Chapter 7
Developing the Mathematical Eye Through Problem-Solving in a Dynamic Geometry Environment

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7.1 Introduction

Images play different roles, but they play a key role in human thinking and, generally speaking, in human mental activities. It is a familiar experience to have images accompanying our thoughts either in fantasying or in trying to solve a difficult mathematical problem. And it is exactly in solving problems that very often we look for help from images, for instance, by sketching a drawing on a sheet of paper; this is what Pólya (1957) wrote in his book *How to Solve It*, opening the discussion on figures in geometry:

> Figures are not only the object of geometric problems but also an important help for all sorts of problems in which there is nothing geometric at the outset. Thus, we have two good reasons to consider the role of figures in solving problems. (p. 103)

Starting with the seminal work of Alan Bishop (1980, 1983), studies on visualization have been developed (Presmeg, 2006) recently focusing on specific aspects of the relationship between images and mathematical thinking. The advent of digital technologies has opened a new direction of investigation on how specific digital environments might affect conceptualization processes and problem-solving in mathematics (Arcavi & Hadas, 2002).

A first fundamental result from the studies on visualization is about reflecting upon the use of a varied and vague set of terms commonly used in the current language both for referring to the internal and the external context – such as visualization,

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visual thinking, mental images, drawing, figures and schema – in order to establish a shared terminology.

According to Presmeg (2006), visualization can be characterized in the following way:

> […] visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (Presmeg, 1997). This characterization is broad enough to include two aspects of spatial thinking elaborated by Bishop (1983), namely, interpreting figural information (IFI) and visual processing (VP). (p. 206)

As Bishop (1989) clearly pointed out, there seems to be a contrast between positive aspects and pitfalls related to visualization. On the one hand, starting from the original work of Krutetskii (1976), mainly based on experts' experience/reports, authors claimed the value and the power of visualization; on the other hand, the first studies concerning students' behaviour have highlighted difficulties related to visualization. Among these are the following (Presmeg, 1986):

> Especially if it is vague, imagery which is not coupled with rigorous analytical thought processes may be unhelpful. (p. 45)

Our aim is that of discussing and clarifying some aspects related to processes of visualization in problem-solving in Euclidean geometry. We claim that specific visualization skills (the *mathematical eye*) that are necessary for solving geometrical problems are actively involved in problem-solving processes that take place in a dynamic geometry environment (DGE). In the following sections, we start discussing and clarifying the meaning of the expression *mathematical eye*. Building on previous studies and on the current literature, we introduce specific cognitive constructs, which we call *visual skills*, involved in the elaboration of visual stimuli and, which are, consequently, fundamental in solving geometrical tasks. As we illustrate how different visual skills are involved in the solution processes of geometrical problems, we will see how a DGE can afford the mobilization of the same visual skills. Therefore, we claim that not only problem-solving activities can be designed with the aim of fostering the student's development of specific visual skills but also that acting within a DGE might strengthen the didactic potential of geometrical problem-solving activities and eventually affect the development of the mathematical eye.

7.2 Selected Skills Involved in Geometrical Problem Solving Using the "Mathematical Eye"

According to the notion of *figural concept* (Fischbein, 1993; Mariotti, 1995), geometrical reasoning consists of a dialectic between figural and conceptual components, so that the solution of a geometrical problem results from a coherent interaction between such components. Indeed, geometrical problem-solving is based on elaborations of images, both external representations (drawings on the paper or drawings on the screen) and internal representations (mental images). This is why we are interested in studying processes involved in treating images and, specifically, in describing and explaining how spatial properties of images are noticed, identified and interpreted geometrically, and eventually linked together logically in a conditional statement.

For instance, how can it happen that, looking at a scribble on a piece of paper (see Fig. 7.1), the observer thinks of "a square"? Or, similarly, looking at a moving image on the screen, the observer suddenly exclaims, "it is a parallelogram!"?

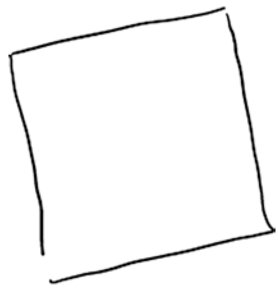
Though these experiences may be considered common to any student who has learned a bit of geometry, other more sophisticated experiences are common for expert mathematicians, thanks to a high level of competence in the treatment of images that supports problem solving in geometry; we can call such competence the *mathematical eye*.

In the following, we will be assuming that the process of perception occurs thanks to specific mental schemes that allow us to interpret visual stimuli, that is, to transform them into a coherent *perceived* image. The internal elaboration that occurs in our brains can be modelled through these mental schemes that we will be talking about in terms of *skills* because of their important role in problem solving and, specifically, in the process of geometrical problem solving. In this section, our intent is not to be exhaustive in describing all skills that allow the mathematical eye to function; instead, we introduce a selection of specific skills, and we identify and describe their roles in processes of problem solving.

The following skills are elaborated based on the theory of figural concepts (Fischbein, 1993), Duval's theory on cognition in geometry (Duval, 1994, 1998) and on the literature in cognitive psychology on visual-spatial abilities (e.g. Cornoldi & Vecchi, 2004). In the following we will refer to these skills as *visual-geometrical skills*, considering them as a possible characterization of what we call the "mathematical eye".

- Identification: immediate identification of a geometrical property of a figure on the plane or in space, with a goal in mind; this skill echoes Duval's perceptual apprehension and the visual-spatial abilities of visual organization and (planned) visual scan applied to the context of geometry.

Fig. 7.1 Scribble that could be recognized as a "square"



- Reconstruction: reconstruction of a figure from parts that are not correctly organized in space or that are not visible. This skill echoes an aspect of Duval's operative apprehension (1995a, 1995b) and the reconstructive visual ability (Cornoldi & Vecchi, 2004).
- Construction: construction of a representation of a figure, taking into account the use of tools and the construction sequence. This skill echoes Duval's sequential apprehension and the visual-spatial ability referred to as reconstructive visual ability.
- Part-whole awareness: abstract a part of the figure and consider it separately from the rest; this skill echoes Duval's attention to relevant subconfigurations and is described by Hadamard (1944, p. 80) as the ability to "abstract some special part of the diagram and consider it apart from the rest".
- Manipulation: manipulating a figure to transform it. This skill echoes the visual-spatial ability of image manipulation and aspects of Duval's operative apprehension.
- Theoretical control: mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry; this skill allows to "see" a figure in relation to verbal statements describing its geometrical properties. Moreover, it allows the solver, in Hadamard's words, to achieve a "simultaneous view of all elements of the argument, to hold them together, to make a whole of them in short", and "understanding the […] proof" (1944, p. 77). Consistently, we will also speak of advanced theoretical control intending to mean how the mathematical eye can bring an expert to automatically make use of sophisticated (for a high school student) mathematical notions to "see" aspects of a figure that are invisible to a less trained eye. Examples would be seeing a configuration "modulus similarity or translation" or declaring certain points "inessential" in the manipulation of a figure. This skill echoes aspects of Duval's discursive apprehension, and operative apprehension, and Fischbein's conceptual component of a figure.

There are two more visual-geometrical skills, which, for their specific role in geometrical reasoning, will be more widely described in the following sections: geometric prediction and crystallization.

7.2.1 The Skill of Geometric Prediction

In solving a geometrical problem, reasoning is heavily guided by the goals the solver has in mind: for example, identifying geometrical properties of a figure and classifying it is a different process from identifying properties that need to remain invariant as a manipulation of the image is performed. Indeed, a process that seems rather frequent in geometric reasoning is to mentally manipulate a figure and imagine how it will change given certain constraints, that is, maintaining certain properties invariant. Such process can be carried out through the use of the various skills listed above, but it is so common for experts to use it as a skill in its own right that we will call it *geometric prediction*[1]. With geometric prediction, we intend the identification of particular properties or configurations of a new figure, arising from a manipulation process. This process does not seem to be precisely described in the psychological literature; however it appears to be coherent with respect to the notions of anticipatory image (Piaget & Inhelder, 1966) and anticipatory schemes (Neisser, 2014), which suggest an ability to make predictions, orienting both perception and imagination, in the presence of a specific goal.

7.2.2 The Skill of Crystallization

Dynamism seems to be a component of experts' reasoning. Indeed, the skills of manipulation and of geometric prediction involve "movement" of the figure. If this movement is imagined by the solver, it may appear in different forms. For example, some may imagine a continuous deformation of the figure and others a "generic" figure that has at the same time infinite realizations that the solver can "move across" selecting the most useful ones. The use of movement (of any type) involves a temporal dimension in this kind of reasoning, which has been well documented in the literature, especially in reference to the use of a DGE during geometrical explorations. These were initially developed and used by mathematicians during their processes of problem solving, thanks to the possibility they offer to "externalize the set of relations defining a figure" (see, e.g., Laborde & Laborde, 1992; Laborde & Straesser, 1990).

Indeed, Laborde, speaking of a specific DGE, Cabri-Géomètre, states:

> The nature of the graphical experiment is entirely new because it entails movement. The movement produced by the drag mode is the way of externalising the set of relations defining a figure. The novelty here is that the variability inherent in a figure is expressed in graphical means of representation and not only in language. A further dimension is added to the graphical space as a medium of geometry: the movement. (1993, p. 56)

In this context, geometrical properties are interpreted as invariants (Laborde, 2005):

> A geometric property is an *invariant* satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions. (p. 22; emphasis added)

Therefore, the identification of invariants is also an important skill, constituting the mathematical eye.

Moreover, as the solver produces conjectures as part of the solution process, s/he can find him-/herself in the need of crystallizing an experimental situation by eliminating the temporal dimension to move to conditional statements. Research has shown that this is not always a spontaneous process. Some research has been conducted on processes of generation of conditionality (PGC) (Boero, Garuti & Lemut, 1999; Boero, Garuti & Mariotti, 1996). We will touch upon this again briefly in the analyses.

[1] This construct has also been used in a recent study by Miragliotta, Baccaglini-Frank and Tomasi (2017) and is being used in the doctoral work of Miaragliotta (Miragliotta and Baccaglini-Frank 2017, 2018).

7.3 Skills Involved in Problem-Solving Processes: Analyses of Two Problems

In this section, we give two examples of problems and provide *a priori* analyses of how the mathematical eye, described through the previous visual-geometrical skills, might guide possible solution processes. The first problem is rather classical, typically found in Italian geometry textbooks (and indeed it is a translation of one such problem), for the fact that it explicitly states what is to be proved, given the described construction. The second is an "open problem", less typical in the curriculum, but with the potential of fostering development of the mathematical eye, because of the skills it is necessary to use to solve it. Moreover, it lends itself quite naturally to be explored within a DGE, which can support the development of such skills thanks to its affordance of specific tools that will be analysed in a later section.

In the following analyses of this section, we imagine working without the support of a DGE.

7.3.1 Visual-Geometrical Skills Used in Solving a "Prove That" Problem

Problem 1: Finding triangles with equal area
Given a triangle ABC and the midpoint D of side BC, consider a point E on segment BD and construct line AE and the parallel to AE through D. This line intersects AC in F. Construct segments AD and EF; these segments intersect at H. Prove that triangles EHD and FHA have the same area.

What are key geometrical problem-solving skills that come into play?

The solver will probably first use the construction skill to produce a representation of the figure described in the problem, and s/he might wonder whether triangles FHA and EHD are congruent in general. Without further questioning, s/he may simply assume this and attempt to prove their congruence by trying to apply the triangle congruence criteria (if the triangles can be proved to be congruent, then of course their areas will be congruent).

However, if the image that is first realized following the construction steps looks like our Fig. 7.2, this (incorrect) assumption may not be made.

The solution process is guided by the aim of finding a relationship between the two triangles that is not their congruence. The process will start with the identification of properties of the figure and in particular of the property "AE parallel to FD". Such property may be identified through a manipulation of the figure: the solver may imagine E moving along BD, and as the figure changes, through geometric prediction, s/he can notice properties of the figure as those that remain unvaried

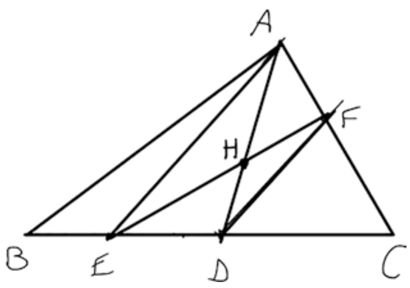


Fig. 7.2 Possible figure obtained by accomplishing the construction described in Problem 1

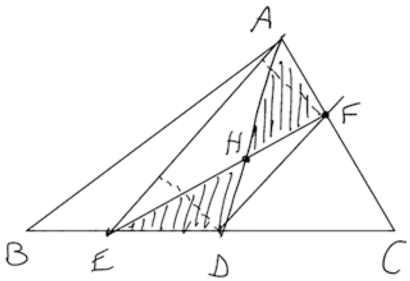


Fig. 7.3 Possible outcome of a (mental) manipulation of the constructed figure for Problem 1 by changing the position of E on BD

throughout the manipulation. Though a changing figure might be imagined, the dynamism is then eliminated through crystallization of the figure into a "generic" configuration. In other words, the mathematical eye will interpret a specific configuration as representing all the characteristic properties and nothing else; in terms of figural concepts, the figural and the conceptual components are fully integrated, and the drawing is properly controlled (Fig. 7.3).

For example, while the property "EHD congruent to AFH" is not always true, the property "AE parallel to FD" is (indeed it is one of the construction properties), as well as the (derived) property "triangles AFE and ADE have congruent heights". Notice that to "see" congruent heights, the solver needs to use the skill of reconstruction, as no segments corresponding to the heights are included in the construction given in the problem. These properties are key in solving the problem, because triangles AFE and ADE, having the same base AE and congruent heights, must have the same area. Identification of these properties is guided by theoretical control over the figure. Indeed, the following theorem needs to be recalled: "Given a triangle ABC, all triangles with the same base AB and the third vertex on a line parallel to AB through C have the same area". Such theorem could be recalled in a dynamic form, for example, with the third vertex "moving" along the parallel line through C.

However, the solution of the problem is possible only through awareness of a specific part-whole relationship, that is, each of the two triangles AFE and ADE can be seen as made of a common part AEH and one of the two triangles AHF and EHD. From such awareness the solver can deduce the equivalence of the two triangles AHF and EHD, as it was to be proved.

There are different reasons why the solver could be led to perceiving the key properties, “common base AE” and “equal heights” of AHF and EHD. The following two reasons we consider come from somewhat opposite directions of reasoning, but they include the same geometrical ingredients and similar activity of the mathematical eye.

1. In an attempt to identify significant (for the problem) properties starting from the configuration, the solver can manipulate the figure, imagining E to vary on segment BD or the triangle ABC to vary. Thanks to geometric prediction, this can lead to noticing that there are very few “special” properties in the figure, so it may be rather straightforward to identify the parallelism of AE and FD (a construction property), seen as constancy of the distance between the two lines and of any other consequent properties; this may be promoted also through theoretical control over the figure.
2. On the other hand, the solver can think about the conclusion (“area EHD = area FAH”) and reason about these surfaces, thinking about how to decompose them or see them as part of greater surfaces, thanks to part-whole awareness (as described above at the end of the first process outlined). In this case, triangles AFE and ADE may be seen as made of the “parts” AHE (common), EHD and FAH; the solver may attempt to search for an argumentation leading to the property “area AFE = area ADE” and in doing so identify their property of sharing a base and having congruent heights (that s/he “sees” thanks to the skill of reconstruction).

At this point, through part-whole awareness, AE can be seen as the common base of two triangles with the same height, and thanks to the same skill, these triangles can then be decomposed into two parts each, of which one (AHE) is common. S/he can conclude that the areas of AFH and HED are equal because these triangles are parts of congruent triangles to which a common region (AHE) is subtracted.

The first form of reasoning seems to be heavily based on identification of relevant (to the problem) properties, thanks to manipulation and geometric prediction, guided by theoretical control over the figure.

The second form of reasoning seems to depend much more on the solver’s skills of recognizing the geometrical rather than the algebraic nature of the area searched for: instead of focusing on the product of two numbers, focusing on the product of two segments (one side and the height relative to it).

7.3.2 Visual-Geometrical Skills in Solving an "Open Problem"

Problem 2: Finding possible types of quadrilaterals a constructed figure can become

Let A, M and K be three points, and construct B as the symmetric point of A with respect to M, and construct C as the symmetric point of A with respect to K. Construct the parallel line l to BC through A and the perpendicular line r to l through C. Let D be the intersection of l and r. Make conjectures about which types of quadrilaterals can ABCD be (Fig. 7.4).

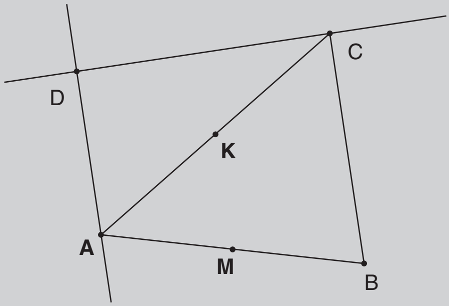


Fig. 7.4 Figure obtained following the construction steps in Problem 2

One way to proceed in solving Problem 2 is to construct a figure and proceed to identify properties of the quadrilateral ABCD. The variable points are A, M and K. Properties can be identified by manipulating mentally the figure, for example, by thinking of moving points A, M or K. The properties constructed will definitely be invariant (the parallelism of AD and BC, the perpendicularity of BC and CD, AM = MB, AK = KC), but also properties that are logical consequences of these in the theory of Euclidean geometry can be identified during any manipulation and process of geometrical prediction (e.g. CD perpendicular to DA, KM parallel to CB, KM = 1/2 CB). Of course, to identify such properties, it is also necessary to have good theoretical control over the figure.

Through manipulation, an expert mathematician may imagine inducing movement on ABCD to explore possible configurations, realizing, as a colleague once reported, that M and K act as a sort of "handles" for moving B and C. The expert might quickly realize this because of the fixed (by construction) relationships between A, M and C (AM/MC = 1/2) and A, K and B (AK/KB = 1/2). Indeed, experts interviewed on this problem have answered that "no matter what fixed ratios" there were between these sets of points, "M and K are inessential" in the manipulation of the quadrilateral. This seems to be an automatic process (the expert says it is "natural and immediate"), characteristic of the mathematical eye of an expert. This automatized process has surely been reached through years of experience in “looking at” and “working with/talking about” mathematical objects. In other words, the trained mathematical eye, thanks to advanced theoretical control acquired over the years, guides the exploration process and the choices of features of the figure that are “relevant”.

Once these properties have been identified, the solver can advance a conjecture such as “ABCD is (always) a right trapezoid”, which can easily be proved using the theorems that allowed him/her to infer “CD perpendicular to DA” from the construction properties. The expert may also make a claim like “The types of quadrilaterals that can be obtained depend only on the choice of K”, arguing that a type of quadrilateral is determined “modulo similarity and rotation”, so A and M can be thought of as fixed. Again, the mathematical eye must be quite experienced and well-trained to be able to see “modulus similarity and rotation”.

What other types of quadrilaterals can ABCD be? Using his/her knowledge of how quadrilaterals can be theoretically classified, the solver can list other possible types of quadrilaterals to test, as subtypes of right trapezoids with a second pair of parallel sides (rectangles, including squares). The solver can perform manipulations of the figure, guided by theoretical control and by part-whole awareness through which specific subconfigurations can be searched for and identified.

At this point, in a possible attempt to construct a figure with the properties listed in the problem plus additional properties that will guarantee the figure’s belonging to a subtype of quadrilateral, the solver can search for conditions under which the subtype may be identified, possibly using geometric prediction.

Conceiving the necessity of the angle at M to be right in order for ABCD to be a rectangle may come from noticing, thanks to (possibly advanced) theoretical control over the figure: (1) triangle AMK is similar to ABC and (2) a right trapezoid becomes a rectangle if either of its non-right angles become right (this is a sufficient condition).

A conjecture might be advanced such as “If AMK is a right triangle (at M), then ABCD is a rectangle”.

7.4 How Development of the Mathematical Eye Can Be Fostered Through Problem Solving in a DGE

In this section, we analyse possible solution processes of the two problems introduced above when these are solved within a DGE. From the analyses, we will show how the tools in the DGE can support the solver’s solution process, either supporting the use or compensating for the weakness of certain visual-geometrical skills and eventually developing the mathematical eye. These analyses will lead into our hypotheses on how the development of certain geometrical exploration modalities can be fostered within a DGE through appropriately designed activities.

7.4.1 *Analysis of a Solution Process of the "Prove That" Problem Within a DGE*

Within a DGE, the first step is generally to realize a construction that incorporates all the properties given in the problem. So the solver will need to use his/her skills of construction, determining which commands to use from the menus in the DGE to incorporate properly each property into the figure described in the problem's hypothesis: in this case D as the midpoint of BC, E as a mobile point "attached to" segment BD and DF as the parallel through D to AE. In order to guide processes of identification and part-whole awareness, the solver may also decide to highlight the surfaces of the two triangles AFH and EHD (Fig. 7.5). In doing this, the solver is supported by the DGE that carries out the constructions correctly and precisely; this is something that is not guaranteed in a hand-made drawing.

The solver might now try to identify properties of the triangles AHF and EHD, as crystallized invariants as s/he manipulates the figure. However, unlike in the previous cases analysed without the DGE, here the solver can physically drag points of the figure; so the DGE becomes responsible for much of the theoretical control over the figure and compensates the solver's geometric prediction ability, showing the result of each manipulation instantly (e.g. see Figs. 7.6 and 7.7).

These manipulations may allow the solver to notice, for example, that the property "triangles AFH and EHD are congruent" is not true in general; indeed, some results of the manipulations (dragging) can be identified as counter examples to such a property.

Through such (physical) manipulations, the solver can search for invariant properties, identifying them through a process of crystallization in which dynamism is eliminated and a particular configuration (product of the crystallization of an invariant) becomes the figural component of a figural concept – an identified property. This may occur even for invariants that were defined by the steps of the construction (we have evidence of this in various students' protocols); in this case, for example, it could occur for the property "AE parallel to FD".

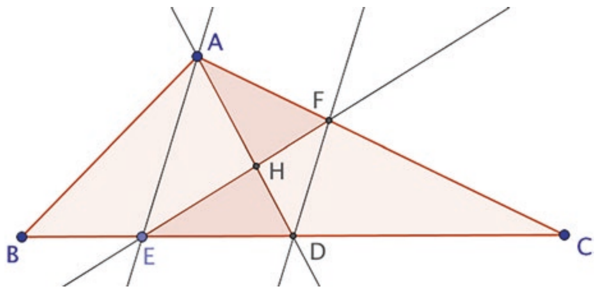


Fig. 7.5 Possible figure obtained by accomplishing the construction described in Problem 1 in a DGE

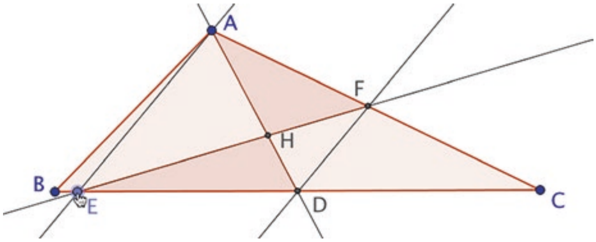


Fig. 7.6 Possible outcome of a manipulation through dragging of the constructed figure for Problem 1 by dragging E along BD

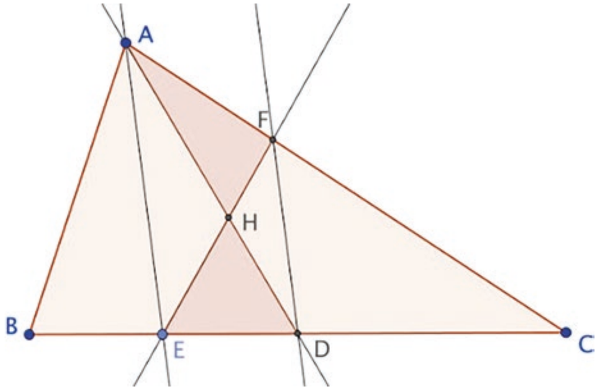


Fig. 7.7 Possible outcome of a manipulation through dragging of the constructed figure for Problem 1 by dragging vertex A of the triangle ABC

Now, the solver might refer to his/her theoretical knowledge and ask him-/herself what can be inferred from segments being parallel, and this could lead to the identification of congruent segments, perpendicular to the parallel lines, seen as an indication of the lines always having a "same distance". This way, through reconstruction, two segments could be seen in the figure and actually constructed in the DGE. Since points A, E, F and D are visible in the figure, it is likely that the solver will choose to construct the segments through two of these points and, possibly, from two points on the same line because of the types of images s/he might be familiar with (Fig. 7.8).

This choice – which may happen without a conscious decision being made by the solver – can guide the part-whole awareness and, thanks to theoretical knowledge, help noticing triangles with a same base and congruent heights and thus identifying properties such as "triangles AFE and ADE have congruent heights", "triangles AFE and ADE have base AE in common" or (if the distances from A and E to FD were drawn) "triangles DFA and DFE have congruent heights" and "triangles DFA and DFE have base FD in common". Moreover, the solver can identify the theorem relating the two properties noticed: "Given a triangle ABC, all triangles with the same base AB and the third vertex on a line parallel to AB through C have the same area". Of course, as described in the preceding solution process for Problem 1, much of the whole solution process may be guided by the solver's identifying a proper configuration of such a theorem within the figure on the screen.

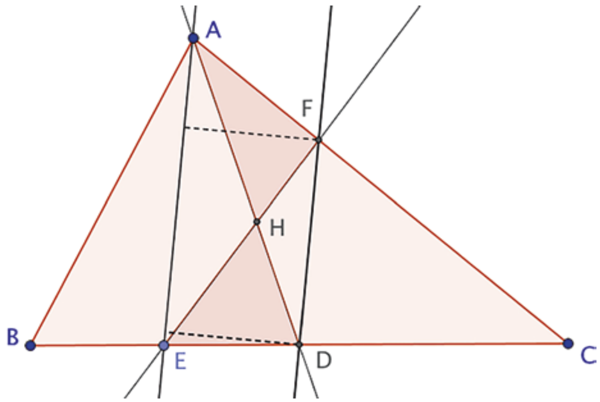


Fig. 7.8 A DGE figure in which segments showing the parallel lines AE and FD having the "same distance" are shown

Finally, as in the previous solution process, the solution can be reached through awareness of a specific part-whole relationship, that is, both of the two triangles AFE and ADE can be seen as made up of a common part AEH and of one of the triangles AHF and EHD. From such awareness, the solver can deduce the equivalence of the two triangles AHF and EHD, as it was to be proved.

Compared to the solution process described previously, we highlight how the skill of geometric prediction plays a much more minor role in the context of the DGE because the software exercises on the figure the theoretical control that the solver would have to otherwise exercise him-/herself. Instead, in the DGE, the solution processes typically involve crystallization, a skill that seems less dominant in geometrical problem-solving with paper and pencil. As a matter of fact, in a DGE, the solution might be achieved, eliminating dynamicity and grasping the invariance of a relation between properties, in other words recognizing the occurrence of a theorem.

7.4.2 Analysis of a Solution Process of the "Open Problem" Within a DGE

The solver can start by constructing a dynamic figure incorporating the properties described in the problem, in this case: C symmetric to A with respect to K, B symmetric to A with respect to M, DA parallel to CB and CD perpendicular to DA (Fig. 7.9).

The solver can manipulate the figure dragging A, M or K and see (on the screen) the effect of the dragging of any of these points. In doing this s/he can identify the property "ABCD is a right trapezoid", as a crystallized invariant emerging from all the configurations that appear. The identification of this property may not require the reconstruction and part-whole awareness skills (to realize that only the angle ADC is right by construction, while angle DCB is right as a consequence of the construction properties), since the quadrilateral ABCD may be perceived as a dynamic whole. A first conjecture like "ABCD is always a trapezoid" may be put forth.

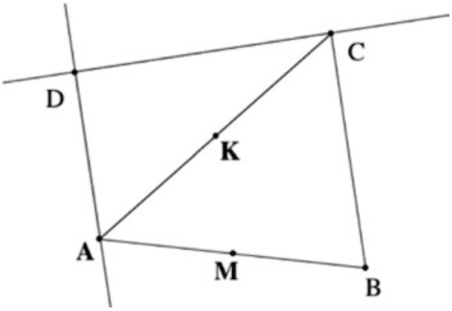


Fig. 7.9 Possible figure constructed for Problem 2 within a DGE

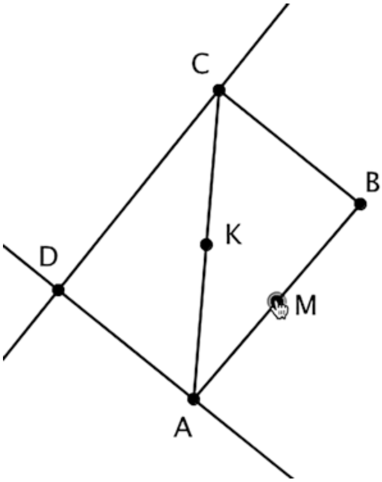


Fig. 7.10 The figure is manipulated by dragging M as the solver searches for "good positions" in which ABCD is a rectangle

Then the solver may ask him-/herself what other types of quadrilateral ABCD can become, theoretically controlling the possibilities by recognizing that the quadrilateral can only become subtypes of a right trapezoid, that is, rectangles or squares.

The exploration (and solution process and products) changes based on which points the solver drags (we are assuming to be in a DGE in which only one point at a time can be dragged). For example, manipulating the figure by dragging M, the solver can search for "good positions", positions in which the property "ABCD rectangle" can be *identified* (Fig. 7.10).

The solver can also search for regularity in the movement of a certain point s/he decides to drag. For example, if K is dragged, and the solver is trying to figure out "when ABCD is a rectangle", s/he will have to drag K along a line, and not just any line, the line through M perpendicular to AM. This line (in this case actually the geometrical locus of the points M such that ABCD is a rectangle) can be discovered without exercising much theoretical control over the figure but instead other skills, including eye-hand coordination and crystallization transforming the regular "straight" movement of K into a geometrical property, such as "K belongs to the perpendicular to AM through M". Such process can be fostered through a tool offered by the DGE that is the trace mark activated on the dragged point (Fig. 7.11).

This particular way of dragging while trying to maintain a desired property has been referred to as maintaining dragging (Baccaglini-Frank & Mariotti, 2010), and we consider it a potential support for mobilizing specific visual skills for the problem-solving in a DGE.

If A is dragged, a similar regularity in its induced movement can be crystallized and another property identified: "A belongs to the line through M and perpendicular to MK". Carrying out the maintaining dragging and identifying that property may stimulate the skill of geometric prediction as, for example, when a path is marked by the trace. Now, the simultaneous presence of two identified properties can be crystallized into the conditional statement (a conjecture): "if A belongs to the line through M and perpendicular to MK, then ABCD is a rectangle".

Finally, if M is the dragged point, the solver can crystallize and identify the property "M belongs to the circle with diameter AK", and the simultaneous presence of two identified properties can be crystallized into the conditional statement (a conjecture): "If M belongs to the circle with diameter AK, then ABCD is a rectangle". In our research, we have witnessed the possibility that solvers, before expressing their conjectures, linking the identified properties, decide to construct the property that will become the premise of their conditional statement in order to obtain a figure in which the whole conjecture, if correct, can be identified and possibly crystallized (Fig. 7.12).

Therefore, based on the point dragged, the solver may make a number of different conjectures that vary in the condition added to the construction to obtain a rectangle and, possibly, in the terminology used to express the conjecture itself. The conditions may be the following: "K belongs to the line perpendicular to AM through M", "A belongs to the line through M perpendicular to MK", "M belongs to the circle with diameter AK" and "AMK is a right triangle (in M)". Based on the results of our research, we can definitely conclude that the last condition was the

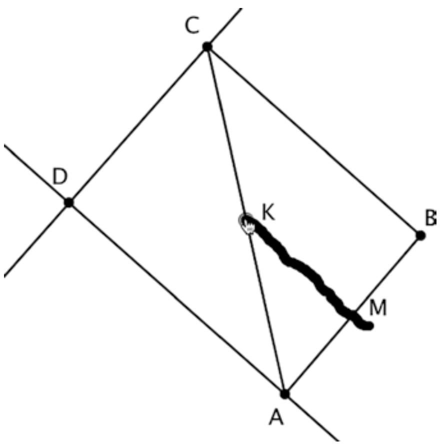


Fig. 7.11 What is shown on the screen in a DGE when the trace mark is active on the dragged point D as the solver tries to maintain the property ABCD rectangle

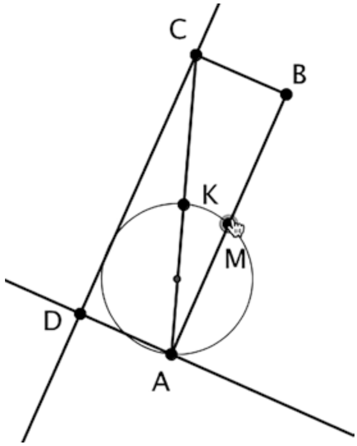


Fig. 7.12 M can be linked to the circle with diameter AK so that the figure embeds properties "M belongs to the circle with diameter AK" and, as a consequence, "ABCD is a rectangle"

least common in students who used maintaining dragging to reach their conjectures in this problem (for further details, see Baccaglini-Frank, 2010a).

The conjecture may not contain only "static", crystallized words; indeed, if dynamic explorations are carried out, our studies (and many others) have shown that students use expressions like "A moves on a line", "ABCD is a quadrilateral when [a certain condition is verified]", "in order for ABCD to remain [a rectangle], the point has to move [in a certain constrained way]" or even "when I move the point [on some path], then ABCD stays a rectangle".

Speaking about the visual-geometrical skills involved in reaching such conjectures, we wish to briefly comment on how crystallization may contribute to the solver's reaching a conjecture as a product of the problem-solving process in a problem like the one described. The skill we have described as crystallization and used in the analyses has important roots in the literature, especially regarding studies of students' "processes of generation of conditionality (PGC)" (Boero et al., 1996, 1999).

In particular, processes like the one we have described echo processes described by Boero and his colleagues, such as the following PGC.

> The conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, that will be subsequently detached from it and then "crystallize" from a logic point of view ("If…, then…"). (Boero et al., 1996, p. 121)

Studying processes of conjecture generation in open problem situations in a DGE, we found that PGCs, as described by Boero and colleagues, seem to be present during the processes analysed (Baccaglini-Frank, 2010a) and also that a new element comes into play: continuity that is induced by the specific kind of motions that occurs in the DGE. Indeed, the examples provided for the PGCs in literature have mostly been of a "discrete" nature, the use of dragging in the processes of conjecture generation attributes a new "continuous" nature to the processes. We note that although dynamism seems to provide support for the processes of conjecture generation like the ones described, making it more "natural", it may turn into an obstacle as far as the aim to formulate conjectures within the "static" theory of Euclidean geometry is concerned, where it becomes necessary to "eliminate" time.

Going back to the analysis of the open problem solved within a DGE, a final consideration can be made in how the role of the mathematical eye can contribute. While for some students, the three premises in the conjectures (1. "K belongs to the line perpendicular to AM through M then ABCD, 2. "A belongs to the line through M perpendicular to MK", 3. "M belongs to the circle with diameter AK") have nothing to do with one another, the mathematical eye, thanks to advanced theoretical control, will allow to condense them into a condition that expresses a general relationship between A, M and K, such as "AMK is a right triangle (at M)" or "the angle AMK is right". Moreover, as we described in the first analysis of this problem, the expert will also exercise advanced theoretical control to reduce the exploration to a minimum, for example, deeming the dragging of A and M as "inessential" as the configurations can be uniquely determined "modulus similarity and rotation".

Similar to what we concluded in the other case, here, too, for the open problem solved in a DGE, manipulation can be carried out through dragging, offloading from the solver much of his/her theoretical control over the figure. Moreover, we highlight how the skill of geometric prediction plays a much more minor role; however it may be stimulated by the solver's use of a tool such as maintaining dragging, as we have described. The use of such tool can be fostered through open problem activities like asking for the production of conjectures, like in the case discussed in the example. In the problem-solving process in a DGE, the crystallization skill seems to emerge much more than in a paper and pencil environment, where, in fact, frequently there seem to be no traces of it at all.

7.5 How Geometric Problem Solving in a DGE Can Foster the Development of Skills Pertaining to the Mathematical Eye

The analyses presented above have highlighted how certain tools offered within the DGE seem to induce the use of particular skills, allowing a less expert solver to take part in explorations in which his/her experience resembles that of an expert, thus fostering the development of his/her mathematical eye. For example, while the expert mentally manipulates the figure and performs geometric prediction, the less expert solver can drag and change the configuration, identifying properties (and relationships between properties) through the crystallization of invariants. In doing this, s/he enriches with dynamism (which cannot be achieved through static images drawn on paper) his/her figural components of the geometric concepts involved; the dynamism represents variability and generality (for the expert). So, we argue that through geometric problem solving in a DGE, it is possible to enhance particular skills supporting development of the mathematical eye. The use of some skills, such as crystallization, and other skills associated to the use of specific dragging modalities like maintaining dragging seems to be mostly present within a DGE, but we have argued that these can be seen as preparatory and supporting for other skills comprising the mathematical eye.

In this section, we take this argument a step further and show a case of how solvers' expert use of skills associated to maintaining dragging actually fostered the development of skills pertaining to the mathematical eye that the solvers were able to use without support from the DGE. For this purpose, we will introduce a third problem, similar to Problem 2, highlighting design aspects that seem to foster the development of desired skills, and then present excerpts from the students' exploration in which we found evidence of the students' strengthened geometric problem-solving skills.

Problem 3
Construct the quadrilateral ABCD following these steps. Construct: a point P and a line r through P, the perpendicular line to r through P, C on the perpendicular line, a point A symmetric to C with respect to P, a point D on the side of r containing A, the circle with centre C and radius CP, point B as the second intersection between the circle and the line through P and D. Formulate conjectures about the possible types of quadrilaterals it can become describing all the ways you can obtain a particular type of quadrilateral.

The design of this problem is similar to that of Problem 2: the task asked of the solver is an open-ended one, in which explorations of the figure are promoted through dragging. The processes of problem-solving induced by these kinds of problems involve the generating of conjectures as an outcome of various kinds of manipulation of the figure. Moreover, our studies have suggested that a request in which the solver is asked to describe "all the ways" in which a certain configuration may be visually verified can foster the use of certain dragging modalities such as maintaining dragging, assuming that this modality is familiar to the solvers (e.g. Baccaglini-Frank, 2010a, 2010b; Baccaglini-Frank & Mariotti, 2010).

Coming to the specific construction proposed in Problem 3, when solving the problem in a DGE, the figure (see Fig. 7.13) can be acted upon by dragging points C, P or D. We will concentrate on dragging D. Among the properties that can be identified, there are the properties described in the steps of the construction.

If the solver tries to obtain the configuration "ABCD parallelogram" through maintaining dragging, new invariants can be crystallized into geometrical properties (e.g. "D lies on a circle C_{AP} with centre in A and radius AP").

As in the analysis of a solution process for Problem 2, here, too, identification of these new properties during dragging can be supported by the use of the trace mark, a functionality in most DGEs. The properties that appear to be visually verified simultaneously as the figure is acted upon through dragging may be crystallized into

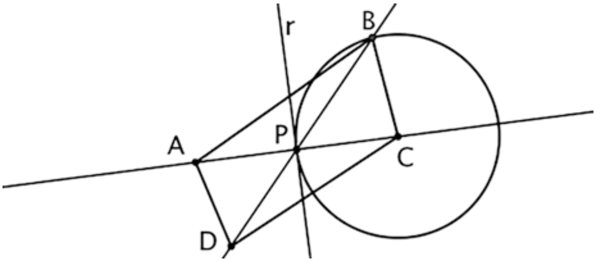
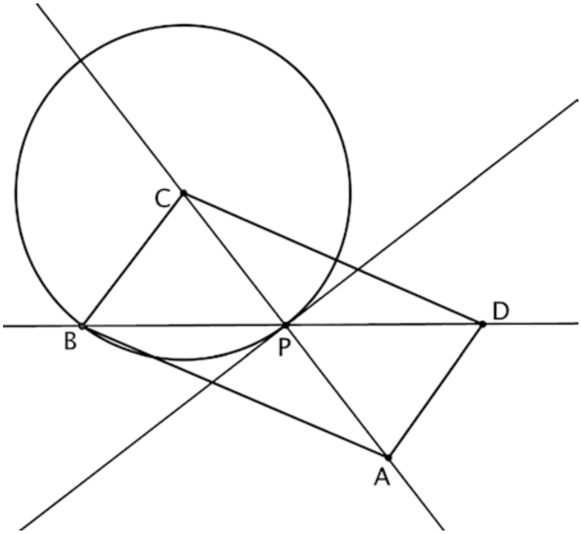


Fig. 7.13 A possible result of the construction in the situation described above

Fig. 7.14 Construction of the figure in Problem 3 with which the students are working



a conditional statement, a conjecture (e.g. "If D belongs to C_{AP}, then ABCD is a parallelogram"), as the outcome of the problem-solving process. Indeed, our research has shown that many solvers who decide to use maintaining dragging perceive a relationship between properties that are simultaneously visually verified on a dynamic figure they are manipulating (e.g. Baccaglini-Frank & Mariotti, 2010; Leung, Baccaglini-Frank & Mariotti, 2013; Mariotti, 2014)[2] (Fig. 7.14).

However, their attempt fails, as shown in the following excerpt[3].

[2] The process is described in further detail by Baccaglini-Frank and Mariotti (Baccaglini-Frank, 2010a, 2010b; Baccaglini-Frank & Mariotti, 2011).

[3] Adapted from Baccaglini-Frank and Mariotti (2010, pp. 238, 240, 241) with permission from International Journal of Computers for Mathematical Learning, copyright 2010, by Springer

What is said [and done]	*Comments*
Gianni: And now what are we doing? Oh yes, for the parallelogram? **Francesco**: Yes [as he drags D with the trace activated] yes, we are trying to see when it remains a parallelogram. Gianni: Yes, okay the usual circle comes out. **Francesco**: Wait, because here...Oh dear! Where is it going? […] so, maybe it's not necessarily the case that D is on a circle so that ABCD is a parallelogram. Because you see, if we then do a kind of circle starting from here, like this, it's good, it's good, it's good, it's good [he drags along a circle he imagines], and then here… see, if I go more or less along a circle that seemed good, instead it's no good…so when is it any good?	Francesco and Gianni seem to have conceived a geometric prediction for the path along which point D should be dragged. This prediction does not seem to fit with the shape of the trace mark appearing on the screen as Francesco performs maintaining dragging. This leads the failure of the students' use of maintaining dragging as a physical tool, so they abandon it.

Suddenly Gianni expresses the result of a geometric prediction he has carried out mentally, thanks to theoretical control he exercises on the figure.

What is said [and done]	*Comments*
Gianni: Eh, since this is a chord, it's a chord right? We have to, it means that this has to be an equal chord of another circle, in my opinion with centre in A. Because I think if you do, like, a circle with centre. **Francesco**: A, you say… Gianni: Symmetric with respect to this one, you have to make it with centre A. […] Gianni: With centre A and radius AP. I, I think… **Francesco**: Let's move D. More or less… Gianni: It looks right doesn't it? **Francesco**: Yes. Gianni: Maybe we found it! (Fig. 7.15)	Gianni, who was not dragging, carries out a geometric prediction. The students proceed to construct the newly conceived circle along which Gianni has imagined D to move. The identified relationship "if D belongs to the symmetric circle then ABCD is a parallelogram" is constructed in the DGE.

The students seem quite satisfied and formulate the following conjecture explicitly: "If D belongs to the circle with centre in A and radius AP, then ABCD is a parallelogram".

What happened to maintaining dragging here? The students continue the exploration mentally as if they were dragging. Gianni seems to be using the maintaining dragging tool mentally (Baccaglini-Frank, 2010a, 2010b). Therefore, the conjecturing process relies entirely on his theoretical control over the figure.

In this case, geometric prediction, paired with theoretical control, plays a key role in the process of problem solving. We conjecture that this way of thinking was fostered by Gianni's extensive experience using maintaining dragging as a physical tool that strengthened his theoretical control and geometric prediction skills to the extent that he was able to take-upon himself to manipulate the figure, controlling its conceptual components and carry out an accurate geometric prediction. In sum-

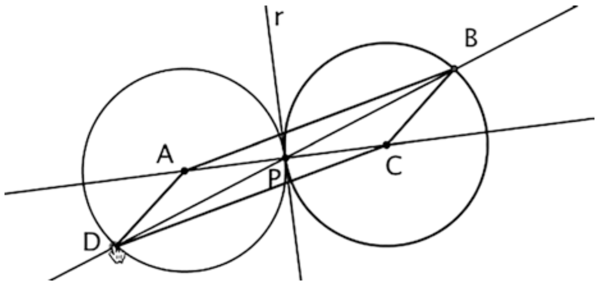


Fig. 7.15 Francesco drags D along the newly constructed circle

mary, looking at the transcript, we can infer that this prediction was possible for him because the combination of visual-geometrical skills comprising his mathematical eye allowed him to “see” the circle with centre in A as the image of the circle in the original construction through reflectional symmetry across *r*.

This case suggests that appropriation of dragging modalities such as maintaining dragging is possible and can lead to strengthening particular visual-geometrical skills and, therefore, to development of the mathematical eye. Moreover, specific analyses of the case have also suggested that use of maintaining dragging as a psychological tool (Vygotski, 1978) can bring continuity between the conjecturing phase and the proving phase (if students are later asked to prove their conjectures), because many theoretical notions become explicit to the solver who can then control them at will (Baccaglini-Frank & Antonini, 2016).

7.6 Concluding Remarks

The aim of this chapter was to discuss some aspects related to visualization processes in problem solving. We started with the idea of the necessity of a mathematical eye to successfully perform geometrical problem solving. In order to clearly describe what the mathematical eye could mean, we introduced a selection of specific skills and elaborated on some theoretical constructs, previously introduced in the literature concerning cognitive aspects of visualization; we called these visual-geometric skills.

Our analyses showed how the problem-solving process can be described through the combination of different visual-geometrical skills that are necessary in the problem-solving process. This raises the educational problem of how it might be possible to improve such skills and eventually develop a mathematical eye.

The general hypothesis presented in this chapter concerns the educational role played by moving into a dynamic geometry environment, that is, proposing and solving a problem with the support of a DGE.

The analyses developed in Sects. 7.3 and 7.4 show how the same visual-geometrical skills are actively involved in the solving processes but also how the functionalities of the DGE can support the emergence and consolidation of these skills. For example, in Sect. 7.4 we have shown how specific DGE skills associated to a way of dragging points, maintaining dragging, can support the development of the skill of geometric prediction.

Moreover, repeated experiences mobilizing the skill of identification and the skill of crystallization seem to contribute to the strengthening of the solver's theoretical control; these visual-geometrical skills guide the mathematical eye during problem-solving processes. In particular, the visual-geometrical skill of theoretical control can be at the basis of the elaboration of a theorem.

Indeed, let us imagine how this could happen if a student who is elaborating the theorem "Given a triangle ABC, all triangles with the same base AB and the third vertex on a line parallel to AB through C have the same area." used the first problem we analysed. The student could be introduced to the theorem in a dynamic form, for example, exploring a figure in which a triangle ABC is constructed as a segment (AB) and a third vertex C attached to a line parallel to AB. As C is moved on the parallel line, the invariant area of the triangle may be identified. Such dynamic configuration, with the invariant relationship between C being on the parallel line and the constant area of ABC, can be crystallized by the student into a statement such as "any triangle with the third vertex C on a line parallel to side AB has a given area" or "any two triangles with a common base and congruent heights have the same area", by eliminating the dynamism in favour of awareness of the being generic of the configuration. Once the process of crystallization is complete, the student will have probably also strengthened his/her theoretical control, since a new fragment of theory is now at his/her disposal. According to the notion of figural concept, the new fragment of theory will not only have a conceptual component – possibly expressed in a verbal text – but also a figural component encompassed in a crystallized configuration. That means that such a theorem, if needed, may be also recalled in a dynamic form, for example, with the third vertex "moving" along the parallel line through C.

The analyses presented and these final considerations support our claim that problem-solving activities can be designed with the aim of fostering students' development of specific visual skills, which can, in turn, contribute to the development of a mathematical eye.

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