An Improved Approximation to the One-Sided Bilayer Drawing

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Abstract. Given a bipartite graph $G = (V, W, E)$, a bilayer drawing consists of placing nodes in the first vertex set V on a straight line L_1 and placing nodes in the second vertex set W on a parallel line L_2. The one-sided crossing minimization problem asks to find an ordering of nodes in V to be placed on L_1 so that the number of arc crossings is minimized. In this paper, we prove that there always exits a solution whose crossing number is at most 1.4664 times of a well-known lower bound that is obtained by summing up $\min\{c_{uv}, c_{vu}\}$ over all node pairs $u, v \in V$, where c_{uv} denotes the number of crossings generated by arcs incident to u and v when u precedes v in an ordering.

1 Introduction

Given a bipartite graph $G = (V, W, E)$, a bilayer drawing consists of placing nodes in the first vertex set V on a straight line L_1 and placing nodes in the second vertex set W on a parallel line L_2. The problem of minimizing the number of crossings between arcs in a bilayer drawing was first introduced by Harary and Schwenk [5,6]. The one-sided crossing minimization problem asks to find an ordering of nodes in V to be placed on L_1 so that the number of arc crossings is minimized (while the ordering of the nodes in W on L_2 is given and fixed).

The problem has many applications such as VLSI layouts [11] and hierarchical drawings [1]. However, the two-sided and one-sided problems are shown to be NP-hard by Garey and Johnson [4] and by Eades and Wormald [3], respectively. Muñoz et al. [10] have proven that the one-sided problem remains to be NP-hard even for sparse graphs such as forests of 4-stars. Recently Dujmović and Whitesides [2] have given an $O(\phi^c \cdot n^2)$ time algorithm to the one-sided problem, where c is the number of crossings to be checked, $n = |V| + |W|$ and $\phi = \frac{1+\sqrt{5}}{2}$, thus showing that the problem is Fixed Parameter Tractable.

There are several heuristics that deliver theoretically or empirically good solutions. The so-called barycenter heuristic finds an $O(\sqrt{n})$-approximation solution or a $(d-1)$-approximation solution, where d is the maximum degree of nodes in the free side V (see [8] for the analysis). Eades and Wormald [3] proposed a simple and theoretically better heuristic, the median heuristic which delivers a 3-approximation solution. They also prove that the performance guarantee of the median heuristic approaches to 1 if graphs become dense. Yamaguchi and

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Sugimoto [13] gave a 2-approximation algorithm if $d \leq 4$. For all the known performance guarantees of these heuristics are based on a conventional lower bound that is obtained by summing up $\min\{c_{uv}, c_{vu}\}$ over all node pairs $u, v \in V$, where c_{uv} denotes the number of crossings generated by arcs incident to u and v when u precedes v in an ordering. An extensive computational experiment of several heuristics including the above two has been conducted by Jünger and Mutzel [7] and by Mäkinen [9]. Jünger and Mutzel [7] reported that most of the heuristics gave good solutions whose crossing numbers are nearly equal to the lower bound. However the theoretically best estimation to the gap between the optimal and the lower bound is 3 due to the heuristic by Eades and Wormald [3].

In this paper, we prove that there always exists a solution whose crossing number is at most 1.4664 times of the lower bound. Our argument is based on a probabilistic analysis, which provides a polynomial randomized algorithm that delivers a solution whose average number of crossings is at most 1.4664 times of the optimal.

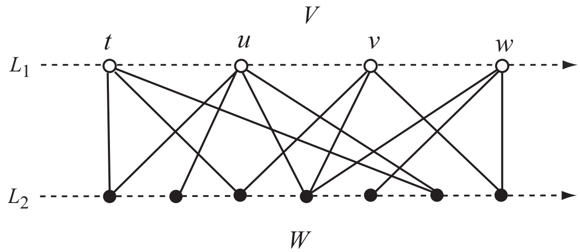
2 Preliminaries

Let $G = (V, W, E)$ be a bipartite graph with a partition V and W of a node set. Assume that G has no isolated node. Let π denote a permutation of $\{1, 2, \ldots, |V|\}$ and σ denote a permutation of $\{1, 2, \ldots, |W|\}$. A pair of π and σ defines a *bilayer drawing* of G in the plane in such a way that, for two parallel horizontal lines L_1 and L_2, the nodes in V (resp., in W) are arranged on L_1 (resp., L_2) according to π (resp., σ) and each arc is depicted by a straight line segment joining the end-nodes, where the directions for traversing L_1 and L_2 are taken as the same one (see Fig. 1(a)). In a bilayer drawing (π, σ) of G two arcs $(v, w), (v', w') \in E$ intersect properly (or create a *crossing*) if and only if $(\pi(v) - \pi(v'))(\sigma(w) - \sigma(w'))$ is negative. In this paper, we consider the following problem.

One-sided Crossing Minimization: Given a bipartite graph $G = (V, W, E)$ and a permutation σ on W, find a permutation π on V that minimizes the number of crossings in a bilayer drawing (π, σ) of G.

Since the permutation σ on $W = \{1, 2, \ldots, |W|\}$ is fixed, we assume throughout the paper that $\sigma(i) = i$ for all $i \in W$. For each node u in G, let $\Gamma(u)$ denote the set of nodes adjacent to u, and let $d_u = |\Gamma(u)|$. For two nodes $u, v \in V$, let $\Delta_{uv} = |\Gamma(u) \cap \Gamma(v)|$. The *crossing number* c_{uv} for an ordered pair of two nodes $u, v \in V$ is the number of crossing generated by an arc incident to u and an arc incident to v when $\pi(u) < \pi(v)$ holds in a bilayer drawing (π, σ). (Fig. 1(b) shows the crossing numbers in the graph in Fig. 1(a).) It is a simple matter to see that

$$d_u d_v = c_{uv} + c_{vu} + \Delta_{uv}, \quad \min\{c_{uv}, c_{vu}\} \geq \frac{\Delta_{uv}(\Delta_{uv} - 1)}{2}.$$



(a)

	t	u	v	w
t	-	5	2	2
u	5	-	3	2
v	6	8	-	2
w	7	9	5	-

(b)

Fig. 1. (a) A bilayer drawing of a bipartite graph. (b) Crossing numbers for each pair of nodes in the top layer.

For a permutation π on V, let

$$cross(u, v; \pi) := \begin{cases} c_{uv} & \text{if } \pi(u) < \pi(v), \\ c_{vu} & \text{otherwise.} \end{cases}$$

Define

$$cross(\pi) := \sum_{u,v \in Y : \pi(u) < \pi(v)} c_{uv} = \sum_{u,v \in V} cross(u, v; \pi).$$

The optimal to the problem is denoted by $opt = \min\{cross(\pi) \mid \text{permutation } \pi$ on $V\}$. For $LB = \sum_{u,v \in V} \min\{c_{uv}, c_{vu}\}$, it holds

$$opt \geq LB.$$

In this paper, we prove the next result.

Theorem 1. *For a bipartite graph $G = (V, W, E)$ with a permutation σ on W, there exists a permutation π on V such that $cross(\pi) \leq 1.4664LB$.* □

Note that the one-sided crossing minimization is a purely combinatorial problem in the sense that the number of crossings is determined by a permutation π, not by the actual positions of nodes in the layers. However, in this paper, we convert the problem into a geometric problem to derive Theorem 1. For this, we first introduce a geometric representation that illustrates how two sets $\Gamma(u)$ and $\Gamma(v)$ determine crossing numbers c_{uv} and c_{vu} in a bipartite graph G.

Rectangles that we treat here are axis-parallel in the xy-coordinate, and they are denoted by the coordinates of the lower-left corner and the upper-right corner, where the x-coordinate increases in the right direction and the y-coordinate increases in the upward direction. For example, $[(0, 0), (1/2, 1)]$ represents the square with four corners $(0, 0), (0, 1), (1/2, 0)$ and $(1/2, 1)$.

Let S denote the square $[(0, 0), (1, 1)]$. For a connected region R in S, we may use R to denote the sets of points in the region R, and let $a(R)$ denote the area size of R. For two points $b, b' \in S$, a line segment connecting b and b' is denoted by bb'. A part of the boundary of a region R may be called an *edge* if it is a line segment. For a line segment (or an edge) e, its length is denoted by $\ell(e)$. We say that edge e *overlaps* with another edge e' if the intersection of e and e' is a line segment of a positive length.

A path P between points $(0,0)$ and $(1,1)$ in S is called *monotone* if none of the x- and y-coordinates of the point on P decreases when we traverse points on P from $(0,0)$ to $(1,1)$ (in general a monotone path is not necessarily piecewise linear).

For two integers $d, d' \geq 1$, the square $S = [(0,0),(1,1)]$ is called (d,d')-*sliced* if it is sliced by $(d-1)$ horizontal line segments and $(d'-1)$ vertical segments so that these line segments give rise to $d \times d'$ congruent rectangles. Each of such rectangles is called a *block*, which has four edges.

We represent the positions of nodes in $\Gamma(u)$ and $\Gamma(v)$ in the permutation σ by using the unit square S in the xy-coordinate. Let $\Gamma(u) = \{u'_1, u'_2, \ldots, u'_{d_u}\}$ and $\Gamma(v) = \{v'_1, v'_2, \ldots, v'_{d_v}\}$. For an ordered pair (u,v) of nodes in V, we consider $d_u d_v$ blocks in the (d_u, d_v)-sliced square S. We denote these blocks by

$$bl(i,j) = [(\tfrac{j-1}{d_v}, \tfrac{i-1}{d_u}), (\tfrac{j}{d_v}, \tfrac{i}{d_u})],\ 1 \leq i \leq d_u \text{ and } 1 \leq j \leq d_v$$

We let $bl(i,j)$ correspond to a pair of arcs (u, u'_i) and (v, v'_j). Note that arcs (u, u'_i) and (v, v'_j) create a crossing in a permutation π with $\pi(u) < \pi(v)$ or $\pi(u) > \pi(v)$ if $u'_i \neq v'_j$, but generate no crossing in any permutation π otherwise. We call a block $bl(i,j)$ with $u'_i \neq v'_j$ an *up-block* if arcs (u, u'_i) and (v, v'_j) creates a crossing in a permutation π with $\pi(u) < \pi(v)$ and an *down-block* otherwise. We call a block $bl(i,j)$ with $u'_i = v'_j$ a *neutral-block*. Observe that the number of up-blocks (resp., down-blocks and neutral-blocks) is equal to c_{uv} (resp., c_{vu} and $\Delta_{uv} = \Delta_{vu}$). We here partition the set of these blocks into two groups UP and DWN as follows (where a neutral-block may be split into two half blocks in the partitioning).

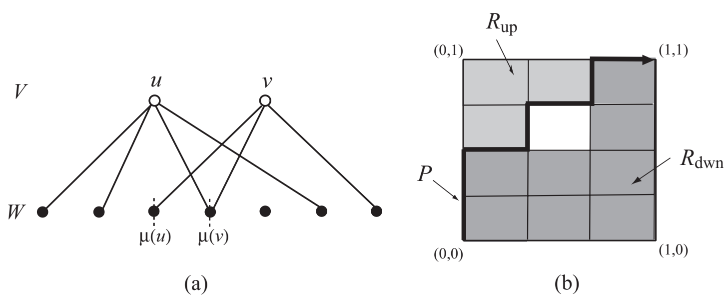


Fig. 2. (a) Two nodes u and v in the top layer, where $c_{uv} = 3$ and $c_{vu} = 8$. (b) A (u,v)-path P of a $(4,3)$-sliced square S in the case of (i).

Definition 1. *For each node $u \in V$, where $\Gamma(u) = \{w_1, w_2, \ldots, w_{d_u}\} \subseteq W$ $(w_1 < w_2 < \cdots < w_{d_u})$, we define the* median index $\mu(u)$ *of its neighbors by*

$$\mu(u) := \begin{cases} w_{\frac{d_u+1}{2}} & \text{if } d_u \text{ is odd,} \\ \frac{1}{2}(w_{\frac{d_u}{2}} + w_{\frac{d_u}{2}+1}) & \text{if } d_u \text{ is even.} \end{cases}$$

(i) *If $\mu(u) < \mu(v)$, then let UP be the set of all up-blocks, and DWN be the set of down-blocks and neutral-blocks (see Fig. 2).*
(ii) *If $\mu(u) > \mu(v)$, then let UP be the set of all up-blocks and neutral-blocks, and DWN be the set of down-blocks (see Fig. 3).*
(iii) *If $\mu(u) = \mu(v)$, then split each neutral-block $[p, q]$ into two parts by the line segment pq, and put the upper-left part into UP and the other in DWN. Then put all up-blocks in the UP, and all down-blocks in the DWN.*

The set of all points in the blocks in UP forms a connected region, which we denoted by R_{up}. Similarly R_{dwn} is defined by DWN. □

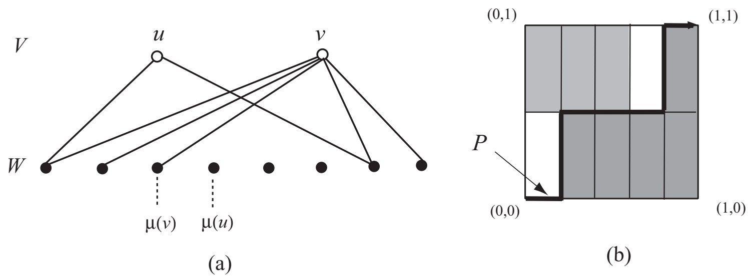


Fig. 3. (a) Two nodes u and v in the top layer. (b) A (u, v)-path P of a $(2, 5)$-sliced square S in the case of (ii).

From the definition, we observe the next property.

Lemma 1. *Let R_{up} and R_{dwn} be the regions defined for an ordered pair of nodes u and v in V. Then there is a monotone path P that separates S into R_{up} and R_{dwn}, and it holds*

$$a(R_{up}) = \begin{cases} \frac{c_{uv}}{d_u d_v} & \text{if } \mu(u) < \mu(v), \\ \frac{c_{uv} + \frac{\Delta_{uv}}{2}}{d_u d_v} & \text{if } \mu(u) = \mu(v), \\ \frac{c_{uv} + \Delta_{uv}}{d_u d_v} & \text{if } \mu(u) > \mu(v). \end{cases}$$

Moreover, R_{up} contains point $(0.5, 0.5)$ if $\mu(u) \geq \mu(v)$. □

Such a path P in the lemma is called the *(u, v)-path* with respect to G and σ.

Lemma 2. *Let $u, v \in V$ be two nodes in (G, σ) such that $1 \leq c_{uv} < c_{vu}$ and $c_{uv} < 2\Delta_{uv}$. Then $\mu(u) < \mu(v)$ holds unless u and v satisfies one of the following conditions* (1), (2) *and* (3)*:*

$$c_{uv} = 3,\ c_{vu} = 4,\ d_u = d_v = 3 \text{ and } \Delta_{uv} = 2, \tag{1}$$

$$c_{uv} = 3,\ c_{vu} = 5,\ \{d_u, d_v\} = \{2, 5\} \text{ and } \Delta_{uv} = 2 \text{ (see Fig. 3)}, \tag{2}$$

$$c_{uv} = 5,\ c_{vu} = 7,\ \{d_u, d_v\} = \{3, 5\} \text{ and } \Delta_{uv} = 3. \tag{3}$$

Proof. Omitted. □

We close this section by showing some technical lemmas.

Lemma 3. *For constants $a > 0, b, c > 0$ and d such that $ad - bc \geq 0$, function $f(x) = (ax + b)(\frac{1}{cx+d} - 2)$ takes the maximum $\frac{1}{c}(\sqrt{a} - \sqrt{2(ad - bc)})^2$ over x with $cx + d > 0$.* □

Lemma 4. *For four positive constants a, b, c and d with $\frac{b}{a} < d \leq \frac{1}{\sqrt{2c}}$, function $f(x) = (ax - b)^2(\frac{1}{cx^2} - 2)$ ($\frac{b}{a} < x \leq d$) takes the maximum at $x = \min\{d, (\frac{b}{2ac})^{\frac{1}{3}}\}$.* □

3 Algorithm and Analysis

Let $\theta : V \to [0, 1]$ be a function from V to the set of reals in $[0, 1]$, where $\theta(u)$ is called the *real key* of node u. Given a real-key function θ, we construct a permutation π_θ of $\{1, 2, \ldots, |V|\}$ by the next procedure.

PERMUTE($\theta; \pi_\theta$)**: Step 1.** For each node $u \in V$, compute $j = \lceil \theta(u) d_u \rceil$, and define an *integer key* $\kappa(u)$ of u by

$$\kappa(u) := w_j \text{ for the } j\text{-th neighbor } w_j \in \Gamma(u),$$

where $\Gamma(u) = \{w_1, w_2, \ldots, w_{d_u}\}$ ($w_1 < w_2 < \cdots < w_{d_u}$).

Step 2. Sort nodes $u \in V$ in the lexicographical order with respect to $(\kappa(u), \mu(u))$, where the ties among nodes u with the same key $(\kappa(u), \mu(u))$ are broken randomly. We denote by π_θ the resulting permutation of $\{1, 2, \ldots, |V|\}$. □

We see the following important property.

Lemma 5. *For two nodes $u, v \in V$, let R_{up} and R_{dwn} be the regions in Definition 1. Then for a given real-key function θ, $\pi_\theta(u) < \pi_\theta(v)$ if point $(\theta(u), \theta(v))$ is inside R_{dwn} and $\pi_\theta(u) > \pi_\theta(v)$ if point $(\theta(u), \theta(v))$ is inside R_{up}.* □

A *scheme* based on which we choose a real-key function θ probabilistically is defined by a set of tuples of reals $\mathcal{S} = \{(s_i, t_i, p_i) \mid i = 1, 2, \ldots, h\}$, such that $0 < s_i \le t_i < 1$ and $0 \le p_i$ for $i = 1, 2, \ldots, h$ and $\sum_{1 \le i \le h} p_i = 1$, where we call each (s_i, t_i, p_i) a *subscheme*. Given a scheme $\mathcal{S}$, we choose a real-key function θ in the following manner.

RANDOM-KEY$(\mathcal{S}; \theta)$**:**
Step 1. Choose a subscheme $(s_i, t_i, p_i) \in \mathcal{S}$ with probability p_i.

Step 2. For each node $u \in V$, choose a real key $\theta(u)$ from $[s_i, t_i]$ uniformly. □

We denote by $E_{\mathcal{S}}[cross(u, v; \pi_\theta)]$ and $E_{\mathcal{S}}[cross(\pi_\theta)]$ respectively the expectations of $cross(u, v; \pi_\theta)$ and $cross(\pi_\theta)$ over all real-key functions θ resulting from RANDOM-KEY. In this paper, we prove the next result.

Theorem 2. *There is a scheme $\mathcal{S}$ such that $E_{\mathcal{S}}[cross(\pi_\theta)] \le 1.4664LB$.* □

By the linearity of expectations, if we have a constant $\alpha \ge 1$ such that

$$E_{\mathcal{S}}[cross(u, v; \pi_\theta)] \le \alpha \min\{c_{uv}, c_{vu}\}, \quad u, v \in V,$$

then it holds $E_{\mathcal{S}}[cross(\pi_\theta)] \le \alpha LB$.

In the rest of this paper, we fix two nodes $u, v \in V$, and analyze $E_{\mathcal{S}}[cross(u, v; \pi_\theta)]$ for a given scheme $\mathcal{S}$. Without loss of generality we assume that $1.46c_{uv} < c_{vu}$ and $d_u \le d_v$ (the case of $\max\{c_{uv}, c_{vu}\} < 1.46 \min\{c_{uv}, c_{vu}\}$ needs no special consideration to prove Theorem 2, and we have $d_u \le d_v$ by renaming u and v after reversing the permutation σ). Note that none of (1) and (3) holds since $1.46c_{uv} < c_{vu}$. Moreover, we can assume that $c_{uv} \ge 1$ since otherwise (i.e., $c_{uv} = 0$) $\pi_\theta(u) < \pi_\theta(v)$ holds in any permutation π_θ computed by PERMUTE due to the comparison of $\mu(u)$ and $\mu(v)$.

For a given scheme $\mathcal{S}$ and a region $R \subseteq S$, let $p_{\mathcal{S}}(R)$ denote the probability that point $(\theta(u), \theta(v))$ falls inside R. By Lemma 5, we observe the next formula.

Lemma 6. $E_{\mathcal{S}}[cross(u, v; \pi_\theta)] = p_{\mathcal{S}}(R_{dwn})c_{uv} + p_{\mathcal{S}}(R_{up})c_{vu}$. □

We are ready to derive an important inequality.

Lemma 7. *Assume that $1 \le c_{uv} < c_{vu}/1.46$ holds. Then it holds*

$$\frac{E_{\mathcal{S}}[cross(u, v; \pi_\theta)]}{\min\{c_{uv}, c_{vu}\}} \le \begin{cases} 1 + p_{\mathcal{S}}(R_{up})(\frac{1}{a(R_{up})} - 2) & \text{if } \mu(u) < \mu(v), \\ 1 + p_{\mathcal{S}}(R_{up})(\frac{1.5}{a(R_{up})} - 2.5) & \text{if } \mu(u) \ge \mu(v) \end{cases}$$

unless (2) *holds.*

Proof. By Lemma 6, we get

$$\frac{E_{\mathcal{S}}[cross(u,v;\pi_\theta)]}{\min\{c_{uv},c_{vu}\}} = \frac{p_{\mathcal{S}}(R_{dwn})c_{uv} + p_{\mathcal{S}}(R_{up})c_{vu}}{c_{uv}}$$
$$= \frac{(1-p_{\mathcal{S}}(R_{up}))c_{uv} + p_{\mathcal{S}}(R_{up})(d_u d_v - c_{uv} - \Delta_{uv})}{c_{uv}} = 1 + p_{\mathcal{S}}(R_{up})(\frac{d_u d_v - \Delta_{uv}}{c_{uv}} - 2).$$

First consider the case of $\mu(u) < \mu(v)$. By Lemma 1, we have $a(R_{up}) = \frac{c_{uv}}{d_u d_v}$. Hence

$$\frac{d_u d_v - \Delta_{uv}}{c_{uv}} - 2 = \frac{1}{c_{uv}}(\frac{c_{uv}}{a(R_{up})} - \Delta_{uv}) - 2 \leq \frac{1}{a(R_{up})} - 2.$$

Next consider the case of $\mu(u) \geq \mu(v)$. By Lemma 1, we have $a(R_{up}) \leq \frac{c_{uv}+\Delta_{uv}}{d_u d_v}$. Since $1 \leq c_{uv} < c_{vu}$ holds but (2) does not holds for the u and v, Lemma 2 implies $\Delta_{uv} \leq c_{uv}/2$. Then

$$\frac{d_u d_v - \Delta_{uv}}{c_{uv}} - 2 \leq \frac{1}{c_{uv}}(\frac{c_{uv}+\Delta_{uv}}{a(R_{up})} - \Delta_{uv}) - 2 = \frac{c_{uv} + \Delta_{uv}(1 - a(R_{up}))}{c_{uv}a(R_{up})} - 2$$
$$\leq \frac{c_{uv} + \frac{1}{2}c_{uv}(1 - a(R_{up}))}{c_{uv}a(R_{up})} - 2 = \frac{1.5}{a(R_{up})} - 2.5.$$

This completes the proof. □

We wish to find a scheme $\mathcal{S}$ that minimizes $\max_{u,v\in V} \frac{E_{\mathcal{S}}[cross(u,v;\pi_\theta)]}{\min\{c_{uv},c_{vu}\}}$ (even though finding such an $\mathcal{S}$ analytically seems a hard problem). For this, we consider the set of all monotone paths P for a given scheme $\mathcal{S}$. Let P be an arbitrary monotone path between points $(0,0)$ and $(1,1)$ in a unit square S (not necessarily a (u,v)-path for particular nodes $u,v \in V$). Define $R_{up}(P)$ and $R_{dwn}(P)$ be the regions obtained by splitting S with P, where we assume that $R_{up}(P)$ is above $R_{dwn}(P)$. Let

$$\beta(\mathcal{S},P) := \begin{cases} p_{\mathcal{S}}(R_{up}(P))(\frac{1}{a(R_{up}(P))} - 2) & \text{if } (0.5,0.5) \notin R_{up}(P), \\ p_{\mathcal{S}}(R_{up}(P))(\frac{1.5}{a(R_{up}(P))} - 2.5) & \text{if } (0.5,0.5) \in R_{up}(P), \end{cases}$$

and $\beta(\mathcal{S}) := \max\{\beta(\mathcal{S},P) \mid \text{ monotone path } P\}$. Given a scheme $\mathcal{S}$, a monotone path P from $(0,0)$ to $(1,1)$ in a unit square S is called $\mathcal{S}$-*maximal* if $\beta(\mathcal{S},P) = \beta(\mathcal{S})$.

Since the choice of monotone paths P is relaxed, we obtain $E_{\mathcal{S}}[cross(\pi_\theta)] \leq (1+\beta(\mathcal{S}))LB$. (Recall that $(0.5,0.5) \in R_{up}$ holds if $\mu(u) \geq \mu(v)$ by Lemma 1.) Therefore, to prove Theorem 2, it suffices to show that there exists a scheme $\mathcal{S}$ such that $\beta(\mathcal{S}) < 0.4664$ (provided that the case of (2) is treated separately to prove Theorem 2).

4 A Scheme $\mathcal{S}$

We consider scheme

$$\mathcal{S} = \{(s_1 = 0.0957, t_1 = 0.5, p_1 = 0.5), (s_2 = 0.5, t_2 = 0.9043, p_2 = 0.5)\}.$$

Denote the squares $S_1 = [(s_1, s_1), (0.5, 0.5)]$ and $S_2 = [(0.5, 0.5), (t_2, t_2)]$, and the corners of these squares by $A_1 = (0.0957, 0.0957)$, $A_2 = (0.5, 0.5)$, $A_3 = (0.9043, 0.9043)$, $B_1 = (0.5, 0.0957)$, $B_2 = (0.9043, 0.5)$, $C_1 = (0.0957, 0.5)$ and $C_2 = (0.5, 0.9043)$. (The constant 0.0957 and others have been determined through some computational experiment.) Fig. 4 illustrates this scheme.

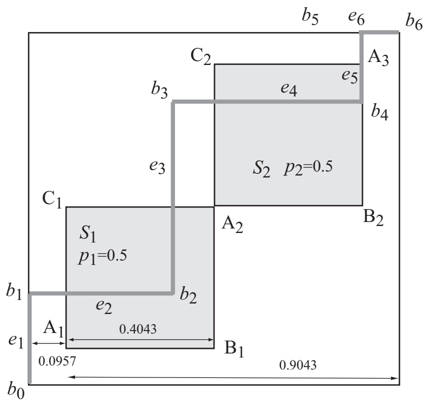


Fig. 4. Illustration of scheme $\mathcal{S} = \{(0.0957, 0.5, 0.5), (0.5, 0.9043, 0.5)\}$, where a grey line indicates an example of a monotone path P.

We first consider the case of (2). In (2), where $c_{uv} = 3$, $c_{vu} = 5$ and $p_{\mathcal{S}}(R_{up}) = p_{\mathcal{S}}(R_{dwn}) = 0.5$ holds, we have by Lemma 6, $E_{\mathcal{S}}[cross(u, v; \pi_\theta)] = p_{\mathcal{S}}(R_{dwn})c_{uv} + p_{\mathcal{S}}(R_{up})c_{vu} = 4 < 1.4664c_{uv}$.

Now consider nodes u and v in the general case. It is not difficult to see that an $\mathcal{S}$-maximal monotone path P consists of axis-parallel line segments, and that the resulting region $R_{up}(P)$ contains at most one convex corner in each subscheme S_i $(i = 1, 2)$. For simplicity, we consider a single subscheme S_i. It should be noted that $p_{\mathcal{S}}(R_{up})$ (or the contribution from S_i to $p_{\mathcal{S}}(R_{up})$) is given by $a(S_i \cap R_{up}(P))/a(S_i)$. As shown in Fig. 5(a), if a monotone path P does not satisfy these properties, then we can modify the path P into another monotone path P' such that $a(S_i \cap R_{up}(P')) = a(S_i \cap R_{up}(P))$ and $a(R_{up}(P')) \le a(R_{up}(P))$. For such an axis-parallel piecewise linear monotone path P, we denote the sequence of the corner points by

$$b_0 = (0, 0), b_1, \ldots, b_k = (1, 1),$$

and the sequence of the edges by

$$e_1 = b_0 b_1, e_2 = b_1 b_2, \ldots, e_k = b_{k-1} b_k$$

(see Fig. 4). Let e be an edge on a path P, where e may be a partial segment of some edge e_i. Without loss of generality we further assume that an $\mathcal{S}$-maximal monotone path P is chosen so that the number of edges of squares in subschemes or of the entire unit square that are overlapped by the edges in P is maximized among all $\mathcal{S}$-maximal monotone paths.

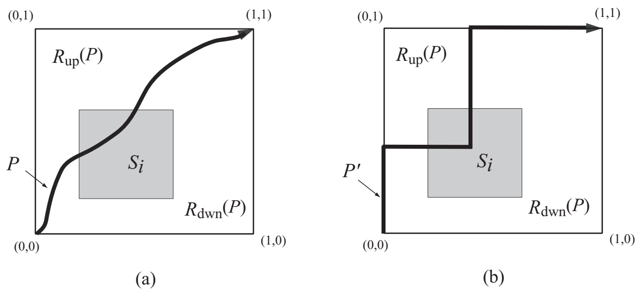


Fig. 5. A monotone path P that passes through a square S_i.

We define the *gain* of edge e at a subscheme $S_i = (s_i, t_i, p_i) \in \mathcal{S}$ as follows. Consider how much amount of $p_{\mathcal{S}}(R_{up})$ changes if we move the line segment L in its orthogonal direction by an infinitely small amount ϵ. The change in $p_{\mathcal{S}}(R_{up})$ is

$$\frac{\epsilon \cdot \ell(e \cap S_i) \cdot p_i}{(t_i - s_i)^2},$$

where $\ell(e \cap S_i)$ means the length of the intersection of e and S_i. On the other hand, the change in $a(R_{up}(P))$ is $\epsilon \cdot \ell(e)$. The *gain* is defined by the ratio of these two, i.e.,

$$g(e) = \frac{\ell(e \cap S_i) \cdot p_i}{(t_i - s_i)^2 \cdot \ell(e)}.$$

A vertical line segment e on a path P is called *incrementable* (resp., *decrementable*) if

- There is a real $\delta > 0$ such that e has the same gain $g(e)$ (with respect to a subscheme S_i) after translating it rightward (resp., leftward) by any amount $\delta' \in [0, \delta]$ (i.e., e remains to be intersecting S_i),
- For the rectangle R formed between e and the translated edge e' and the current path P, there is a monotone path P' such that $R_{up}(P') = R_{up}(P) \cup R$ (resp., $R_{up}(P') = R_{up}(P) - R$).

Analogously, the incrementability (resp., decrementability) of a horizontal line segment e is defined by replacing "rightward" with " downward" (resp., "leftward" with "upward").

An edge e_i between two corners in a path P is called a *free edge* if it does not overlap with any edge of square S_i in a subscheme or of the entire unit square S. For example, in Fig. 4, e_2, e_3 and e_4 are free edges, and e_5 is decrementable, but not incrementable.

By definition, we observe the following.

Lemma 8. *For an $\mathcal{S}$-maximal monotone path P, let e and e' be respectively an incrementable edge and a decrementable edge such that $(0.5, 0.5)$ is not an internal point in any of e and e'. Then if e and e' are not adjacent, then $g(e) < g(e')$. If e and e' are adjacent, then $g(e) = g(e')$.*

Proof. Otherwise we would have another monotone path P' such that $\beta(\mathcal{S}, P') > \beta(\mathcal{S}, P)$ or such that $\beta(\mathcal{S}, P') = \beta(\mathcal{S}, P)$ and P' overlaps with more edges of the squares than P does. □

In particular, there is no pair of non-adjacent free edges in an $\mathcal{S}$-maximal monotone path P.

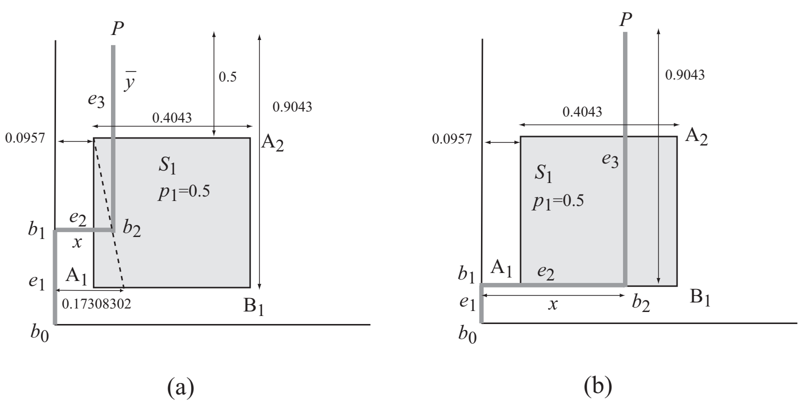


Fig. 6. Illustration for Case-(A,1), where (a) indicates the case where a corner point of P is inside S_1, and (b) indicates the case where edge A_1B_1 of S_1 is overlapped by edge e_2 of P.

In the sequel, P is assumed to be an $\mathcal{S}$-maximal monotone path, and for simplicity $R_{up}(P)$ is written by R_{up}. To prove that $\beta(\mathcal{S}, P) \leq 0.4664$ holds for our scheme $\mathcal{S}$, we distinguish the following cases.

Case-A: point $(0.5, 0.5)$ is not on the boundary of R_{up} or inside R_{up}; $\beta(\mathcal{S}, P)$ is given by $p_{\mathcal{S}}(R_{up})(\frac{1}{a(R_{up})} - 2)$.

Case-B: point $(0.5, 0.5)$ is on the boundary of R_{up} or inside R_{up}; $\beta(\mathcal{S}, P)$ is given by $p_{\mathcal{S}}(R_{up})(\frac{1.5}{a(R_{up})} - 2.5)$.

Case-1: R_{up} contains an internal point from exactly one of S_1 and S_2; Without loss of generality R_{up} contains no internal point in S_2.

Case-2: R_{up} contains an internal point from each of S_1 and S_2.

In the following, we only treat Case-(A,1) due to the space limitation.

In this case, R_{up} has no convex corner in S_2, and exactly one convex corner b_3 in S_1 (see Fig. 6(a)). Consider edges $e_2 = b_1b_2$ and $e_3 = b_2b_3$ in P. By $(0.5, 0.5) \in R_{up}$, e_3 does not overlaps with B_1A_2, and thereby e_3 is a free edge. Let $x = \ell(e_2)$ and $\bar{y} = \ell(e_3)$.

First consider the case where e_2 does not overlaps with A_1B_1, i.e., e_3 is a free edge. Then $g(e_2) = \frac{0.5}{(0.4043)^2} \times \frac{x-0.0957}{x}$, and $g(e_3) = \frac{0.5}{(0.4043)^2} \times \frac{\bar{y}-0.5}{\bar{y}}$. Since P is $\mathcal{S}$-monotone, it must hold $g(e_2) = g(e_3)$ for two free edges. Thus we have $\bar{y} = \frac{0.5}{0.0957}x$. Hence by $\bar{y} \leq 0.9043$, we have $x < 0.9043 \times \frac{0.0957}{0.5} = 0.17308302$. By $\bar{y} = \frac{0.5}{0.0957}x$, we have $\bar{y} - 0.5 = \frac{0.5}{0.0957}x - 0.5 = \frac{0.5}{0.0957}(x - 0.0957)$. We have $a(R_{up}) = x\bar{y}$ and $p_{\mathcal{S}}(R_{up}) = 0.5 \times \frac{(x-0.0957)(\bar{y}-0.5)}{(0.4043)^2} = \frac{0.5\times 0.5}{(0.4043)^2 \times 0.0957}(x - 0.0957)^2$. Then $\beta(\mathcal{S}, P) = p_{\mathcal{S}}(R_{up})(\frac{1}{a(R_{up})} - 2) = \frac{0.5\times 0.5}{(0.4043)^2\times 0.0957}(x-0.0957)^2(\frac{1}{\frac{0.5}{0.0957}x^2} - 2)$. By Lemma 4 with $a = 1, b = 0.0957$ and $c = \frac{0.5}{0.0957}$, this takes the maximum at $x = \min\{0.17308302, (\frac{0.0957}{2\times\frac{0.5}{0.0957}})^{\frac{1}{3}}\}$ (the latter is at least 0.209).

Cases-(A,2), (B,1) and (B,2) can be treated similarly. This establishes $\beta(\mathcal{S}) < 0.4664$ and hence Theorems 2 and 1.

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