have now to make some assumption about the function $f(s)$. We could assume alternatively that.

(a) $f(s)$ is zero up to a definite value s_o (the rupture point) and is then constant and equal to f_o, say.
(b) $f(s)$ is zero up to s_o, and then increases, as $c(s - s_o)^n$ say.
(c) $f(s)$ is never zero, but increases rapidly in the neighbourhood of the rupture point, as $Ae^{\gamma s}$ say.

The hypothesis (c) is the most attractive, for reasons that will be given in the next section; but they all lead to somewhat similar conclusions about the fragmentation.

An idea of the order of magnitude of the constants involved can be obtained from the behaviour of steel in tensile tests, if we make the assumption that the behaviour in static tests is similar to that at high rates of strain. In tensile tests, steels nearly always fracture after necking; the reduction of area thus gives a measure of the strain at which fracture occurs. Specimens prepared from one sample of carbon steel show a certain seatter in the measured values of the reduction of area; thus, if a steel fractures on the average for a reduction of area of 50%, individual specimens will show values between 49 and 51 approximately. The following, for instance, are values[14] for a normalised 0.4% carbon steel:

$$58 \quad 56\frac{1}{2} \quad 59\frac{1}{2} \quad 59 \quad \text{per cent}$$

Now according to our assumptions, the chance that a specimen of length l will fracture before the strain reaches a value s is

$$1 - \exp\left[l \int_0^s f(s)ds\right] \tag{3.10}$$

In case (a) this gives

$$1 - \exp\left[-f_o l(s - s_o)\right] \tag{3.11}$$

and in case (c), to a sufficient approximation

$$1 - \exp\left[-\frac{lA}{\gamma} e^{\gamma s}\right] \tag{3.12}$$

Suppose that we assume that an increase in s by Δs increases the chance that fracture has taken place from 10 to 90%. Then we find from (3.11) and (3.12)

$$f_o l = 2.2/\Delta s \quad \text{(case a)}$$
$$\gamma = 3.1/\Delta s \quad \text{(case c)}$$

In case (a) it is not clear what value of l should be taken, since the maximum strain only occurs at the neck. In case (c), however, l does not occur in the formula for γ; if, in accordance with the experimental values given above, we take $\Delta s = 0.02$, we obtain

$$\gamma = 155$$

A plot of the functions (3.11) and (3.12), showing the chance that a fracture has occurred when the strain (reduction in area) is s, is given in Fig. VI for $lf_o = 100$ and for $\gamma = 150$. The origin of s for curve (c) is arbitrary[15].

Experiments on the extent to which the reduction in area at the breaking point fluctuates from specimen to specimen, carried out for a sufficiently large sample, would shed light on the nature of the function $f(s)$.

With any of these form of $f(s)$, an estimate of the order of magnitude of the breadth a can be made as follows: At each crack, after a time t, a breadth

[14] N.P.L. Report to A.R. Committee, Paper 4755

[15] A Fig. VI does not appear in the original graphs however discussions in this paragraph refer to the lower curves plotted in the present Fig. IV.

$$2\left(\frac{2T_o r}{V\rho}\right)^{1/2} t^{1/2}$$

has stopped expanding and is thus "safe" from cracking. Since the strain increases as Vt/r, when the strain has increased by Δs after the formation of a given crack, a breadth round it equal to

$$\beta(\Delta s)^{1/2}, \quad \beta = 2^{3/2}\left(\frac{T_o}{\rho}\right)^{1/2}\frac{r}{V}$$

is "safe" also. If we neglect the overlapping of "safe" areas, a proportion

$$\beta\int^{s} f(s')\sqrt{s-s'}\,ds'$$

is safe when the strain is s. When this approaches unity, the break up is complete. Thus a, the average breadth, is given as regards its order of magnitude, by eliminating s between

$$\beta\int^{s} f(s')\sqrt{s-s'}\,ds' \simeq 1 \tag{3.13}$$

$$\int^{s} f(s')\,ds' \simeq 1/a$$

With the forms for $f(s)$ suggested above we obtain the following:

(a) Equations (3.13) lead to

$$a = \left(\frac{2}{3}\right)^{2/3} f_o^{-1/3}\, 2\sqrt{\frac{T_o}{\rho}}\left(\frac{r}{V}\right)^{2/3}$$

This gives the same power of (r/V) as the author's previous theory, and with $f_o = 100\,\text{cm}^{-1}$, values of a of the order $0.5\,\text{cm}$

(b) Equation (3.13) give

$$a = (n+1)\left[\frac{\Gamma(n+1)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{5}{2}\right)}\right]^{1-\frac{1}{2n+3}} C^{\frac{-1}{2n+3}}\,\beta^{1-\frac{1}{2n+3}}$$

It will be seen that the fragment size is proportional to $(r/V)^{1-\frac{1}{2n+3}}$ and thus to some power of r/V between 1 and 2/3.

(c) With $f(s) = Ae^{\gamma s}$, the (3.13) give us

$$A\beta e^{\gamma s}\int_0^{\infty'} e^{-\gamma s'} s^{1/2}\,ds' = 1$$

$$Ae^{\gamma s}\int_0^{\infty} e^{-\gamma s'}\,ds' = 1/a$$

and hence

$$a = \sqrt{\frac{2\pi T_o}{\rho}\frac{1}{V}\frac{1}{\gamma^{1/2}}}$$

With $\gamma = 100, T_o = 60$ tons/sq. inch, this gives 0.7 cm for a normal shall of the calibres considered here.

It will be seen that a is now proportional to r/V.

Our formulae suggest, then, that the mean width of fragment will be proportional to

$$\text{const.}\ (r/V)^s ,$$

where s lies between 2/3 and 1, the constant will depend on the nature of the steel; it may depend on the thickness of the case and pressure of the explosive, but consideration of the next section suggests that it will not.

We have not been able to find an analytical expression for the number of fragments with breadth between a and $a + da$, but our equations for the break-up enable a distribution to be found graphically. We limit ourselves to the form (c) for $f(s)$. The theory is at present one-dimensional; we are considering the division of a line (a circumference of the shell.) by random fracture. Let l be the length of this line; then as before where each crack is formed, a space on each side of it equal to

$$\left(\frac{2T_o}{\rho}\right)^{1/2} \frac{r}{V}(\Delta s)^{1/2}$$

is safe from further cracking when s has increased by Δs. If N is the number of cracks already formed then the rate of increase of N is given by

$$\frac{dN}{dS} = Aple^{\gamma S}$$

where p the proportion of the line where cracks can still form. The first crack will form, on the average, when

$$Ale^{\gamma s}/\gamma = 1$$

If the value of s given by this equation be denoted by s_o, and a new variable σ defined by

$$\sigma = \gamma(s - s_o) ,$$

then the rate of increase in the number of cracks is given by the equation

$$\frac{dN}{d\sigma} = pe^{\sigma}$$

Also, if a crack is formed when $\sigma = \sigma_1$, the region round it where subsequent cracking is impossible is at any subsequent instant

Table 3.5. Values of γ deduced from observed distributions of fragment breadths

	88 mm shell	3.7″ shell	Rocket head* (thick case)	Rocket head (thin case)
x_o (cm) observed	0.37	0.56	0.44	0.31
$2r$ (cm)	11.4	12.2	12.8	12.8
V (cm/sec)	64,000	63,000	76,000	110,000
$\gamma = \frac{2T_o}{\rho}\left(\frac{r}{Vx_o}\right)^2$	230	105	125	124

* The values of x_o for 0.15 and 0.45% carbon are about the same.

$$2x_o(\sigma - \sigma_1)^{1/2} \tag{3.14}$$

where

$$x_o = \left(\frac{2T_o}{\rho\gamma}\right)^{1/2} \frac{r}{V}$$

A line drawn on paper can now be cut at random, using playing cards or dice. Initially σ is supposed to be zero; after each successive cut is made σ is supposed to increase by $d\sigma$ where

$$d\sigma = 1/pe^{\sigma}$$

After each new cut is made, the "safe" region round all cuts made earlier must be increased according to formula (3.14). Any arbitrary value of the ratio l/x_o may be taken. We took $l/x_o = 20$. The line is repeatedly cut until the whole region is "safe" from further cracking. The lengths of all intervals are then measured and recorded, and the process repeated a number of times until enough data are obtained to draw a histogram, in which the numbers of "fragments" (i.e. intervals) are plotted against their lengths. The results are shown in Fig. II(c)[16]. The similarity to the distributions of fragment breadths observed in Figs. I and II (a) and (b) is satisfactory.

By comparing Fig. V with the observed fragment distributions and especially the values of their upper limits, we have estimated in Table 3.5 the value of x_o for the projectiles investigated[17]. The values are not correct to more than $\pm 10\%$.

From these values we have attempted to deduce γ. For this we require the radius of the shell at the moment of break-up (r), the velocity of the casing and the true ultimate tensile strength, T_o. The two former quantities are deduced from the values given in Part I. To deduce T_o from a tensile test we require the stress at the moment of rupture at the base of the neck, which

[16] This theoretical curve is an inset identified by Mott as "(c) Theory" in the graph provided in the present Fig. II.

[17] Again Fig. V refer to the curves provided in the upper plot in the present Fig. IV.

is of course considerably greater that the U.T.S. given in engineering tables. For steels the following values are given by Korber and Rohland, (Mitt. d. K. Wilhelm Inst. f. Eisenforschung, 5 (1924) 55).

Carbon	Reduction in Area	True Ultimate Stress	
(%)	(%)	kg/mm^2	tons/sq.inch
0.13	70	78	51
0.25	63	80	52
0.45	57	82	53
0.55	50	87	57

These will probably be somewhat higher for high rates of strain;[18] we have thus assumed

$$\begin{aligned} T_o &= 80\,\text{tons/sq. inch} \\ &= 100 \quad '' \qquad '' \end{aligned}$$

for British (0.45% carbon) and German (0.7% carbon) shell steels respectively.

For the values of γ we cannot claim an accuracy greater than $\pm 30\%$; within these limits the British shells (0.45% carbon) show the same value, which is of the order expected. The German shell shows a higher value, which we assume to be due to the higher carbon content of the steel.

3.8 Dependence on Thickness and Pressure

We have seen that the hypothesis

$$f(s) = Ae^{\gamma s} \qquad \gamma \sim 100$$

fits the facts well both for the fragmentation of shells and for the consistency of the rupture point, and seems a priori more likely than the other hypotheses. We have now to consider the following points:

(a) Is γ likely to depend on the thickness of the casing, or the pressure of the gases at the moment of rupture?
(b) Why is γ larger for steels with high carbon content?
(c) Can we deduce a factor γ of this order from any known property of the metal?

It has not at present been possible to answer point (b); to the others an answer can be given:

Let us make the following assumptions about fracture in ductile metals:

[18] cf. G.I. Taylor, Stress Strain Relationship on Impact. Civil Defence Research Committee. R.C. 36.

(i) Cracks can start at a limited number of points or regions in the metal of which we assume that there are n per unit volume.
(ii) Cracks will start at these points, on the average, when the strain has increased to a value s_1
(iii) The strains at which cracks will form at the individual points of weakness show a certain scatter about the value s_1; it is natural to represent this scatter by a Gaussian distribution. We thus assume that the number of points per cm^3 at which a crack will form as the strain increases from s to $s + ds$ is

$$\frac{n}{s_2\sqrt{2\pi}} \exp\left[\frac{-(s-s_1)^2}{s_2^2}\right] ds \ ,$$

For a tensile specimen of cross sectional area A, this gives us for our function $f\,(s)$

$$f(s) = \frac{nA}{s_2\sqrt{2\pi}} \exp\left[\frac{-(s-s_1)^2}{s_2^2}\right] cm^{-1} \tag{3.15}$$

We are interested only in the tail end of this curve where $f(s)$ first becomes appreciable; let us then define the rupture point s_o as the strain for which one crack per cm is expected, so that

$$\int^{s_o} f(s)ds = 1 \ , \tag{3.16}$$

and write

$$s = s_o + s'$$

Then we obtain from (3.15)

$$f(s) \simeq \frac{nA}{s_2\sqrt{2\pi}} \exp\left[\frac{-(s_1-s_o)^2}{s_2}\right] e^{\gamma s'}$$

with

$$\gamma = 2(s_1 - s_o)/{s_2}^2 \tag{3.17}$$

Also from (3.16)

$$\frac{nAs_2}{2\sqrt{2\pi}(s_1-s_o)} \exp\left[\frac{-(s_1-s_o)^2}{s_2}\right] = 1 \ ,$$

whence

$$\left(\frac{s_1-s_o}{s_2}\right)^2 = \log_e\left[\frac{nAs_2}{2\sqrt{2\pi}(s_1-s_o)}\right] \tag{3.18}$$

Hence from (3.17) we obtain finally

$$\gamma = 2\log_e\left[\frac{nAs_2}{2\sqrt{2\pi}(s_1-s_o)}\right]/(s_1-s_o) \tag{3.19}$$

Since n comes within the logarithm, its exact value is not important. For a number of reasons we expect the distance between the points where rupture can start to be of the order 10^{-4} to 10^{-5} cm. This is for instance the distance between the slip bands[19] in a metal, the "dislocations" in G.I. Taylor's theory of slip,[20] or the "crystallites" whose existence has been suggested in cold worked metals.[21] We thus take n of the order 10^{15}; the other terms within the square bracket are negligible in comparison and we obtain

$$\begin{aligned} \gamma &= 2\log_e 10^{15}/(s_1 - s_o) \\ &= 69/(s_1 - s_o) \end{aligned} \tag{3.20}$$

From formulae (3.19), (3.20) we deduce:

(a) That γ is practically independent of the cross section of the specimen, and thus of the thickness of the shell casing.
(b) That γ is practically independent of the pressure of the explosive at the moment of rupture, because (cf. footnote 12) the pressure must vanish at the outside surface, and if the formation of cracks were confined to a small layer near the surface only, it would not affect γ appreciably.
(c) The properties of the steel affect the value of γ only through the value of $s_1 - s_o$, and if s_1 is of the order unity, as is not unlikely, values of γ in agreement with observation are obtained.

3.9 Lengths of Fragments

Up till this section we have discussed only the breadths of fragments, believing that splitting parallel to the axis is the primary process in fragmentation. We have now to discuss the factor determining their lengths.

Observation on fragments of marks cut on the surface of the case shows that shell casings do not stretch parallel to their axis; we must therefore look for an explanation of rupture at the ends of the fragments different from that given for the longitudinal cracks.

If cracks start at A and B and spread to the right, and from C and D and spread to the left, then as Professor Andrew[22] has pointed out, when the cracks bounding two fragments meet, there will be a tendency to split, as at E. According however to the hypothesis on which this paper is based, a split like this is only likely to take place if the steel between the cracks A

[19] cf. for example, Orowan, Nature, 147, 452 (1941) or the beautiful photographs of worked steel obtained with the electron microscope by Heidonreich and Peck, J. Applied Physics, 14, 24 (1943).
[20] Proc. Roy. Soc. A. 145, 362 (1934).
[21] Smith and Wood. Proc. Roy. Soc. A. 178, 93 (1941).
[22] Report R.C. 342 from the Dept. of Metallurgy of the University of Sheffield (31.8.42).

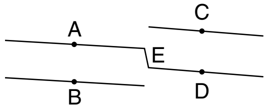

Fig. 3.7.

and B has already stopped flowing before the crack D reaches it; otherwise the crack D will be unaware of the presence of the cracks A and B and will penetrate between them. If however plastic flow has stopped, the different directions in which the two fragments are moving will, we consider, lead to their separation.

Let u be the velocity with which each crack extends. As soon as a crack has formed, the region spreads in which flow has stopped, so that after a time t its width a is given by

$$a = 2\left(\frac{2rT_o}{\rho V}\right)^{1/2} t^{1/2}$$

Thus a crack starting at 0 in Fig. 3.8 and which has spread to a length 2b is surrounded by a region bounded by two parabolas, in which plastic flow has stopped; the breadth PQ of this region is

$$2\left[\frac{2rT_o b}{\rho V u}\right]^{1/2}$$

As a rough criterion for the condition that the region between two cracks should be no longer in flow, we write a, the width of the crack, equal to half this;

$$a = \left[\frac{2rT_o b}{\rho V u}\right]^{1/2}$$

Thus the ratio, length to breadth, is equal to

$$\frac{2b}{a} = \left[\frac{\rho V u a}{rT_o}\right] \qquad (3.21)$$

According to (3.14), a for the average fragment is proportional to r/V; we obtain

$$\frac{2b}{a} = 2\sqrt{\frac{\pi\rho}{2T_o}}\,\frac{u}{\gamma^{1/2}} \qquad (3.22)$$

With $T_o = 60\,\text{tons/sq. inch} = 9 \times 10^9$ c.g.s. units, $\rho = 8, \gamma = 100$, this gives

$$2b/a = 0.7 \times 10^{-5} u$$

If we equate u to the velocity of sound in steel, 5×10^5 cm/sec., we obtain

$$2b/a \simeq 3.5$$

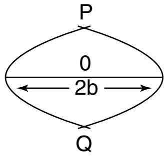

Fig. 3.8.

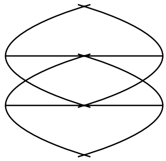

Fig. 3.9.

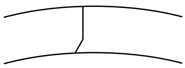

Fig. 3.10.

in fair agreement with experiment. The hypothesis that cracks spread with the speed of sound is not unlikely to be correct, if the atomic cohesion only has to be overcome, and no plastic deformation is involved.

Formula (3.18) suggests that the length/breadth ratio of the average fragment is independent of the calibre of capacity of the projectile, but will be less for the German high carbon steel (large γ) than for the British steel. These conclusions seem to be born out by the figures of Table 3.4.

3.10 Shape of Cross Section of Fragments

We have already remarked on the types of rupture observed, and pointed out that the type of rupture shown in Fig. 3.10 is usual, with a brittle crack on the outside of the casing and shear rupture at 45° on the inside. In this section we attempt an explanation of this double type of rupture. For this purpose we calculate the stresses in the case during plastic expansion.

According to G.I. Taylor's calculations, the pressure at various stages in the expansion of a long cylindrical cased charge are given by the following figures, where r is the radius of the inner surface of the case and r_o its initial value:
At the moment of break-up, therefore, the pressure is of the same order as the yield stress, and both will be of comparable importance in determining the stresses in the material for thick casings.

Table 3.6.

r/r_o	1.0	1.05	1.1	1.3	1.54	2.4
pressure dynes/ $cm^2 \times 10^{-9}$	150	49	25	8.4	4.0	2.0
pressure tons/ sq. in.	1000	320	160	55	26	13

In a cylindrical tube subject to an internal pressure just great enough to cause flow, the stresses have been worked out.[23] The radial and tangential stresses are, at distance r from the axis

$$S_r = -T_o \log \frac{b}{r}$$

$$S_t = T_o \left(1 - \log \frac{b}{r}\right)$$

where b, a are the external and internal radii; the pressure necessary to cause flow is

$$T_o \log \frac{b}{a}$$

Here $T_o = 2\ S_o/\sqrt{3}$ where S_o is the shearing stress. If p is the actual pressure of the gases, we have an additional pressure at the surface

$$p - T_o \log \frac{b}{a}$$

giving a hydrostatic pressure at a distance r from the axis equal to

$$\left(p - T_o \log \frac{b}{a}\right) \frac{a}{b-a} \left(\frac{b}{r} - 1\right)$$

The stresses can thus be resolved into

(1) A tangential stress T_o
(2) A hydrostatic pressure equal to

$$\left(p - T_o \log \frac{b}{a}\right) \frac{a(b-r)}{(b-a)r} + T_o \log \frac{b}{r} \ ,$$

which vanishes at the outside surface and reaches the value p at the inner surface.

Now it is known that hydrostatic pressure makes fracture more difficult, while having little effect on the resistance to glide. For nonplastic materials, where fracture starts from a microscopic crack, the following account of the

[23] Nadai, Plasticity, McGraw Hill Book Co., p. 188:

effect of hydrostatic pressure has been given by A.A. Griffiths in a well-known paper.[24] Suppose elliptical cracks are acted on by a stress T and a hydrostatic pressure P; the angle made by the plane of any crack to the normal to T is denoted by θ, and θ is distributed over all values Fig. 3.11. Then T will be great enough to cause cracks to spread under the following conditions:

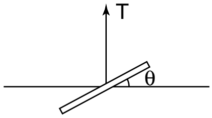

Fig. 3.11.

(a) $4p < 3T$ If this condition is fulfilled, cracks for which $\theta = 0$ will be the first to spread, where T reaches a value $k + p$, where k depends on the elastic constants and surface tension of the metal, and the dimensions of the crack.
(b) $4p > 3T$ Under these conditions cracks for which $\theta = 0$ will not be the first to spread, but those for which

$$\cos 2\theta = \frac{1}{2}\frac{T}{2p - T}$$

At the critical pressure given by $4p = 3T$, this gives $\theta = 45°$.

In plastic materials it is probable that the high tensile stress T near the apex of a crack will cause cracks to form in crystal grains near to it. As the apex of the crack travels inwards, if a point is reached where $4p$ exceeds $3T$, the crack should abruptly change its direction by 45°. This is just what is observed.

Since T is certainly greater than T_o, an necessary condition for such a change of direction will be

$$\frac{4}{3}p > T_o$$

where p is the pressure exerted by the explosive. For casings that break up at 30 and 50% expansions respectively, the calculated values of $4p/3$ are 73 and 35 tons per sq. inch, which are of the same order as T_o, though they are somewhat less than the values that we have assumed to hold for the metal at high rates of strain.

For this reason we put forward the above explanation somewhat tentatively.

[24] Proc. Int. Congress for Applied Mechanics. Delft (1924), p. 55.

3.11 Comparison with Observed Fragmentation of Service Projectiles

In this memorandum we have reached the following conclusions:

For a given type of steel

(a) The ratio of length to breadth of fragments is constant.
(b) The average fragment area is proportional to $(r/V)^{2s}$, where s lies between 2/3 and 1, probably nearer the latter value.
(c) The weight distribution is given approximately by formula (3.3)

We may thus equate M_A of formula (3.3) to

$$\text{const } t^{1/2}(r/V)^s$$

where the value of the constant depends on the properties of the steel, or, making use of formula (3.1) for the velocity

$$M_A = \text{const } t^{5/6}{d_2}^{1/3}\left(1+\frac{t}{d_2}\right) \qquad s=\frac{2}{3}$$
$$M_A = \text{const } t{d_2}^{1/2}\left(1+\frac{t}{d_2}\right) \qquad s=1$$

where the constant depends on the type of explosive and steel, d_2 is the internal diameter and t the thickness of the casing. The first of these formulae has already been compared with experiment in 2), in which M_A was determined for a number of service weapons.

Comparison with fragmentation of observed projectiles should show whether $s = 1$ or $s = 2/3$ or some intermediate value gives the best fit. Ursell[25] has determined the best value of M_A for three model bombs fragmented by Payman,[26] with thicknesses 0.018, 0.125 and 0.3 inches (diameter 2″). He comes to the conclusion that M_A is proportional to $1/V^{1.2}$. The casings of these bombs were of mild steel and gave shear fracture, and so are not directly comparable with our theory. Unfortunately the range of values of r and v available in British shells of carbon steel for which detailed information is available is not great enough to allow any certain conclusion to be drawn.

Observed values of M_A for a number of projectiles filled with TNT are shown in Table 3.7; we have limited ourselves to those with a reasonably cylindrical cross section. It looks as though $s = 1$ gave rather a better fit than $s = 2/3$.

[25] A.W.A.S. Report No. 46; Ministry of Supply No. A.C. 3817
[26] loc. cit.,

Table 3.7.

Projectile	d_1 inches	t inches	M_A (oz)$^{1/2}$ observed	$\frac{M_A}{t^{5/6}d_2^{1/3}\left(1+\frac{t}{d_2}\right)}$	$\frac{M_A}{td_2^{1/2}\left(1+\frac{t}{d_2}\right)}$
3″ U.P.	3.25	0.265	0.134	0.265	0.27
95 mm shell	3.7	0.425	0.23	0.29	0.26
U.P. (thick cased)	3.5	0.50	0.30	0.32	0.29
3.7″ shell	3.7	0.60	0.36	0.32	0.275 ± .015
25 pr. shell	3.43	0.65	0.35 ± 0.03	0.29 ± .025	0.245 ± .02

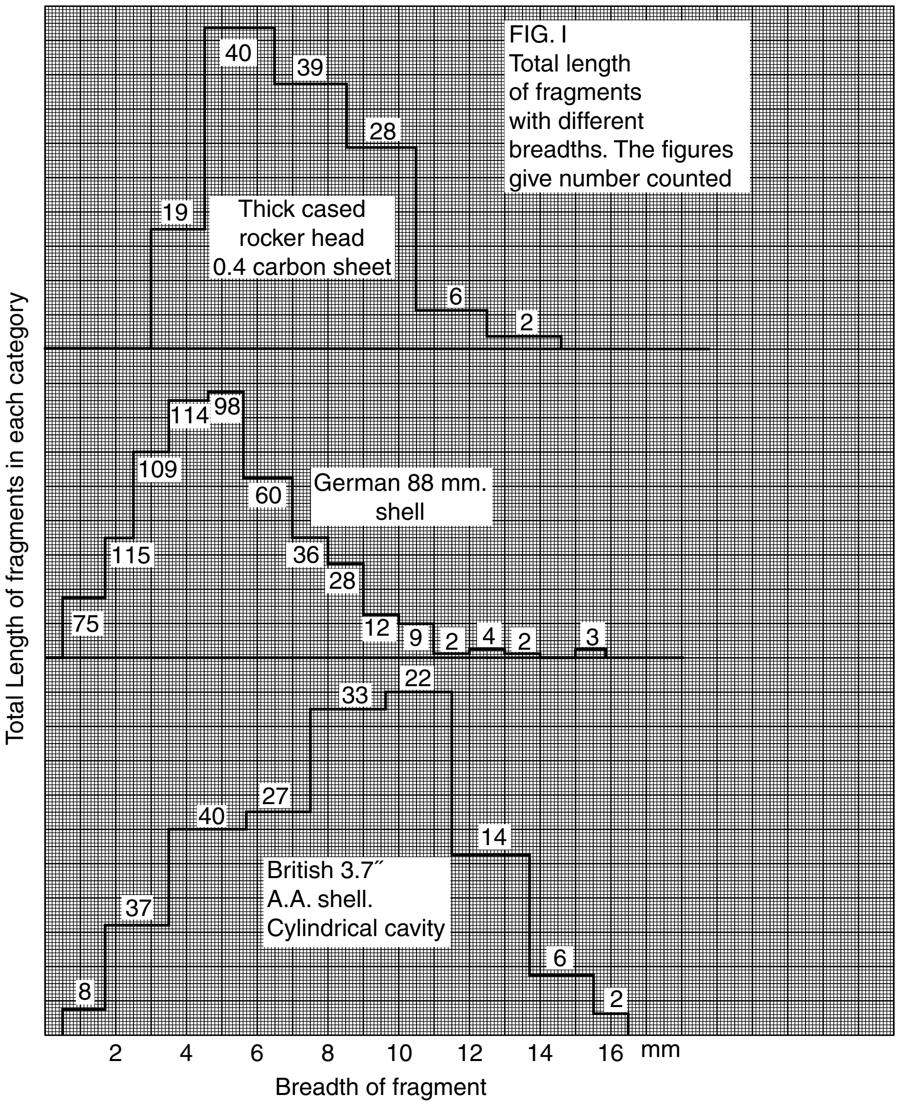
FIG. I
Total length of fragments with different breadths. The figures give number counted
Thick cased rocker head 0.4 carbon sheet
40
39
28
19
6
2
German 88 mm. shell
114
98
109
60
115
36
28
75
12
9
2
4
2
3
British 3.7″ A.A. shell. Cylindrical cavity
22
33
27
40
14
37
6
8
2
Total Length of fragments in each category
2
4
6
8
10
12
14
16
mm
Breadth of fragment

Fig. I.

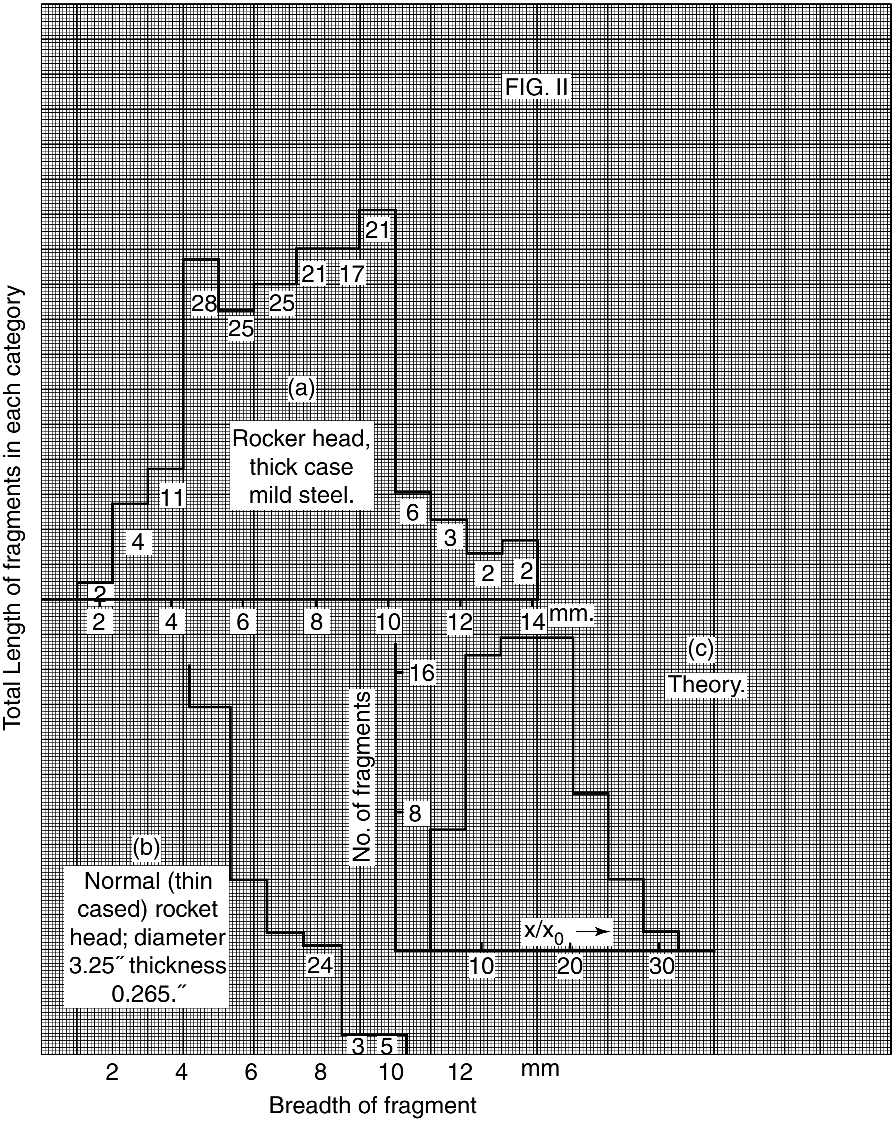
FIG. II
21
21
17
28
25
25
(a)
Rocker head,
thick case
mild steel.
11
4
2
6
3
2
2
2
4
6
8
10
12
14
mm.
(c)
Theory.
16
8
No. of fragments
(b)
Normal (thin
cased) rocket
head; diameter
3.25″ thickness
0.265.″
x/x0 →
24
10
20
30
3
5
Total Length of fragments in each category
2
4
6
8
10
12
mm
Breadth of fragment

Fig. II.

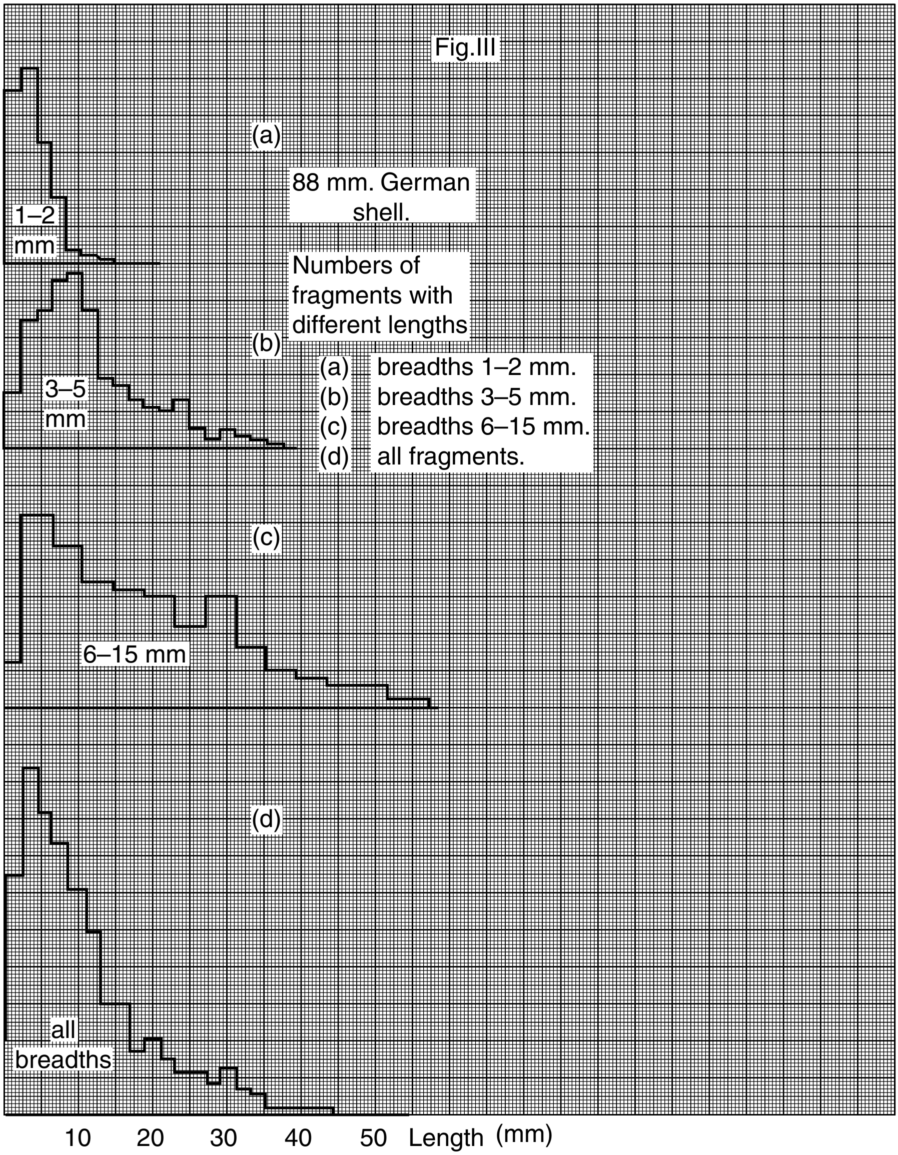
Fig.III
(a)
88 mm. German shell.
1–2 mm
Numbers of fragments with different lengths
(b)
(a) breadths 1–2 mm.
(b) breadths 3–5 mm.
(c) breadths 6–15 mm.
(d) all fragments.
3–5 mm
(c)
6–15 mm
(d)
all breadths
10
20
30
40
50
Length (mm)

Fig. III.

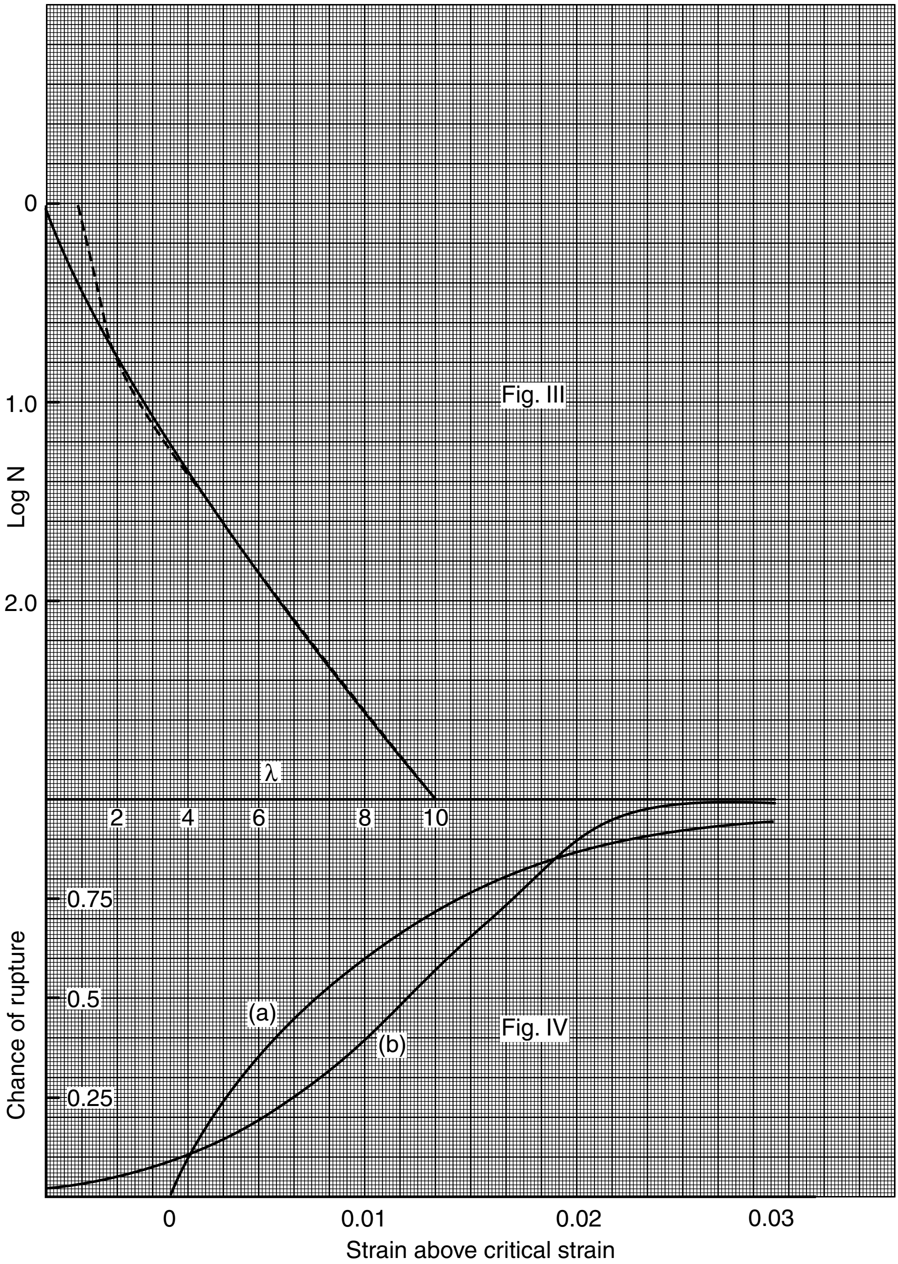
0
1.0
2.0
Log N
Fig. III
λ
2
4
6
8
10
0.75
0.5
0.25
Chance of rupture
(a)
(b)
Fig. IV
0
0.01
0.02
0.03
Strain above critical strain

Fig. IV.

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MINISTRY OF SUPPLY

ADVISORY COUNCIL ON SCIENTIFIC RESEARCH
AND TECHNICAL DEVELOPMENT

FRAGMENTATION PANEL OF THE
STATIC DETONATION COMMITTEE

A theory of the fragmentation of shells and bombs

by

Professor N.F. Mott

Received May 27th, 1943.

E.G.

A THEORY OF THE FRAGMENTATION OF SHELLS AND BOMBS.

By N. F. MOTT.

1. In a recent report on this subject[1], a tentative theory was put forward to account for the sizes of the fragments obtained from steel projectiles. In a further note[2], the theory was compared with the observed fragmentation of service shells. In this report an attempt is made to extend and to improve the theory, as far as is possible without a satisfactory theory of rupture in metals, which does not exist at present.

Before discussing the theory of fragmentation in Part II of this paper we shall give in Part I a summary of the information available about the velocities, weights and shapes of fragments and the mechanisms by which the explosive transfers its energy to them. We shall confine ourselves as far as possible to cylindrical projectiles of uniform diameter, both internal and external; shells with conical cavities are obviously less suitable for the deduction of theoretical conclusions. The rocket head is particularly suitable from this point of view, as is also the German 88 mm. shell, and a special British 3.7" shell recently fragmented by C.S.A.R., Millersford.

PART I

2. EXPANSION OF THE CASING

It is well known that steel casings expand considerably before rupture; this can be seen most clearly by examining the larger fragments which contain part of the inner and outer surfaces; the case has become thinner by an amount which varies very little from one fragment to another[3]. The present author has examined fragments from the following projectiles which have a uniform case thickness : A German 88 mm. shell, a special British 3.7" shell with cylindrical cavity, and three rocket shells fragmented in the Safety in Mines Research Station, Buxton. The filling was TNT in each case; the results are as follows :

TABLE I

Type of shell	Carbon content of shell %	External diameter (mm)	Thickness of Casing (mm)	Thickness of Fragment (mm)	Extension %
German AA 88 mm.	0.7	88	15	11.8	27
British A.A. 3.7" (cylindrical cavity)	0.4 - 0.5	94	16.5	12.8	30
Service A.A. rocket head	"	85	6.75	4.5	50
Thick cased rocket head	"	85	12.8	9	42
"	0.15	85	12.8	8.5	50

Further evidence is available from photographic records of the explosions of model bombs obtained at the Safety in Mines Research Station, Buxton[4].

1) A Theory of Fragmentation, by N.F.Mott and E.H.Linfoot, D.S.R. Extra-Mural Report A.C. 3348.

2) A.O.R.G. Memo. No. 24. "Fragmentation of H.E.Shells; a theoretical formula for the distribution of weights of fragments".

3) Report R.C. 282 from Dept. of Metallurgy, University of Sheffield.

4) Report R.C.236 from the Safety in Mines Research Station.

According to these, model bombs 2" in dia. with mild steel casings filled with tetryl expanded by the following amounts before breaking up :

Thickness of case (inches)	Expansion (%)
0.125	67
0.30	100

The result obtained that the thicker cased bomb expands further may however be due to end effects; it is not confirmed by the two rocket heads in Table I.

3. FRAGMENT VELOCITIES

A theoretical treatment of the expansion of the casing of a long cylindrical cased charge of TNT has been given by G.I. Taylor[5]). Apart from the unknown end effect at the base of the shell, his results should be applicable to the nose-fuzed projectiles considered here.

According to Taylor the velocity of the casing can be expressed by the following formula :

$$V = V_0 \, d_2 \Big/ \sqrt{d_1^2 - d_2^2} \qquad (1)$$

where d_1, d_2 are the external and internal diameters of the casing before expansion, and V_0 is given for different degrees of expansion in Table II Actual velocities calculated for certain shells are also given :

TABLE II
Velocities in ft/sec.

% expansion	11	30	67	124	200
V_0	2000	2400	2700	3000	3100
V (88 mm. shell)	1750	2100	2400	2700	2800
V (3" U.P)	2750	3500	3700	4100	4250

These figures neglect the work done in deforming the case; assuming a constant* resistance to elongation T_0 (poundals/sq.ft) and a density ρ for the steel, a short calculation gives for the reduction in velocity due to this cause

$$\delta V = \frac{T_0}{\rho V} \log (1+\varepsilon) \qquad (2)$$

Assuming T_0 to be 30 tons/sq.in., we obtain the following values :

TABLE III

% expansion	11	30	67	124	200
V (88 mm. shell)	1700	2000	2300	2500	2550

The work done against the plastic forces does not decrease the fragment velocity appreciably, except perhaps for projectiles of very low charge-weight ratio (A.P. shells). The work done in rupturing the case is probably quite negligible.

It cannot be assumed that the fragments are projected from the shell with the velocity of the casing at the moment of break-up; the following observations show this :

(1) According to (unpublished) results obtained at Buxton, model bombs of similar dimensions made of steel and cast iron give fragments of about the same velocity. The cast iron gives very fine fragmentation and probably breaks up without plastic expansion.

(2) By grooving the charge, controlled fragments can be obtained of a

5) Report to M. of H. S. No. R.C. 193.

* In steels the resistance is, of course, not constant, but increases somewhat as the metal hardens.

desired size from U.P. casings. These fragments do not show thinning, but have the original thickness of the case. The case must therefore have broken before expansion. Nevertheless the velocity of the fragments is appreciably the same as for the normal shell without a grooved charge (unpublished results with model bomb.)

Both these results show that the explosive must continue to exert pressure on the fragments after break-up, and up to about 20 or 30% expansion the pressure cannot depend much on whether the case has broken or not.

Evidence about fragment velocities is contradictory; at Buxton all fragments from a given model bomb are found to have approximately the same speed, except for a few very small ones of high velocity, probably acquired from the expanding gases after break-up; at Millersford, on the other hand, whilst most of the fragments from shells of the 88 mm or 3.7" type have fragments with speeds in the range 2000-2500 ft/sec., there are a considerable number with much lower speeds down to 1000 ft/sec., and thus with speeds less than the calculated velocity of the casing before break-up. The origin of these is unexplained.

Photographic measurements of the velocity with which the casing of a model bomb expands have been made at Buxton; surprisingly enough, the velocity of the case comes out in one case to be greater than that of the fragments*.

In view of these contradictory results we shall take theoretical values for the velocities of the casing, calculated as in Table III; these agree at any rate as regards order of magnitude with observed fragment velocities.

4. TYPES OF FRAGMENTATION OBSERVED

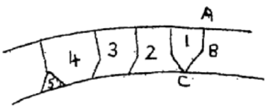

Fig. 1

The cross sections of the large fragments from a cylindrical shell are usually of one or other of the types shown in fig.1; on the outside of the case (along AB) the rupture is brittle, with shear rupture from B to C. Types 1 and 4 are the commonest, with small pieces of triangular cross section frequently shearing off (as in type 5 in fig.1).

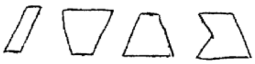

Fig. 2

In some casings the rupture is by shear only, fragments of the types shown in fig. 2 being observed. This has been observed both for mild steel and carbon steel casings. In the theories of Part II we have limited ourselves to rupture which is at least partly brittle.

Fragments are commonly five to ten times as long as they are wide.

5. WEIGHTS OF FRAGMENTS

The most usual classification is by weighing. The present writer has pointed out** that for many shells and bombs the weight distribution satisfies the following formula; the number of fragments with weights between m and m + dm is equal to

$$C e^{-M/M_A} \, dM, \qquad M = m^{1/2} \qquad (3)$$

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* cf. Reference 4); values given on pp. 2 and 5 for a model bomb with 0.018" casing.

** Reference 1).

where C, M_A are constants. Since C depends on the total weight of the casing, the fineness of the fragmentation is given by a single parameter M_A. Apart from any theoretical significance of formula (3), it provides a convenient practical method of comparing the fragmentation of different projectiles.

Using $M_A{}^2$ as a measure of the mean fragment weight, the following factors appear to affect it in the following ways :

1. Type of Steel : there is little evidence that the tensile strength or yield point affects the fragmentation, but the carbon content certainly does. Thus two similar projectiles, the German 88 mm. and the British 3.7" shell give the following values of M_A :

	Diameter (mm)	Thickness (mm)	Steel, carbon %	M_A (ounce)$^{\frac{1}{2}}$
88 mm. shell	88	15	0.7	0.19
3.7" "	94	16.5	0.4-5	0.36

Also 3.25" rocket heads of carbon (0.4%) and mild steels, thickness 0.5 gave the following values of M_A :

Carbon	0.30 (ounce)$^{\frac{1}{2}}$
Mild steel	0.33

2. Calibre of shell : for given charge-weight a big shell undoubtedly gives bigger fragments. For example, values of M_A for a large and for a small shell of similar capacities are (U.B.Proc. 21099 and 21051)

		M_A
95 mm. shell	Amatol 50/50	0.26
5.5" (80 lb. shell)(140 mm.	Amatol 50/50	0.46

3. Charge-weight ratio : this affects both the thickness of the casing and its velocity at the moment of break-up. That the velocity at the moment of break-up has a profound influence on the fragmentation is shown by two facts :

(a) That a 250 lb. bomb fragmented in water gives only about a quarter as many fragments as when exploded in air 6).

(b) The well known gross fragmentation of that part of an H.E.shell with direct acting fuze which is in contact with the ground at the moment of explosion; large pieces can be picked from the crater.

Apart from its influence on the velocity, a thin casing will of course give thinner fragments than a thick one. Whether it affects the other dimensions will be discussed below.

Values of M_A for two otherwise similar rocket heads with thicknesses 0.265" and 0.5" are (Reference 2))

Thickness	0.265"	0.5"
M_A	0.134	0.255 (ounce)$^{\frac{1}{2}}$

The velocity of expansion could be altered at will without affecting the size or thickness of the casing by putting a lead covering round the outside of the shell. Experiments to determine the effect of this on the fragmentation would be of great interest. The pressure distribution within the case would also be altered (cf. § 11).

6. DIMENSIONS OF FRAGMENTS

The primary process in fragmentation must be splitting parallel to the axis of the shell, with subsequent rupture at the ends, and production of secondary fragments of type 5 in fig. 1. Assuming that cracking

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6) Compilation of data on Trials on Explosive Effects of Aircraft Bombs. R.D. Woolwich, 1936.

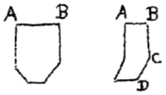

Fig. 3

(e.g. along BC in fig. 3) precedes shear rupture (e.g. along CD), the first task of any theory of fragmentation must be to account for the distance AB in fig. 3 between the edges of the average fragment. The observed distributions of the breadth AB are shown in figs. I and II at the end of this paper. It is of course true that the length AB often varies considerably along the length of a fragment, and a visual estimate of the mean breadth is subject to error; nevertheless the general shape of the curves is significant. We plot against fragment breadth not the total number of fragments, but the total length of all fragments (placed end to end) in each category.

The following points will be noted :

(a) The rather sharp cut-off for large breadths.

(b) The much narrower fragments obtained with the German 88 mm. shell (0.7% carbon steel) than with the British 3.7" shell or thick cased rocket head, (0.45% steel but similar diameter and casing thickness).

(c) The narrower fragments obtained with the thin cased (high capacity) rocket head than with the thick cased projectiles of similar steel.

The lengths of fragments from the German 88 mm. shell are shown in fig. III; the curve does not show the same cut-off at high values. In fig. IV we show the length distribution for fragments of different breadths there is obviously a rough correlation, broad fragments being longer. The average length of fragments in different categories is given in Table IV.

TABLE IV
Lenghts in mm.

Breadth (mm.)	2 – 3	4	5	6	7	8	9	10
Thick-cased U.P. (carbon steel)		59	39	56	44	37	50	36
Thick-cased U.P. (mild steel)		34	35	33	36	47	58	54
Service U.P.			27	30	28	29		
German 88 mm.	5.5	10	14.8			21.7		

Evidence for correlation between breadth and length is not marked except for the German shell. For the British shells a ratio of length to breadth of the order 5 seems to be normal, for the German shell a somewhat smaller value.

7. WEIGHT DISTRIBUTION OF FRAGMENTS

The formula (3) was derived[1)] by the author on the assumption of some sort of random break-up; figs. 1 to 3 show however that neither the break-up parallel or perpendicular to the axis can be considered random as would be the case if the breadths were distributed according to the law : number of fragments with breadths between a and a + da is proportional to $\exp(-a/a_0)da$. It therefore seems worth while to attempt a derivation of (3) from different assumptions.

Let us assume :

(a) that the casing is broken into strips and that the number of strips with breadths between x and x + dx is

$$C x \exp(-x/x_0)\, dx \tag{4}$$

This does not represent the facts exactly, but gives a nearer approximation than the random fracture.

(b) that each strip is broken up according to the same law, and that the average length of fragment is proportional to the thickness x of the strip. Thus from a strip of length ℓ the number of fragments of length between y and y + dy is

$$y\, e^{-y/px}\, \ell\, dy/(px)^3 \qquad (5)$$

where p is a factor (of the order 5).

Then the number of fragments of area greater than a^2 is

$$\frac{C\ell}{p^3} \iint_{xy>a^2} \frac{y}{x^2} \left[\exp\left(-\frac{x}{x_0} - \frac{y}{px}\right)\right] dx\, dy$$

This reduces to

$$\text{const}\ \lambda \int_0^\infty \left(1+\frac{1}{z^2}\right) \exp\left(-\lambda z - \frac{1}{z^2}\right) dz, \qquad \lambda = a/x_0 p^{\frac{1}{2}}$$

and thus the number of fragments with area such that a(= $\sqrt{\text{area}}$) lies between a and a + da is

$$\text{const}\ f(\lambda)\ d\lambda$$

where

$$f(\lambda) = \int_0^\infty \left\{\left(1+\frac{1}{z^2}\right) - \lambda\left(z+\frac{1}{z}\right)\right\} \exp\left(-\lambda z - \frac{1}{z^2}\right) dz \qquad (6)$$

This function is plotted logarithmically in fig. V over all values of λ from 0 to 10, i.e. over a range of $\log_{10} f$ equal to 3, which is about the range over which the fragment distribution is usually plotted. It will be seen that the deviation from a straight line is not very large.

Weight distributions of actual fragments are likely to deviate from this theoretical curve for the following reason : the narrower fragments frequently break as shown in fig. 4, thus having a smaller depth than they should. Moreover the removal of the triangular pieces from the base of the smaller fragments will obviously make a greater proportional difference to their weight. This will result in a shift of the whole upper part of the curve in fig. 5 somewhat to the left. On the other hand, on reaching the weight categories of the small triangular fragments, a large number of new fragments appear which are not included in the analysis given above. Thus the curve should appear as the dotted curve in fig. V, which is very similar to those observed.

Fig. 4

PART II

THEORY OF THE MEAN FRAGMENT SIZE

8. DEPENDENCE ON VELOCITY

We consider that the fragmentation will be determined by the properties of the casing at the moment of break-up, and will not depend, for instance, on the pressures to which the case has been subjected during the expansion. The factors that may be of importance are thus

(a) Properties of the steel at the moment of rupture - for example the true ultimate tensile strength rather than the yield point.

(b) The rate of increase of plastic strain; this is equal to V/r, where V is the velocity of the case and r its radius.

(c) The thickness of the casing.

(d) The pressure of the explosive at the moment of break-up; according to Taylor's calculations this is from 60 - 25 tons/sq. in. for casings that break up after a 25 to 50% expansion; this is much less than the initial pressure, which is of the order 1000 tons/sq. in.

The theory which we shall develop suggests that (c) and (d) are of minor importance in determining the breadths and lengths of fragments. As in the author's previous report, we take the point of view that it is the kinetic energy of the case which tears it to pieces; the fragmentation would be almost the same if the expanding explosive could be miraculously removed just before the case broke up, leaving it to fly into pieces under its own momentum.

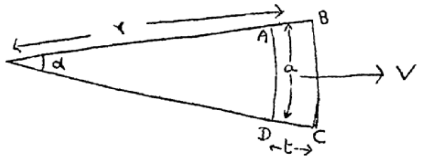

Fig. 5

In the author's previous report[1] the following derivation of the fragment breadth was given. Suppose that ABCD in fig. 5 is the cross section of a fragment which has just broken along BA, CD. The fragment is still in a state of plastic flow, the rate of increase of plastic strain being V/r. The kinetic energy of this flow of metal is

$$\frac{1}{2}\rho t V^2 \int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} r\theta^2 \, d\theta = \frac{1}{24} V^2 t \rho a^3/r^2$$

It was argued that if this were greater than the energy Wt required to rupture the metal, the fragment would split in half. Thus the value

$$a = \left[\frac{24 r^2 W}{\rho V^2}\right]^{1/3} \qquad (7)$$

would give an upper limit to the possible breadth of a fragment.

Agreement with observation, i.e. values of a of the order 1 cm., was obtained with values of W given by the notched bar impact test for a brittle steel, i.e. 40 ft/lbs. per sq. inch*. Since W occurs only as $W^{1/3}$, the values obtained are not very sensitive to W.

Equation (7) will certainly give a lower limit to the maximum fragment breadth, if W is the true fracture energy under the conditions existing in an H.E. shell. It is doubtful however if this bears any relation to the energy expended in the notched bar test, most of which is probably due to plastic deformation of the metal in the neighbourhood of the notch until the formation of a true crack of atomic width at its apex, leading to brittle rupture. The actual work necessary to separate two planes of atoms in a metal is of course much less, of the order 10^{-3} ft.lbs/sq.inch.

We shall therefore attempt a theory of fragmentation based on the assumption that the energy of fracture is negligible. In addition we shall make the following assumption : fracture can start at any one of a number of places on the surface or in the body of the casing, and once started will rapidly spread across it. During the initial stages of the expansion, it is very unlikely (or even impossible) that a crack will start anywhere; as the expansion increases the chance of a crack forming in any

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* Measurements were made at the N.P.L. of the Izod value of test pieces cut from a 3.7" H.E. shell casing which had been extended 20% in the direction originally circumferential to the shell, to represent the state of the steel at the moment of rupture; values obtained for specimens with the usual 10 x 8 mm. section at the notch were, for the energy absorbed to fracture

5.0 5.9 5.0 ft. lbs.

This gives 43 ft.lbs/sq. inch. (Ref. Eng. Dept/OYY/RE/B.104 A, 5.3.43).

part of the case increases. We introduce a function $f(s)ds\,dx$, which gives the chance that a crack will form on a length dx of the circumference of the casing as the strain increases from s to $s + ds$. We may take $f(s)$ to be zero up to a certain value of s (the rupture point), or we may assume a very rapid increase of $f(s)$ in the neighbourhood of the rupture point. We shall find that the form of $f(s)$ determines the mean fragment size.

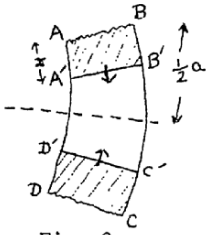

Fig. 6

As before we consider a fragment that has just broken along the lines AB, CD (fig.6), and ask whether it is likely to break again. As soon as a fracture has formed along AB, for instance, the metal in the neighbourhood of AB will stop flowing. A boundary A'B' between the part of the metal which is still in plastic flow and the metal which has stopped flowing will move downwards with a velocity that can be calculated. It will soon reach the boundary C'D' moving upwards from the lower crack; when this has happened, no further crack can form. For a fragment of average width, therefore, the chance of a new crack forming before A'B' and C'D' meet each other must be small. This chance can be calculated by comparing the function $f(s)$, giving the rate of formation of cracks, with the time available before the surfaces join.

The velocity with which the surface A'B' moves can be determined as follows, if we assume that this is small compared with the velocity of sound in steel : Let $\underline{a}$ be the breadth of the fragment, x the breadth of the part that has stopped flowing, and T_0 the stress required to cause plastic flow. The velocity upwards of all material above A'B' is then

$$\frac{V}{r}\left(\tfrac{1}{2}a - x\right)$$

Therefore the equation of motion of the block ABB'A' is

$$T_0 = -\rho x \frac{d}{dt}\left\{\left(\tfrac{1}{2}a - x\right)\frac{V}{r}\right\}$$

which gives

$$T_0 = \frac{\rho V}{r}\, x \frac{dx}{dt} \tag{8}$$

Thus

$$\tfrac{1}{2}x^2/t = r T_0/\rho V, \tag{9}$$

and the time which the fragment takes to stop expanding is

$$a^2 \rho V/8 T_0 r,$$

which is of order 10^{-6} secs. if $a \sim 7$ mm. The increase in the strain s of the material during this time is of the order 10^{-2}.

From equation (9) we find

$$\dot{x} = T_0 r/\rho V x \sim 6 \times 10^4/x \quad \text{cm/sec}$$

so the velocity, except for very thin fragments, is considerably less than that of sound in steel (5×10^5 cm/sec.)

We have now to make some assumption about the function $f(s)$. We could assume alternatively that

(a) $f(s)$ is zero up to a definite value s_0 (the rupture point) and is then constant and equal to f_0, say.

(b) $f(s)$ is zero up to s_0, and then increases, as $c(s - s_0)^n$ say.

(c) $f(s)$ is never zero, but increases rapidly in the neighbourhood of the rupture point, as $Ae^{\gamma s}$ say.

The hypothesis (c) is the most attractive, for reasons that will be given in the next section; but they all lead to somewhat similar conclusions about the fragmentation.

An idea of the order of magnitude of the constants involved can be obtained from the behaviour of steel in tensile tests, if we make the assumption that the behaviour in static tests is similar to that at high rates of strain. In tensile tests, steels nearly always fracture after necking; the reduction of area thus gives a measure of the strain at which fracture occurs. Specimens prepared from one sample of carbon steel show a certain scatter in the measured values of the reduction of area; thus, if a steel fractures on the average for a reduction of area of 50%, individual specimens will show values between 49 and 51 approximately. The following, for instance, are values* for a normalised 0.4% carbon steel :

58 $56\frac{1}{2}$ $59\frac{1}{2}$ 59 per cent

Now according to our assumptions, the chance that a specimen of length ℓ will fracture before the strain reaches a value s is

$$1 - \exp\left[\ell \int_0^s f(s)\,ds\right] \tag{10}$$

In case (a) this gives

$$1 - \exp\left[-f_0 \ell (s - s_0)\right] \tag{11}$$

and, in case (c), to a sufficient approximation

$$1 - \exp\left[-\frac{\ell A}{\gamma} e^{\gamma s}\right] \tag{12}$$

Suppose that we assume that an increase in s by Δs increases the chance that fracture has taken place from 10 to 90%. Then we find from (11) and (12)

$$f_0 \ell = 2{\cdot}2/\Delta s \qquad \text{(case a)}$$

$$\gamma = 3{\cdot}1/\Delta s \qquad \text{(case c)}$$

In case (a) it is not clear what value of ℓ should be taken, since the maximum strain only occurs at the neck. In case (c), however, ℓ does not occur in the formula for γ; if, in accordance with the experimental values given above, we take $\Delta s = 0.02$, we obtain

$$\gamma = 155$$

A plot of the functions (11) and (12), showing the chance that a fracture has occurred when the strain (reduction in area) is s, is given in fig. VI for $\ell f_0 = 100$ and for $\gamma = 150$. The origin of s for curve (c) is arbitrary.

Experiments on the extent to which the reduction in area at the breaking point fluctuates from specimen to specimen, carried out for a sufficienrly large sample, would shed light on the nature of the function f(s).

With any of these forms of f(s), an estimate of the order of magnitude of the breadth a can be made as follows : At each crack, after a time t, a breadth

$$2\left(\frac{2T_0 r}{V\rho}\right)^{\frac{1}{2}} t^{\frac{1}{2}}$$

gas stopped expanding and is thus 'safe' from cracking. Since the strain increases as Vt/r, when the strain has increased by Δs after the formation of a given crack, a breadth round it equal to

$$\beta(\Delta s)^{\frac{1}{2}}, \qquad \beta = 2^{3/2}\left(\frac{T_0}{\rho}\right)^{\frac{1}{2}} \frac{r}{V}$$

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* N.P.L. Report to A.R. Committee, Paper 4755

is "safe" also. If we neglect the overlapping of "safe" areas, a proportion

$$\beta \int_0^s f(s') \sqrt{s - s'}\, ds'$$

is safe when the strain is s. When this approaches unity, the break up is complete. Thus a, the average breadth, is given as regards its order of magnitude, by eliminating s between

$$\beta \int_0^s f(s') \sqrt{s - s'}\, ds' \simeq 1 \qquad (13)$$
$$\int_0^s f(s')\, ds' \simeq 1/a$$

With the forms for f(s) suggested above we obtain the following :

(a) Equations (13) lead to

$$a = \left(\frac{2}{3}\right)^{2/3} f_0^{-1/3}\, 2\sqrt{\frac{T_0}{\rho}} \left(\frac{r}{V}\right)^{2/3}$$

This gives the same power of (r/V) as the author's previous theory, and with $f_0 = 100\ \mathrm{cm}^{-1}$, values of a of the order 0.5 cm.

(b) Equations (13) give

$$a = (n+1)\left[\frac{\Gamma(n+1)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{5}{2}\right)}\right]^{1-\frac{1}{2n+3}} C^{\frac{-1}{2n+3}}\, \beta^{1-\frac{1}{2n+3}}$$

It will be seen that the fragment size is proportional to $(r/V)^{1-\frac{1}{2n+3}}$ and thus to some power of r/V between 1 and 2/3.

(c) With $f(s) = A e^{\gamma s}$, the equations (13) give us

$$A\beta e^{\gamma s} \int_0^\infty e^{-\gamma s'} s'^{\frac{1}{2}}\, ds' = 1$$
$$A e^{\gamma s} \int_0^\infty e^{-\gamma s'}\, ds' = 1/a$$

and hence

$$a = \sqrt{\frac{2\pi T_0}{\rho}}\, \frac{r}{V} \frac{1}{\gamma^{\frac{1}{2}}}$$

With $\gamma = 100$, $T_0 = 60$ tons/sq. inch, this gives 0.7 cm. for a normal shell of the calibres considered here.

It will be seen that a is now proportional to r/V.

Our formulae suggest, then, that the mean width of fragment will be proportional to

$$\text{const.}\ (r/V)^s,$$

where s lies between 2/3 and 1; the constant will depend on the nature of the steel; it may depend on the thickness of the case and pressure of the explosive, but consideration of the next section suggests that it will not.

We have not been able to find an analytical expression for the number of fragments with breadths between a and a + da, but our equations for the break-up enable a distribution to be found graphically. We limit ourselves to the form (c) for f(s). The theory is at present one-dimensional; we are considering the division of a line (a circumference of the shell) by random fracture. Let ℓ be the length of this line; then as before when each crack is formed, a space on each side of it equal to

$$\left(\frac{2T_0}{\rho}\right)^{1/2} \frac{r}{V} (\Delta s)^{\frac{1}{2}}$$

is safe from further cracking when s has increased by Δs. If N is the number of cracks already formed then the rate of increase of N is given by

$$\frac{dN}{ds} = A p \ell e^{\gamma s}$$

where p is the proportion of the line where cracks can still form. The first crack will form, on the average, when

$$A l e^{\gamma s}/\gamma = 1$$

If the value of s given by this equation be denoted by s_o, and a new variable σ defined by

$$\sigma = \gamma(s - s_o),$$

then the rate of increase in the number of cracks is given by the equation

$$\frac{dN}{d\sigma} = p e^{\sigma}$$

Also, if a crack is formed when $\sigma = \sigma_1$, the region round it where subsequent cracking is impossible is at any subsequent instant

$$2 x_o (\sigma - \sigma_1)^{1/2} \tag{14}$$

where

$$x_o = \left(\frac{2T_o}{\rho\gamma}\right)^{\frac{1}{2}} \frac{Y}{V}$$

A line drawn on paper can now be cut at random, using playing cards or dice. Initially σ is supposed to be zero; after each successive cut is made σ is supposed to increase by $d\sigma$ where

$$d\sigma = 1/pe^{\sigma}$$

After each new cut is made, the "safe" region round all cuts made earlier must be increased according to formula (14). Any arbitrary value of the ratio l/x_o may be taken. We took $l/x_o = 20$. The line is repeatedly cut until the whole region is "safe" from further cracking. The lengths of all intervals are then measured and recorded, and the process repeated a number of times until enough data are obtained to draw a histogram, in which the numbers of "fragments" (i.e. intervals) are plotted against their lengths. The results are shown in fig. II(c). The similarity to the distributions of fragment breadths observed in figs. I and II (a)and (b) is satisfactory.

By comparing fig. V with the observed fragment distributions and especially the values of their upper limits, we have estimated in Table V the value of x_o for the projectiles investigated. The values are not correct to more than ± 10%.

From these values we have attempted to deduce γ. For this we require the radius of the shell at the moment of break-up (r), the velocity of the casing and the true ultimate tensile strength, T_o. The two former quantities are deduced from the values given in Part I. To deduce T_o from a tensile test we require the stress at the moment of rupture at the base of the neck, which is of course considerably greater that the U.T.S. given in engineering tables. For steels the following values are given by Korber and Rohland, (Mitt. d. K. Wilhelm Inst. f. Eisenforschung, 5 (1924) 55).

Carbon (%)	Reduction in area (%)	True ultimate stress kg/mm²	True ultimate stress tons/sq.inch
0.13	70	78	51
0.25	63	80	52
0.45	57	82	53
0.55	50	87	57

These will probably be somewhat higher for high rates of strain*; we have thus assumed

$$T_o = 80 \text{ tons/sq. inch}$$
$$= 100 \text{ tons/sq. inch}$$

for British (0.45% carbon) and German (0.7% carbon) shell steels respectively.

* cf. G. I. Taylor, Stress Strain Relationship on Impact. Civil Defence Research Committee. R.C. 36.

TABLE V
Values of γ deduced from observed distributions of fragment breadths

	88 mm shell	3.7" shell	Rocket head* (thick case)	Rocket head (thin case)
x_0 (cm) observed	0.37	0.56	0.44	0.31
2r (cm)	11.4	12.2	12.8	12.8
V (cm/sec)	64,000	63,000	76,000	110,000
$\gamma = \frac{2T_0}{\rho}\left(\frac{r}{Vx_0}\right)^2$	230	105	125	124

For the values of γ we cannot claim an accuracy greater than ± 30%; within these limits the British shells (0.45% carbon) show the same value, which is of the order expected. The German shell shows a higher value, which we assume to be due to the higher carbon content of the steel.

9. DEPENDENCE ON THICKNESS AND PRESSURE

We have seen that the hypothesis

$$f(s) = A\, e^{\gamma s} \qquad \gamma \sim 100$$

fits the facts well both for the fragmentation of shells and for the consistency of the rupture point, and seems a priori more likely than the other hypotheses. We have now to consider the following points :

(a) Is γ likely to depend on the thickness of the casing, or the pressure of the gases at the moment of rupture?

(b) Why is γ larger for steels with high carbon content?

(c) Can we deduce a factor γ of this order from any known property of the metal?

It has not at present been possible to answer point (b); to the others an answer can be given :

Let us make the following assumptions about fracture in ductile metals :

(i) Cracks can start at a limited number of points or regions in the metal of which we assume that there are n per unit volume.

(ii) Cracks will start at these points, on the average, when the strain has increased to a value s_1

(iii) The strains at which cracks will form at the individual points of weakness show a certain scatter about the value s_1 ; it is natural to represent this scatter by a Gaussian distribution. We thus assume that the number of points per cm^3 at which a crack will form as the strain increases from s to s + ds is

$$\frac{n}{s_2\sqrt{2\pi}} \exp\left[-\frac{(s-s_1)^2}{s_2^2}\right] ds .$$

For a tensile specimen of cross sectional area A, this gives us for our function f(s)

$$f(s) = \frac{nA}{s_2\sqrt{2\pi}} \exp\left[-\frac{(s-s_1)^2}{s_2^2}\right] \ cm^{-1}. \qquad (15)$$

---

* The values of x_0 for 0.15 and 0.45% carbon are about the same.

We are interested only in the tail end of this curve where f(s) first becomes appreciable; let us then define the rupture point s_0 as the strain for which one crack per cm. is expected, so that

$$\int^{s_0} f(s)\, ds = 1, \tag{16}$$

and write

$$s = s_0 + s'$$

Then we obtain from (15)

$$f(s) \simeq \frac{nA}{s_2\sqrt{2\pi}} \exp\left[-\left(\frac{s_1 - s_0}{s_2}\right)^2\right] e^{\gamma s'}$$

with

$$\gamma = 2(s_1 - s_0)/{s_2}^2 \tag{17}$$

Also from (16)

$$\frac{nAs_2}{2\sqrt{2\pi}(s_1 - s_0)} \exp\left[-\left(\frac{s_1 - s_0}{s_2}\right)^2\right] = 1,$$

whence

$$\left(\frac{s_1 - s_0}{s_2}\right)^2 = \log_e\left[\frac{nAs_2}{2\sqrt{2\pi}(s_1 - s_0)}\right] \tag{18}$$

Hence from (17) we obtain finally

$$\gamma = 2\log_e\left[\frac{nAs_2}{2\sqrt{2\pi}(s_1 - s_0)}\right]\Big/(s_1 - s_0) \tag{19}$$

Since n comes within the logarithm, its exact value is not important. For a number of reasons we expect the distance between the points where rupture can start to be of the order 10^{-4} to 10^{-5} cm. This is for instance the distance between the slip bands* in a metal, the "dislocations" in G.I. Taylor's theory of slip**, or the "crystallites" whose existence has been suggested in cold worked metals***. We thus take n of the order 10^{15}; the other terms within the square bracket are negligible in comparison and we obtain

$$\begin{aligned}\gamma &= 2\log_e 10^{15}/(s_1 - s_0)\\ &= 69/(s_1 - s_0)\end{aligned} \tag{20}$$

From formulae (19), (20) we deduce :

(a) That γ is practically independent of the cross section of the specimen, and thus of the thickness of the shell casing.

(b) That γ is practically independent of the pressure of the explosive at the moment of rupture, because (cf.§ 12) the pressure must vanish at the outside surface, and if the formation of cracks were confined to a small layer near the surface only, it would not affect γ appreciably.

(c) The properties of the steel affect the value of γ only through the value of $s_1 - s_0$, and if s_1 is of the order unity, as is not unlikely, values of γ in agreement with observation are obtained.

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* cf. for example, Orowan, Nature, 147, 452 (1941) or the beautiful photographs of worked steel obtained with the electron microscope by Heidenreich and Peck, J. Applied Physics, 14, 24 (1943).

** Proc. Roy. Soc. A. 145, 362 (1934).

*** Smith and Wood. Proc. Roy. Soc. A. 178, 93 (1941).

10. LENGTHS OF FRAGMENTS

Up till this section we have discussed only the breadths of fragments, believing that splitting parallel to the axis is the primary process in fragmentation. We have now to discuss the factor determining their lengths.

Observation on fragments of marks cut on the surface of the case shows that shell casings do not stretch parallel to their axis; we must therefore look for an explanation of rupture at the ends of the fragments different from that given for the longitudinal cracks.

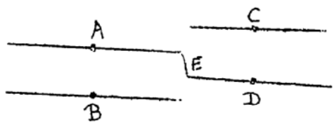

FIG. 7

If cracks start at A and B and spread to the right, and from C and D and spread to the left, then as Professor Andrew* has pointed out, when the cracks bounding two fragments meet, there will be a tendency to split, as at E. According however to the hypothesis on which this paper is based, a split like this is only likely to take place if the steel between the cracks A and B has already stopped flowing before the crack D reaches it; otherwise the crack D will be unaware of the presence of the cracks A and B and will penetrate between them. If however plastic flow has stopped, the different directions in which the two fragments are moving will, we consider, lead to their separation.

Let u be the velocity with which each crack extends. As soon as a crack has formed, the region spreads in which flow has stopped, so that after a time t its width a is given by

$$a = 2\left(\frac{2\gamma T_0}{\rho V}\right)^{\frac{1}{2}} t^{\frac{1}{2}}$$

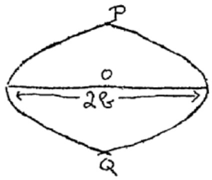

Fig. 8

Thus a crack starting at O in fig. 8 and which has spread to a length 2b is surrounded by a region bounded by two parabolas, in which plastic flow has stopped; the breadth PQ of this region is

$$2\left[\frac{2\gamma T_0 b}{\rho V u}\right]^{\frac{1}{2}}$$

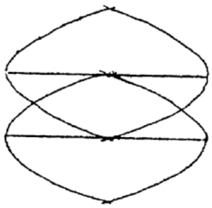

Fig. 9

As a rough criterion for the condition that the region between two cracks should be no longer in flow, we write a, the width of the crack, equal to half this;

$$a = \left[\frac{2\gamma T_0 b}{\rho V u}\right]^{\frac{1}{2}}$$

Thus the ratio, length to breadth, is equal to

$$\frac{2b}{a} = \frac{\rho V u a}{\gamma T_0} \qquad (21)$$

According to equation (14), a for the average fragment is proportional to r/V; we obtain

$$\frac{2b}{a} = 2\sqrt{\frac{\pi\rho}{2T_0}} \quad \frac{u}{\gamma^{\frac{1}{2}}} \qquad (22)$$

With T_0 = 60 tons/sq. inch = 9 x 10^9 c.g.s. units, ρ = 8, γ = 100, this gives

$$2b/a = 0.7 \times 10^{-5}\,u$$

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* Report R.C. 342 from the Dept. of Metallurgy of the University of Sheffield (31.8.42).

If we equate u to the velocity of sound in steel, 5×10^5 cm/sec.,we obtain

$$2b/a \simeq 3.5$$

in fair agreement with experiment. The hypothesis that cracks spread with the speed of sound is not unlikely to be correct, if the atomic cohesion only has to be overcome, and no plastic deformation is involved.

Formula (18) suggests that the length/breadth ratio of the average fragment is independent of the calibre or capacity of the projectile, but will be less for the German high carbon steel (large γ) than for the British steel. These conclusions seem to be born out by the figures of Table IV.

11. SHAPE OF CROSS SECTION OF FRAGMENTS

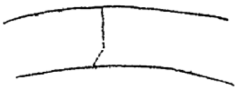

Fig. 10

We have already remarked on the types of rupture observed, and pointed out that the type of rupture shown in fig. 10 is usual, with a brittle crack on the outside of the casing and shear rupture at 45° on the inside. In this section we attempt an explanation of this double type of rupture. For this purpose we calculate the stresses in the case during plastic expansion.

According to G.I. Taylor's calculations, the pressure at various stages in the expansion of a long cylindrical cased charge are given by the following figures, where r is the radius of the inner surface of the case and r_0 its initial value :

TABLE VI

r/ro	1.0	1.05	1.1	1.3	1.54	2.4
pressure dynes/ $cm^2 \times 10^{-9}$	150	49	25	8.4	4.0	2.0
pressure tons/ sq. in.	1000	320	160	55	26	13

At the moment of break-up, therefore, ~~the~~ pressure is of the same order as the yield stress, and both will be of comparable importance in determining the stresses in the material for thick casings.

In a cylindrical tube subject to an internal pressure just great enough to cause flow, the stresses have been worked out*. The radial and tangential stresses are, at distance r from the axis

$$S_r = -T_0 \log \frac{b}{r}$$

$$S_t = T_0 \left(1 - \log \frac{b}{r}\right)$$

where b, a are the external and internal radii; the pressure necessary to cause flow is

$$T_0 \log \frac{b}{a}$$

Here $T_0 = 2\, S_0/\sqrt{3}$ where S_0 is the shearing stress. If p is the actual pressure of the gases, we have an additional pressure at the surface

$$p - T_0 \log \frac{b}{a}$$

giving a hydrostatic pressure at a distance r from the axis equal to

$$\left(p - T_0 \log \frac{b}{a}\right) \frac{a}{b-a} \left(\frac{b}{r} - 1\right)$$

* Nadai, Plasticity, McGraw Hill Book Co., p.188.

The stresses can thus be resolved into

(1) A tangential stress T_0
(2) A hydrostatic pressure equal to

$$\left(p - T_0 \log \frac{b}{a}\right)\frac{a(b-r)}{(b-a)r} + T_0 \log \frac{b}{r} ,$$

which vanishes at the outside surface and reaches the value p at the inner surface.

Now it is known that hydrostatic pressure makes fracture more difficult, while having little effect on the resistance to glide. For non-plastic materials, where fracture starts from a microscopic crack, the following account of the effect of hydrostatic pressure has been given by A.A.Griffiths in a well-known paper*. Suppose elliptical cracks are acted on by a stress T and a hydrostatic pressure P; the angle made by the plane of any crack to the normal to T is denoted by θ, and θ is distributed over all values. Then T will be great enough to cause cracks to spread under the following conditions :

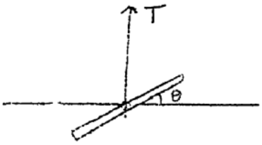

Fig. 11

(a) $4p < 3T$ If this condition is fulfilled, cracks for which $\theta = 0$ will be the first to spread, where T reaches a value $K + p$, where K depends on the elastic constants and surface tension of the metal, and the dimensions of the crack.

(b) $4p > 3T$ Under these conditions cracks for which $\theta = 0$ will not be the first to spread, but those for which

$$\cos 2\theta = \frac{1}{2}\frac{T}{2p - T}$$

At the critical pressure given by $4p = 3T$, this gives $\theta = 45°$.

In plastic materials it is probable that the high tensile stress T near the apex of a crack will cause cracks to form in crystal grains near to it. As the apex of the crack travels inwards, if a point is reached where 4 p exceeds 3 T, the crack should abruptly change its direction by 45°. This is just what is observed.

Since T is certainly greater than T_0, a necessary condition for such a change of direction will be

$$\frac{4}{3} p > T_0$$

where p is the pressure exerted by the explosive. For casings that break up at 30 and 50% expansions respectively, the calculated values of 4p/3 are 73 and 35 tons per sq. inch, which are of the same order as T_0, though they are somewhat less than the values that we have assumed to hold for the metal at high rates of strain.

For this reason we put forward the above explanation somewhat tentatively.

12. COMPARISON WITH OBSERVED FRAGMENTATION OF SERVICE PROJECTILES.

In this memorandum we have reached the following conclusions :

For a given type of steel

(a) The ratio of length to breadth of fragments is constant.

(b) The average fragment area is proportional to $(r/V)^{2s}$, where s lies between 2/3 and 1, probably nearer the latter value.

(c) The weight distribution is given approximately by formula (3)

* Proc. Int. Congress for Applied Mechanics. Delft (1924), p.55.

We may thus equate M_A of formula (3) to

$$\text{const } t^{\frac{1}{2}} \left(\frac{Y}{V}\right)^s$$

where the value of the constant depends on the properties of the steel, or, making use of formula (1) for the velocity

$$M_A = \text{const } t^{5/6} d_2^{1/3} \left(1 + \frac{t}{d_2}\right) \qquad s = \frac{2}{3}$$

$$M_A = \text{const } t\, d_2^{\frac{1}{2}} \left(1 + \frac{t}{d_2}\right) \qquad s = 1$$

where the constant depends on the type of explosive and steel, d_2 is the internal diameter and t the thickness of the casing. The first of these formulae has already been compared with experiment in 2), in which M_A was determined for a number of service weapons.

Comparison with fragmentation of observed projectiles should show whether s = 1 or s = 2/3 or some intermediate value gives the best fit. Ursell* has determined the best value of M_A for three model bombs fragmented by Payman **, with thicknesses 0.018, 0.125 and 0.3 inches (diameter 2"). He comes to the conclusion that M_A is proportional to $1/V^{1.2}$. The casings of these bombs were of mild steel and gave shear fracture, and so are not directly comparable with our theory. Unfortunately the range of values of r and V available in British shells of carbon steel for which detailed information is available is not great enough to allow any certain conclusion to be drawn.

Observed values of M_A for a number of projectiles filled with TNT are shown in Table VII; we have limited ourselves to those with a reasonably cylindrical cross section. It looks as though s = 1 gave rather a better fit than s = 2/3.

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* A.W.A.S, Report No. 46; Ministry of Supply No. A.C.3817

** loc. cit.,

TABLE VII

Projectile	d_1 inches	t inches	$M_A (oz)^{\frac{1}{2}}$ observed	$\frac{M_A}{t^{5/6} d_2^{1/3}(1+\frac{t}{d_2})}$	$\frac{M_A}{t d_2^{1/2}(1+\frac{t}{d_2})}$
3" U.P.	3.25	0.265	0.134	0.265	0.27
95 mm.shell	3.7	0.425	0.23	0.29	0.26
U.P.(thick-cased)	3. 5	0.50	0.30	0.32	0.29
3.7" shell	3.7	0.60	0.36	0.32	0.275±.015
25 pr. shell	3.43	0.65	0.35±0.03	0.29±.025	0.245±.02

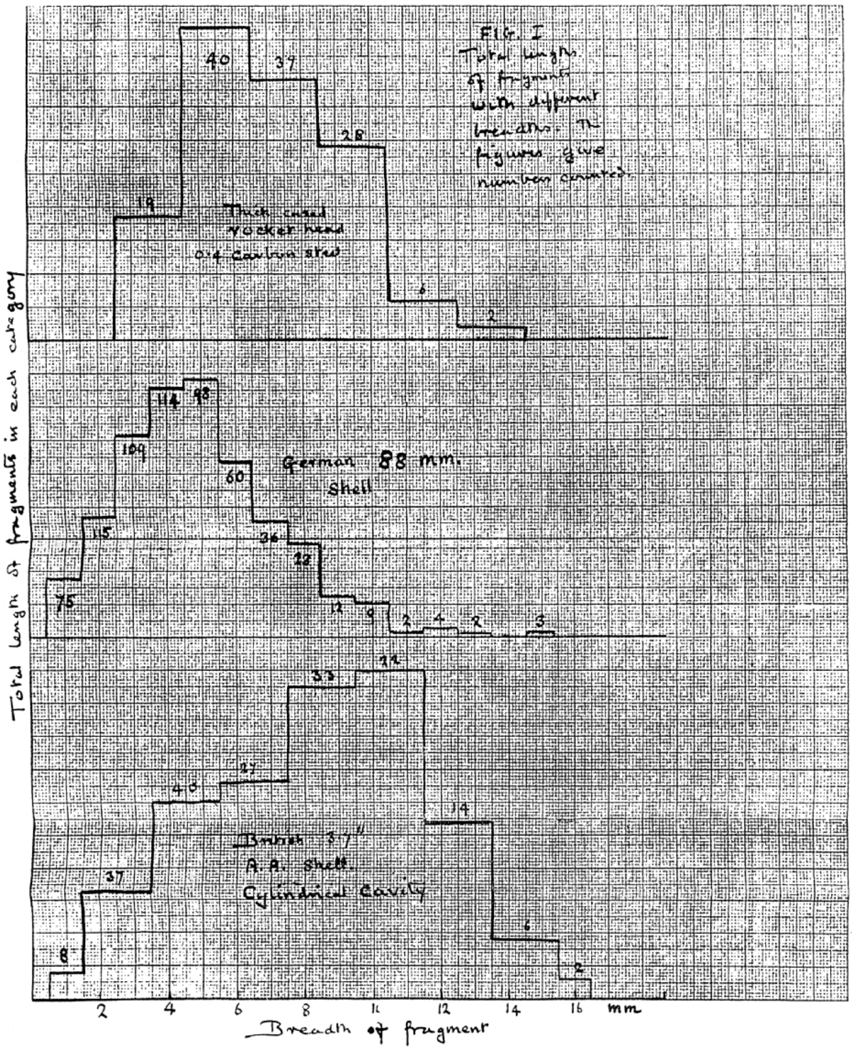


FIG. I
Total length of fragments with different breadths. The figures give numbers counted.
Thick cased rocket head
0·4 Carbon steel
German 88 mm. Shell
British 3.7" A.A. Shell. Cylindrical Cavity
Total length of fragments in each category
Breadth of fragment
mm

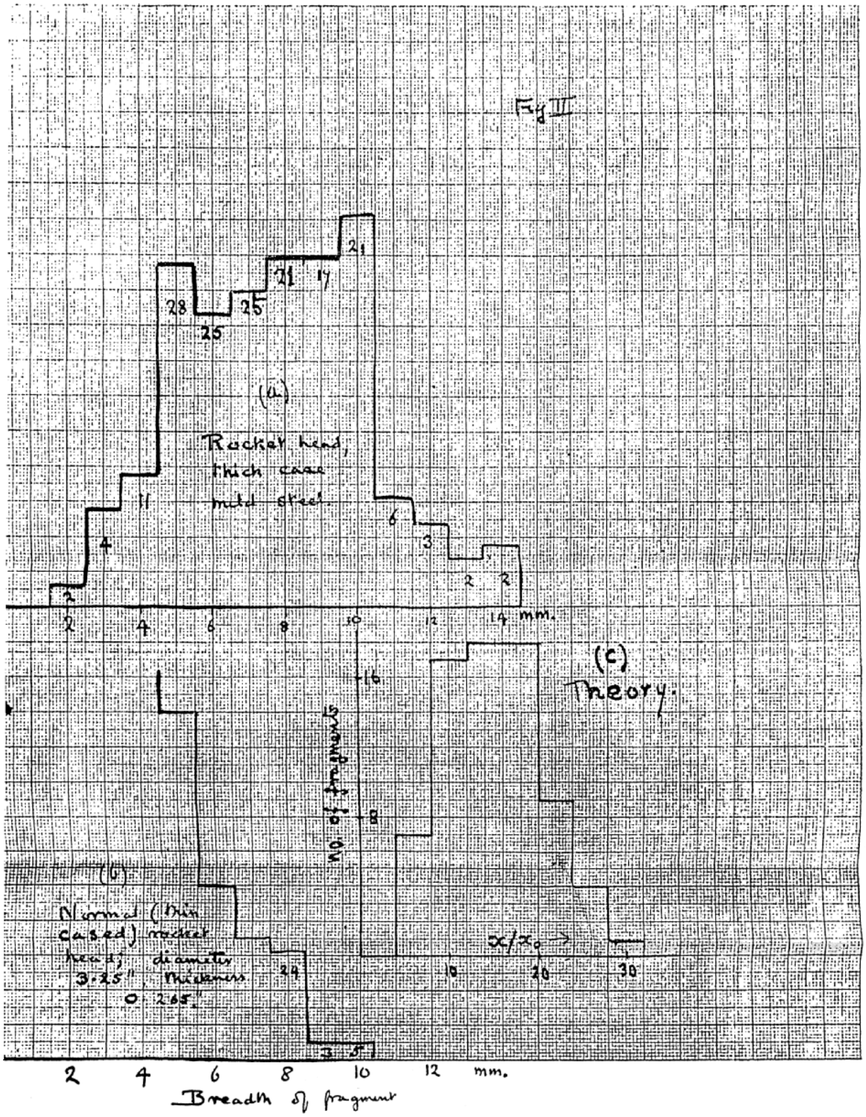


Fig II
(a)
Rocket head, thick case mild steel.
(b)
Normal (thin cased) rocket head, diameter 3·25" thickness 0·265"
(c)
Theory.
No. of fragments
x/x₀
mm.
Breadth of fragment

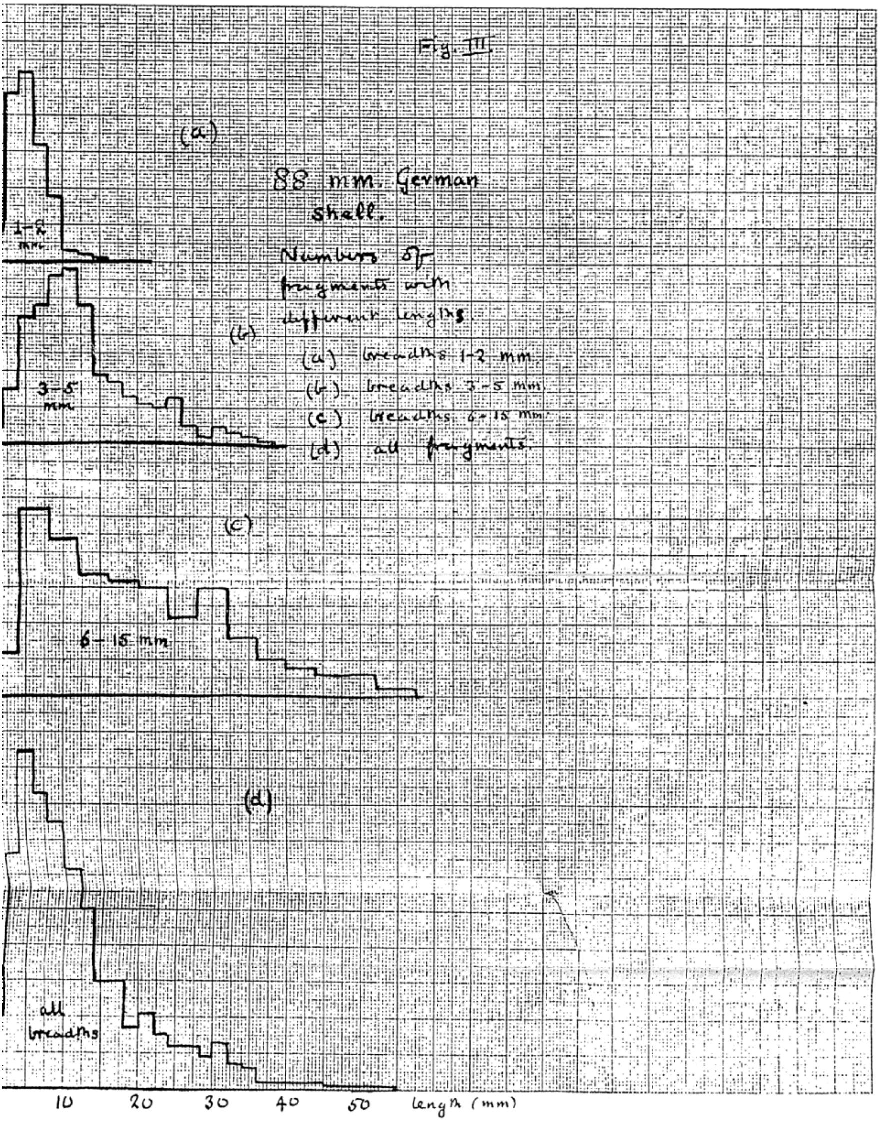
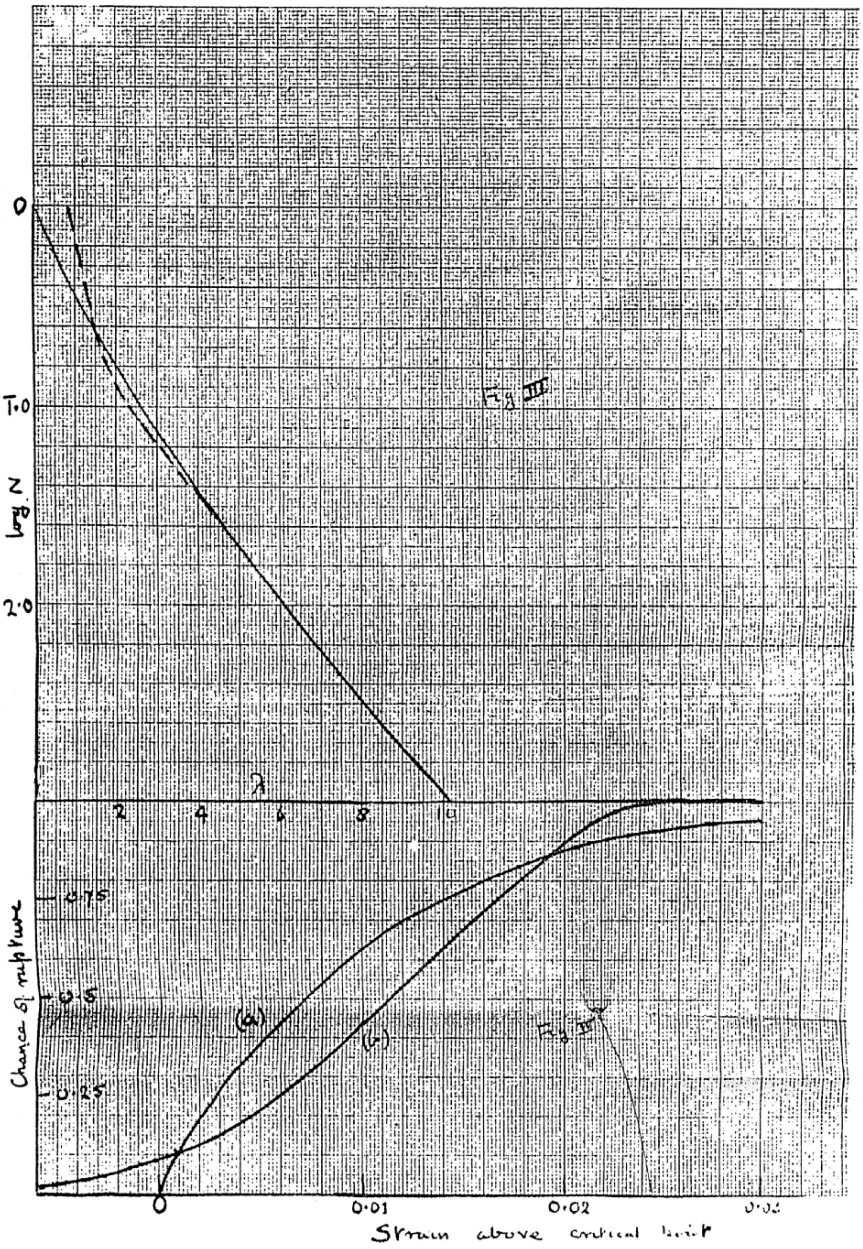


Fig. III
(a)
88 mm. German shell.
Numbers of fragments with different lengths.
(a) breadths 1-2 mm.
(b) breadths 3-5 mm
(c) breadths 6-15 mm
(d) all fragments.
1-2 mm
(b)
3-5 mm
(c)
6-15 mm
(d)
all breadths
10
20
30
40
50
length (mm)



0
1.0
log N
2.0
Fig III
λ
2
4
6
8
10
0.75
0.5
0.25
Chance of rupture
(a)
(b)
Fig IV
0
0.01
0.02
Strain above critical limit