Research in Mathematics Education
Series Editors: Jinfa Cai · James A. Middleton

Nélia Amado · Susana Carreira
Keith Jones *Editors*

Broadening the Scope of Research on Mathematical Problem Solving

A Focus on Technology, Creativity and Affect

Springer

Research in Mathematics Education

This series is designed to produce thematic volumes, allowing researchers to access numerous studies on a theme in a single, peer-reviewed source. Our intent for this series is to publish the latest research in the field in a timely fashion. This design is particularly geared toward highlighting the work of promising graduate students and junior faculty working in conjunction with senior scholars. The audience for this monograph series consists of those in the intersection between researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications ready at hand to help them make decisions regarding the improvement of teaching, learning, policy, and practice. With this vision, our mission of this book series is:

1. To support the sharing of critical research findings among members of the mathematics education community;
2. To support graduate students and junior faculty and induct them into the research community by pairing them with senior faculty in the production of the highest quality peer-reviewed research papers; and
3. To support the usefulness and widespread adoption of research-based innovation.

More information about this series at http://www.springer.com/series/13030

Nélia Amado • Susana Carreira • Keith Jones
Editors

Broadening the Scope of Research on Mathematical Problem Solving

A Focus on Technology, Creativity and Affect

Springer

Editors
Nélia Amado
Universidade do Algarve and UIDEF
Instituto de Educação
Universidade de Lisboa
Lisbon, Portugal

Susana Carreira
Universidade do Algarve and UIDEF
Instituto de Educação
Universidade de Lisboa
Lisbon, Portugal

Keith Jones
School of Education
University of Southampton
Southampton, UK

ISSN 2570-4729 ISSN 2570-4737 (electronic)
Research in Mathematics Education
ISBN 978-3-319-99860-2 ISBN 978-3-319-99861-9 (eBook)
https://doi.org/10.1007/978-3-319-99861-9

Library of Congress Control Number: 2018962920

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This book has its origins in the *Problem@Web International Conference*, subtitled *Technology, creativity, and affect in mathematical problem solving*, which took place in 2014 in Portugal[1] (Carreira, Amado, Jones, & Jacinto, 2014). The conference was the culmination of the *Problem@Web* research project conducted in Portugal (with the support of FCT, the Portuguese research funding agency) and was also one of the project outcomes (Carreira, Jones, Amado, Jacinto, & Nobre, 2016). The event offered an opportunity to share the project results, as well as to connect researchers and experts from various parts of the world involved in the study of problem solving, taking into account the multiple aspects of this area of study. One major aim of the conference was to assemble emerging and cutting-edge knowledge on what has been a highly relevant and fundamental subject in Mathematics Education research for several decades. The awareness that this field of research is invigorating, and has today new extensions, developments, theoretical approaches, practical implications, and international relevance, was one of the great motivations of the *Problem@Web* conference.

To a certain extent, following up on the developments and challenges in research on mathematical problem solving was an impetus for the *Problem@Web* project that was launched to investigate youngsters' (aged 10–14) mathematical problem solving activity in a context beyond school, namely in the sphere of two mathematical competitions.[2] These competitions are inclusive problem solving competitions, involving the use of digital technologies, bringing families and teachers closer to the mathematical activity of young people, and generating opportunities for the manifestation of mathematical creativity through original approaches to moderate mathematical challenges (Carreira et al., 2016).

At the *Problem@Web* conference, we sought submissions in the three strands of technology, creativity, and affect in mathematical problem solving, by considering the school context but also others such as mathematical competitions.

[1] http://www.fctec.ualg.pt/problemweb2014/

[2] http://fctec.ualg.pt/matematica/5estrelas/

At the end of the *Problem@Web* conference, on a beautiful sunny afternoon in the Algarve, a range of conference participants gathered to discuss the prospect of publishing the research that had been shared at the conference in the form of an English-language edited book. The discussion around that enterprise continued and evolved in the following months until a general plan was reached. The plan was based on the idea of broadening the outlines of the research on specific facets of mathematical problem solving seen as pertinent for advancing the field.

The 25 chapters that compose the book are the result of a hard, prolonged, and very committed work of a group of authors to whom we, as editors, want to express our deep thanks. The contributing authors include people that were presenters at the *Problem@Web* conference, together with other researchers who were later invited to join, and who enthusiastically and diligently brought their knowledge, and their own research, into the whole. The set of manuscripts denotes the inclusive spirit of this publication rooted in a highly participated conference in Portugal. The book does this through the diversity of authors' nations, the participation of young researchers as coauthors of chapters, as well as through the variety of research perspectives and wealth of ideas put into place and discussed.

The fact that we have brought the "footprint" of the *Problem@Web* conference, and the driving energy of the *Problem@Web* research project, into the book justifies the decision to begin each of the first three parts by a chapter that has its source in the keynote addresses that were presented at the conference. To complete each of the first three parts of the book, we invited researchers who are well known for the excellence of their research, and for their influential work, to act as discussants. They accepted the challenging task of writing a critical commentary on the set of chapters in each part. In view of their kind and full dedication to our proposal, and the results achieved, which add to each of the themes of the book a careful, synthesizing, and insightful reading, we owe words of praise and thanks to Arthur Powell, Pietro Di Martino, and Roza Leikin.

Finally, we counted on the invaluable collaboration of a well-known author who agreed to undertake the final part that aims to link, articulate, and integrate the three topics that the book develops throughout the first three parts. To our colleague Viktor Freiman, for his dedication, generosity, and commitment, our sincere thanks for his valuable work.

We wish to emphasize our appreciation for the extensive and continuous collaboration of all the authors and reviewers of the chapters, who patiently responded to our requests and who strove to ensure that each manuscript reached its final version, and encouraged us to conclude this project, which, although sometimes delayed by different contingencies, never ceased to be an ambition of us all.

Our thanks go to the staff at Springer, and the editors of the *Research in Mathematics Education* book series, Jinfa Cai and James A. Middleton, who stimulated and supported us and enabled our proposal to go ahead and become this book.

Lisbon, Portugal Nélia Amado
Lisbon, Portugal Susana Carreira
Southampton, UK Keith Jones

References

Carreira, S., Amado, N., Jones, K., & Jacinto, H. (2014). *Proceedings of the Problem@Web International Conference: Technology, creativity and affect in mathematical problem solving*. Faro, Portugal: Universidade do Algarve. http://hdl.handle.net/10400.1/3750.

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematical problems with technology: The results and implications of the Problem@Web project*. Cham, CH: Springer.

Contents

Chapter 1
Broadening Research on Mathematical Problem-Solving: An Introduction

Nélia Amado, Susana Carreira, and Keith Jones

1.1 Introduction

Problem-solving continues to be an increasingly key focus for education in countries all around the world. Internationally, for example, the OECD (the Organisation for Economic Co-operation and Development) has, for some years, been assessing students' problem-solving skills; in 2003 there was the first PISA (Programme for International Student Assessment) assessment of cross-curricular problem-solving; in the 2012 PISA, there was assessment of creative problem-solving skills; and the 2015 PISA featured the first international assessment of collaborative problem-solving skills (Herde, Wüstenberg, & Greiff, 2016).

This increasing emphasis on problem-solving is, at least in part, because educational systems are being implored not only to embrace a wider range of goals and exploit a wider range of educational technologies than may have happened previously but also in ways that prepare learners to be confident to tackle new and emerging challenges. Griffin (2018, p. iv), for example, captures the current impetus:

> Education must now focus on the preparation of a workforce demanding new ways of thinking and working that involve creativity, critical analysis, problem solving and decision making. Citizens need to be prepared for new ways of working that will call upon their communication and collaboration skills. They will need to be familiar with new tools that include the capacity to recognise and exploit the potential of new technologies. In addition, they will need to learn to live in this multifaceted new world as active and responsible global citizens.

N. Amado (✉) · S. Carreira
Universidade do Algarve and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal
e-mail: namado@ualg.pt; scarrei@ualg.pt

K. Jones
University of Southampton, Southampton, UK
e-mail: d.k.jones@soton.ac.uk

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_1

Although there is debate over the extent to which such goals are new (Suto, 2013), the current stimulus for education to address a wider range of goals does mean that mathematical problem-solving continues to be a key element of the mathematics curriculum across countries and a central aim of mathematics teaching and learning. Not only that, but, in meeting these "new" ways of thinking and working, the manner in which mathematical problem-solving is taught and learnt needs to embed the increasingly widespread use of digital technologies, the promotion of creativity, and the recognition of the affective and aesthetic dimensions.

This is the stimulus for this book, with its focus on mathematical problem-solving in different educational environments in the early decades of the twenty-first century. The book gathers together contributions from leading researchers in mathematics education (and related research areas such as the psychology of education, technology education, mathematics popularisation, and other relevant fields) to capture the broadening of research on mathematical problem-solving that has been taking place in recent years in different educational environments. The book presents the three major themes of technology, creativity, and affect, within which new pedagogical and research perspectives are presented. Each of the three themes represents crucial issues that are being solidly embedded in the activity of problem-solving in the teaching and learning of mathematics, both within the school setting and beyond the school. The book illustrates the broadening of the scope of research on mathematical problem-solving by not only examining each of the three themes of technology, creativity, and affect separately but also, in the final section, moving the field of research forward by evidencing a strong interconnection between the facets of technology, creativity, and affect, each of which has consistently shaped problem-solving. Thus, it reinforces the challenge to endorse a more comprehensive approach to research on mathematical problem-solving.

In this chapter, we provide a short overview of research on mathematical problem-solving. We use this as a springboard to introduce the central themes of the book: technology in mathematical problem-solving, creativity in mathematical problem-solving, and affect and aesthetics in mathematical problem-solving. We conclude by introducing the final section of the book on making evident the need to investigate in a comprehensive way how technology, creativity, and affect and aesthetics in mathematical problem-solving are being brought together. We begin by outlining some of the central trends in research on mathematical problem-solving over the past decades.

1.2 Trends in Research on Mathematical Problem-Solving

As Santos-Trigo (2014, p. 497) states in the entry on problem-solving in mathematics education in the *Encyclopedia of Mathematics Education*, mathematical problems and their solutions "are a key ingredient in the making and development of the discipline" of mathematics. As such, mathematical problems and their solutions are a vital component of all the elements of school mathematics.

In a review of some of the main research on mathematical problem-solving conducted between 1970 and 1994, Lester (1994, p. 663) suggests that from the early 1970s until the early 1980s, "quite a lot of attention was devoted to the study of determinants of problem difficulty". He goes on to suggest that beginning in the late 1970s and continuing until the mid-1980s, problem-solving research focused primarily on "individual problem-solving competence and performance" (p. 665). During this latter period, early research on the role of affect on mathematical problem-solving began to appear. Into the 1990s, as research on social influences on mathematical problem-solving (and ideas of situated problem-solving) began to appear, Lester says he began to detect that mathematical problem-solving was "beginning to be of less interest to mathematics education researchers" (p. 667). Amongst possible reasons for this decline, Lester suggests that "other issues have drawn attention away from problem solving" or perhaps "problem solving is even more complex than first thought" (pp. 667–669). That problem-solving is highly complex is also one of the conclusions of a slightly earlier retrospective review of mathematical problem-solving research by Kilpatrick (1985).

Whatever the reasons for the decline in interest in researching mathematical problem-solving that began at that time, Lester concluded his review with a spirited call for more research on mathematical problem-solving. Amongst the issues identified by Lester as benefitting from further research was the role of affect in problem-solving. While Lester makes reference to Kilpatrick's (1969) review of problem-solving and creative behaviour, there is no mention of research on creativity and mathematical problem-solving in Lester's review. Likewise, while Lester makes reference to Ponte, Matos, Matos, and Fernandes (1992), Lester makes no mention of technology and mathematical problem-solving in his review.

Ten years later, Lester (2013) published an update reflecting on 40 years of research on mathematical problem-solving. In this revised review, Lester suggests that a major issue is that "although research on mathematical problem solving has provided some valuable information about problem-solving instruction, we haven't learned nearly enough" (p. 251). As with Lester's earlier review (Lester, 1994), there is no mention of technology and mathematical problem-solving, no mention of research on creativity and mathematical problem-solving, and only passing reference to research on affect and mathematical problem-solving. Most recently, Lester and Cai (2016) make the case that while the past 30 years were "an especially productive period in the history of problem solving in school mathematics" (p. 117), there is much that remains to be learned. Amongst the issues they identify is that further research is needed on the influence of mathematical tools on problem-solving, on learning to be creative in mathematical problem-solving, and on affect and mathematical problem-solving.

In terms of tools, Santos-Trigo (2014) devotes a section of his encyclopaedia entry to the influence on mathematical problem-solving of using digital tools, concluding that the appropriation of digital tools for use in problem-solving activities involves "extending previous frameworks" and developing "different methods to explain mathematical processes that are now enhanced with the use of those tools" (p. 500). While there is this consideration of digital tools, Santos-Trigo (2014) makes no mention of creativity and mathematical problem-solving and only a passing mention of affect and mathematical problem-solving. Of course the encyclopaedia entry by Santos-Trigo must, necessarily, be short. Nevertheless, it illustrates that while there has been much research that has involved creativity and mathematical problem-solving and much research on affect and mathematical problem-solving, such recent research tends to be found in publications about creativity in mathematics education and affect in mathematics education and not always in research on mathematical problem-solving. For example, the book edited by Singer (2018) on mathematical creativity and mathematical giftedness makes many mentions of mathematical problem-solving. Likewise, Goldin et al. (2016) make many mentions of mathematical problem-solving in their review of developments in research on attitudes, beliefs, motivation, and identity in mathematics education. In the same way, Liljedahl et al. (2016) contain separate sections on creative problem-solving and on digital technologies and mathematical problem, while Felmer et al. (2016) contain separate chapters on affect in problem-solving and the use of digital technology to frame and foster learners' problem-solving experiences.

The research project *Mathematical Problem Solving: Perspectives on an interactive web-based competition*, usually referred to as the *Problem@Web* project, studied youngsters engaged in mathematical problem-solving using digital technologies of their choice and to which they had personal access (see Carreira, Jones, Amado, Jacinto, & Nobre, 2016). While Carreira et al. (2016) primarily report on strategies for, and ways of thinking in, mathematical problem-solving when using digital technology (such as forms of representation and expression of mathematical thinking and technology-supported problem-solving approaches), the *Problem@Web* project also focused on creativity manifested in the expression of mathematical solutions to problems and its relation to the use of digital technologies and on the beliefs, attitudes, and emotions related to mathematics and mathematical problem-solving, both in school and beyond school, considering students, parents, and teachers.

The international conference organised by the *Problem@Web* project team at the end of the project (see Carreira, Amado, Jones, & Jacinto, 2014) kept true to these foci by identifying three strands of research as needing attention from the international research community. The three strands are the basis for this book: technology in mathematical problem-solving, creativity in mathematical problem-solving, and affect and aesthetics in mathematical problem-solving. It is to each of these strands that we turn now. The three strands, each making up a section of the book, lead to the fourth section of the book in which technology, creativity, and affect and aesthetics in mathematical problem-solving are brought together.

1.3 Technology in Mathematical Problem-Solving

As exemplified by Carreira et al. (2016), research on technology and mathematical problem-solving seeks to encompass the strategies and representations used in technology-based problem-solving approaches. This includes research addressing the impact and the effects of using technology in problem-solving activity, the mediational role of technology in problem-solving, the relationship between strategies and the use of digital and visual representations, the nature of digitally based communication and expression of the results of mathematical problem-solving, and so on. As noted by Santos-Trigo (2014), such research is likely to entail extending previous frameworks for mathematical problem-solving and developing different methods to explain mathematical processes that are now enhanced through the use of digital tools. It is some of these issues that are tackled in this section of this book.

In Chap. 2, Jacinto, Nobre, Carreira, and Amado (2018) analyse the representational and conceptual expressiveness that youngsters (aged 10–14 years old) exhibit when they use digital tools to shape and support their reasoning and solutions to mathematical problems. Through their analysis, Jacinto et al. offer a way of identifying levels of sophistication and robustness of technology-based solutions to problems, according to the characteristics of the tool used and its connection to the conceptual models underlying students' thinking on the problems. For Chap. 3, Parzysz (2018) uses the didactical framework of *Mathematical Working Spaces* (MWS) (Kuzniak, Tanguay, & Elia, 2016) to inquire into the cognitive and epistemological aspects of using digital technology to tackle probability problems. In doing so, he illustrates how the role of technology is not just to make it easier to get a large data set in a short time but constitutes an important element for mathematical problem-solving.

Analysing the ways of reasoning exhibited by high school teachers when using a specific technology, in this case, dynamic geometry software is the focus of Chap. 4 by Santos-Trigo, Camacho-Machín, and Olvera-Martínez (2018). They report that the use of the particular technology not only offered a set of affordances for the teachers to test the pertinence of their initial ideas but also a family of examples that could be examined in order to detect patterns or possible relationships between relevant parameters. Canavarro and Reis (2018), in Chap. 5, focus on analysing the contributions that a specific technology, that of the interactive whiteboard, provides for the development of dialogical interaction in the context of mathematical problem-solving. They argue that the multimodality and interactivity available with this particular technology simultaneously stimulate and support students' mathematical problem-solving.

For Fernandes, Lopes, and Martins (2018), in Chap. 6, the need is to understand how the use of a specific technology, in this case robots, acts as a mediating artefact of learning that contributes to mathematical problem-solving. They found that using the particular technology had a transformative impact on the way students negotiated the meaning of mathematical knowledge for their problem-solving. Mariotti and Baccaglini-Frank (2018), in Chap. 7, illustrate how, when using a particular technology, there are specific visualisation skills that are differently involved in mathematical problem-solving processes. They show how problem-solving processes can be described through the combination of different visual-mathematical skills that are necessary in problem-solving.

In the final chapter of this section, Chap. 8, Powell (2018) reflects on the use of digital technology in mathematical problem-solving situations investigated in the six preceding chapters. Through his analysis of the six chapters, Powell shows how the chapters capture how learners develop powerful ways of seeing and acting mathematically in the context of technology-enhanced mathematical problem-solving. He concludes that, while the six chapters highlight a diversity of perspectives on how technology shapes mathematical problem-solving, there still remain research questions to address.

1.4 Creativity in Mathematical Problem-Solving

As can be inferred from some of the chapters in the book edited by Singer (2018), research on creativity in students' mathematical problem-solving seeks to address the relationship between problem-solving activity and mathematical creativity. This is likely to include research on how problem-solving promotes the development of creativity across student attainment levels, the analysis of creative solutions in terms of strategies and expressive media, the relationship between creativity and inventiveness in solving mathematical problems, the creativity involved in exposing and communicating solutions and processes, the fostering creativity in school mathematics and beyond (including mathematical competitions), and so on. It is some of these issues that are tackled in this section of this book.

In Chap. 9, Carreira and Amaral (2018) analyse and examine the close and complex relationship between mathematical creativity and mathematical problem-solving. They propose an analytical tool that guides the identification of features of mathematical problem-solving pertaining to the three dimensions of *originality*, *knowledge activation*, and *representational means activation*. For Chap. 10, Tabach and Levenson (2018) focus on the mathematical creativity elicited by mathematical problems that can be solved in many different ways. On the basis of their analysis, they argue that creative mathematical problem-solving can involve a combination of convergent and divergent thinking and that following one line of thought to an eventual end may actually lead to a new direction and new solution path.

Analysing the links between the use of visual solutions to problems and mathematical creativity is the focus of Chap. 11 by Vale, Pimentel, and Barbosa (2018). Based on their analysis, they argue that problem-solving approaches that use visual representations bring out the development of creative thinking. Vanegas and Giménez (2018), in Chap. 12, focus on developing the proficiency of future teachers in what might be called creativity-directed teaching. While they found that the future teachers that they studied had difficulties in identifying authentic traits of creativity in children's problem-solving activity, the future teachers did think that it would be possible for children to learn to improve their mathematical creativity.

For Bokhove and Jones (2018) in Chap. 13, mathematical creativity can be stimulated through tasks that have *constraints* rather than necessarily requiring open-ended mathematical tasks. They show how tasks with suitable constraints can also serve as creativity-inducing problem-solving tasks. Moore-Russo and Demler (2018), in Chap. 14, report on an exploration of educators' conceptions of creativity.

Through their study, they found how different views of creativity impact on the way educators consider methods for fostering creativity in mathematics classrooms.

In Chap. 15, Gómez-Chacón and de la Fuente (2018) aimed at analysing the nature of mediation in student's creative processes. They show, on the basis of high school students' reports, how the process of generating a mathematical research project from a specific given problem not only requires creativity on the part of the students but also the mediation of the teacher for the establishment of a creative suitable working space. For Chap. 16, Gontijo (2018) explains the *systems perspective* on creativity (Csikszentmihalyi, 1988, 1999) that considers creativity as a result of the interaction of three systems: person (genetic background and personal experiences), domain (culture and scientific production), and field (social system).

In the final chapter of this section, Chap. 17, Leikin (2018) reflects on the studies of creativity in mathematical problem-solving contained in the preceding eight chapters in this section of the book. Through her analysis of the eight chapters, Leikin formulates a categorisation of creativity-directed mathematical tasks in terms of openness, constraints, and mathematical insight. She argues that using this range of tasks in the classroom can provide a springboard for the development of children's cognitive, intrapersonal, and interpersonal skills and that such use of creativity-directed mathematical tasks warrants further investigation on a larger scale.

1.5 Affect and Aesthetics in Mathematical Problem-Solving

As can be surmised from some of the sections of the review by Goldin et al. (2016), research on affect and aesthetics in mathematical problem-solving embraces research on attitudes, aesthetics, emotions, and so on of students, parents, and teachers regarding mathematical problem-solving. This is likely to include research addressing the role and influence of affect and emotion in problem-solving activity; how emotions shape mathematical problem-solving; the question of enjoyment in mathematical problem-solving; the perceptions of students, teachers, and parents about the emotional dimensions of problem-solving; the notion of help-seeking in mathematical problem-solving activity; problem-solving as an enrichment activity connecting school, parents, family, and home (including mathematical competitions); and so on. It is some of these issues that are tackled in this section of this book.

In Chap. 18, Amado and Carreira (2018) use an analytical model to study attitudes towards mathematical problem-solving. They report that the emotional dimension of mathematical problem-solving is related to the type of problems proposed and to the encouragement of expressing the process that leads to the solution of a problem. For Chap. 19, Presmeg (2018) focuses on some of the significant aspects of both aesthetic sense and affective issues, which she describes as two under-researched domains, in solving nonroutine mathematical problems. She concludes that the positive emotions generated under the umbrella of aesthetic experiences can provide a strong motivational effect in solving nonroutine mathematical problems and that this may encourage students to persist with the broader context of mathematics.

The role of the aesthetic in the problem-solving experiences of pre-service teachers is the focus of Chap. 20 by Sinclair and Rouleau (2018). They report how the pre-service teachers referred to two considerations that align with research on mathematicians' aesthetic: the problem should be adequately difficult, and it should be surprising. Ferreira and Moreira (2018), in Chap. 21, focus on the interrelationship between affect and cognition in mathematical problem-solving. They found that enabling students' spontaneous expression of activating emotions can serve as a mediating tool in promoting students' positive global affect towards mathematics.

For Liljedahl (2018), in Chap. 22, students' *engagement* in mathematical problem-solving is sometimes overlooked in efforts to improve students' problem-solving. He reports that students with higher than expected perseverance in the face of challenge *and* tolerance in the face of the mundane can use these attributes as safeguards while autonomously correcting any imbalance between the two. Daher, Swidan, and Masarwa (2018), in Chap. 23, report on a study of students' emotions in learning to tackle mathematical problems with technology. They report how students' emotions can be related to their "positioning" as group leader or collaborator and the success or difficulty of proceeding with the problem-solving activity.

In the final chapter of this section, Chap. 24, Di Martino (2018) reflects on the studies of affect and aesthetics in mathematical problem-solving contained in the preceding six chapters in this section of the book. Amongst the issues he raises is whether aesthetics is seen as a part of affect or whether aesthetics and affect are closely intertwined but different domains with different functions. While also discussing the methodological difficulties in conducting studies of affect and aesthetics in mathematical problem-solving, Di Martino points to the new theoretical approaches, new methodologies, and new critical aspects of research about affect, aesthetic, and problem-solving that are evident in the six chapters in this section of the book.

1.6 Broadening Research on Mathematical Problem–Solving

While the main three sections of the book address the central themes of technology in mathematical problem-solving, creativity in mathematical problem-solving, and affect and aesthetics in mathematical problem-solving, the final section of the book emphasises the broadening of research on mathematical problem-solving in which technology, creativity, and affect and aesthetics in mathematical problem-solving are brought together. This endeavour of broadening the research on mathematical problem-solving, by looking at its multiple dimensions in a comprehensive way, is exemplified by Freiman (2018), in Chap. 25.

As Powell (2018) says in Chap. 8, the six chapters on the use of digital technology in mathematical problem-solving situations illustrate how the use of technology shapes mathematical problem-solving in formal and informal settings and has implications for mathematics teachers' content and pedagogical knowledge. Furthermore, the emergence of new collaborative technologies occasions opportunities and challenges (Jones, Geraniou, & Tiropanis, 2013) both for the teaching of mathematical problem-solving and, in tandem, for mathematics teacher education.

The emerging collaborative technologies highlight the importance of interpersonal skills that Leikin (2018), in Chap. 17, argues are characteristic of creative mathematical problem-solving illustrated by Chaps. 9, 10, 11, 12, 13, 14, 15, and 16. She makes the case that creativity-directed activities are challenging for students due to the openness, constraints, and mathematical insight embedded in creativity-directed mathematical tasks. Such tasks, she goes on to say, are not only effective tools for developing mathematical skills but also have a role in the development of intrapersonal and interpersonal competences. Engaging students in such tasks primes student motivation.

This issue of the priming of student motivation leads Di Martino (2018), in Chap. 24, to argue how the studies reported in Chaps. 18, 19, 20, 21, 22, and 23 provide much-needed evidence of the dynamic progression of motivational and emotional states through the problem-solving process. Here, at least for some of the studies, student participation is voluntary, and the problems are moderately challenging and intended for all.

To complete the book, Freiman (2018), in Chap. 25, explores the sort of approach that broadens the scope of research on mathematical problem-solving by which technology, creativity, and affect and aesthetics in mathematical problem-solving are brought together. He does this by using the example of recreational puzzles that may occupy the problem-solver for more than the equivalent of a single school, or a school day, or longer. By choosing the example of recreational puzzles, he brings attention not only to the origins of the puzzles and details of their analysis by renowned mathematicians but also, importantly, to the use of multiple representations, rigorous investigations, and observations that go beyond finding a solution to a particular problem towards elaboration of new methods and theories while experiencing and sharing emotions and tribulations, along with endless pleasures of continued engagement in the intellectual endeavour of mathematical problem-solving.

1.7 Conclusion

The collective efforts of all the authors of chapters in this book show the richness of contemporary research in mathematical problem-solving. The empirical, theoretical, and methodological contributions of the chapters in the book are symptomatic of a vigorous field of research that is succeeding in broadening its scope and embracing the themes of technology in mathematical problem-solving, creativity in mathematical problem-solving, and affect and aesthetics in mathematical problem-solving. The book details solid research findings that allow the building of further theoretical foundations about mathematical problem-solving. At the same time, the book contains open questions and insufficiently explored avenues that are ripe for additional research. We hope that a broad audience of researchers, educators, curriculum designers, pre-service teachers, and doctoral students find the book challenging and useful for their current and future research work on mathematical problem-solving and the closely related research fields of educational technologies, creativity, and affect and aesthetics.

References

Amado, N., & Carreira, S. (2018). Students' attitudes in a mathematical problem solving competition. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Bokhove, C., & Jones, K. (2018). Stimulating mathematical creativity through constraints in problem solving. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Canavarro, A. P., & Reis, C. S. P. (2018). The interactive whiteboard and the development of dialogic interaction in the context of problem solving. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Carreira, S., Amado, N., Jones, K., & Jacinto, H. (Eds.). (2014). *Proceedings of the Problem@ Web International Conference: technology, creativity and affect in mathematical problem solving*. Faro, Portugal: Universidade do Algarve.

Carreira, S., & Amaral, N. (2018). Mathematical problem solving beyond school - a tool for highlighting creativity in children's solutions. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematical problems with technology: The results and implications of the Problem@Web project*. Cham, CH: Springer.

Csikszentmihalyi, M. (1988). Society, culture, and person: A systems view of creativity. In R. J. Sternberg (Ed.), *The nature of creativity* (pp. 325–339). Cambridge: Cambridge University Press.

Csikszentmihalyi, M. (1999). Implications of a systems perspective for the study of creativity. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 313–335). Cambridge: Cambridge University Press.

Daher, W., Swidan, O., & Masarwa, A. (2018). Positioning and emotions in tackling algebra problems with technology: The case of high and middle achieving students. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Di Martino, P. (2018). Affect and aesthetics in mathematical problem solving: New trends, methodologies, results and critical aspects. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Felmer, P., et al. (2016). *Posing and solving mathematical problems: Advances and new perspectives*. Cham, CH: Springer.

Fernandes, E., Lopes, P., & Martins, S. (2018). Learning scenarios with robots leading to problem solving and mathematics learning. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Ferreira, R. A. T., & Moreira, M. D. D. (2018). Facebook and WebQuests as tools for engagement with mathematical problem solving: The emotions experienced in the Math@XXI gymkhana. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Freiman, V. (2018). A-musing with chess and the eight queens math puzzle: looking for connections between problem solving, technology, creativity, and affect in mathematics education. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Goldin, G. A., Hannula, M. S., Heyd-Metzuyanim, E., Jansen, A., Kaasila, R., Lutovac, S., et al. (2016). *Attitudes, beliefs, motivation and identity in mathematics education. An overview of the field and future directions*. Cham, CH: Springer.

Gómez-Chacón, I. M., & De la Fuente, C. (2018). Problem solving and mathematical research projects: Creative processes, actions, and mediations. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Gontijo, C. H. (2018). Mathematics education and creativity: a point of view from the systems perspective on creativity. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Griffin, P. (2018). Preface. In E. Care, P. Griffin, & M. Wilson (Eds.), *Assessment and teaching of 21st century skills: Research and applications* (pp. v–vi). Dordrecht: Springer.

Jacinto, H., Nobre, S., Carreira, S., & Amado, N. (2018). Different levels of sophistication in solving-and-expressing mathematical problems with digital tools. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Jones, K., Geraniou, E., & Tiropanis, T. (2013). Patterns of collaboration: towards learning mathematics in the era of the semantic web. In D. Martinovic, V. Freiman, & Z. Karadag (Eds.), *Visual mathematics and cyberlearning* (pp. 1–21). Dordrecht, NL: Springer.

Kilpatrick, J. (1969). Problem solving and creative behavior in mathematics. In J. W. Wilson & L. R. Carey (Eds.), *Reviews of recent research in mathematics education* (pp. 153–187). Stanford, CA: School Mathematics Study Group.

Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–16). Hillsdale, NJ: Lawrence Erlbaum.

Kuzniak, A., Tanguay, D., & Elia, I. (2016). Mathematical working spaces in schooling: An introduction. *ZDM Mathematics Education, 48*(6), 721–737.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25*(6), 660–675.

Lester, F. K. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast, 10*(1&2), 245–278.

Lester, F. K., & Cai, J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems*. Cham, CH: Springer.

Leikin, R. (2018). Openness and constraints associated with creativity-directed activities in mathematics for all students. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Liljedahl, P., et al. (2016). *Problem solving in mathematics education*. Cham, CH: Springer.

Liljedahl, P. (2018). On the edges of flow: Student problem solving behaviour. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Herde, C. N., Wüstenberg, S., & Greiff, S. (2016). Assessment of complex problem solving: What we know and what we don't know. *Applied Measurement in Education, 29*(4), 265–277.

Mariotti, M. A., & Baccaglini-Frank, A. (2018). Developing the mathematical eye through problem solving in a dynamic geometry environment. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Moore-Russo, D., & Demler, E. L. (2018). Mathematical creativity: Views from the field. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Parzysz, B. (2018). Solving probabilistic problems with technologies in middle and high school: The French case. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Ponte, J. P., Matos, J. F., Matos, J. M., & Fernandes, D. (Eds.). (1992). *Mathematical problem solving and new information technologies: Research in contexts of practice*. Berlin, Germany: Springer-Verlag.

Powell, A. (2018). Reflecting on digital technology in mathematical problem-solving situations. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Presmeg, N. (2018). Roles of aesthetics and affect in mathematical problem solving. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Santos-Trigo, M. (2014). Problem solving in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 496–501). Dordrecht, NL: Springer.

Santos-Trigo, M., Camacho-Machín, M., & Olvera-Martínez, C. (2018). High school teachers' use of a dynamic geometry system to formulate conjectures and to transit from empirical to geometric and algebraic arguments in problem solving approaches. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Sinclair, N., & Rouleau, A. (2018). On choice, collaboration and closeness in problem solving: Aesthetic experiences of pre-service teachers. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Singer, F. M. (Ed.). (2018). *Mathematical creativity and mathematical giftedness: Enhancing creative capacities in mathematically-promising students*. Cham, CH: Springer.

Suto, I. (2013). *21st century skills: Ancient, ubiquitous, enigmatic. Research Matters* (Vol. 15, pp. 2–8).

Tabach, M., & Levenson, E. (2018). Solving a task with infinitely many solutions: Convergent and divergent thinking in mathematical creativity. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Vale, I., Pimentel, T., & Barbosa, A. (2018). The power of seeing in problem solving and creativity: An issue under discussion. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Vanegas, Y., & Giménez, J. (2018). Creativity and problem solving with early childhood future teachers. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect*. Cham, CH: Springer.

Part I
Technology in Mathematical Problem Solving

Chapter 2
Different Levels of Sophistication in Solving and Expressing Mathematical Problems with Digital Tools

Hélia Jacinto, Sandra Nobre, Susana Carreira, and Nélia Amado

2.1 Introduction

The research community has been acknowledging the significant role of beyond school and extracurricular activities where youngsters throughout the world are engaging in, which reveals a growing interest on studying its implications for mathematics learning contexts and also the impact of the formal curricular learning on beyond school activities (Barbeau & Taylor, 2009; Wijers, Jonker, & Drijvers, 2010).

Presently, there are many organizations that support the development of mathematical problem-solving skills by promoting competitions, clubs or summer schools, among others. This is the case of the Mathematical Competitions SUB12® and SUB14®, brought to young students in the south of Portugal (aged 10–14-year-olds) by the University of Algarve.

Our research agenda has been focused on uncovering the skills these young competitors use and master by taking advantage of everyday digital tools and their representational expressiveness to shape and support their reasoning in the development of a solution approach (Carreira, 2015; Jacinto, 2017; Jacinto & Carreira, 2013; Nobre, Amado, & Carreira, 2012). One emergent outcome deals with the fact that

H. Jacinto (✉)
Group of Schools Poeta Joaquim Serra and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal
e-mail: hjacinto@campus.ul.pt

S. Nobre
Group of Schools Paula Nogueira and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal

S. Carreira · N. Amado
Universidade do Algarve and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_2

there are different levels of robustness in the solutions produced by the participants in the competitions, mainly in terms of the strategies they devise by means of a particular technological tool while solving the problems.

In this chapter, we will focus on identifying and understanding the underlying aspects of these different ways of solving a problem with the same tool. As our main source of evidence, we will analyse a selection of solutions of two problems produced by participants in the SUB12 and SUB14 competitions, who resorted to GeoGebra, in one case, and to Excel, in the other. We will also identify levels of sophistication in the solutions produced with these tools.

Our main goal is to unveil how these youngsters are capable of creating effective ways of reaching the solution to a problem and communicate it mathematically, based on the digital resources at their disposal, in their daily lives and, most of them, in their home environment (e.g. the tools available on their computers such as the text editor, the electronic spreadsheet, the image editor or free access tools such as GeoGebra). In the context of our research, solving mathematical problems means to find ways of thinking about challenging situations where a mathematical stance is suitable and useful, though it does not necessarily or exclusively encompass the use of school mathematics. It is assumed that the concepts generated from intuitive and common sense-based approaches are fundamental in problem-solving.

Within this kind of beyond-school problem-solving activity, the conceptual model generated from the mathematization process may not have the sophistication of a mathematical model. The conceptual models developed by the youngsters often have an unsophisticated semblance, an informal tone and a look that strongly clings on to the context suggested in the problem. Thus, we intend to understand the main characteristics of these conceptual models when the solution of a problem is supported by the use of a digital tool, namely, the ways in which their different levels of sophistication may be related to a specific tool for solving the problem and expressing the underlying conceptual model.

2.2 Solving and Expressing Mathematical Problems with Digital Tools

Mathematical problem-solving is often regarded as a mathematization activity in which the solver develops productive ways of thinking about the situation (Lesh & Zawojewski, 2007), adopting a mathematical stance while developing a conceptual model (Carreira, Jones, Amado, Jacinto, & Nobre, 2016; Lester & Kehle, 2003).

It has been fully acknowledged that mathematization is not a straightforward result of applying a formal method that leads directly from the given data to a result; on the contrary, concepts produced from intuitive and common sense approaches are fundamental anchors in problem-solving. Hence, the often informal character, the unsophisticated appearance and the adherence to the situation are compatible with the development of early models of thinking about the problem (Gravemeijer, 1997). To consider these conceptual models as mathematical models or mathematical skeletons, which are fundamental in finding a solution, supports the idea of a progressive mathematization activity: from a 'model of' into a 'model for' and from an informal into a formal model (Gravemeijer, 2005). Even though it may not have a full robustness and formal appearance, the mathematical skeleton supports the understanding of the situation as well as the actions taken in solving and in expressing the solution.

Obtaining the solution of a problem and communicating the result are often seen as separated phases in the problem-solving process. We argue that these are two closely related aspects in solving problems and that this connection is deeper when the use of digital tools is made available to support the expression of mathematical thinking. This means that every description, illustration, explanation and all the material incorporated into the final product are considered as an integral part of the whole process, as timely argued by Lesh and Doerr:

> …descriptions, explanations, and constructions are not simply processes students use on the way to "producing the answer", and, they are not simply postscripts that students give after the "answer" has been produced. They ARE the most important components of the responses that are needed. (Lesh & Doerr, 2003, p. 3)

In this sense, problem-solving is here regarded as a synchronous process of mathematization and of expressing mathematical thinking mediated by technological tools, hereafter termed problem-solving-and-expressing (Carreira et al., 2016).

Hegedus and Moreno-Armella (2009a) have also discussed the use of representational and communicational infrastructures in the mathematics classroom, pointing out the mathematical expressiveness embedded. A key point in the technological environments lies in the ways in which expressiveness is transformed: there is a new representational expressiveness that students may profit from by using the various functionalities of software that allow many more ways to explain and interpret mathematically, through colours, symbols, gestures, metaphors, etc. For example, actions directly related to the use of digital devices may be the strategically colouring or the use of visual elements such as inserting points, lines or objects, in highlighting the mathematical system in action. As Hegedus and Moreno-Armella (2009a) point out, 'the students express themselves in vivid forms, informally and formally' (p. 405), hence the need of a common ground on understanding the meaning of mathematical representations in digital environments distinctive of the twenty-first-century technological world.

Borba and Villarreal (2005) emphasize the fact of being impossible to offer a clear separation between people and technology and suggest conceiving them as a single entity – *humans-with-media*. From this standpoint, claiming that human beings are constituted by technologies means to acknowledge that they change and reshape ways of thinking and acting, but, at the same time, human beings are continuously producing, developing and transforming technology itself.

One other significant proposal by Borba and Villarreal (1998) is that personal experiences with specific digital media, either in the present or in the past, becomes integrant part of the unit of humans-with-media, even if that experience may not be occurring at the moment. Experience then integrates cognition; for instance, the use of a computer is not limited to assist or support mathematical procedures in a particular context; rather it transforms the nature of what one can do in another moment, i.e. it changes the essence of mathematical activity itself. Furthermore, research also suggests that different collectives of humans-with-media may produce different ways of knowing and thinking: the mathematical knowledge produced by humans-with-paper-and-pencil is qualitatively different from that of humans-with-GeoGebra (Villarreal & Borba, 2010).

2.2.1 Understanding the Interaction Between the Individual and the Digital Tools

The construction of different ways of knowing can be explained by the recognition of possibilities for action in the tools, i.e. their *affordances*. According to Gibson (1979), the affordances are a set of aspects attributed to a certain tool that invite the individual to operate an action upon it. Later, Chemero (2003) argued that 'perceiving affordances is placing features, seeing that the situation allows a certain activity' (p. 187). In a way, this stand contradicts Gibson's original perspective since it sustains that the possibilities for action emerge from the interactions between the agent and the object (Chemero, 2003; Greeno, 1994), and they are not merely properties of potentials of the tool. Yet, there seems to be a consensus regarding the fact that the perception of affordances is a sine qua non condition for activity, even if activity may not actually occur. In fact, Greeno (1994) argues that if the notion of *affordance* refers to whatever there may be in the system that contributes to the existing interaction, then it becomes necessary to define another expression to designate whatever there is in the agent that also contributes to the very same interaction, thus proposing *ability* or *aptitude*.

This intrinsic relationship translates into an impossibility of separating the affordances of the tool from the agent's aptitude, that is, affordances and aptitude are not specifiable in the absence of each other, which to some extent echoes the metaphor of humans-with-media. It therefore seems appropriate to understand problem-solving as a production of mathematical knowledge that is mediated by digital tools, assuming that the subject is immersed, socially and culturally, in a technological environment that shapes and is shaped by its activity.

As we deepen this interplay between the individual and the digital tool, we see how the way of expressing mathematical thinking – that is, how to convey the process of obtaining the solution of a problem (the conceptual models) – is an activity in which intentionality is confused with the feasibility of digital representations (Hegedus & Moreno-Armella, 2009b; Moreno-Armella & Hegedus, 2009). Thus, to understand the subject and the tool as a single unit in solving and expressing problems, it is essential to pay attention to the role of visual methods for solving or other experimental approaches afforded by technologies. This view elucidates the pairing of individuals' intentionality with the feasibility of digital representations, in the sense in which Hegedus and Moreno-Armella (2009b) describe it: 'Today, besides their traditional spaces, symbols are digital and that means that intentionality becomes blended with the executability of their digital representations' (p. 388).

To understand this problem-solving-and-expressing activity involves to recognize the new opportunities found today in the most diverse technological devices related to image and modes of visualization, interactivity and manipulation of objects, with its expressiveness in the use of colours, icons and formats and with communication through sound, video, words and figures. Thinking and communicating seem to become increasingly linked by the increasing plasticity of digital artefacts.

The evolution of static digital objects into dynamic digital objects has been discussed and analysed based on the concept of co-action between the subject and the digital medium that it acts upon (Hegedus & Moreno-Armella, 2009a, 2009b, 2010, 2011; Moreno-Armella & Hegedus, 2009; Moreno-Armella, Hegedus, & Kaput, 2008). The idea of co-action suggests that the individual can be guided or driven by a digital tool selected to perform a given activity and simultaneously can guide and direct the tool to achieve the desired goals. For example, the co-action with the spreadsheet while solving and expressing problems starts with the need to structure the conditions in columns or rows. This procedure allows students to identify a set of numbers with a single name, making it look like a variable, thus enhancing the understanding of its mathematical meaning (Wilson, 2006; Wilson, Ainley, & Bills, 2005). Students then go on to introduce numerical data in the various cells, which may or may not include formulas expressing the relations involved. At this stage, they can analyse the immediate feedback provided by the spreadsheet and readjust their actions in a permanent flow of interactions with the tool (Nobre & Amado, 2013).

This means that the use of a digital tool, such as a spreadsheet or GeoGebra, to solve a problem, may suggest different ways of dealing with the problem, and, at the same time, the way to approach the problem can be a consequence of the guidance provided by the tool to express the thinking that will lead to the solution. The solution is, then, the result of collaborative work between the student and the tool.

Intentionality is another central element of co-action, which is carried by both the individual and the media. Much of the intentionality in the dynamic tools is to react to the subject, thus becoming malleable and adaptable environments. The subject and the environment act and react one over the other, and it is such co-action that constitutes a distinctive feature of the unity between the individual and the tool.

The human-with-media reveals itself and becomes visible through the development of co-actions. The idea that tools and human beings evolve together is essential to understand the importance of co-action in a digital environment, such as GeoGebra or Excel, in which solving mathematical problems occurs, as is the case in the empirical context of our research. In rejecting the separation between humans and the media, as we look at the ways in which students approach mathematical problems with digital tools, we are assuming that students and tools are agents in the development of mathematical knowledge.

We have argued that the problem-solving-and-expressing activity is an outcome of a co-action between the solver, who perceives useful affordances, and the digital tool, which also demonstrates the combination between the mathematical knowledge embedded in the tool and the solver's knowledge. The resulting knowledge is a synthesis of both and becomes a unit rather than an aggregate of both or an extension of either.

The diverse conceptual models developed by students while solving and expressing the solutions may provide relevant clues to understand how mathematical concepts can be used under a variety of (mathematical) points of view and additionally may reveal the mathematical thinking and mathematical representations that stem from the use of digital technologies.

2.3 A Way of Addressing the Data

Considering the nature of the phenomenon and our research goal, namely, the search for a clearer understanding of the participants' ways of solving problems in a particular technological and social environment (SUB12 and SUB14), we assume an interpretative positioning in the development of a qualitative methodological approach (Quivy & Campenhoudt, 2008). We selected qualitative techniques for data collection, organization and treatment, foreseen appropriate to the data at hand, which are composed of digital solutions submitted by several participants to two problems posed by the competitions. The data selection was grounded on the existence and diversity of solutions produced by means of GeoGebra to the problem *How many?*, posed by SUB12, and of solutions produced by means of a spreadsheet to the problem *The passion fruit crop of Mr. Tomás*, posed by SUB14.

The research work was carried out independently by the researchers working in pairs but maintaining a close cooperation in discussing the organization and categorization of the data as well as the potentials of the theoretical concepts mobilized in the analysis. The solutions collected were initially organized according to their original formats, and then a selection was made containing only the ones produced with GeoGebra, in the first case, and with Excel, in the other. The following sections will provide details regarding each set of solutions.

In each case, the samples included similar solutions so we chose the ones who could be representative of one same group and analysed them in depth. The analysis was carried out considering the theoretical concepts and arguments, following descriptive and inductive processes of the steps taken by the solvers in the development of their approaches to the solutions. In each case, our intention is to offer a palette of solutions, that is, from the initial differentiation, we look for progression directly related to the robustness of the conceptual model involved and with its expression by means of GeoGebra and Excel.

Grounded on the intermediary analysis conducted by each pair of researchers, we then identified patterns transversal to the solutions that could assist in explaining and characterizing the differentiation found in the solutions, namely, in terms of the degree of generality in the underlying conceptual models and in the way they are adapted to a certain use of the technological tools.

It is timely to note that both GeoGebra and Excel, as with other interactive tools, offer access to the ways the students interact with them. In the spreadsheet, it is possible to see how the relationships between cells, columns or lines were designed and what formulas were introduced to establish such relationships, and when intermediate relations are used, one can access the order in which they were produced. GeoGebra, on the other hand, includes a particular tool – the Construction Protocol – which records, step by step, the interactions with the software, that is, the procedures taken by the student in obtaining a construction. Although the Construction Protocol does not keep a record of the cognitive processes or other tools that may have been used in the solving-and-expressing activity, it allows access to the particular order in which the geometrical and other objects were included in the construction. This feature of these particular interactive environments is very important for the analysis of solutions whose construction was not observed, since there is evidence of the kind of co-action that may have existed during the activity between the student and the tool.

2.4 A Palette of Solutions Produced with GeoGebra

In this section, we describe and analyse how SUB12 participants (10–12 years old) solve and express mathematical problems with GeoGebra and discuss the ways in which the perception of affordances in this tool triggers the development of different conceptual models of the situation.

The empirical data that support this segment of our research were gathered from the digital solutions submitted by the participants to a geometry problem that aims at determining the total of rectangles that may be inscribed in a dodecagon (Fig. 2.1). The complete and correct solutions were collected and organized according to their original electronic format. The following stage consisted of selecting the solutions produced with GeoGebra and organizing them according to the strategies developed by the participants, which led to three major approaches.

Three solutions, representative of each of these approaches, were selected for a deeper analysis, considering the notions that constitute the theoretical frame for

How many?
How many rectangles can you build inside the polygon, by connecting its vertices?

Don't forget to explain your problem solving process!

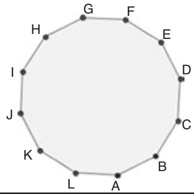


Fig. 2.1 Problem #3 from the SUB12 competition (edition 2012/2013)

understanding them. This detailed analysis is focused on identifying the affordances that are set forth in the development of the approaches and in their relationship with the mathematization level within the conceptual models, grounded on the description of the solutions. The steps taken by the participants while constructing their solution with GeoGebra are possible due to the records included in the Construction Protocol, a tool that allows replicating the sequence of constructions performed.

This problem was proposed in the qualifying stage of the competition SUB12. A total of 215 participants submitted correct solutions and 184 youngsters submitted incorrect or incomplete answers. The participants solved the problem and expressed their reasoning resorting to a multitude of digital tools, ranging from digitalization of paper-and-pencil solutions, using text editors, presentation editors and GeoGebra. From the correct solutions submitted, 24 participants resorted to GeoGebra: 6 of them from the fifth grade and 18 from the sixth grade.

From the analysis of the correct solution produced with GeoGebra, there are three emergent approaches. Thus, we selected one of each type: the solution produced (i) by David, (ii) by Débora and Isabel and (iii) by Greg. The latter was the single one to present the conceptual model of that particular kind.

2.4.1 Setting the Type of Rectangle and Displaying all

David, attending sixth grade, submitted a GeoGebra file (Fig. 2.2) containing three polygons, each one with a certain number of rectangles with similar characteristics. He started his construction by placing the points A and B freely on the GeoGebra plane and resorting to the tool 'regular polygon' used those two points to obtain the dodecagon. He continued by using the tool 'polygon' to represent rectangles of type 1, that is, the ones whose smaller side coincides with the side of the dodecagon – [JEDK], [JCDI], [IBCH], [HABG], [GLAF] and [FELK]. David changed the colour

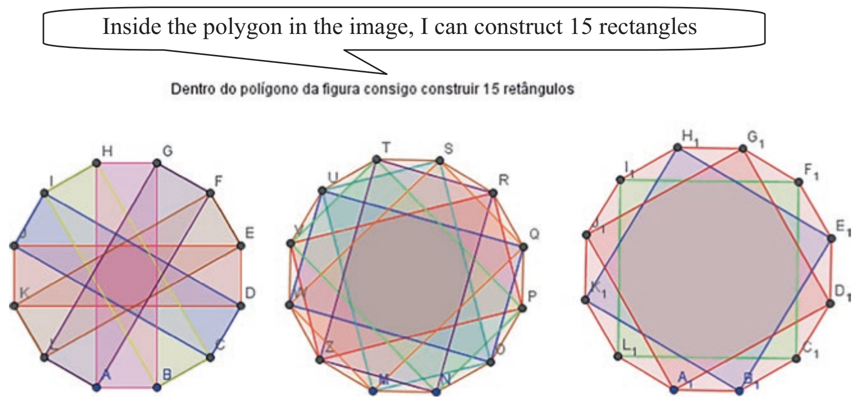


Fig. 2.2 Solution sent by David (sixth grader)

of the quadrilaterals using the ‘object properties’ tool, even though the Construction Protocol does not include information regarding when it was carried out.

He proceeded by placing another two points, M and N, repeating the previous procedure to obtain rectangles of type 2 – [VZPR], [WUQO], [VTPN], [USOM] and [TRNZ]. He also changed the colours in this set of rectangles. On the right side, he continued his approach by representing another dodecagon and then the rectangles of type 3, which are actually squares – [ILCF], [KBEH] and [JADG], also changing their colours resorting to GeoGebra’s tools. At this point, David realized there was a rectangle missing in the previous construction, so he added the quadrilateral [SQMW]. Lastly, using the ‘insert text’ tool, he presented his answer to the problem: in those conditions, it is possible to construct a total of 15 rectangles, which he possibly obtained by adding the rectangles represented in each construction.

David’s way of looking mathematically at this problem entails a conceptual model organized around the three different types of rectangles that are possible to be built. In each step, the youngster sets one of these types and systematically represents them all, recognizing several affordances in GeoGebra, namely, placing points, constructing regular polygons by two given points, constructing quadrilaterals from their vertices, changing properties of objects and inserting text. His conceptual model, a ‘model of’, comprises a rudimentary mathematization extremely close to the context presented, based on the exhaustive representation of every possible case and culminating in counting those cases to obtain the solution.

2.4.2 Setting a Pair of Opposite Vertices and Displaying All

Débora and Isabel, attending fifth grade, were participating in SUB12 as a team. They also solved this problem with GeoGebra, but used a slightly different process from the previous one, which is also possible to document through the analysis of the Construction Protocol in the file they submitted.

As seen in Fig. 2.3, below the constructions, these students included short descriptions of their processes. They began by placing two points on the plane, A and B, using them to obtain the dodecagon by means of the tool ‘regular polygon’. As they mention, they copied and pasted the initial dodecagon five times and started to represent rectangles by setting a vertex and its opposite – in the first case, H and B. The construction started by the rectangle of side HG and continued clockwise: the rectangle of side HF, then HE, HD and HC, all of them containing the vertices H and B. Resorting to the ‘object properties’ tool, they changed the colours of the rectangles and the thickness of their sides. By then, they included the label beneath the first construction using the tool ‘insert text’.

On the right, they repeated the process now considering G_1 e A_1 as the opposite vertices and represented every possible rectangle passing through them and also by another two vertices of the dodecagon. When they completed this step, they realized there were only four possibilities left, as the fifth was already included in the

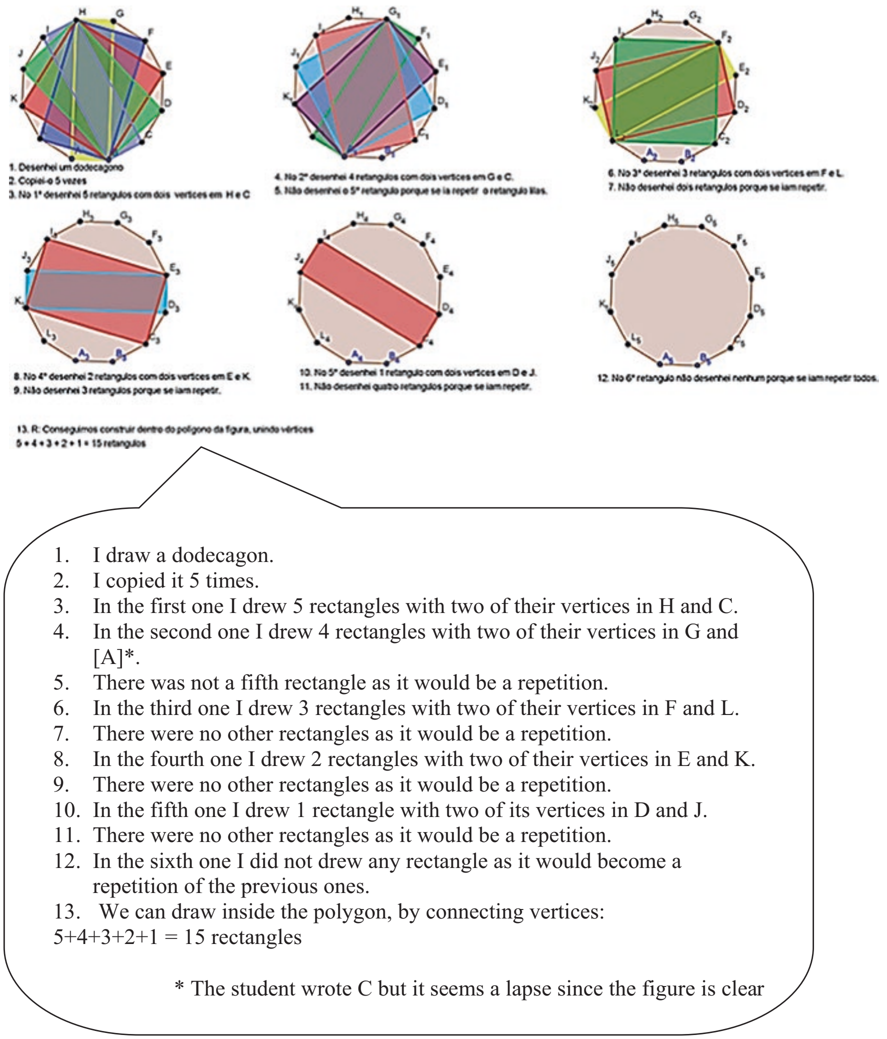


Fig. 2.3 Solution sent by Débora and Isabel (fifth graders)

previous construction. This reasoning guided the remaining of the constructions: on the third one, they represented three rectangles; on the fourth one, they represented two rectangles; and on the fifth dodecagon, there was only one rectangle missing. Underneath each construction, they included a short description of their procedure. The answer to the problem is given by the analysis of the number of represented rectangles in each case which means Débora and Isabel understood that, by maintaining this organized representation approach, they could exclude rectangles previously considered, thus supporting a systematized counting process.

The conceptual model underlying this approach is developed with the perspective that it is possible to find the requested number of rectangles by setting a pair of opposite vertices and representing every possible rectangle containing those vertices of the dodecagon. The method is developed by varying, intentionally and systematically, the positions of the chosen vertices. The affordances identified in GeoGebra's tools support the easy and quick representation of the rectangles, and favour its distinction by using colouring or thickness of the borders, changing the properties of those objects. They also use the tools that permit copy/paste GeoGebra objects, taking advantage of a kind of knowledge that is common in other technological tools, thus allowing them to replicate easily the dodecagon. The lines of text that they include contain detailed descriptions associated with each construction, which are also incorporated in their conceptual model.

However, besides providing meaning to the constructions, these inscriptions are a lever for a more comprehensive look, for thinking mathematically about what they are observing: from one construction to another, there are repeated rectangles – on the second construction, one; then, two; three and so on. The answer they present, whose representation is associated with experimentation and observation and can be considered a horizontal mathematization of the situation, has great potential and can almost be considered a 'model for', where the youngsters identify a particular mathematical pattern: 5 + 4 + 3 + 2 + 1 = 15.

2.4.3 Setting a Vertex and Generalizing

A third approach was identified in the work of Greg, a sixth grader, exemplifying another way of approaching and thinking about this problem that stands out for not needing to exhaustively represent the rectangles to obtain the solution.

Greg initiated his construction (Fig. 2.4) by placing two points on the plane, A and B, and using them to construct a twelve-sided regular polygon using

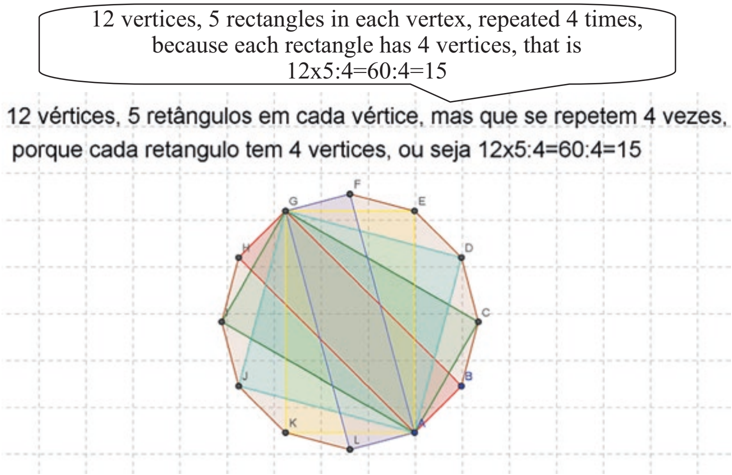


Fig. 2.4 Solution sent by Greg (sixth grader)

GeoGebra tools. Afterwards he represented the rectangle [ABGH] and all the others that contain the vertex A, following the anti-clockwise direction. Similar to the previous solutions, Greg edited the colours of the rectangles using the 'object properties' tool.

This initial approach to the problem seemed to be sufficient for Greg to visualize a way of obtaining the solution without having to construct the remaining rectangles. As he explains, if we think that the polygon has 12 vertices and that in each vertex it is possible to construct 5 rectangles, we would have a total of 60 rectangles to construct. But since each rectangle has 4 vertices inscribed in the dodecagon, in this total count, there are sets of 4 rectangles that, in fact, represent the same. Therefore, it is necessary to divide the total, 60, by 4 to obtain the number of different rectangles that can be inscribed in the vertices of a dodecagon, that is, 15.

This approach comprehends a conceptual model that also combines schemas with textual and mathematical inscriptions, as the ones before. But this strategy differs from previous ones because it is not based on an exhaustive list of possible cases, or, rather, the constructed scheme – which reveals the perception of essential affordances – favours a concrete first experience with the problem that can be considered a model of this specific context. Nonetheless, and despite revealing an elementary mathematization, this 'model of' seems to be responsible for triggering a more elaborate mathematical thinking, with a deeper understanding, almost a generalization, aimed at obtaining the solution and, at the same time, at finding mathematical arguments to justify it.

2.4.4 Discussion

The diversity of digital solutions that were submitted in response to this problem exposes a great disposition of the young participants with this technological tool. Similar to other cases already reported within the same context of mathematical problem-solving with technologies, these youngsters reveal characters of youngsters-with-GeoGebra (Jacinto & Carreira, 2013, 2017) in the sense that they are able to perceive the adequacy and usefulness of this tool and to use it effectively in the development of a solution and in expressing their procedures.

The data analysed indicate that the choice of GeoGebra to solve this particular problem can be grounded on the double role that the tool plays in this activity. If, initially, GeoGebra supports the 'materialization' of the situation, it is afterwards based on this embodiment that the mathematization underlying the strategy and the solution emerges. The free but intentional choice of GeoGebra reveals that these youngsters find in it the most suitable tool to experiment, solve or express themselves, activities that are not in any way parallel to other programmes or tools.

Representing a 12-sided regular polygon can be a demanding and time-consuming task, depending on the tools. But this is radically simplified when GeoGebra is available since it suffices to indicate two points and the number of sides of the polygon to construct it. The Euclidean geometry system of rules incorporated in the programme, which the users may not be familiar with, guarantees that the construction is robust; that is to say, although the dimensions of the polygon may vary, it remains a regular polygon with 12 sides. Building a dodecagon with precision with GeoGebra is easy; these youngsters are aware of it and have the skills to do so.

Similarly, representing the several rectangles, which is vital for the counting of their total number in the first two cases, is also simplified with GeoGebra, using the polygon tool. Although it is a tool that allows free constructions, participants quickly realize whether the choice of vertices they are doing corresponds to a rectangle. Moreover, the way in which they choose to organize, identify and separate the different constructions is an indication that solving and expressing the solution are intentionally simultaneous: the first experiences allow to foresee a number of figures which, when overlapped, may hinder a clear and immediate reading of the strategy by providing lengthy or detailed descriptions. For this reason, these youngsters also resort to setting properties to change the colour of the rectangles or the thickness of their sides. Nevertheless, these formatting choices bring forth a 'star-shaped polygon' that may drive the identification of a visual pattern, which, in turn, favours an organized counting process of the several rectangles. Thus, building all the rectangles with sufficient clarity is also very easy with GeoGebra; these youngsters are aware of it and have the skills to do so.

As the literature suggests, these results show that youngsters perceive GeoGebra's affordances in creating a model of the problematic situation. In some cases, this model is so close to the context that it uncovers all possibilities establishing itself as a model characterized by a horizontal mathematization, somewhat elementary. In another case, recognizing affordances also drives the development of a *model of* the situation, but there is already an intention in providing a more general meaning to the solution. This initial experience can be identified as an attempt to find a pattern or a possibility of generalization, which has the outlines of a *model to*, since it has a more formal structure that allows the solution to be obtained without exhaustively listing all possibilities.

These solutions illustrate how the construction undertaken impels a mathematization approach: to set the type of rectangle and determine all possibilities lead the first participant to explain the solution through the addition $6 + 6 + 3$; setting two opposite vertices and determining all possibilities lead the other two participants to *see* a different sequence so that the solution arises from adding $5 + 4 + 3 + 2 + 1$; while setting a vertex, experimenting and generalizing lead the last participant to another mathematical perspective: $12 \times 5 \div 4$. By further analysing these mathematical structures, we observe that they differ greatly in terms of their potentialities of generalization, which can also be understood in different degrees of sophistication.

Let us test these models' robustness by using them to conjecture the number of rectangles that can be inscribed in a 20-sided regular polygon or an n-sided regular polygon. The first mathematical structure is very dependent on the context and does not allow uncovering relationships between the facts that there are 6 rectangles of type 1 but only 3 rectangles of type 3 when a dodecagon is considered. There is a certain impossibility of moving away from the context to obtain the solution based on this structure. The second model presents a higher degree of sophistication in the sense that the addition of a sequence of natural numbers is very visible so it would be expected that, in a regular polygon of 20 sides, the number of rectangles arises from adding $9 + 8 + \cdots + 2 + 1$. The first term in this sequence is a unit less than half of the number of sides in the regular polygon. Although this is a structure that allows a greater abstraction from the context, it loses its friendliness at this school level, when it requires a large number of calculations and, in itself, impels the need to work with a model of another level of generalization.

The last model shows that by one of the vertices of a dodecagon, it is possible to represent 5 rectangles, so if the polygon has 20 sides, it would be possible to represent 9 different rectangles (half minus 1). Following the same reasoning as the youngster, we can determine the number of rectangles by multiplying the number of vertices of the regular polygon, 20, by the number of possible rectangles in a single vertex, 9, and dividing by the number of times each rectangle repeats, 4. Thus, one would obtain $20 \times 9 \div 4$ or a total of 45 different rectangles. This model is more robust than the previous ones, since it does not require the exhaustive displaying of all rectangles, and it is strongly promoting generalization: with an n-sided polygon, there would be $(n/2) - 1$ rectangles, which means that its algebraic expression could easily be obtained from this.

These mathematical productions expose youngsters-with-GeoGebra mathematization's activity and show that the set of affordances perceived and put into action, besides being diverse, is a key factor throughout the process of finding a solution, from the understanding of the context to the development of a strategy and to its expression. The free choice of the tool and the approach that begins to sketch while each participant perceives possibilities for action in the tool unleashes the progressive development of mathematical structures that differ in terms of their robustness and can be organized in different levels of sophistication. However, the various approaches are channelled into useful conceptual models, since all competitors are able to undertake effective strategies to solve and express the problem.

2.5 A Palette of Solutions Produced with Excel

In this section, we describe and analyse how SUB14 participants (12–14 years old) solve and express mathematical problems with the spreadsheet and discuss the ways in which the co-action between the solvers and the tool triggers the development of different conceptual models of the situation. The data analysed refer to a numerical problem proposed in the qualifying stage of SUB14 (Fig. 2.5).

Of the 93 students who answered correctly, 35 resorted to the spreadsheet to solve the problem. After collecting all the solutions with Excel attachments, we analysed how students used the spreadsheet to solve the problem: how the columns were generated, how relationships were established (with or without formulas) and the diversity of complementary representations used. From the analysis of the various solutions, we found typical solutions that can be assumed as representative.

The passion fruit crop of Mr. Tomás

Mr. Tomás has a passion fruit plantation from where he produces concentrated passion fruit juice. From his last harvest, he managed to get 620 litres of passion fruit concentrate that were bottled in bottles of 5.5 litres and 7 litres. The entire concentrate was packaged and transported to quality control.

What were the possible ways that Mr. Tomás had to bottle all the passion fruit concentrate, using bottles of the two sizes?

Don't forget to explain your problem solving process!

Fig. 2.5 Problem #5 from SUB14 competition (edition 2009/10)

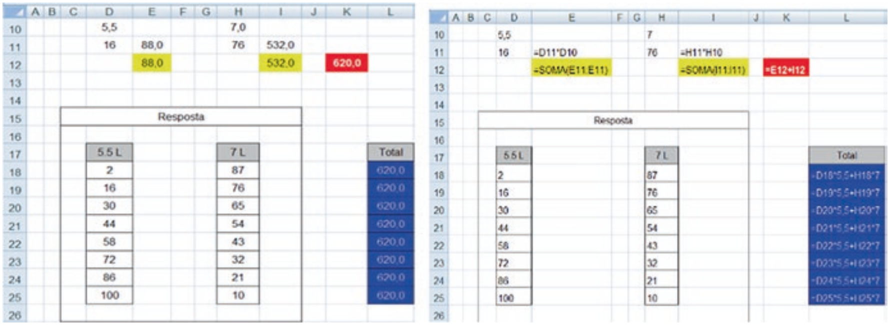


Fig. 2.6 Production sent by Bruno (seventh grader)

From these set of solutions, we selected 3 that are illustrative and show the different paths that the students undertook to obtain the solutions with this digital tool.

This is a numerical problem that has several solutions. Although it is not explicitly stated in the problem, one has to admit that the bottles (of any of the capacities) to be used will be completely filled. From a mathematical point of view, the problem can be solved by the following equation:$5.5x + 7y = 620$ and $x, y \in \mathrm{IN}_0$. If the number x is odd, it is obvious that the amount of juice bottled in the smaller bottles will not be whole number; therefore the remaining juice will not fill a whole number of larger bottles. Thus, the number x will necessarily be even ($x = 2t$, $t \in \mathrm{IN}_0$), which originates an equation such as $11t + 7y = 620$ and $t, y \in \mathrm{IN}_0$. As this is a linear Diophantine equation, it has infinite integer solutions (given that 11 and 7 are relative prime numbers), of which we only want the positive integers. However, the spreadsheet offers other avenues of exploring these solutions, alternative to solving the equation using formal methods.

2.5.1 The Strategy of Guess and Check with Variable Cell

The first example refers to the solution of Bruno, a seventh grader participant, and is summarized in an excerpt of his Excel file (Fig. 2.6). He makes several experiments using variable cells for the two types of bottles. In the cells D10 and H10, he sets the capacities of 5.5 L and 7 L, respectively. The two cells below, D11 and H11, work as containers where Bruno inserts different values to perform experiments on the number of bottles. Cells E11 and I11 are dedicated to compute the amount of bottled concentrate in each of the two types of bottles; for that purpose, Bruno uses formulas where he multiplies the capacity of the bottle by the number of bottles. The results of these two cells are then duplicated to the row immediately below; then he adds those two values in the cell K12. This way, he obtains the total of bottled concentrate. This works as a control cell whose output is used by the student to make adjustments in the number of bottles in order to get a total of 620.

Bruno also constructs a table to expose the solutions he obtained where he presents a column for each type of bottle. On a side column 'total', he shows that in each case, the total of juice is 620. It is likely that Bruno has inserted the solutions on the table as he got them. Looking at the organization of the solutions, it is also likely that he started with a small number of 5.5 L bottles, and while increasing this number, he decreases the number of 7 L bottles.

This student's production shows a great co-action with the spreadsheet, especially in the experiments carried out. In the email message he sent, Bruno explained his procedure in the spreadsheet:

> […] Actually, I did not use any specific method to solve the problem – I got the result only through experimentation, as demonstrated in the Excel sheet (prob5.xls), where if we change some number in the cell for the answer, the result changes instantly.

Bruno's solution is structured in terms of constructing variable cells in the spreadsheet. The variable cells correspond to a simple but very important mechanism for obtaining the outputs of a calculation that involves performing numerical operations between values that are fixed in particular cells (which may be called parameters). The use of variable cells is associated with the idea that in the same cell, a new result can be obtained by simply changing the *input* in a container cell, i.e. the cell that is free to enter different values. As explained by the participant himself, the strategy used was to search for the various solutions to the problem by inspection, i.e. assigning values to the variables x and y and verifying whether or not they satisfy the equality, by checking whether the quantities bottled in the two types of bottles total 620 L.

One relevant aspect of this approach is the perception that the problem has more than one solution. The second aspect to emphasize is that there is an inverse relationship of growth between the variable x and the variable y, that is, the larger the number of one type of bottles, the smaller the number of the other type, since they will have to compensate for the total to be 620 L.

The model underlying Bruno's solution can, therefore, be described as a model of inspection of the solutions of the equation, based on the clear notion of covariation between the variables x and y and the fact that they constitute an ordered pair of variables with materialized meaning in the two types of bottles to be considered. In terms of how this model is transported to the spreadsheet, we are looking at with a good combination with the technological tool that performs the calculations efficiently and automatically and that transforms the systematic and laborious computational work (assign values to the variables and to verify if they satisfy the equation) in an expeditious process. There is an action with the tool (insert two numerical inputs in the variable cells) and a reaction from the tool (output of the calculations performed with the introduced values); then it follows a reaction from the individual (controlling the result and concluding whether or not a solution was found in the cell that shows the output of the linear combination) and if necessary a new action (trying new inputs).

An obvious feature of the conceptual model involved is that of experimentation that could be described as trial and refinement. It is a way of thinking that requires a clear notion of the role of the variable and simultaneously of the meaning of the unknown in an equation. Bruno's model relies on this knowledge and on the ability to recognize the role of the variables x and y in the domain of the problem. But, first and foremost, it is a model for clever 'guessing' of the solutions. The spreadsheet plays an important role in this intention, especially since it clearly supports the work of testing the hypotheses by alleviating the laborious and time-consuming task of computation and verification. In a sense, the tool is adequate to have cells assigned to parameters and variable cells through which the computations are readily assured. This affordance of the tool is, moreover, ascertained by the solver himself, when he states that 'if we change the values in the "answer cells", the result instantly changes'. He names as 'answer cells' those that we know as variable cells, which represents one of tool's affordances that effectively serves the purposes of implementing the student's conceptual model of the situation.

2.5.2 The Use of an Equation with Variable Columns

Ivo is an 8th grader who sent a spreadsheet file and also a Word file where he explained his process:

> To get my answer I used an Excel file (which is attached). First I created the multiples of 5.5 and took out all those that were not integers. Then, for each row, I made the subtraction: 620 - multiples of 5.5. After getting those values, I tried to locate them in a column made of the multiples of 7 (this means finding the number of 7 litter bottles to add to the number of 5.5 litter bottles so that the sum gives 620 according to the expression: $5.5x + 7y = 620$). These numbers are indicated in red in the spreadsheet. The following are the hypotheses of Mr. Tomás to bottle the concentrate using the two [types of] bottles. (...).

This participant writes the equation $5.5x + 7y = 620$ in the spreadsheet (Fig. 2.7). He names the column x, which refers to the number of 5.5 L bottles where he only considers even numbers, because he deletes the multiples that are not integers. In a column labelled as $5.5x$, Ivo obtains the amount of concentrate that is bottled in the 5.5 L bottles. In the next two columns, labelled as y and $7y$, Ivo represents the number of 7 L bottles and the respective amount of concentrate (multiples of 7). In the construction of all the columns, Ivo resorted to variable columns using the Autofill of linear sequences.

The participant then creates a new column named $620\text{-}5.5x$ and applies a formula, which, although unusual, leads to the same result as would be obtained with

	A	B	C	D	E	F	G	H	I
2						5,5x+7y=620			
3									
4		x	5.5x	y	7y	620-5,5x		x	y
5		2	11	1	7	609		2	87
6		4	22	2	14	598		16	76
7		6	33	3	21	587		30	65
8		8	44	4	28	576		44	54
9		10	55	5	35	565		58	43
10		12	66	6	42	554		72	32
11		14	77	7	49	543		86	21
12		16	88	8	56	532		100	10
13		18	99	9	63	521			
14		20	110	10	70	510			
15		22	121	11	77	499			
16		24	132	12	84	488			
17		26	143	13	91	477			
18		28	154	14	98	466			
19		30	165	15	105	455			
54		100	550	50	350	70			
55		102	561	51	357	59			
56		104	572	52	364	48			
57		106	583	53	371	37			
58		108	594	54	378	26			
59		110	605	55	385	15			
60		112	616	56	392	4			
61		114	627	57	399	-7			
62				58	406				
63				59	413				

	A	B	C	D	E	F	G	H	I
2						5,5x+7y=620			
3									
4		x	5.5x	y	7y	620-5,5x		x	y
5		2	11	1	7	=620-(C5:C61)		2	87
6		4	22	2	14	=620-(C6:C62)		16	76
7		6	33	3	21	=620-(C7:C63)		30	65
8		8	44	4	28	=620-(C8:C64)		44	54
9		10	55	5	35	=620-(C9:C65)		58	43
10		12	66	6	42	=620-(C10:C66)		72	32
11		14	77	7	49	=620-(C11:C67)		86	21
12		16	88	8	56	=620-(C12:C68)		100	10
13		18	99	9	63	=620-(C13:C69)			
14		20	110	10	70	=620-(C14:C70)			
15		22	121	11	77	=620-(C15:C71)			
16		24	132	12	84	=620-(C16:C72)			
17		26	143	13	91	=620-(C17:C73)			
18		28	154	14	98	=620-(C18:C74)			
19		30	165	15	105	=620-(C19:C75)			
54		100	550	50	350	=620-(C54:C110)			
55		102	561	51	357	=620-(C55:C111)			
56		104	572	52	364	=620-(C56:C112)			
57		106	583	53	371	=620-(C57:C113)			
58		108	594	54	378	=620-(C58:C114)			
59		110	605	55	385	=620-(C59:C115)			
60		112	616	56	392	=620-(C60:C116)			
61		114	627	57	399	=620-(C61:C117)			
62				58	406				
63				59	413				

Fig. 2.7 Production sent by Ivo (eighth grader)

the simple formula '=620-C5'. This way, the student obtains the amount of concentrate that will be bottled in the 7 L bottles. By inspecting all the values in this column, he looks for those that are multiples of 7, by comparing them with the values in the '7*y*' column. When he detects that a particular value appears in the two columns, he is able to find the value of *x* and the value of *y*. The value of *x* is in the row corresponding to the '620-5.5*x*', whereas the value of *y* is in the row corresponding to the '7*y*'. Finally, Ivo records all the solutions found, the pairs of values *x* and *y*.

In his activity, the participant develops a co-action with the spreadsheet in the generation of the variable columns and in the establishment of relations between variables. However, the search for solutions is done through the comparison of the values in two columns. Ivo, in his solution, explains quite clearly the relationship between the algebraic equation and the resolution in the spreadsheet. His approach consists in defining a variable column to obtain the amount of juice in any even number of 5.5 L bottles and then calculating the difference for the total of 620 L. The search for the solutions is done by checking whether this difference is a multiple of 7. As an illustration, the idea of Ivo can be summarized in the diagram of Fig. 2.8, based on the equality $620 - 5.5x = 7y$.

This solution portrays a strong interaction with the spreadsheet aiming that all solutions to the problem are found. It should also be noted that this solution is based on the multiples of 5.5 and 7 but introduces a way of treating them quite differently

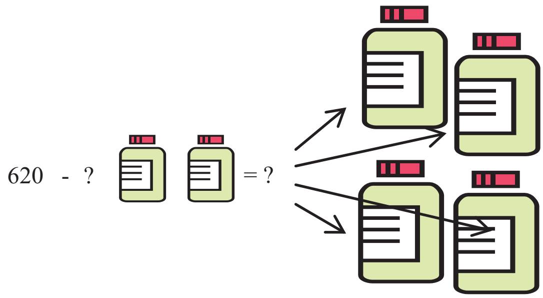


Fig. 2.8 A diagram depicting the conceptual model produced by Ivo (eighth grader)

and more effectively than the previous solution, which used columns to generate these multiples but did not take advantage of the spreadsheet to express the desired relationship between the possible pairs formed by these multiples.

2.5.3 The Use of Predefined Functions with Variable Columns

Luís is another participant, attending the 8th grade, who also used the spreadsheet as can be seen in Fig. 2.9. In his solution, Luís names five columns: the first, for the number of 7 L bottles; the second, for the amount of concentrate bottled in the 7 L bottles; the third column shows the number of litres to be stored in the smaller bottles in order to make a total of 620 L; the fourth column presents the remainder of the division of values in the third column by 11; and the last column makes a test on the values of the remainders to see when they are null, which means finding a solution.

In his activity, Luís uses the generation of variable columns through the introduction of formulas and the Autofill function. He also uses predefined functions, such as the MOD function (RESTO in Portuguese), and resorts to a logical function where in case the remainder is 0, it returns the quotient (number of 5.5 L bottles); otherwise it returns the word 'False'. In the email sent, Luís explains his procedures:

> Answer:
> x = number of bottles of 7 L, y = number of bottles of 5.5 L
>
> Since the equation has 2 variables, it is not possible to solve it in a 'normal' way (by hand). I used Excel and created the right formulas.
>
> In the first column, there is the number of 7 L bottles used.
>
> In the second column there is the volume contained in the 7 L bottles (1st column times 7). In the third column, there is the remaining volume to be put in the 5.5 L bottles. Then the number of 5.5 L bottles must be a multiple of 2 because otherwise there is always half a litter left. So, to be a whole number the amount stored in the 5.5 L bottles is always a multiple of 11.
>
> In the fourth column, I have the remainder of the division of the remaining volume by 11 (2 bottles of 5.5). In the fifth column I checked if the remainder of the division is 0; if so, it returns the number of 5.5 L bottles. There are 8 possible solutions.

	A	B	C	D	E
1	Nº de garrafas de 7 ltrs.	Vol. nas garrafas de 7ltrs.	Vol. Guardado (restante)	divisão do volume restante por 11	Nº de garrafas de 5,5 ltrs.
2	1	7	613	8	FALSO
3	2	14	606	1	FALSO
4	3	21	599	5	FALSO
5	4	28	592	9	FALSO
6	5	35	585	2	FALSO
7	6	42	578	6	FALSO
8	7	49	571	10	FALSO
9	8	56	564	3	FALSO
10	9	63	557	7	FALSO
11	10	70	550	0	100
12	11	77	543	4	FALSO
13	12	84	536	8	FALSO
14	13	91	529	1	FALSO
15	14	98	522	5	FALSO
16	15	105	515	9	FALSO
76	74	518	102	3	FALSO
77	75	525	95	7	FALSO
78	76	532	88	0	16
79	77	539	81	4	FALSO
80	78	546	74	8	FALSO
81	79	553	67	1	FALSO
82	80	560	60	5	FALSO
83	81	567	53	9	FALSO
84	82	574	46	2	FALSO
85	83	581	39	6	FALSO
86	84	588	32	10	FALSO
87	85	595	25	3	FALSO
88	86	602	18	7	FALSO
89	87	609	11	0	2
90	88	616	4	4	FALSO

	A	B	C	D	E
1	Nº de garrafas de 7 ltrs.	Vol. nas garrafas de 7ltrs.	Vol. Guardado (restante)	divisão do volume restante por 11	Nº de garrafas de 5,5 ltrs.
2	1	=A2*7	=620-B2	=RESTO(C2;11)	=SE(D2=0;C2/5,5)
3	2	=A3*7	=620-B3	=RESTO(C3;11)	=SE(D3=0;C3/5,5)
4	3	=A4*7	=620-B4	=RESTO(C4;11)	=SE(D4=0;C4/5,5)
5	4	=A5*7	=620-B5	=RESTO(C5;11)	=SE(D5=0;C5/5,5)
6	5	=A6*7	=620-B6	=RESTO(C6;11)	=SE(D6=0;C6/5,5)
7	6	=A7*7	=620-B7	=RESTO(C7;11)	=SE(D7=0;C7/5,5)
8	7	=A8*7	=620-B8	=RESTO(C8;11)	=SE(D8=0;C8/5,5)
9	8	=A9*7	=620-B9	=RESTO(C9;11)	=SE(D9=0;C9/5,5)
10	9	=A10*7	=620-B10	=RESTO(C10;11)	=SE(D10=0;C10/5,5)
11	10	=A11*7	=620-B11	=RESTO(C11;11)	=SE(D11=0;C11/5,5)
12	11	=A12*7	=620-B12	=RESTO(C12;11)	=SE(D12=0;C12/5,5)
13	12	=A13*7	=620-B13	=RESTO(C13;11)	=SE(D13=0;C13/5,5)
14	13	=A14*7	=620-B14	=RESTO(C14;11)	=SE(D14=0;C14/5,5)
15	14	=A15*7	=620-B15	=RESTO(C15;11)	=SE(D15=0;C15/5,5)
16	15	=A16*7	=620-B16	=RESTO(C16;11)	=SE(D16=0;C16/5,5)
76	74	=A76*7	=620-B76	=RESTO(C76;11)	=SE(D76=0;C76/5,5)
77	75	=A77*7	=620-B77	=RESTO(C77;11)	=SE(D77=0;C77/5,5)
78	76	=A78*7	=620-B78	=RESTO(C78;11)	=SE(D78=0;C78/5,5)
79	77	=A79*7	=620-B79	=RESTO(C79;11)	=SE(D79=0;C79/5,5)
80	78	=A80*7	=620-B80	=RESTO(C80;11)	=SE(D80=0;C80/5,5)
81	79	=A81*7	=620-B81	=RESTO(C81;11)	=SE(D81=0;C81/5,5)
82	80	=A82*7	=620-B82	=RESTO(C82;11)	=SE(D82=0;C82/5,5)
83	81	=A83*7	=620-B83	=RESTO(C83;11)	=SE(D83=0;C83/5,5)
84	82	=A84*7	=620-B84	=RESTO(C84;11)	=SE(D84=0;C84/5,5)
85	83	=A85*7	=620-B85	=RESTO(C85;11)	=SE(D85=0;C85/5,5)
86	84	=A86*7	=620-B86	=RESTO(C86;11)	=SE(D86=0;C86/5,5)
87	85	=A87*7	=620-B87	=RESTO(C87;11)	=SE(D87=0;C87/5,5)
88	86	=A88*7	=620-B88	=RESTO(C88;11)	=SE(D88=0;C88/5,5)
89	87	=A89*7	=620-B89	=RESTO(C89;11)	=SE(D89=0;C89/5,5)
90	88	=A90*7	=620-B90	=RESTO(C90;11)	=SE(D90=0;C90/5,5)

Fig. 2.9 Production sent by Luís (eighth grader)

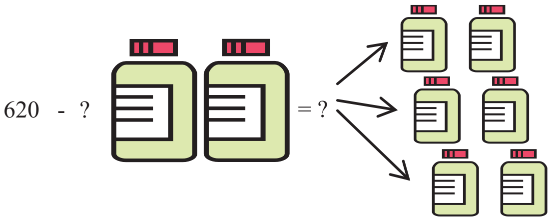


Fig. 2.10 A diagram depicting the conceptual model produced by Luís (eighth grader)

As Luís explains, the spreadsheet allows an alternative way of solving an equation with two variables, for which he lacks the formal knowledge. His co-action with the spreadsheet consists of establishing relationships between variables by creating variable columns and searching the solutions by using a logical function representing the criterion to identify the solutions to the problem.

Luis' solution reveals a degree of sophistication and awareness of the affordances of the tool even greater than in the previous cases. His conceptual model is schematized in Fig. 2.10. In this case, the variable x is defined as a function of y. By changing the value of y, the amount of juice used in the 7 L bottles is calculated, and then the remaining juice of the total of 620 L is calculated, which is to be distributed by the 5.5 L bottles. To obtain the number of 5.5 L bottles, this student performs a test that consists of dividing the remaining juice by 11 (two times 5.5 L). When the remainder is zero, it means that the student found a solution to the problem.

The way the test is accomplished deserves special attention. The student uses a predefined function in the spreadsheet, which returns the remainder of the division of a number by another. The idea of using the MOD function is a more sophisticated alternative to check whether a number is or is not a multiple of 11, therefore avoiding the creation of an auxiliary column of multiples to perform that checking. Luis's activity goes even further regarding the co-action with the tool. He uses a logical function to perform the test and introduces a conditional formula that returns the number of 5.5 L bottles when the remainder is zero.

From the point of view of the conceptual model used to interpret and solve the problem as well as of the use of the digital tool, this solution shows the highest degree of sophistication and robustness, because it provides a general method that can be replicated for any other situation in which the given constants (capacity of each type of bottles and total amount of juice) are arbitrary.

2.5.4 Discussion

In reviewing the previously analysed solutions, we observe a diversity of ways of solving the problem with the spreadsheet. Firstly, we can identify the two main types of conceptual models. In one of them, students generate the multiples of 5.5 (or 11) and the multiples of 7 and look for the cases where their sum is 620, thus obtaining the solutions (Bruno's solution). In the other, the students define one of the variables as a function of the other, which leads them to find the solutions more easily (Ivo and Luís' solutions). The second type of model is more sophisticated than the first, in what concerns the mathematical reasoning involved.

In the analysis carried out, there are evidences that more sophisticated conceptual models are associated with a stronger co-action of the participant with the spreadsheet. In the case of the last solution, the spreadsheet is used in a way that the solver does not need to make trials or go looking for the solutions. This case shows that the participant knows how to take advantage of the tools' affordances to achieve the solution more efficiently and quickly.

At the opposite end, the spreadsheet is only used to make trials through the use of variable cells. The student still needs to go searching for the solutions and his co-action with the spreadsheet is weaker.

In this way, the co-action between the individual and the spreadsheet is a relevant criterion for establishing a gradual progression of mathematically expressed conceptual models through this digital tool.

2.6 Conclusions

In this chapter we discuss mathematical problem-solving with digital tools in terms of the development of conceptual models, taking as grounding arguments the impossibility of offering a clear boundary between solving a problem and expressing the underlying mathematical thinking and the impossibility of separating the solver from the digital tools in the activity of solving and expressing the problem.

In the theoretical background, we argued that the construction of conceptual models emerges from the individuals' ability in dealing productively with the mathematics embedded in the situations. In other words, this means to be in the presence of explicit descriptive and explanatory systems that offer visibility and expressiveness to the understanding of the mathematizable situation and to the search for and achievement of a solution. Thus, to distinguish different ways of approaching a problem and to consider the emergent conceptual models seem not to be enough; it is also important to understand how different conceptual models emerge and are developed in parallel with the way the digital tools, nowadays accessible to all, are used in such activity. In fact, these conceptual models can have several degrees of robustness, may it be embryonic or completely developed, and even if complete, they may be more or less robust.

This is partially why we have used a set of empirical data from the large hundreds of responses to mathematical problems in digital format produced by many young participants in the SUB12 and SUB14 Competitions. This opportunity to face a vast amount of digital solutions enables a data acquisition in which quantity and diversity are valuable ingredients for searching and discussing different degrees of sophistication of conceptual models and ways of using digital technologies. Among several possibilities, we decided to focus on two problems: one of SUB12 and one of SUB14; the first is related to the use of GeoGebra and the second related to the use of Excel. In each case, we analysed a set of solutions of different students in order to find evidence of different degrees of sophistication of the conceptual models and, at the same time, try to characterize the way in which the use of the tool was incorporated in the development of these models.

In the first case, attention was paid to each participant's ability in recognizing affordances on GeoGebra, so we sought to interpret the humans-with-media entity regarding the affordances and the actions triggered, which led to a particular solving-and-expressing approach to the problem. In the versions reported, the common ground seems to lie in the construction of figures, exploration of properties and in the possibility of visualizing and recognizing some kind of systematization or organization in the given situation. The use of multiple affordances in GeoGebra was relevant and productive in each solution. The construction of regular polygons, the definition of polygons based on all their vertices, as well as the use of formatting options, such as the colour of the polygons or the thickness of the segments, and the use of text and inscriptions were decisive in solving and expressing the problem.

The recognition and mobilization of affordances by these youngsters show a differentiation in the role played by the visualization ability, endorsed by GeoGebra. Although distinct from each other, David's solution and the one produced by Débora and Isabel take advantage of an attribute of visualization: they visualize in order to verify. Every possible rectangle is displayed in the course of solving the problem, and in each case there is a certain way of thinking that guides the construction of all the rectangles to be considered. In contrast, Greg's solution takes advantage of another attribute of visualization: he visualizes in order to imagine. This variation is consistent with the fact that digital geometry environments can generate reasoning mediated by the software but may also elicit mathematical explanations of the geometric situation (Jones, 2000). Between the visualization to verify and the visualization to imagine, it seems to be a differentiation that is explained by the way the interaction between the agent and the tool occurs (Chemero, 2003). Greg's construction of the various rectangles that can be defined maintaining one of its vertices on a particular dodecagon vertex led him to find the total number of rectangles. His conceptual model is robust, sophisticated and generalizable.

In the first two cases, the display of every rectangle is a result of different ways of thinking about its construction and how to guarantee the total number without repetitions. The use of colour has a very important meaning because it highlights and clarifies the configurations and the spatial characteristics of the various rectangles. The solution of Débora and Isabel has, in addition, the advantage of referring to a numerical regularity and, as such, also allows imagining beyond the image, which is not plausible with David's solution.

Therefore, it is possible to relate a nuance in the sophistication or robustness of these conceptual models ('model of' and 'model for') with the recognition of affordances of the tool. Figure 2.11 seeks to illustrate this progression on a continuous scale and proposes a view of the relative position between the three modes of thinking about this particular situation, in conjunction with the distinguishing criterion of the perception of affordances in the digital tool.

Regarding the analysis of the solutions produced with Excel in a problem involving a linear equation with two integer variables, we also observed a range of ways of thinking about the situation that allows discussing and interpreting a progression in the conceptual models developed.

There is a fundamental distinction between two approaches: one that assumes the two variables to take independent values and the other that assumes one of the variables to be depending on the other. The first approach entails a less sophisticated or robust conceptual model, while the spreadsheet becomes a means of experimentally investigating the outcomes of assigning values to one variable through organized attempts involving numerical properties of the solutions, such as being multiples of particular numbers.

It should be noted that the use of colour in the various versions of the solving-and-expressing process is extremely relevant. In fact, the use of colour and its

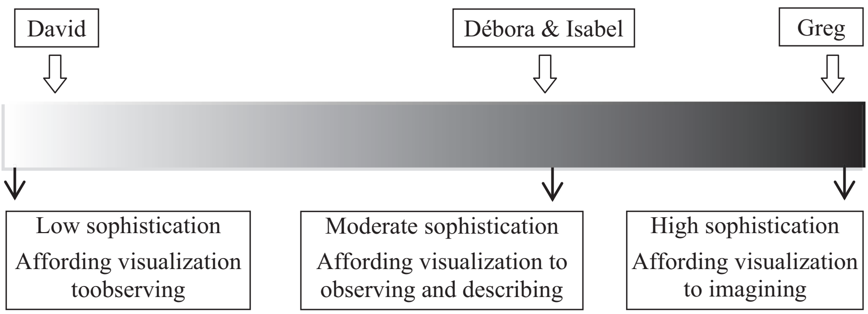


Fig. 2.11 Different levels of sophistication in conceptual models and its correspondence with the perception of affordances for visualization in the tool

relevance were also reported on the solutions to the problem *How many?*, particularly useful as a strategy of organization and clarification of the given situation and its expression through the digital tool. The use of textual and other types of inscriptions, in addition to those that result immediately from the use of cells and formulas in Excel, is equally relevant in these solutions. 'Translating' the actions performed in Excel by means of long descriptions may be a way of compensating for the opacity of the spreadsheet's language regarding the expression of mathematical thinking.

Considering this set of solutions, while there are informal conceptual models, essentially rooted in numerical modes of thinking, others have a more formal aspect since they have an increasingly algebraic character, although they may apparently be numerical since they are expressed in the spreadsheet through the use of variable cells or variable columns. The increasingly algebraic solutions portray a transfer of the goal of materializing the variables into Excel. They focus on the identification of a free variable, exhibiting the other variable as a function of the first one. In a sense, they approach the numerical relational nature of this tool and incorporate a functional or covariational way of thinking.

One of the relevant aspects of this progressive mathematization activity is closely linked to the visual feedback that the spreadsheet provides to the user's actions. This feedback tends to be more informative in cases where the type of relationships between columns or between cells is progressively more relational and less informative when cells or columns are mainly numerical and the type of approach is based on trials and checking the solutions. In this sense, it is possible to differentiate the co-action between the student and the tool according to the problem-solving-and-expressing activity they engage in.

By detecting, in each case, indicators of how solvers guided and were guided by the opportunities and attributes they recognized in the digital tool, we clarify the existing co-action, which portrays the interaction linking intentionality and representational expressiveness to the conceptual models produced by the participants while developing a solution. The most sophisticated conceptual models, namely, those in which the possibility of defining one variable as a function of the other prevails, reflect a high degree of co-action between the tool and the user. In the case of Luís' solution, for example, not only is the conceptual model expressed through variable columns representing the relationship between the variables x and y, but the spreadsheet itself has the role of testing and returning the results that correspond to admissible solutions. Therefore, the students' reliance on the feedback from the tool is evident either when they use numerical relations or when they represent functional relations.

From the analysis of the solutions produced with the spreadsheet, we propose a way of ordering them, in terms of the robustness or sophistication of the conceptual model that prevails in the organization and materialization of the operations and actions executed with Excel, in line with the intensity of the co-action between the solver and the digital tool (Fig. 2.12).

By comparing the independent analyses of these two sets of data, it is possible to distinguish the greater or lesser sophistication of the students' conceptual models

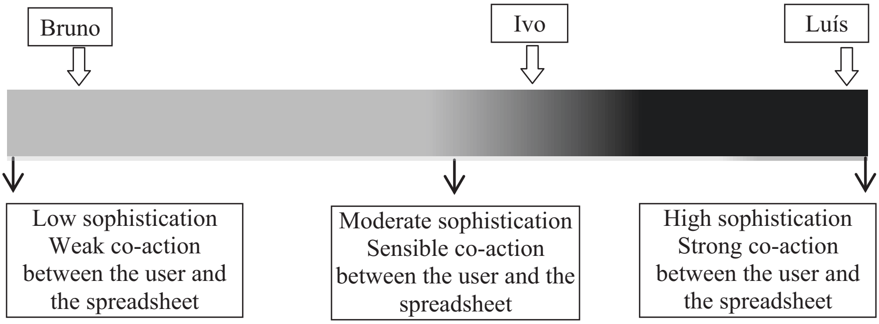


Fig. 2.12 Different levels of sophistication in conceptual models and its correspondence with the intensity of the co-action between the user and the tool

and to understand how this emerges linked to a certain way of using a digital tool in solving a problem and expressing the underlying conceptual model. Two independent forms of analysis were used, based on different theoretical concepts, although consistent with one another: (i) the recognition of the tool's affordances to develop productive work on the problem and (ii) the type of co-action between the individual and the tool in the problem-solving-and-expressing process.

Our results reveal that more or less sophisticated conceptual models are directly involved in the understanding of the task and in the development of an approach to the solution. This is neither a new result nor a new way of understanding and explaining problem-solving and its relation to the production of mathematical knowledge. However, these results amplify the idea that there is a gradual continuous between distinct versions of conceptual models, characterized by details, variations, specificities, possible to be described by means of a gradual scale. We are not, of course, advocating a distinction between worse and better conceptual models, let alone between unacceptable and acceptable models; rather, we are emphasizing their legitimacy as being in a continuous development. Our data highlight how we can understand its development.

Moreover, our results also place a particular emphasis on the way mathematical problem-solving takes place with the use of generally available digital tools. Bearing in mind the theoretical notion of humans-with-media, thus assuming the unity between the individual and the tool it uses, we found ways of establishing an elucidative parallel between the students' conceptual models and patterns of use of the digital tools (GeoGebra and Excel). Both of them indicate analogous, compatible and congruent types of progression. Thus, the use of a digital tool can be seen as a vehicle for the expression of different conceptual models, in the same way that the construction of conceptual models seems to have as counterpoint the different ways of recognizing the tools' affordances and the corresponding interactions.

From this point of view, we argue that to-think-with-GeoGebra or to-think-with-Excel means to incorporate the possibilities of mathematical 'expressiveness' conveyed by the digital tools, which invite the student to develop mathematical thinking. Solving-and-expressing by means of digital tools is therefore something that comes as a result of introducing a mathematical point of view to address a situation combined with recognizing a mathematical point of view in the tool to deal with the same situation. The relevance of this conclusion seems to lie in the fact that a digital tool is itself a tool of *conceptual expressiveness*, even in seemingly insignificant aspects such as allowing formatting or editing, organizing and materializing objects, just as it happens with the *conceptual nature* of the use of colour, i.e. when it adds meaning to a mathematical construction.

References

Barbeau, E., & Taylor, P. (Eds.). (2009). *Challenging mathematics in and beyond the classroom. The 16th ICMI study*. New York, NY: Springer.

Borba, M., & Villarreal, M. (1998). Graphing calculators and the reorganization of thinking: The transition from functions to derivative. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 136–143). Stellenbosch, South Africa: PME.

Borba, M., & Villarreal, M. (2005). *Humans-with-media and the reorganization of mathematical thinking*. New York, NY: Springer.

Carreira, S. (2015). Mathematical problem solving beyond school: Digital tools and students' mathematical representations. In S. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 93–113). Cham, Switzerland: Springer.

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematical problems with technology*. New York, NY: Springer.

Chemero, A. (2003). An outline of a theory of affordances. *Ecological Psychology, 15*(2), 181–195.

Gibson, J. (1979). *The ecological approach to visual perception*. Hillsdale, NJ: Lawrence Erlbaum.

Gravemeijer, K. (1997). Solving word problems: A case of modelling? *Learning and Instruction, 7*(4), 389–397.

Gravemeijer, K. (2005). What makes mathematics so difficult, and what can we do about it? In L. Santos, A. P. Canavarro, & J. Brocardo (Eds.), *Educação matemática: Caminhos e encruzilhadas* (pp. 83–101). Lisboa, Portugal: APM.

Greeno, J. (1994). Gibson's affordances. *Psychological Review, 101*(2), 336–342.

Hegedus, S., & Moreno-Armella, L. (2009a). Intersecting representation and communication infrastructures. *ZDM Mathematics Education, 41*, 399–412.

Hegedus, S., & Moreno-Armella, L. (2009b). Introduction: The transformative nature of "dynamic" educational technology. *ZDM Mathematics Education, 41*, 397–398.

Hegedus, S., & Moreno-Armella, L. (2010). Accommodating the instrumental genesis framework within dynamic technological environments. *For the Learning of Mathematics, 30*(19), 26–31.

Hegedus, S., & Moreno-Armella, L. (2011). The emergence of mathematical structures. *Educational Studies in Mathematics, 77*, 369–388.

Jacinto, H. (2017). A atividade de resolução de problemas de matemática com tecnologias e a fluência tecno-matemática de jovens do século XXI (Unpublished doctoral dissertation). Instituto de Educação, Universidade de Lisboa, Lisboa, Portugal.

Jacinto, H., & Carreira, S. (2013). Beyond-school mathematical problem solving: A case of students-with-media. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th conference of the IGPME* (Vol. 3, pp. 105–112). Kiel, Germany: PME.

Jacinto, H., & Carreira, S. (2017). Mathematical problem solving with technology: The techno-mathematical fluency of a student-with-GeoGebra. *International Journal of Science and Mathematics Education*, *15*(6), 1115–1136. https://doi.org/10.1007/s10763-016-9728-8

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*, 55–85.

Lesh, R., & Doerr, H. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism – Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–33). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Zawojewski, J. (2007). Problem Solving and Modeling. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Charlotte, NC: Information Age Publishing and National Council of Teachers of Mathematics.

Lester, F., & Kehle, P. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism – Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 501–517). Mahwah, NJ: Lawrence Erlbaum Associates.

Moreno-Armella, L., & Hegedus, S. (2009). Co-action with digital technologies. *ZDM Mathematics Education, 41*, 505–519.

Moreno-Armella, L., Hegedus, S., & Kaput, J. (2008). From static to dynamic mathematics: Historical and representational perspectives. *Educational Studies in Mathematics, 68*, 99–111.

Nobre, S., & Amado, N. (2013). Combining the spreadsheet with paper and pencil: a mixed environment for learning algebraic methods. In E. Faggiano & A. Montone (Eds.), *11th International Conference on Technology in Mathematics Teaching – Conference Proceedings* (pp. 226–231). Bari, Italy: Università degli Studi di Bari Aldo Moro.

Nobre, S., Amado, N., & Carreira, S. (2012). Solving a contextual problem with the spreadsheet as an environment for algebraic thinking development. *Teaching Mathematics and Its Applications, 31*, 11–19. https://doi.org/10.1093/teamat/hrr026

Quivy, R., & Campenhoudt, L. (2008). *Manual de Investigação em Ciências Sociais*. Lisboa, Portugal: Gradiva.

Villarreal, M., & Borba, M. (2010). Collectives of humans-with-media in mathematics education: Notebooks, blackboards, calculators, computers and… notebooks throughout 100 years of ICMI. *ZDM Mathematics Education, 42*(1), 49–62.

Wijers, M., Jonker, V., & Drijvers, P. (2010). MobileMath: Exploring mathematics outside the classroom. *ZDM Mathematics Education, 42*(7), 789–799.

Wilson, K. (2006). Naming a column on a spreadsheet. *Research in Mathematics Education, 8*, 117–132.

Wilson, K., Ainley, J., & Bills, L. (2005). Spreadsheets, pedagogic strategies and the evolution of meaning for variable. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 321–328). Melbourne, Australia: PME.

Chapter 3
Solving Probabilistic Problems with Technologies in Middle and High School: The French Case

Bernard Parzysz

3.1 Introduction

Through the last century statistics and probability used to be separate chapters in the French secondary mathematics curriculum (Parzysz, 2003), but a new approach of probability, together with the development of technologies for teaching, threw a bridge between them from the beginning of this century onwards. In high school curricula, for instance, the teaching of probability has evolved a great deal, shifting from a purely 'cardinalist' standpoint on the concept of probability to a more experimental and less dogmatic approach, integrating a 'frequentist' standpoint.

To some extent, this situation results from the fact that nowadays technological tools play a large role in the teaching/learning of mathematics and in particular for solving probability problems. However, 'it is not good enough to only consider which technology to use, but (…), in order for effective learning to take place, it is how the technology is integrated into the curriculum and learning process and how the teacher uses it that are vital' (Pratt, Davies, & Connor, 2008, p. 98). As Trouche pointed out:

> The development of computer tools has largely influenced the development of some mathematical domains […] and given a new status to experimental features in research […]. These effects apply to classroom mathematics as well. (…) Tools also have a great impact on the way students work. […] Finally, the tools put into play in teaching have thorough effects on conceptualization. (Trouche, 2005 p. 267)[1]

[1] My translation.

B. Parzysz (✉)
Bernard Parzysz, Université d'Orléans, Orléans, France

Laboratoire de Didactique André Revuz, Université Paris-Diderot, Paris, France
e-mail: parzysz@irem.univ-paris-diderot.fr

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_3

These recent changes were not without consequence on students' understanding; for example, as I show below, they may favour a tendency to blur the distinction between similar notions in one domain and the other. As Girard (2008, p. 2) states, 'teaching probability by modelling and simulation is not easy', also adding that 'the link between statistics and probability is still to be clarified'. Borovcnik (2008, p. 79) advocates supplying students with a wider and clearer view on probabilistic notions, noting that 'clarification of the mutual dependencies between frequentist, Bayesian, and mathematical conceptions and intuitive thought makes the concept of probability flexible and robust'.

Although dealing with the specific case of the French current situation at middle and high school levels, this paper may be considered a survey of the main types of problems to be solved with the help of technological devices, some results being surely applicable to other countries since many students meet with such problems some time.

3.2 Didactical Frameworks for Teaching/Learning Mathematics in France

3.2.1 The Tradition: TDS, ATD and TCF

In France, contrary to many other countries, there is, for historical reasons, almost no research labelled 'problem-solving', i.e. aiming at developing specific skills. This has to be related with some features of the didactical frameworks developed by French researchers from 1970 onwards, the two best known and best used among them being TDS (Theory of Didactic Situations, initiated by Brousseau) and ATD (Anthropological Theory of Didactics, initiated by Chevallard).

TDS postulates that mathematical knowledge comes out of solving mathematical problems within didactical situations, and hence problem-solving is a fundamental element of the theory,[2] a condition for a suitable problem being that the aimed knowledge is an optimal tool for solving it. Another condition is that mathematical learning can only be achieved if the problem is not much dependent on the teacher, hence the importance of the notions of devolution (i.e. the teacher's problem becoming the students' problem) and milieu (i.e. symbolic and material artefacts with which teacher and students interact when they behave as epistemic subjects). A consequence is that a problem is never isolated but is an element in the progression of knowledge in a given domain of mathematics, the development of general solving skills being not an aim for the teaching.

ATD considers institution and praxeology as fundamental elements[3]: mathematical activity – like any human action – takes place within an institution and com-

[2] See Brousseau (1997) and Warfield (2006).

[3] See Chevallard (1999), and Chevallard (2002).

prises two main components, a practical one (praxis) and a theoretical one (logos), interacting one with the other. In this framework mathematical knowledge comes out of solving types of tasks, together with associated techniques and a discourse justifying these techniques within the institution (called technology), and in turn technology being possibly justified by a theory. In this framework, a problem is embedded in a problematic question allowing the elaboration of technique and theoretical elements (Bosch, Chevallard, & Gascón, 2005).

A third framework, TCF (Theory of Conceptual Fields, initiated by Vergnaud[4]), includes a cognitive perspective in mathematical learning, putting the stress on the organization of problems into classes inside a given conceptual domain. Thus, one can see that even if the cognitive dimension is not missing in French didactics, the emphasis is rather on the epistemological side. Until recently, the French situation was what Artigue and Houdement (2007) described: 'Research that could more or less be connected with problem-solving perspectives or metacognition has been maintained in a marginal position' (p. 367).

In accordance with what was said above, problem-solving does not appear explicitly as a major goal of teaching mathematics but as a point among others, the stress being put on getting familiar with semiotic representation registers (figural, graphical, numerical) and on skills and methods linked with technologies (working out algorithms, simulation, experimentation…). However, solving problems is considered the basis of whole classroom work, under the designation 'activity', and general cognitive skills are taken into account, for instance, in the 11th grade syllabus:

> The activities proposed in the classroom and outside lean on solving problems, purely mathematical or coming from other matters. Of various kinds, they must train the students to:
>
> – search experiment, model, especially with technological tools;
> – choose and apply calculation techniques;
> – put algorithms into play;
> – reason, demonstrate, find out partial results and put them in perspective;
> – explain orally a process communicate a result orally or in writing. (Official Bulletin, 2010/09/30)[5]

3.2.2 A Newcomer: MWS

Some years ago, the Geometrical Working Spaces framework (Houdement & Kuzniak, 2006) was extended into a general framework, MWS (Mathematical Working Spaces), including an explicit interaction between the epistemological and cognitive sides of mathematical activity (Kuzniak, 2011). It does not conflict with other frameworks but, on the contrary, is intended to be open and adaptable in interaction with other approaches in mathematics education:

[4] See Vergnaud (1990).

[5] My translation.

> The MWS construction is an object of a very different nature (…). Its logic seems more that of an assembly that would incorporate, possibly with adaptation, a diversity of constructs and perspectives developed in the field, without privileging any of them. This gives the MWS structure a plasticity that big theories (…) do not have, and certainly contributes to its accessibility and attractivity. (Artigue, 2016, 936)[6]

Its aim is to describe the various dimensions of the work put in play by anybody who is engaged in solving a mathematical problem; therefore, it takes into account not only mathematical knowledge, of course, but also cognitive activity. More precisely, within the MWS framework, two distinct 'planes', one epistemological and one cognitive, interacting with each other, are considered (Kuzniak, 2011).

The epistemological plane has three components:

– A real space, including a set of semiotic representations (Duval, 1995)
– A set of instruments (or artefacts), allowing to perform actions on the representations
– A theoretical reference system, consisting of a set of mathematical knowledge

The cognitive plane is composed of three processes:

– Visualization (this word being taken in a quite broad sense)
– Construction, regarding the use of instruments
– Proof, in relation to the reference system

The elements of this plane are connected with those of the epistemological plane along three dimensions:

– Semiotic, related to the production of representations and visualization
– Instrumental, regarding constructions using instruments (e.g. software)
– Discursive, in relation to proof

Figure 3.1 shows the articulation between the two planes and their components.

The hypothesis underlying the MWS framework is that a good connection between the three dimensions fosters a better grasping of mathematical knowledge.

In relation to various possible uses and settings, three levels of MWS are distinguished in problem-solving:

– Personal MWS, that of the student solving the problem
– Suitable MWS, concerned with what is planned by the teacher to insure an effective work in the classroom
– Reference MWS, i.e. the knowledge wished by the institution to be acquired by the student (e.g. official texts defining the syllabus).

Another feature of the MWS is that it takes into account the notion of paradigm. This word is taken from Kuhn (1962): a scientific paradigm consists of 'universally recognized scientific achievements that, for a time, provide model

[6] My translation.

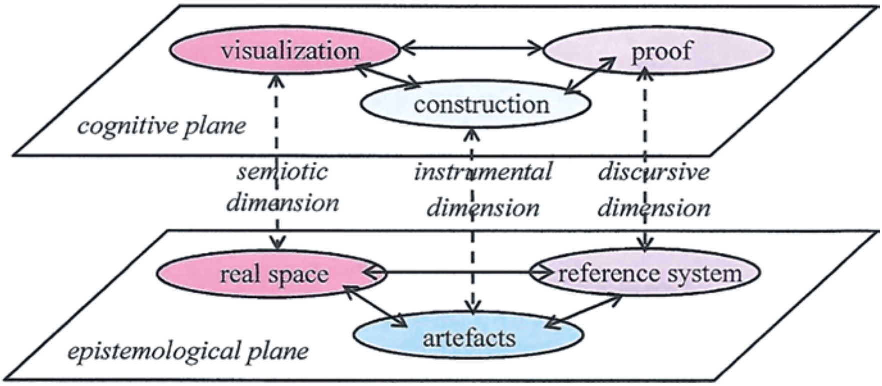


Fig. 3.1 The MWS model. (After Kuzniak, 2011)

problems and solutions for a community of practitioners' (Kuhn, 1996, p. 10). A paradigm includes which phenomena are to be observed, the sensible associated questions, the predictions which can be expected and how to carry out experiments for such purposes. In the present case, the 'community' is any given level of school institution.

In this chapter, the question of solving probability problems with technological devices is analysed from the MWS standpoint, with an addition of some features coming from other frameworks, namely:

- Semantic congruence (Duval, 1995), i.e. a match between units belonging to two different semiotic registers and a similar organization of these elements
- Instrumental genesis (Rabardel, 1995), i.e. the appropriation process of an artefact by its user, making a mere 'tool' become an 'instrument' through use

3.2.3 Application to Stochastics

A parallel between the teaching/learning of probability and geometry had been brought to the fore some years ago (Henry, 1999; Parzysz, 2011). At school level, two geometrical paradigms can be distinguished:

- In one of them (G1), the objects of study are lines drawn on a physical support with instruments (ruler, compasses, protractor…,), and the proof is established by these instruments.
- In the other one (G2), the objects of study are immaterial entities (represented by drawings), and the proof is established by a logical discourse.

Similarly, in the case of probability, several paradigms may also be distinguished:

- A ‘realistic’ paradigm, i.e. the concrete experiment put into play using material objects (e.g. real dice, coins, lottery wheel, balls inside a bag, etc.).
- A second paradigm (P1), resulting from a modelling in which the concrete experiment will result in a list of outcomes and a precise experimental protocol (‘pseudo-concrete’ experiment) ensuring that the experiment can be repeated in the same conditions, the repetition leading to observe phenomena to which a chance of occurrence is ascribed.
- A third paradigm (P2), resulting from a ‘probabilistic’ view on the experiment, in which the generic random experiment and the notion of probability are defined. Then a study of the properties of probability is undertaken, i.e. an algebraic structure is defined about the events, illustrated in classical model laws (binomial, exponential, geometrical, Gaussian…). The related tools are mathematical proof, usual calculation techniques, representation registers (two-way table, probabilistic tree, set diagram, various statistical graphs…); this ‘standard’ paradigm can possibly be extended by integrating continuous distributions (P2+), which is currently done in France during the last year of high school.

It is also necessary to consider statistics, since this domain is also frequently at play in current probabilistic problems. In this case, two paradigms are classically distinguished:

- A descriptive statistical paradigm (DS), at play in computer simulation, which is also used to establish links with some probabilistic notions such as relative frequency/probability, arithmetic mean/expected value…. The current French syllabus for 11th grade encourages such a relation: ‘By using simulations and a heuristic approach to the law of large numbers, the link will be made with the mean and variance of a series of data’.[7]
- An inferential statistical paradigm (IS), including estimations and tests and making use of probabilistic tools.

To these paradigms must also be added a subdomain of calculus: integral calculus (IC), which is used to calculate probabilities with Gaussian distributions.

3.3 Current Reference MWS Associated with Probability

Even if developing methods and skills of problem-solving among students is not a specific aim of mathematics teaching in France, solving problems is the ground on which mathematical knowledge is acquired. Hence studying the types of tasks put into play by students engaged in solving probability problems provides insights on *why* and *how* technologies are used in order to develop their knowledge in this domain. For this purpose, it appears useful to begin with pointing out the main

[7] My translation.

features of the current syllabi, which are in relation to stochastics in middle and high school. When solving probability problems, the student's MWS would of course include his/her knowledge and skills – and he/she would get new ones – not only in probability but possibly also in statistics and in the use of technologies, including algorithmics.

It is worth to be noticed that, contrary to many other countries, after having been for years limited to the end of high school (11th–12th grades), the French probability syllabus, following other countries, is now conceived as a continuous progress from 7th to 12th grade.

3.3.1 Middle School

In France the teaching of descriptive statistics begins at the start of middle school (6th–9th grades), the aim being to provide the students with means for organizing and managing statistical data, namely, using graphical representations and parameters. At the end of middle school, the student is supposed to be able to:

> use a spreadsheet to calculate indicators and represent graphically data. Have a critical look on numerical information (…), organize and process results coming from measures (…), question the relevance of the way by which the data are collected, read, interpret or build a diagram (…) (Official Bulletin, 2015/11/26)[8]

At this same level, through the observation of the stabilization of relative frequencies, the students are supposed to establish a link between relative frequency and probability, either by hand or with a spreadsheet, and calculate probabilities in simple cases.

Regarding the aim of the teaching of technologies, the syllabus for 'cycle 4' (7th–9th grades) states:

> The student develops his/her intuition by shifting from one form of representation to another (…). Such changes of registers are fostered by the use of polyvalent software, such as a spreadsheet and dynamic geometry software. The use of a spreadsheet and calculator is necessary for managing real data (…).
>
> The teaching of computer science at cycle 4 does not aim at training expert students but at providing them with keys for deciphering a numerical world constantly evolving. It enables the gaining of methods for constructing an algorithmic thought and developing abilities in representing and processing information, solving problems, controlling results. (ibid.)[9]

At the end of middle school, the student should 'conceive parametrize and program computerized applications for mobile devices' (ibid.).[10] Since algorithmics is now included in the curriculum, the student should be able to:

---

[8] My translation.

[9] My translation.

[10] My translation.

– break up a problem into sub-problems in order to structure a program, recognize schemes.
– write down, settle (test, correct) and run a program in answer to a given problem. (ibid.)[11]

3.3.2 High School

In high school (10th–12th grades), the panel of statistical parameters and representations is extended, helped, of course, by computer or calculator. At 10th grade level, the aim of the teaching of probability is now to enable students to:

– study and model experiments coming within equal probability (…);
– propose a probabilistic model from observing frequencies in simple situations;
– interpret events using sets;
– see through probabilistic calculations. (Official Bulletin, 2009/07/23)[12]

'Scientific' students of 11th grade carry on the study of finite probability spaces, together with the binomial distribution, which is then used to determine a fluctuation interval for a sample of a $B(n,p)$ distribution, defined as 'the interval centred around p in which the relative frequency of a sample of size n is situated with a 0.95 probability' (Resource document, June 2011).[13] At 12th grade conditional probability and continuous probability distributions are introduced; they are, namely, applied to the study of random walk, exponential and normal laws. To end with, the de Moivre–Laplace theorem (accepted) is used for introducing an asymptotic fluctuation interval and a confidence interval at a 95% level.

At all three levels of high school, the aim with algorithmics is to train students to:

– describe some algorithms in natural language or in a symbolic language;
– run some of them with spreadsheet or a small program implemented in a calculator (…);
– interpret more complex algorithms. (ibid.)[14]

Regarding technologies, like in other countries their use has been included in the math curricula for years, first through calculator and then computer. The syllabus for 10th grade goes further:

> The use of software (calculator or computer), of tools of visualization and representation, calculation (numerical or formal), simulation, programming develops the possibility for experimenting, opens widely the dialectics between observation and proof and changes deeply the nature of teaching.(ibid.)[15]

[11] My translation.

[12] My translation.

[13] My translation.

[14] My translation.

[15] My translation.

At 11th grade level, a noticeable consequence of the use of technology is mentioned:

> In particular, when solving problems, the use of formal calculation software can limit the time spent in very technical calculation, in order to concentrate on the setting up of reasoning. (Official Bulletin, 2010/09/30)

This remark is applicable to probability as well, making, for instance, obsolete browsing through double-entry tables to find out a probability.

The syllabus also indicates three ways for using them in the classroom:

– By the teacher using a collective visualization device
– By the students working in groups (guided practical work)
– By the students working separately outside the classroom

3.4 'Virtual' Suitable MWS: Textbooks

The reference MWS is given by the curricula, but the suitable and personal spaces can only be known by observing extended real classes. Nevertheless, an idea of suitable MWS can be given by textbooks, since they are much used by the teachers when planning their work, the more so that many of them are not familiar with the domain; for this reason textbooks can be considered as giving 'virtual' suitable MWS. The same goes for the so-called resource documents elaborated by a group of teachers and trainers and edited by the Ministry of Education; they are not, strictly speaking, part of the syllabus but are intended to give guidelines for teachers, including problems to be solved in class under various forms.

In France, ten or so different textbooks are published for each level – some of them by the same publisher. In relation to the purpose of this chapter, they constitute a sample of how trained teachers conceive the teaching of probability and give examples of problems to be solved by students. In particular, some problems ('activities') provide details on the processes that are to be undertaken by the student, giving thus an insight on the suitable MWS intended by the authors for 'ordinary' classes. Some of the textbooks have a digital version, intended for teachers and/or students, including the contents of the paper textbook (which can be projected), additional exercises (some of them in English and Spanish), animations, video tutorials, simulations of random experiments, MQC tests, worksheets, Internet links, and so on.

At high school level, many problems present a random experiment and ask questions about the probabilities of associated events. In order to answer these questions, students are asked to experiment, either 'by hand' or by simulation, with calculator or computer; and, on this basis, they are asked to formulate conjectures about the required probabilities. Then they are asked to search for mathematical answers to the initial questions (see Sect. 3.6.1).

In some cases, the first step is not to simulate the experiment but to carry it out a number of times, the initial purpose being the devolution of the problem to the students (Sect. 3.6.1). Since the limitation of the number of tries prevents making sensible guesses on probabilities from just observing frequencies, the next step is always to simulate the experiment. Anyway, in all cases a 'frequentist' standpoint has to be put into play for giving an answer.

On the contrary, in nearly all cases giving a mathematical solution implies identifying elementary events having the same probability, i.e. a 'cardinalist' standpoint.

3.5 Two Standpoints on Probability

3.5.1 Random Generator

The first question about technologies being used to simulate random experiments for solving probability problems is: why not perform the experiment with real objects such as coins, dice, cards, etc.? Of course, a machine works incomparably faster and easier than any human being. Performing a single real experiment and writing down its outcome takes at least several seconds. Within 1 h, it might be possible to reach 1000 tries at the best, whereas a calculator can give several thousands in the same time span and a computer several thousands in 1 min or so. Another reason to use technology is that after the simulation it is possible to use the results for various calculations. These are in fact the main reasons for using random generators.

Random generators do not produce randomness, since they generate deterministic sequences of numbers (Parzysz, 2005); this fact is not questioned by students (no more than about trigonometric or logarithmic functions). In any case, their teachers can tell them that, as far as they are concerned, they can take it for granted that the machine produces randomness and use it freely for simulating random experiments, once they have learnt how to do it.[16] Indeed, how to link a decimal number belonging to [0;1] to a random experiment has to become part of the personal MWS of the students.

The main limitation of random generators for simulating random experiments is that they can only be used for some experiments, but not for all; for instance, it is not possible to simulate the tossing of a pushpin, which may fall on its head or on its point (Fig. 3.2), because it is not possible to assign a sensible probability to each of these two outcomes. In this case, a solution would be to toss it a great number of times and decide to assign the observed relative frequency of each outcome as its probability (see below).

[16] Although in fact, when simulating a random experiment, what one actually does is testing the quality of the generator.

Fig. 3.2 The two possible outcomes when throwing a pushpin

3.5.2 Frequentist and Cardinalist Standpoints

The example of pushpin introduces a dual point of view on probability that is the very base on which many probability problems asked of French students rely. Being given a random experiment and an event related to this experiment, then:

- A first ('cardinalist') standpoint assumes from a priori considerations (e.g. on the symmetry of the issues) that all the issues of the experiment have the same probability to occur; then the probability of the event is the ratio of the number of outcomes producing it to the total number of outcomes (known as Laplace's formula).
- A second ('frequentist') standpoint is based on experimentation; to assign a value to the probability of an event, the experiment is performed a great number of times, and the relative frequency of the event is noted down; then, a posteriori, a value linked with (equal or close to) this frequency is assumed to be the probability of the event.

This duality was clearly expressed for the first time by Jacob Bernoulli (1654–1705), in his posthumous book *Ars conjectandi* (Bernoulli, 1713):

> But to tell the truth another way to get what we are looking for is offered to us. What is not possible to be obtained a priori can at least be a posteriori, i.e. it will possibly be extracted from the observation of numerous similar examples, because it has to be assumed that each issue can happen and not happen afterwards following the same number of cases which previous observation showed it happen or not happen, in a similar state of things.[17]

Until recently, the cardinalist standpoint was the only one taken into account in the teaching of probability, but – as mentioned above – the development of technology made quite easy and fast, thanks to random generators, to get very great numbers of tries of a random experiment through simulation, and thus made possible for students having little or no knowledge about probabilistic paradigm P2 to obtain a fair approximation of the probabilities of events linked to this experiment, probabilities which would be difficult, if not impossible, for them to calculate directly. The condition being for them to be able to implement the experiment in the machine,[18] hence the need for some specific knowledge both on probability and software. This is the kind of problems by which the frequentist standpoint made its appearance in French high school.

---

[17] My translation.

[18] This implies that a model of the experiment is available, that is, a probability can be assigned to each issue.

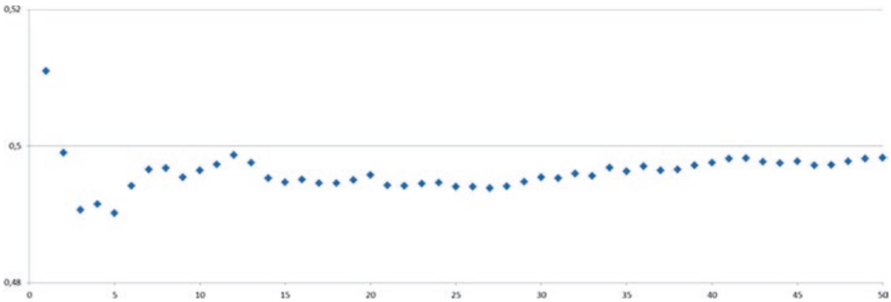


Fig. 3.3 Heads and tails. Evolution of the relative frequencies of 'heads' up to 50,000 tosses

At 9th grade, students become sensitive to the frequentist standpoint by solving simple problems through carrying out and/or simulating elementary random experiments with only two issues. They are asked to make a guess about the probability of a given event from observing on a related diagram produced by the machine how the relative frequency of successes fluctuates and stabilizes when the number of tries increases more and more. Figure 3.3 shows an example of the fluctuation of the frequency of 'heads' in a game of heads and tails simulated with Excel, for an increasing number of 1000, 2000, …, 50,000 tosses.

3.6 Technologies to Simulate Random Experiments

3.6.1 A Classroom Experiment

Although students seldom ask questions about how the random generator works, the shift from real to simulated random experiments is not so obvious and should be an initial aim in the teaching of probability. Then starting with performing real random experiments is unavoidable if one wants to show that probability is anchored in reality, and this can be done quite soon (sooner than it used to be currently done in France, anyway).

The choice of the experiment is important, because the task must make sense for the students and not be too easy to solve. It is the case with the so-called d'Alembert's problem,[19] which can be summarized as follows:

Toss a well-balanced coin; if you get 'head' you win.
If you get 'tail' toss the coin again; then if you get 'head' you win, and if you get 'tail' you lose.
What is your chance to win at this game?

[19] The origin of this problem is the entry 'Croix ou Pile' in the *Encyclopédie* (1751–1772) supervised by Jean le Rond d'Alembert and Denis Diderot.

Table 3.1 d'Alembert's game: frequencies of successes on 80 tries

Group	1	2	3	4	5	6	7	Total
Frequency	65	69	53	53	64	58	62	424
Relative frequency	0.81	0.86	0.66	0.66	0.80	0.72	0.77	0.76

Some years ago, this task was given to 'non-scientific' 10th graders (Parzysz, 2007).[20] After discussing this before any experimentation, the students fell into two main equal categories:

– Some believed that the chances of winning are two out of three, since three issues are possible: Head (→ win), tail + head (→ win), and tail + tail (→ lose).
– Some others believed that one has three chances of winning out of four: One out of two at the first toss plus one out of four at the second.

They could not find any agreement between them – at that time, they were not familiar with tools like double-entry tables or tree diagrams– and, taking coins out of their pockets, some of them began to play the game. This feature having been anticipated by the teacher, all students were given coins and a planned experimentation began. Table 3.1 shows the frequencies of successes on 80 tries for a class in which the students were divided into seven working groups.

After gathering the results obtained by all the students, it appeared that the relative frequency of wins was nearer 3/4 than 2/3, but one could not be quite sure because there were not enough tries. Then simulation with calculator (a computer was not available then), proved useful and consequently was undertaken (Fig. 3.4). It showed the students that along with its rapidity, a random generator is really trustworthy and efficient for simulating random experiments.

3.6.2 Simulation: A Triple-Sided Process

The French syllabus for 10th grade states:

> On the occasion of setting up a simulation, one may:
> use the logical functions of a spreadsheet or calculator,
> set up conditional instructions in an algorithm. (Official Bulletin, 2009/07/23)[21]

What is meant here by 'simulation' seems clear: it is using a calculator or a computer to simulate a random experiment. The authors of the syllabus have indeed taken into account the expansion of technology, but by this they restrict the meaning of the word, since as a 10th grade textbook states: 'simulating an experiment is replacing it by another experiment which enables one to get results similar to those of the first experiment; moreover, the new experiment has to be simpler to carry

[20] The students were in fact given the original text, in which d'Alembert gives both the solution given 'by all authors' (3/4) and his own solution (2/3).

[21] My translation.

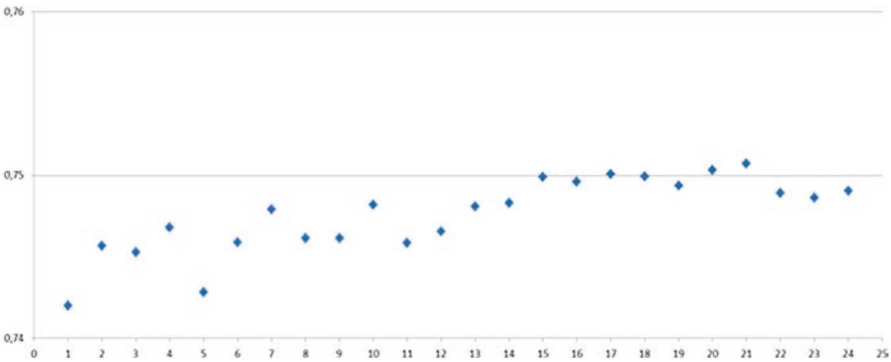
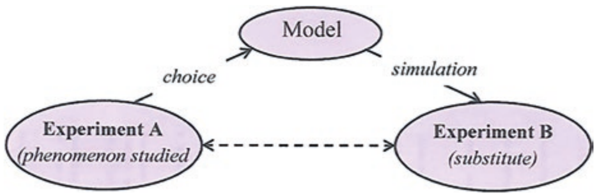


Fig. 3.4 d'Alembert's game: relative frequencies of successes up to 25,000 games (with spreadsheet)

Fig. 3.5 From a random experiment to its simulation



out'.[22,23] The essential difficulty lies in ascertaining that the results of the simulation are 'similar' to those of the experiment. In fact, the similarity between the experiment and its substitute can only come from the fact that they refer to the same probabilistic model. For instance, throws of a well-balanced coin to play heads and tails may be simulated with a die, by agreeing that getting an even number is a 'head' and an odd number is a 'tail'. Similarly, in order to study how boys and girls are distributed in families with three children, black and white balls can be put into a bag in equal numbers (e.g. ten) and three balls removed one at a time, putting each ball back in the bag afterwards and studying how many balls of each colour are obtained. In that case, the underlying model supposes that the births of a girl or a boy are independent and occur with the same probability. One may also assign different probabilities, for instance, use the current world sex ratio, 107 boys to 100 girls.

Figure 3.5 shows that, although seeming double-sided, the simulation process is in fact triple-sided, requiring a theoretical model (possibly implicit) as a mediator between the real experiment and its simulation: Bernoulli's law in the case of the die and binomial law in the case of the families. As Batanero, Henry, and Parzysz (2005) argue:

> Using computers as simulation tools requires characterizing a model of the simulation and makes it still more necessary to explicitly state the working hypotheses. Finally, we remark that a pure experimental approach is not sufficient in the teaching of probability. Even when

[22] My translation.

[23] *Deledicq, A. (dir.) Mathématiques classe de Seconde. Collection Indice.* Bordas, Paris 2000, 138.

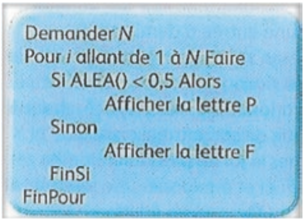


Fig. 3.6 An algorithm for tossing a coin

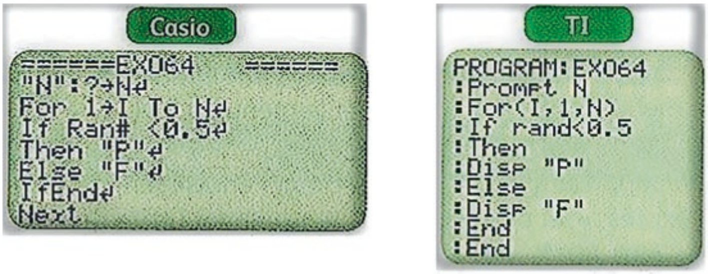


Fig. 3.7 Implementations of the algorithm

> simulation can help to find a solution to a probability problem arising in a real world situation, the simulation cannot prove that this is the most relevant solution, because it depends on the hypotheses and the theoretical setting on which the model is built. (p. 33)[24]

3.6.3 Simulating with a Technological Device

In order to carry out a simulation of a random experiment making use of technology, the machine must first be told what to do, i.e. a conversion needs to be made from natural language to that of the machine. Most often, this is achieved through an algorithm which makes it necessary to declare the variables, enter or initialize the data, process them and display the results. Figure 3.6 shows the example of an algorithm simulating N tosses of a well-balanced coin, taken from a 11th grade French textbook[25].

This algorithm is then 'translated' into a specific language applicable to the technological device (Fig. 3.7).

An algorithm may be written in natural language, but specialized software have been developed (Algobox, Larp…). Figure 3.8 shows another example from another textbook relating to the following problem:

In a factory a product may present two defects:

– Defect A with probability 0.1,
– Defect B with probability 0.2.

---

[24] My translation.

[25] Beltramone et al., 2015

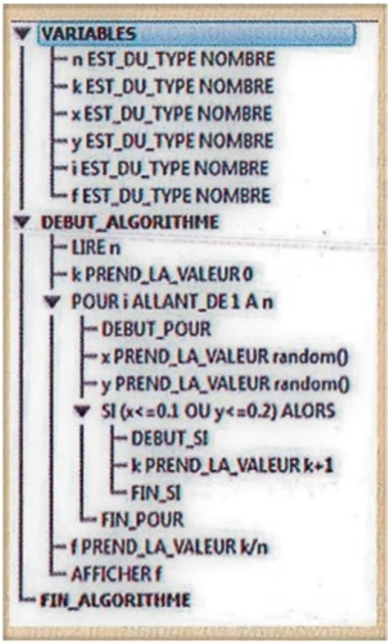


Fig. 3.8 Another algorithm

The two defects are independent from each other. The choice of a sample of size *n* of this product is simulated by a program and the relative frequency of the products presenting at least one defect is displayed. (…)

(a) Explain what is represented by each variable of this program.
(b) Implement and carry out this program for samples with $n = 1000$, $n = 5000$, $n = 10{,}000$. (…).[26,27]

One may notice that this part of the solution involves the three dimensions of the MWS: discursive to explain an algorithm, semiotic to shift from one language to another and instrumental to carry out the simulation; a study of the problems given in textbooks shows that, even if all three dimensions are present, the discursive dimension is limited to give simple justifications.

3.6.4 Analysis of a Simulation Problem

A most frequent type of problem to be found in French 10th and 11th grade courses and textbooks includes a simulation of a real situation in which several outcomes can occur in an unpredictable way, according to the following process:

---

[26] My translation.

[27] Malaval et al. (2011), p. 399.

1. Simulate the situation (with calculator or spreadsheet).
2. Observe the results of this simulation.
3. Express a conjecture about the probability of one of the possible issues of the experiment or about the possible value of a parameter of the situation.
4. Solve the problem using the probability theory.
5. Compare the theoretical results with the results of the simulation.

The implicit aim of such problems is that the student becomes aware of certain processes for solving a 'real' probability problem:

(a) When you have no clear idea of the solution, look for an algorithm describing the experiment.
(b) Implement this algorithm in a machine, and carry out a 'great' number of tries.
(c) Make a guess about the solution.
(d) With the help of the algorithm and your knowledge in probability, search for a mathematical solution.

In these processes, the student will observe that:

(a) Repeating the experiment a great number of times and observing the stabilization of the relative frequencies of given events allow to get an idea on the probability of these events.
(b) The number of tries being as high as he/she wants, one does not reach a unique value (Table 3.2 shows an example for 9994,…, 10,000 tries of heads and tails); hence this method is not able to give 'the' solution.
(c) The solution can only be obtained by a mathematical reasoning.

As stated by Batanero et al.:

> Finally, we remark that a pure experimental approach is not sufficient in the teaching of probability. Even when a simulation can help to find a solution to a probability problem arising in a real world situation, the simulation cannot prove that this is the most relevant solution, because it depends on the hypotheses and the theoretical setting on which the model is built. (Batanero et al., 2005, p. 33)

In this process, technology is used to simulate the experiment; that is, to find and implement an algorithm, this step, although insufficient to reach a solution, is nevertheless important, not only to make a guess on the solution but also to find a path towards it, since the algorithm is in relation with a theoretical model (possibly implicit).

Table 3.2 Stabilization of relative frequencies

9994	0.49889934
9995	0.49894947
9996	0.4989996
9997	0.49904971
9998	0.49909982
9999	0.4990499
10,000	0.4991

In order to make the purpose clearer, a problem taken from a French 11th grade textbook well used in classrooms is instructive.[28] In fact, by now, this task has become a stereotype, since numerous examples of this same 'activity' are to be found in textbooks of that level:

(*Part 1*[29]) You have three perfect tetrahedral dice: one blue, one red and one green. You throw these three dice and observe how many 4s you obtain.

An outcome is a three-digit number, for instance 1 4 3. It is assumed that the probability is the same for all the outcomes.

Let us consider the following game: if you get the number 4 three times you win 36 €; if you get it twice you win 2 €; otherwise you lose 1 €.

The aim of this exercise is to estimate the average profit, which can be expected from a series of 2500 tries.

(*Part 2*) <u>Simulate with a spreadsheet</u> (…)

(*Part 3*) <u>Conjecture</u>

Simulate this experiment several times. (…)

How does the mean profit evolve with the number of tries?

Which mean profit can be expected at the end of 2500 tries?

(*Part 4*) <u>Proof</u>

(a) How many outcomes are there for a try?
(b) Count the outcomes giving three 4s; two 4s.
(c) Let G be the random variable giving the algebraic profit (in €) of the player. Set up the table of the law of probability of G, then calculate E(G).
(d) (*Part 5*) What link do you establish between the result of the simulation and the expected value of G?

In *Part 1*, the first sentence describes the 'concrete' situation, material objects at play (shape, colour) and rule of the game, and a question is asked about this game.

---

[28] *Barra, R. et al. Mathématiques classe de Première. Collection Transmath*. Nathan, Paris 2011, 303.

[29] This division into five parts is mine; it is intended to make the following discussion easier.

The student's intended MWS is now composed of these objects and rules, together with actions (2500 tries) and subject of interest (mean profit); it stands in the discursive dimension.

Then a hypothesis is made ('It is assumed that the probability is the same for all the outcomes'). In fact, it introduces a probabilistic model of the situation. This going through P1 is necessary, since to simulate an experiment you need to model it (cf.Sect. 3.6.2.). Making the model explicit ensures that the *same* experiment will be repeated, since in each try the *same* protocol occurs (Parzysz, 2009). However, this necessary step is often skipped in teaching, and one shifts directly from the real situation to its simulation. This is made easy because for most of the random experiments studied in classrooms, a canonic model can be thought of, which is then considered 'transparent'. But, by so doing, the students may be induced to simulate the real situation at first, i.e. without being aware that they are implicitly making hypotheses.

In the model presented here, 'an outcome is a three-digit number', and, since each die can show 1, 2, 3 or 4, there are 64 possible outcomes (4^3) which, from the hypothesis, are assigned 1/64 as probability. The referent workspace (in P1) is composed of the outcomes with their probabilities, and the student is supposed to work within the domain of rational numbers.

The next step of the task (*Part 2*) is a simulation. In this part, the student's MWS shifts from P1 to DS: the probability hypotheses have to be converted into instructions for the software, through an implicit algorithm. The work stands in the semiotic-instrumental plane, the corresponding semiotic registers being natural language and the symbolic language implemented in the spreadsheet. Then he/she works within DS, using notions he/she already knows of (sum, mean). Implementing the rule of the game in the worksheet is trickier, but the textbook helps him/her by giving the formula to be used. The student has to elaborate a strategy, and for this, he/she is given clues at each of the three steps, which can be distinguished in the strategy:

1. Dealing with the three dice individually.
2. Putting the three results in common.
3. Repeating the experiment.

At this point, a major difference with using geometry dynamic software appears in many of these problems: the student is much more guided, and he/she is confined to carry out elementary tasks, a characteristic fact which has already been mentioned by educators:

> A […] feature of the [computer aided] teaching is the tendency to develop a solution strictly step by step […] The aspect being important for my concern is not that an order is put on mathematical task solving […] but that the participants restrict their considerations to the actual mathematical step. (Jungwirth, 2009, p. 3)

This is indeed the case for the above problem, in which Part 2 goes:

	A	B	C	D	E	F	G
1	*Try n*	*Blue die*	*Red die*	*Green die*	*Issues of 4*	*Profit*	*Average profit*
2							
3							

(a) Open a worksheet and fill cells A2 to D2 with the appropriate formulas.
(b) In E2 count the occurrences of 4 in the block of cells B2:D2. [help: Use the instruction COUNTIF].
(c) In F2 have the profit of try #1 displayed.
(d) Enter the conditional instruction: =IF(E2=3;36;IF(E2=2;2;-1)). Justify that it suitably describes the rule of the game.
(e) In column H you want to display the mean profit as a function of the number of tries. Which formula do you have to enter?
(f) Select the block of cells A2:H2, then paste downwards until row 2501.
(g) Select columns A and H, then represent the mean profit as a function of the number of tries, *n*.

Since 'making the handling of software easier does not ensure the necessary ability for succeeding in more complex tasks' (Rabardel, 1995),[30] a diagram visualizing the algorithm at work can be most useful.

The justification required at the end of question (c) is supposed to make sure that the student understands what he/she is doing. Several textbooks pay attention to this question and have devoted paragraphs on how to use specific calculators, spreadsheet and dynamic geometry software in relation with the tasks required in the curriculum.

In *Part 3* the spreadsheet is used for repeating the experiment and making conjectures based on the results displayed in various registers (numbers and diagram), i.e. in the semiotic dimension. This dynamic aspect is an important feature of the task, because it establishes a link between DS and P2.

Until now, the spreadsheet had been used in turn as a random generator (for throwing dice), a logical tool (conditional instruction), a copying machine (in order to get 2500 tries), a calculator (for the average profit) and finally a plotter (to show its evolution). Then it is used for making conjectures. The student's task is now to observe how the results evolve through several series of experiments and then conclude by making a guess about a 'limit' for the parameter. An important feature of the software for teaching is indeed its dynamic aspect (using the F9 key) which shows the sample fluctuations and the convergence of the relative frequencies; nevertheless it has some limits, since the – postulated – limit of the series of frequencies is not accessible. The student's personal MWS is mostly DS, with a – more or

[30] My translation.

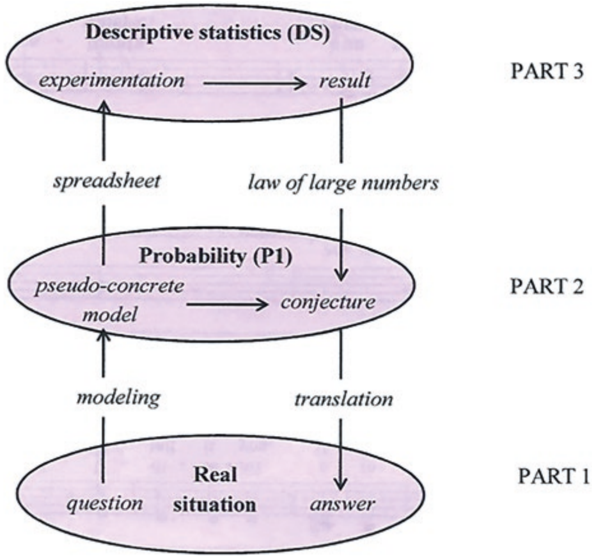


Fig. 3.9 Processes involved in the task

less implicit – recourse to a 'naive wording of the law of large numbers' which refers to probability.[31]

Finally, this process can be illustrated by Fig. 3.9: starting from the real situation, the student shifts to probability (to implement a model in the spreadsheet) then to statistics (to calculate parameters of the distribution), then again to probability (using the law of great numbers to conjecture limits) and finally back to the real situation to express his/her answer.

Regarding the dimensions of the MWS, in parts 1–3 the semiotic-instrumental plane is greatly favoured, and the discursive dimension has only the rhetorical role to ensure that a subtask has been correctly implemented.

Part 4 is quite another problem, in which simulation is left aside and a theoretical solution asked within the traditional cardinalist standpoint. Its structure and wording are that of a 'classical' probability problem to solve in paper-and-pencil environment (P2). The student's first task is to determine which theoretical space he/she has to work in and then to set up the table of the law of probability of G and calculate its expected value.

The subjacent model is the initial one, in which an outcome is a three-digit number: this is the reason why the dice must have different colours, the aim of such a 'pedagogical artefact' being to guide the student towards the correct solution.

Contrary to the previous parts, in this part the task stands in the semiotic-discursive plane of the MWS.

In the final part of the problem (*Part 5*), the student is asked to establish a link between the 'statistical' part (simulation) and the 'probabilistic' part (proof) of the

[31] 'When the number of experimentations grows up the relative frequency gets nearer the probability'.

activity. The justification is to be based on a 'naive formulation of the law of large numbers'. But here E(G) = $\sum x_i.\ p(x_i)$, whereas the average profit is $\sum x_i.\ f(x_i)$ ($1 \leq i \leq 3$). Therefore, account has to be taken of not only the convergence of a particular frequency but also the convergence of the frequency *distribution* towards the probability distribution. This is indeed a consequence of the law, but it makes it necessary to consider the *whole* distribution.

[*Remark.* The purpose of the authors was certainly to introduce the binomial law, since three independent Bernoulli random variables $B(1;\ 0.5)$ can be attached to the dice. Moreover, the values of the random variable were chosen so that its expected value is 0 (fair game).]

Here a question arises about the model, which is far from unusual in probability, especially when spreadsheet is used (Parzysz, 2009): the model implemented in the software does *not* fit with the hypothesis. The table shows that the assumed equal probability concerns the four vertices of each die, since the hypothesis states that it is about the 64 possible three-digit numbers. A more suitable implementation would have been to display each three-digit number together with the corresponding number of 4s and/or the associated profit. Of course, in the present case the two models, although different, give the same results. According to the hypothesis, there are 54 outcomes with no or one 4, 9 with two 4s and 1 with three 4s. Hence the distribution is:

$$P(G=36)=1/64, P(G=2)=9/64, P(G=-1)=27/32$$

(and, as noticed above, $E(G) = (1/64) \times 36 + (9/64) \times 2 + (27/32) \times (-1) = 0$).

But, according to the model implemented, for each die, $p(1) = p(2) = p(3) = p(4) = ¼$. Assuming that the dice are mutually independent, a binomial law with parameters $n = 4$ and $p = 0.25$ gives the same distribution as above:

$$P(G=36)=\binom{3}{3}\left(\tfrac{1}{4}\right)^3=1/64,\quad P(G=2)=\binom{3}{2}\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)=9/64,$$

$$P(G=-1)=\binom{3}{1}\left(\tfrac{1}{4}\right)\left(\left(\tfrac{3}{4}\right)^2\right)+\binom{3}{0}\left(\tfrac{3}{4}\right)^3=27/32$$

However, the first model would have been more difficult to implement with the software, and this is certainly the reason why the authors chose another one, as they knew it would lead to the same results.

Replacing a model by another one is quite 'transparent' for the teacher, since he/she *knows* that they are equivalent. But the students could rightly be puzzled by this substitution. In the present case a possible better solution would be to study another experiment, in which a *same* die is thrown three times, the first throw giving the first digit of the number and so on (Table 3.3).[32] The associated table would then be:

[32] In this case, the 'pedagogical artefact' consisting of colouring the dice is no use.

Table 3.3 An alternative simulation

	A	B	C	D	E	F	G
1	Try $n°$	1st throw	2nd throw	3rd throw	Outcomes of 4	Profit	Average profit
2							
3							

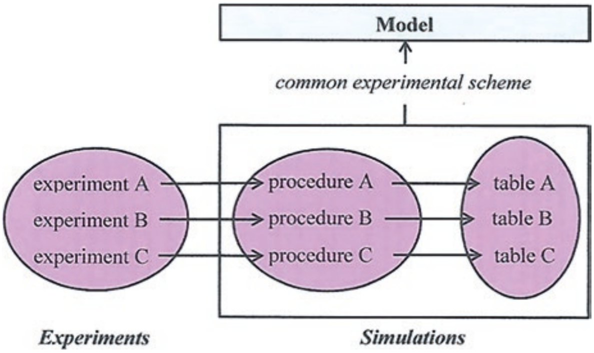


Fig. 3.10 Emergence of a model of the experiment

At any rate, a problem appears when students are beginning to study probability, since they do not know what a probabilistic model is. The use of technological simulation can help them grasp this notion, in the following way (Parzysz, 2009): after simulating various random experiments, they may become aware that for some of them the procedures implemented are very similar, since the corresponding algorithms have the same structure (e.g. see the above table and the table given in the original wording). This common experiment scheme can then be used to justify that one experiment may be replaced by another one that will 'give the same results' (Fig. 3.10):

With regard to the present activity, the task looks at first composed of two different exercises, the first one making use of a spreadsheet in a frequentist approach and the second one standing within a traditional cardinalist paper-and-pencil type. In fact, the link between them comes from the algorithm implemented in the spreadsheet and represented in Fig. 3.11, which, in a 'natural' way, can be associated with a probabilistic tree; using then the properties of this semiotic register, one can deduce the probability distribution and the average profit.

The processes can be schematized by the diagram in Fig. 3.12, showing that the algorithm is a crucial point: the upper branch leading to an estimation of the profit (→ frequentist standpoint) and the lower one to its theoretical value (→ cardinalist standpoint).

Another feature which makes it interesting from a pedagogical point of view is the final comparison between the two standpoints: it makes explicit the fundamental link between statistics and probability and shows that the two domains are distinct – probability is about *theoretical* objects, whereas statistics studies a series of data

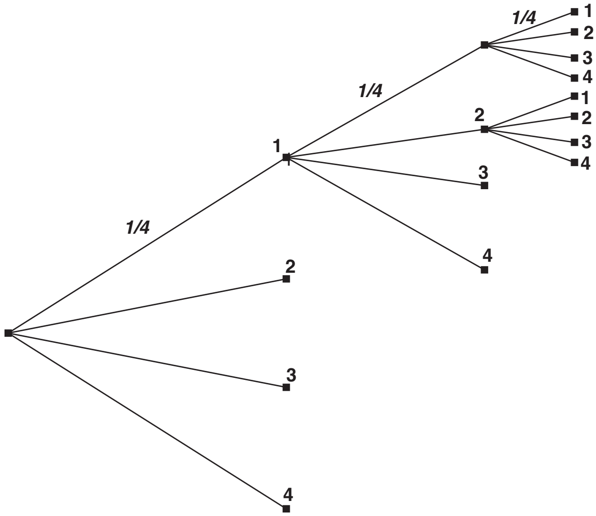


Fig. 3.11 Probabilistic tree for the dice problem

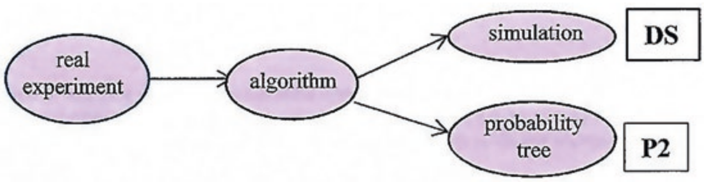


Fig. 3.12 The crucial point leading to both standpoints

coming from an *observed* phenomenon, either real (experiment) or virtual (simulation). There are of course strong links between them, the more so that some concepts and formulas are very similar, for instance, arithmetic mean (statistics) and expected value (probability), standard deviation (both), etc. Moreover, at any level, take statistics as a starting point to solve probability problems – for instance, conditional probability (Chow & Van Hanegham, 2016)–may help students become more performant in probabilistic problem-solving.

In the present case, the average profit observed in 2500 tries may be 0.0426, 0.0018 or 0.0235, but the expected value of profit is exactly 0. The wording of the problem is confusing, since 'average profit' is used both in a probabilistic context (*Part 1*, 'estimate the average profit which can be expected') and a statistical context (*Part 2*, 'In column H you want to display the average profit'). Consequently, the meaning of 'average profit' in the question 'Which average profit can be expected at the end of 2500 tries?' is really puzzling: if it really means 'average profit', after having carried out 2500 tries, you *know* it because you can calculate it; but if it means 'expected value', you do not need 2500 tries to know it, since it is 0. This

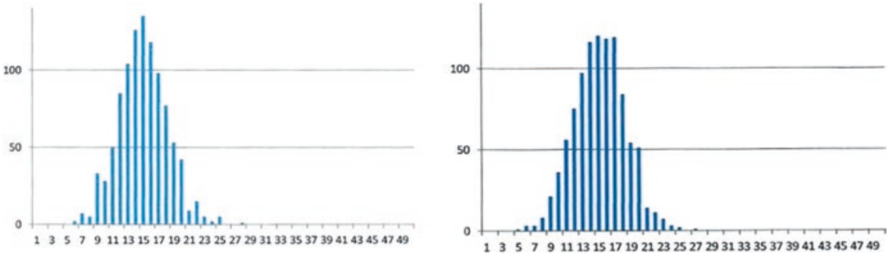


Fig. 3.13 Statistical fluctuation of simulations of a binomial distribution

may give a clue to making the difference between the two domains appear. For instance, when you compare on a screen several experimental graphs issued by simulations of a binomial law with the theoretical graph of this same law, you notice that the statistical graphs are different from one another and different from the probabilistic graph as well (although they are close to it). For instance, Fig. 3.13 shows two successive diagrams obtained for the frequency distribution of successes on 1000 simulations of $B(50;0.3)$.

Finally, in such problems the student is faced with three different experiments: a real experiment (dice), a model experiment (to be implemented in the technological device) and a virtual experiment (simulation with the device). These experiments are associated with different reference domains, and thus with different MWS, and they interact with one another. This is somewhat similar with what happens with geometry problems when using dynamic geometry systems.

3.7 Giving a Statistical Solution to an 'Everyday Life' Problem

As said above (Sect. 3.3.2), the French high school curricula include studying fluctuation intervals, i.e. an 'interval centred around p in which, the relative frequency of a sample of size n is situated with a 0.95 probability'.[33] This interval is studied under three forms, according to each level:

– At 10th grade, an approximate simple form : $\left[p-\frac{1}{\sqrt{n}}, p+\frac{1}{\sqrt{n}}\right]$.
– At 11th grade, a form corresponding with the binomial distribution
– At 12th grade, a form ('asymptotic') corresponding with the normal distribution

At 11th grade level, for instance, a typical task consists of determining a fluctuation interval for a given binomial distribution. Here is an example:

[33] My translation.

Table 3.4 Using a table for determining the fluctuation interval (excerpts)

X	P(X)	
80	0.012974222	FALSE
81	0.017469525	FALSE
82	0.023212001	FALSE
83	0.030441985	TRUE
84	0.039414912	TRUE
85	0.050393551	TRUE
…	…	…
115	0.956588024	TRUE
116	0.965632562	TRUE
117	0.973053721	TRUE
118	0.979075551	FALSE
119	0.983908197	FALSE
120	0.987744109	FALSE

> A large hotel in a seaside resort propose a "thalassotherapy" option to their guests (…) The manager of the hotel wishes that 20% of his guests choose this option (…) To test if this objective is reached a sample of 500 guests has been taken randomly, showing that 70 guests chose the "thalassotherapy" option.
>
> Using a fluctuation interval, what should you recommend the manager to do? [34,35]

Using a technological device, the student displays the corresponding distribution $B(500,\ 0.2)$ under its cumulative version (left two columns of Table 3.4) and then, for X increasing from 0 to 500, searches for the first value of $P(X)$ verifying $P(X) > 0.025$ and the first value verifying $P(X) > 0.975$. For this, he/she can either observe each value of the second column one by one or have the help of a logical function (for instance, AND($B1$ > 0.025; $B1$ > 0.975)).[36] Then, once he/she has determined the fluctuation interval for the relative frequency, he/she has just to check whether the observed value belongs to this interval or not.

In the last century, such a problem would have been solved with the help of a table (with the major inconvenience that such tables did not exist for any values of n and p). So, in the present case, technology allows to save a lot of time and makes some techniques obsolete, in this case reading tables and interpolation.

In the student's MWS statistics appears under both the DS and the IS paradigms, in the initial sample (DS) and the fluctuation interval (IS); probability (P2) is also present with the binomial law (Fig. 3.14). The three dimensions occur: semiotic in implementing an algorithm, instrumental for displaying the distribution and finding out the fluctuation interval, and discursive to draw a conclusion.

[34] My translation.

[35] After Beltramone et al. (2015).

[36] The use of a logical function is not necessary; it just entrusts the conclusion to the device.

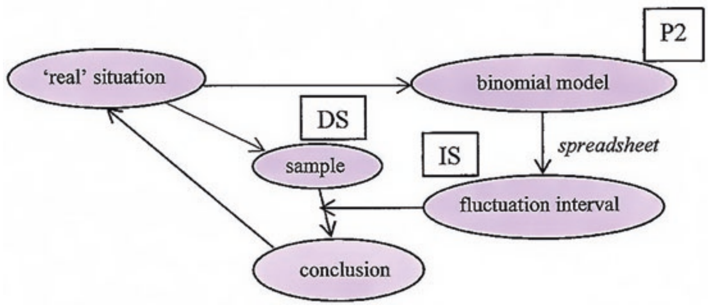


Fig. 3.14 Diagram of the process

3.8 Using Technologies with Continuous Distributions

[The research work for this section recently took place in France (Derouet, 2016; Derouet & Parzysz, 2016.]

3.8.1 Posing the Question

The question of introducing continuous laws in probability teaching is most important, not only because of the popularity of the normal law (the famous 'bell curve') in the mass media but also for being able to suggest and use other model laws (e.g. exponential, almost as popular). Major differences with discrete laws appear at first, such as:

– The probability of any value of the variable is nil
– The need for a density function.

The subjacent idea is indeed to link a random variable *X*, not with a set of isolated probability values $p(x_i)$ but with a function *f* (density) verifying, for any *a* and *b*

$$(a \leq b) : p(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

In 2002, the French resource document for teachers of 12th grade suggested a possible path for this introduction: start with a statistical series corresponding to a continuous character (as it happens, a sample of 50,000 heights of adult men), gather the data in classes of equal lengths and draw a corresponding histogram and then search for 'a function f, the curve of which fits the histogram, the area under the curve being equal to 1'.[37]

[37] My translation.

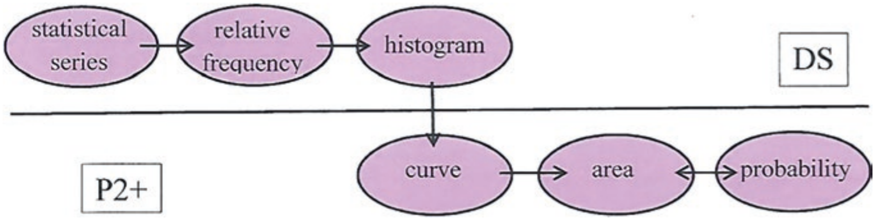


Fig. 3.15 Diagram of the process

In this process, situated mostly in the semiotic-instrumental plane of the MWS, the student engaged in solving the problem stands first in the DS paradigm, and the modelling process leads him/her towards P2+, according to Fig. 3.15.

This idea looks interesting but, as it rests on the notion of histogram, it makes necessary to have some knowledge about it.

The notion of density is commonly met in other matters, such as geography (population density), physics (voluminal mass), biology (cell density), and so on. In statistics, it is implicitly at play in histograms with unequal classes, but this feature remains often implicit or is not taken into account, the more so that spreadsheet cannot produce such histograms and hence can be no help.

Namely, the interest of histogram is that, if only one does not stick to classes of equal lengths, it implicitly contains the notion of density, the density of each class being proportional to its height.

This results in French 12th graders having not a clear notion of that type of diagram, especially if the help of software comes too early, before their constructing 'true' histograms 'by hand'. All the more so since 10th grade textbooks are not clear about them, and in particular none of them gives a reason (i.e. visual) for the rule: height = frequency/width. A consequence is that teachers do not spend much time on the subject (a French study showed a meantime of 1 h per school year): 'almost half [of them] deem that teaching histograms does not contribute to the students' mathematical education' (Roditi, 2009, p. 131).[38]

3.8.2 *Problems Introducing Continuous Distributions*

When histogram has been thus worked and become part of the student's MWS, the question is to move towards continuous laws. As Henry writes:

> One can see that the bet of introducing continuous distributions at the end of high school is daring, to say the least. Here the point of view of modelling is unavoidable. Continuous distributions are then tools enabling, in the corresponding models, to calculate the probability of some types of events. These continuous models are chosen and their parameters determined (model hypotheses) to fit pretty well with a described discrete reality (working hypotheses), grounding on heuristic considerations and statistical estimations. (Henry, 2003, p. 6)[39]

[38] My translation.

[39] My translation.

The modelling point of view looks quite sensible. For instance, students could be given a *real* set of statistical data and been asked to represent it with a histogram and then find a continuous curve approximating its upper edge.

Here is an example posed in several scientific 12th grade classes by Derouet. This problem is based on a set of statistical data, inspired from a 12th grade textbook and completed through the Internet:

> The Aso volcano, situated on the Kyushu Island in Japan, is one of the most active in the world. The dates of its eruptions are available from the 13th century on. The following list gives the successive years of the eruptions, from 1229 to 1897 (for later years, the data collected are different).
>
> 1229 1239 1240 1265 1296 … 1874 1884 1894 1897
>
> The last eruption of the volcano took place in September 2015. How do you propose to assess the probability for the next eruption to occur during the year 2030? (after Derouet, 2016)[40]

In a previous problem, in order to model a statistical distribution, using spreadsheet (GeoGebra) the students had approximated a histogram of relative frequencies with a straight line, one condition for finding this line being that the area between it and the x axis had to be equal to 1. In the case of the volcano, working in groups again with GeoGebra, they began with implementing a table showing the number of years between two successive eruptions, together with histograms of the waiting times with various steps. Then they undertook to approximate the histogram with curves drawn by the software. Obviously, a straight line was not suitable, so they tried various positive decreasing functions (Fig. 3.16), taken among the types they knew, experimenting different values of the parameters: $\frac{a}{x+b}$, $\frac{a}{x^2+b}$, …, $a.\ e^{-bx}$.

The last family of functions looked the most suitable, and the conditions led the students to take a same value for a and b (for instance, 0.1).[41]

The diagram (Fig. 3.17) can describe the whole process.

This problem, in which the software has an essential part, became the starting point for a discussion about the continuous nature of a random variable, and the research of a trend function for a histogram helped in giving sense to the notion of density.

A problem presented in a 12th grade textbook starts with asking a simulation of the operation time of an electronic device with the GeoGebra software,[42] using the function $t \to f(t) = 0,07{*}e^{-0,07t}$. The data are grouped into classes, and the students are asked to estimate some probabilities of the $P(X \leq a)$ and $P(a \leq X \leq b)$ types and then to compare them with $\int_0^a f(x)dx$ and $\int_a^b f(x)dx$. Finally, an explanation of the similarity of the results is asked.

[40] My translation.

[41] In particular, for the area condition GeoGebra can display the area under the curve of a function.

[42] Le Yaouanq (2012), p. 399.

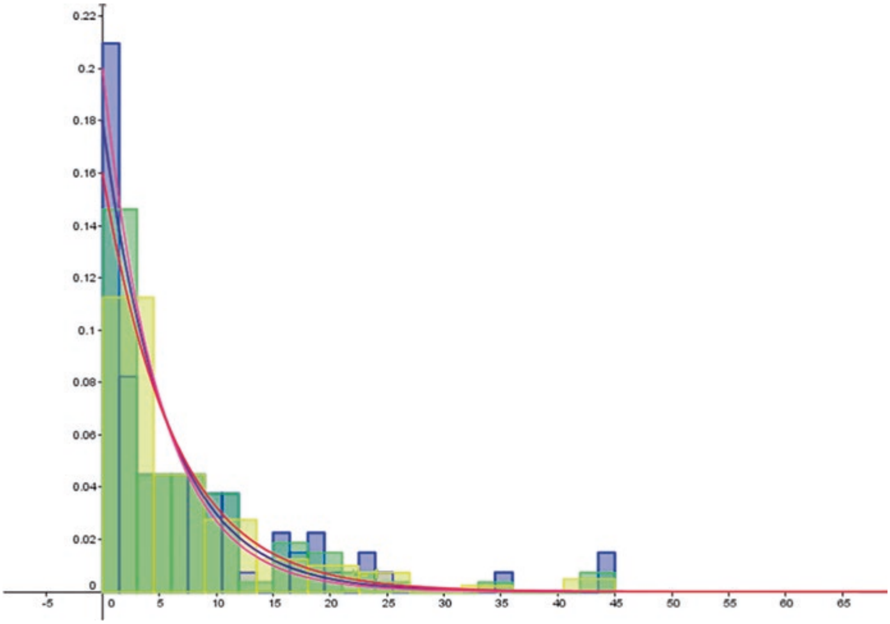


Fig. 3.16 Volcano problem. Various tries for the model curve

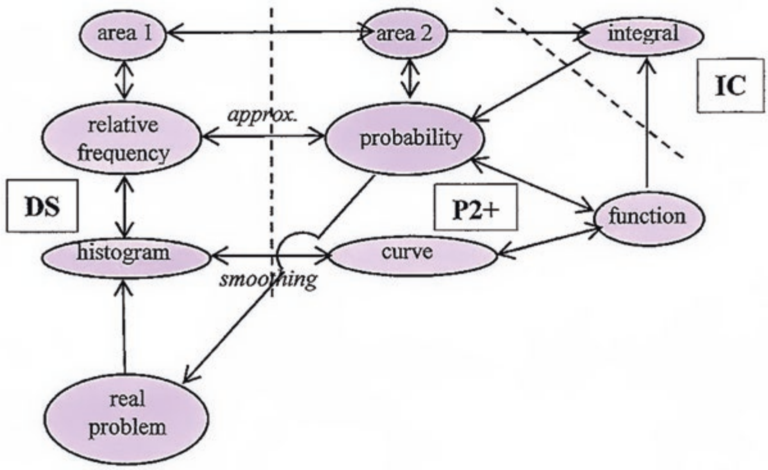


Fig. 3.17 Process of the volcano problem

Here the process consists of starting from a density (neither defined nor named as such) which is used to simulate a probability distribution and then getting the histogram of a sample ($N = 5{,}000$) which is superimposed with the density curve. A relative frequency is calculated, interpreted as an area of the histogram and compared

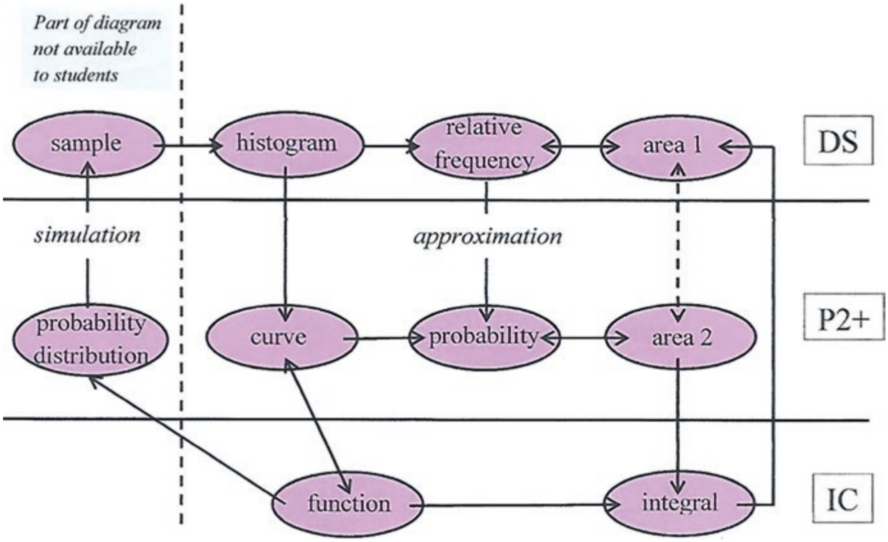


Fig. 3.18 Process of the textbook practical work. (From Derouet & Parzysz, 2016)

with the corresponding area of the theoretical density curve according to the scheme. Since the area of the histogram is 'naturally' interpreted as a relative frequency for the sample, the area under the curve will be similarly interpreted as a probability for the theoretical distribution, according to the diagram in Fig. 3.18.

In this case, real data are replaced by a simulation, in order to make the probabilistic situation more experimental (Biehler, 1991). For the teacher, the intended MWS begins within IC (function) and then moves to DS (sample). But the student's personal MWS starts from calculus (software) and then moves to DS (sample), P2+ (probability), and finally to IC (integral). It includes subtle to and fro shifts between DS and P2+, based on a graphic analogy between histogram and curve, i.e. within the semiotic and instrumental dimensions of the MWS. Yet these shifts, although an essential part of the process, are not highlighted, as if they were self-evident (Parzysz, 2011).[43] In particular, the question of using a continuous model to account for a discrete distribution should be posed.

3.8.3 Gaussian Distribution

The French curriculum for 12th grade recommends an introduction of the Gaussian distribution by observing the variations of the random variable

[43] We may notice that the big size of the sample results in a histogram very close to the theoretical curve and consequently makes the analogy easier to conceive and accept.

$Z_n = (X_n - np)/\sqrt{np(1-p)}$, X_n being a binomial variable with parameters n and p (n a large number), and using the – assumed – result of the de Moivre-Laplace theorem.

This introduction of the normal distribution is in accordance with the historic development, since the research work of Jacob Bernoulli (at the turn of the eighteenth century), followed by Abraham de Moivre and later Pierre-Simon Laplace, on the behaviour of a binomial law when the number of tries becomes very large finally resulted in the discovery of the limit law by Carl Friedrich Gauss at the very beginning of the nineteenth century.

A binomial variable is discrete, since a normal one is continuous, so one is brought back to the previous problem: approximate a discrete distribution by a continuous distribution, although in a particular case. A study of the French 12th grade textbooks on that subject shows a common framework in accordance with the official recommendation (even if only one of them mentions the historical origin of the question). Problems introducing the normal law start from a binomial variableX_n, which is first centred and reduced (hence another variable, Z_n); then, using a progression similar to the previous one, the histogram is approached by a continuous curve. Thus, the entire process remains within P2+.

Of course, the change of variable is operated with computer help, but, although the activity deals with particular cases for n and p, the general formula for Z_n is given, no reason being given for the shift from X_n to Z_n, except in one textbook, which undertakes a comparison of several distributions by 'standardizing' them:

> By centring and reducing the variables, one makes them independent from the unit chosen for their values, with standardized mean and standard deviation equal to 0 and 1. By so doing, comparing several variables is made easier.[44,45]

Three types of diagrams are successively put into play, as shown in Fig. 3.19: bar chart (for Z_n), histogram (or, rather, as seen above, 'pseudo histogram') and 'bell curve' of the normal law $N(0;1)$. Like in the general case, the histogram appears a mere 'transitional' artefact, created for allowing a comparison between the other two diagrams.

A difficulty appears in the comparison between pseudo histogram and curve: the ordinates are proportional, but not equal – since the distance between two successive values of Z_n is not 1 but $1/\sqrt{np(1-p)}$ – and the textbooks solve it more or less awkwardly, by imposing a change of scale; this drawback would not appear with a real histogram, since a rectangle of the histogram would correspond to a similar area under the curve.

In all textbooks, a computer is much used, first for making it possible to observe a number of various cases (namely, with great values of n). Some of them use spreadsheet, while some others prefer other software (like GeoGebra or OpenCalc). Not surprisingly (Sect. 3.6.4), the instructions given to students for that purpose are again either vague ('compare the diagrams') or of push-button type ('calculate', 'complete the table'). The work is limited to the semiotic-instrumental plane of the

[44] My translation.

[45] Le Yaouanq (2012), p. 400.

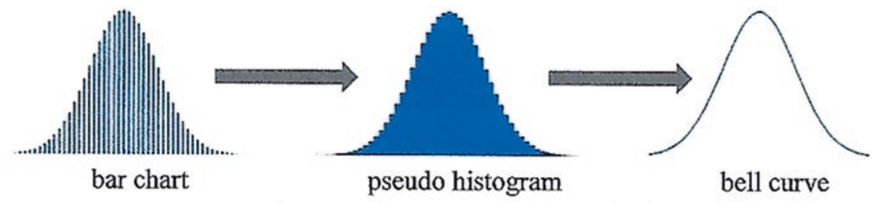


Fig. 3.19 The three diagrams for $B(200;0.4)$

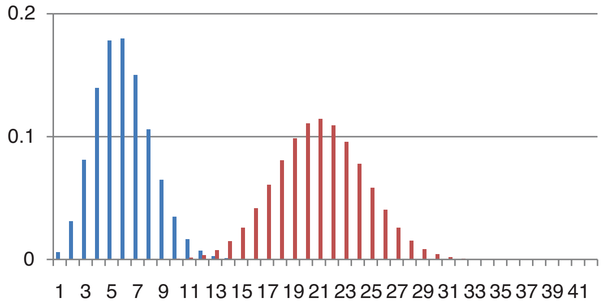


Fig. 3.20 Bar charts of $B(100;0.05)$ and $B(50;0.4)$

MWS, and even the instrumental dimension is quite limited; in particular, the reason for centring and reducing the binomial random variable is never questioned; the need for it appears, for instance, when you want to compare the overall shape (symmetry) of the bar charts of two binomial distributions which are not located in the same area of the Cartesian plane and have not the same height (Fig. 3.20); you may then think of a 'calibration' changing the units on the axes in order to give them same mean and height.

Similarly, the reason for the choice of a continuous function approximating the histogram is never questioned. For instance:

> Move the n cursor. The more n takes great values, the more the upper edges of the rectangles get nearer a curve, an equation of which seems to be of the y = $k.e^{-.5x^2}$ form. Conjecture the value of k. Draw this curve with GeoGebra.[46]

However, this drawback could be overcome by including such a function among other examples of density function when the students begin to study continuous laws.

Finally, in the present case like in the previous one, the software is mostly used for its' dynamic' power, i.e. to provide a number of various graphs in a short time. The discursive dimension appears only fleetingly and mostly for little demanding or

[46] My translation.

a posteriori justifications. One reason for such a feature is certainly that the authors of the textbooks want the student go through the whole activity without being held up at any moment; in a real classroom, the situation would possibly be different, and time would certainly be left to students for research and experimentation, with help produced only on requirement.

Twelfth graders are also requested to use a computer to calculate probabilities associated with Gaussian distributions $N\left(\mu,\sigma^2\right)$. This has now become quite an easy task, comparatively with the use of a table for $N\left(0,1\right)$ that costs students a lot of time in the past years. Besides, computers can also be used in the classroom for other purposes related to probability, e.g. use a Monte Carlo method – mentioned in the syllabus – for estimating areas by plotting points at random within a surface (see, for instance, Kroese, Taimre, & Botev, 2011). The French Ministry of Education has recently put a set of computer animations online (mostly using GeoGebra and R software) visualizing some points of the syllabus: centring-reducing a binomial variable, convergence of binomial distribution towards normal distribution, fluctuation and confidence intervals, use of Monte Carlo method for calculating integrals, and so on.

3.9 Conclusion

Technologies are involved in multiple ways in the solving of probability problems, sometimes in the same ways as in other domains but sometimes specifically, thanks to their random generator.

In this chapter, the case is made that, in tasks dealing with simulation, the semantic congruence between the real situation and the model implemented in the technological device should be taken explicitly into account in teaching. Moreover, in such problems the process which has to be put into play by students is complex, since they have to deal with several domains and shift from one to another: reality, probability and statistics. The student's personal MWS is constantly changing, first through a modelling process – often implicit – then through a simulation on spreadsheet or calculator and finally through a return back to the model. In this process the computer acts as a versatile help, since it is used in turn as random generator, calculator, logical tool and plotter, both to calculate (and then it produces *results*) and to 'show' theoretical results (and then it produces *conjectures*), which may not be so clear for students. In these process three different 'experiments' are carried out in turn, each of them belonging to a different domain, and teachers would certainly be aware of the risk of 'blurring' the distinction between them:

> A dissymmetry between students and teachers must be noted. The latter, who are experts in their domain, directly decode the results which they are presented, without even thinking about it. It is not the case with students in the process of learning, who usually do not have the required knowledge at their disposal. (Bruillard, 1997)[47]

[47] My translation.

For that reason the role of technology is not just to make it easier to get a large number of experiments in a short time (i.e. *tool*), even if this is a major feature, but it constitutes an important element for modelling (i.e. *instrument*). This 'instrumentation' process needs time, and students must be given a conceptual background enabling them to become familiar with the modelling process, in order to fulfil this goal (Rabardel, 1995); technology is a means to set up such a background, as already stated by Santos-Trigo, Moreno-Armella, and Camacho-Machin (2016) about geometry:

> Indeed, [digital technologies] offer a rich diversity of ways to represent and explore the tasks. It is not the material object by itself that produces this transformation; it is the process of appropriation led by the teacher and the students that eventually transform the digital artefact into a mathematical instrument. (p. 829)

Obviously, the computer has now become a very useful – in fact essential – help in solving probability problems. Its versatility results indeed in numerous advantages, among which:

- Access to the internet, to obtain real data
- Statistical software, to process the data (parameters, diagrams, etc.)
- Random generator, to simulate random experiments
- Table, to display and visualize any probability distribution
- Calculus software, to visualize and calculate continuous probabilities

But this must be accompanied by a concern for possible teaching problems which may arise from using technologies. As shown in the chapter, some teaching problems can be overcome if teachers train their students to become aware of the various domains brought into play, putting the stress on what comes from (real or computer-aided) experiment and what comes from theory. In this chapter, we could see in particular the need for:

- Carefully designing the association between real random experiment and simulation
- Making explicit the various domains of mathematical knowledge which take place when solving a probability problem

Research should be undertaken to specify how this can be achieved in 'ordinary' classrooms. Besides, when some points of the syllabus appear to put into play notions which may seem out of reach for high school students (for instance Gaussian distributions), we could see that problems are frequently split up into elementary tasks, the students' activity being bound to stick to the semiotic-instrumental plane of the MWS and the discursive dimension quite poor, if not left aside. This is why some researchers recently recommended 'to postpone the study of the normal law to university […], to concentrate in high school on discrete laws (binomial, Poisson, geometric…) and to use this probabilistic knowledge to study statistics' (Perrin, 2015, p. 63[48]). In many countries, solving probability problems is nowadays becoming an important element

[48] My translation.

of the students' mathematical activity, and there is still a lot of research work to be undertaken in this domain. Young researchers in math education will not lack work for the years to come if they decide to get an interest in it.

References

Artigue, M. (2016). Mathematical working spaces through networking lens. *ZDM Mathematics Education, 48*(6), 935–939.

Artigue, M., & Houdement, C. (2007). Problem solving in France: Didactic and curricular perspectives. *ZDM Mathematic Education, 39*(5–6), 365–382.

Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability). In G. A. Jones (Ed.), *Exploring probability in school. Challenges for teaching and learning* (pp. 15–37). Berlin: Springer.

Beltramone, J.-P., et al. (2015). *Mathématiques Première S. Collection Déclic*. Paris: Hachette.

Bernoulli, J. (1713). *Ars conjectandi*. Basel: Impensis Thurnisiorum. http://www.sheynin.de/download/bernoulli.pdf

Biehler, R. (1991). Computers in probability education. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters - probability in education* (pp. 169–221). Dordrecht: Kluwer.

Borovcnik, M. (2008). Strengthening the role of probability within statistics curricula. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds.), *Teaching statistics in school mathematics. Challenges for teaching and teacher education. Joint ICMI/IASE study* (pp. 71–83). Berlin: Springer.

Bosch, M., Chevallard, Y., & Gascón, J. (2005). Science or magic? The use of models and theories in didactics of mathematics. In M. Bosch (Ed.), *Proceedings of the 4th Congress of the European Society for Research in Mathematics Education* (pp. 1254–1263). Barcelona: University Ramon Llull.

Brousseau, G. (1997). *Theory of didactical situations in mathematics (1970–1990)*. Dordrecht: Kluwer.

Bruillard, E. (1997). L'ordinateur à l'école: de l'outil à l'instrument. In L. O. Pochon & A. Blanchet (Eds.), *L'ordinateur à l'école: de l'introduction à l'intégration* (pp. 97–118). Neuchâtel: IRDP.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique de la didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude. In J. L. Dorier et al. (Eds.), *Actes de la 11e Ecole d'été de Didactique des Mathématiques* (pp. 3–22 & 41–56)). Grenoble: La Pensée Sauvage.

Chow, A. F., & Van Hanegham, J. P. (2016). Transfer of solutions to conditional probability problems: Effects of example problem format, solution format, and problem context. *Educational Studies in Mathematics, 93*(1), 67–85.

Derouet, C. (2016). *Les espaces de travail mathématiques relatifs au calcul intégral et aux lois de probabilité en terminale scientifique* (PhD thesis, Université Paris-Diderot, Paris).

Derouet, C., & Parzysz, B. (2016). How can histograms be useful for introducing continuous probability distributions? *ZDM Mathematics Education, 48*(6), 757–773.

Duval, R. (1995). *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*. Bern: Peter Lang.

Girard, J.-C. (2008). The interplay of probability and statistics in teaching and in training the teachers in France. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds.), *Teaching statistics in school mathematics. Challenges for teaching and teacher education. Joint ICMI/IASE study* (pp. 1–2). Berlin: Springer.

Henry, M. (1999). L'introduction des probabilités au lycée: un processus de modélisation comparable à celui de la géométrie. *Repères-IREM, 36*, 15–34. http://www.univ-irem.fr/exemple/reperes/articles/36_article_245.pdf

Henry, M. (2003). Des lois de probabilités continues en terminale S, pourquoi et pour quoi faire? *Repères-IREM, 51*, 5–25. http://www.univ-irem.fr/exemple/reperes/articles/51_article_356.pdf

Houdement, C., & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie. Strasbourg. *Annales de Didactique et de Sciences Cognitives, 11*, 175–193. https://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/Annales_didactique/vol_11_et_suppl/adsc11-2006_007.pdf

Jungwirth, H. (2009). Explaining a technologically shaped practice: A sociological view on computer based mathematics teaching. In C. Bardini, D. Vagost, & A. Oldknow (Eds.), *Proceedings of the ICTMT 9*. France: Metz: University of Metz.

Kroese, D., Taimre, T., & Botev, Z. (2011). *Handbook of Monte Carlo methods*. New York: Wiley & Sons.

Kuhn, T. S. (1962).(1996). *The structure of scientific revolutions* (3rd ed.). Chicago: University of Chicago Press.

Kuzniak, A. (2011). L'espace de travail mathématique et ses genèses. *Annales de didactique et de sciences cognitives, 16*, 9–24. https://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/Annales_didactique/vol_15/adsc15-2010_003.pdf

Le Yaouanq, M.-H. (dir.).(2012). *Math'x. Term S. Enseignement spécifique.* Paris: Didier.

Malaval, J., et al. (2011). *Mathématiques Terminale S. Collection Hyperbole*. Paris: Nathan.

Parzysz, B. (2003). L'enseignement de la statistique et des probabilités en France: évolution au cours d'une carrière d'enseignant (période 1965-2002). In B. Chaput & M. Henry (Eds.), *Probabilités au lycée* (pp. 9–34). Paris: ADIREM/APMEP.

Parzysz, B. (2005). Quelques questions à propos des tables et des générateurs aléatoires. In B. Chaput & M. Henry (Eds.), *Statistique au lycée I* (pp. 181–199). Paris: APMEP.

Parzysz, B. (2007). Expérience aléatoire et simulation: le jeu de croix ou pile. Relecture actuelle d'une expérimentation déjà un peu ancienne. *Repères-IREM, 66*, 27–44. http://www.univ-irem.fr/exemple/reperes/articles/66_article_453.pdf

Parzysz, B. (2009). Simulating random experiments with computers in the classroom: Indispensable but not so simple. In C. Bardini, D. Vagost, & A. Oldknow (Eds.), *Proceedings of the ICTMT 9 conference*. Metz: University of Metz.

Parzysz, B. (2011). Quelques questions didactiques de la statistique et des probabilités. Strasbourg. *Annales de Didactique et de Sciences Cognitives, 16*, 127–147. http://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/Annales_didactique/vol_16/adsc16-2011_006.pdf

Perrin, D. (2015). Remarques sur l'enseignement des probabilités et de la statistique au lycée. *Statistique et enseignement, 6*(1), 51–63. http://www.statistique-et-enseignement.fr

Pratt, D., Davies, N., & Connor, D. (2008). The role of technology in teaching and learning statistics. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds.), *Teaching statistics in school mathematics. Challenges for teaching and teacher education. Joint ICMI/IASE study* (pp. 97–107). Berlin: Springer.

Rabardel, P. (1995). *Les hommes et les technologies. Une approche cognitive des instruments contemporains*. Paris: Armand Colin.

Roditi, E. (2009). L'histogramme: à la recherche du savoir à enseigner. Spirale. *Revue de recherches en éducation, 43*, 129–138. http://spirale-edu-revue.fr/IMG/pdf/8_roditi_spi43f.pdf

Santos-Trigo, M., Moreno-Armella, L., & Camacho-Machin, M. (2016). Problem solving and the use of digital technologies within the mathematical working space framework. *ZDM Mathematics Education, 48*(6), 827–842.

Trouche, L. (2005). Des artefacts aux instruments, une approche pour guider et intégrer les usages des outils de calcul dans l'enseignement des mathématiques. In *Actes de l'Université d'été de Saint-Flour - Le calcul sous toutes ses formes* (pp. 265–289). Saint-Flour, France: Académie de Clermont-Ferrand. https://hal.archives-ouvertes.fr/hal-01559831

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques*, 10(2–3), 133–170.

Warfield, V. M. (2006). Invitation to didactique. Retrieved from https://www.math.washington.edu/-warfield/Didactique.html

Chapter 4
High School Teachers' Use of a Dynamic Geometry System to Formulate Conjectures and to Transit from Empirical to Geometric and Algebraic Arguments in Problem-Solving Approaches

Manuel Santos-Trigo, Matías Camacho-Machín, and Carmen Olvera-Martínez

4.1 Introduction

Nowadays, it is recognized that students access not only a variety of information regarding mathematical courses but also consult and review online materials and even presentations of concepts or contents provided by other instructors or experts (e.g. www.khanacademy.org). Furthermore, the learners' easy access to multiple online resources does not ensure that they can select and use them efficiently in their learning experiences. They need to critically assess and monitor what information is pertinent or relevant and to discuss ways in which they can use online resources. "Educational uses of, for example, e-books, digital videos, podcasts, social networking, cloud computing, and many other mobile apps have been adopted by different groups of innovative educators and institutions around the world" (Churchill, Fox, & King, 2016, p. 21). As a consequence, curriculum contents and learning environments need to be analysed and redesigned, so that students rely on digital tools affordances to construct and apply mathematical knowledge (Kereluik, Mishra, Fahnoe, & Terry, 2013). Likewise, teachers need to incorporate several digital tools

M. Santos-Trigo (✉)
Center for Research and Advanced Studies, Cinvestav-IPN, Mathematics Education Department, Mexico City, Mexico
e-mail: msantos@cinvestav.mx

M. Camacho-Machín
Departamento de Análisis Matemático, Universidad de La Laguna, San Cristóbal de La Laguna, Spain
e-mail: mcamacho@ull.edu.es

C. Olvera-Martínez
Universidad Juárez del Estado de Durango, Facultad de Ciencias Exactas, Durango, Mexico
e-mail: carmen.olvera@ujed.mx

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_4

in their teaching practice and to analyse not only possible changes in what is taught but also discuss what types of reasoning to develop in problem-solving environments that enhance the use of those technologies.

In education, technology is challenging traditional curriculum proposals and learning scenarios in terms of offering teachers some flexible ways to organize and approach contents and their learning routes. Dick and Hollebrands (2011) recognize that "technology is an integral part of modern society and the workplace… But while mathematics has been an indispensable tool for advancing research and development of new technologies, the mathematics classroom itself has been slow to take advantages of using technology in the service of advancing mathematics learning" (p. xi). Recently, main institutions around the world have launched online learning activities that include online courses and a platform to investigate how the use of technology can transform learning (e.g. www.edx.org). Thus, it becomes important to discuss ways to structure and design online learning activities and the types of learning outcomes that learners can achieve during their interaction with the tasks. Churchill et al. (2016) propose a framework (RASE) to design technological learning environments around four interrelated elements: (i) resources that include online materials, (ii) activity that refers to a conceptual or theoretical perspective to orient learners' actions (problem-solving, for instance), (iii) support that includes ways to assist and foster students learning (forums, email, chats, etc.) and (iv) evaluation to monitor and inform about both teachers' and students' progress and learning outcomes. The core issue in mathematics education is to analyse the extent to which the systematic use of several digital technologies offers students and teachers novel opportunities to understand and develop mathematical knowledge.

Hoyles and Noss (2008) suggest that the incorporation of technologies in the students' mathematical learning goes hand in hand with the identification of possible changes in the mathematics curriculum and the process involved in the learners' appropriation of the tool. "[The uses of] digital technologies disrupt many taken-for-granted aspects of what it means to think, explain and prove mathematically and to express relationships in different ways" (p. 87). Thus, it becomes important to discuss and explain what type of mathematical activities, including ways of reasoning, that learners engage in and exhibit when they use digital tools in their problem-solving experiences.

Problem-solving activities that involve the systematic use of digital technologies offer teachers and students an opportunity to examine mathematical tasks from different perspectives that include the use of diverse concepts, resources and problem representations. Likewise, the coordinated use of various digital technologies in the classroom represents a challenge for teachers in terms of extending their knowledge about the use of digital tools and analysing the purposes, potentialities and limitations that each technology brings into the curriculum and the learning environments.

In this study, we focus on analysing ways in which a group of eight high school teachers develops and uses both subject and technological knowledge (Mishra & Koehler, 2006) to construct and extend both mathematical and didactical knowledge. Specifically, we analysed the participants' ways of reasoning and problem-solving competencies that they exhibit during the process of solving a task that involved exploring and analysing mathematical properties of linear functions and its product. In this context, they construct cognitive schemata to transform a technological artefact into a useful instrument to represent, explore and solve mathematical tasks or problems (Santos-Trigo & Moreno-Armella, 2016).

The aim of this study is to analyse and characterize ways in which high school teachers rely on a Dynamic Geometry System's affordances to represent and reason about mathematical tasks in a problem-solving environment. Thus, research questions that guide the development of the study include:

To what extent do high school teachers rely on digital technology affordances to represent and explore mathematical tasks and what ways of reasoning they exhibit in their problem-solving approaches?

To what extent do the participants, via *the use of digital technologies, reconcile visual and empirical evidences with geometric and algebraic arguments to identify and support conjectures that emerge during all problem-solving phases?*

To delve into these questions, the participants worked on a series of mathematical tasks individually, in pairs and as a group, and consistently they use a Dynamic Geometry System (GeoGebra) throughout problem-solving episodes. These phases or episodes include making sense of tasks or comprehension phase, initial representation and exploration of concepts, discussion of ideas and resources involved in the task statement, solution planning and implementation (looking for different ways to approach and solve a task), contrasting task approaches and looking for extensions and the formulation of new problems.

4.2 Conceptual Framework

Different digital technologies might offer students and teachers distinct opportunities to engage in mathematical activities. What types of digital technologies are important for learners to develop mathematical knowledge and problem-solving competencies? How can teachers incorporate in their lessons the systematic use of several digital technologies? What task contexts are important for teachers to take into account in order to engage their students in the use of digital technologies in problem-solving approaches? High school teachers need to develop a broad conceptualization of mathematics to frame learning activities within contexts that involve dealing with daily problems (mathematics for life), facing and developing problem-solving skills (mathematics for the workplace) and developing ways of thinking consistent with mathematics practices and the use of technology (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016).

The National Council of Teachers of Mathematics (NCTM) (2009) recognizes the importance for learners to focus on reasoning and sense-making activities as a way to understand concepts and to develop the foundations and cornerstone of the high school mathematics curriculum. In characterizing reasoning, the NCTM (2009) points out that "…[R]easoning can be thought of as the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4). It is also recognized that reasoning in mathematics can take many forms ranging from informal or empirical explanations to formal justifications and proofs. Sense-making is a key activity for learners to develop conceptual comprehension and problem-solving proficiency: "We define sense making as developing understanding of a situation, context, or concept by connecting it with existing knowledge" (NCTM, 2009, p. 4). Thus, reasoning and sense-making activities are essential for guiding learners in the development of interrelated mathematical processes such as problem-solving, representations and communication during their learning experiences. Likewise, Santos-Trigo and Reyes-Rodríguez (2016) argue that learners should always look for different ways to represent and solve problems and to reflect on what each solution process entails in terms of concepts and strategies.

Schoenfeld (2009) reviewed *the NCTM Focus in High School Mathematics: Reasoning and Sense Making* document and pointed out that the examples discussed throughout it are useful for teachers to illustrate ways of approaching the high school content and the organization of students' classroom interactions. He goes on saying that the reasoning and sense-making approach is consistent with his definition of teaching for problem-solving: "The mathematics studied should emerge as a reasoned and reasonable, rather than arbitrary" (Schoenfeld, 2009, p. 170). How can we frame the systematic use of digital technology in mathematics instruction that focuses on a reasoning and sense-making approach? Delving into this question involves paying attention to both: (i) the appropriation process shown by learners to transform an artefact or device into an instrument to comprehend concepts, to represent, to explore and to solve problems. Trouche (2016) pointed out the importance of distinguishing "between an artefact (a product of human activity that a subject can appropriate for performing a given task) and an instrument (resulting from this appropriation process)" (p. 246) and (ii) the characterization of ways of reasoning that subjects develop as a result of using particular tools.

> Using technology to display multiple representations of the same problem can aid in making connections…When technology allows multiple representations to be linked dynamically, it can provide new opportunities for students to take mathematically meaningful actions and immediately see mathematically meaningful consequences—fertile ground for sense-making and reasoning activities. (NCTM, 2009, p. 14)

Similarly, Dick and Hollebrands (2011) pointed out the importance for students to use technology as a means to foster problem-solving habits during the entire solution process that involves:

- *Analysing a problem*: identifying which technology tools are appropriate to use and when to use them.
- *Implementing a strategy*: making purposeful use of the technology and monitoring progress toward a solution.
- *Seeking and using connections*: especially looking across different representations.
- *Reflecting on a solution to a problem*: considering the reasonableness of technology-derived results, recognizing the limitations of the technology, reconciling different approaches (both with and without technology), and interpreting the results in the context of the problem (Dick & Hollebrands, 2011, p. xiii).

Thus, to characterize the participants' ways of approaching the tasks, we focused on analysing kinds or types of reasoning and sense-making activities that high school teachers exhibit when they systematically use digital tools throughout problem-solving episodes. These include task comprehension, design and implementation of a solution plan, connections and task extension and looking back at the entire solution process. To this end, Santos-Trigo and Camacho-Machín (2009) propose a framework to foster learners' use of digital technologies in problem-solving approaches. Similarly, Santos-Trigo, Moreno-Armella, and Camacho-Machín (2016) identify ways in which high school students rely on technology affordances to represent and explore geometric and calculus tasks.

Thus, the existence of several types of technologies makes necessary to identify what types of technology learners can use to both represent and solve mathematical tasks and to share and discuss ideas and problem solutions with peers and others beyond class settings (Santos-Trigo, Reyes-Martinez, & Aguilar-Magallón, 2016). The selection of a set of technologies to engage learners in mathematical activities implies that teachers develop certain skills and expertise that lead them to rely on technology affordances to represent, explore and analyse concepts and to solve problems. That is, teachers need to reflect on what the incorporation of digital technologies brings into their practice in terms of ways of representing and exploring mathematical concepts. They need to characterize ways of reasoning involved in the use of technology to deal with mathematical tasks and to also discuss what tools are important to foster teachers' discussion of mathematical problems within and beyond classrooms.

In this context, a bricolage perspective that integrates problem-solving approaches, the systematic use of digital technologies and the subjects' construction of mathematical knowledge (Santos-Trigo, 2014) was used to frame the study. In terms of problem-solving, our interest lies on orienting and explaining teachers about the importance of focusing on reasoning and sense-making activities to deal with concepts and mathematical problems (what do mathematics learning and problem-solving activities involve?), orientations that influence their ways of implementing their activities (what do learning experiences, beliefs, values and resources teachers or learners bring into learning environments?) and the decision-making process involved in dealing with mathematical tasks (Schoenfeld, 2011, 2012, 2015). In the field of digital technologies, we focus on analysing the process that learners show during the transformation of an artefact into an instrument or tool to solve mathematical problems (Trouche, 2016). In particular, we emphasize and make explicit the potential and opportunities that the coordinated use of digital technologies can offer to subjects in order to reason about mathematical concepts and problems. In terms of the subjects' understanding of mathematical ideas and the development of problem-solving competencies, learning environments are structured around an inquiring or inquisitive approach in which learners pose and pursue questions as a means to understand concepts and to develop problem-solving competencies. In this process, learners rely on technology affordances to construct dynamic models of tasks, explore and identify relations and conjectures and look for arguments to validate and communicate results (Leung, 2011). Santos-Trigo, Reyes-Martínez, and Aguilar-Magallón (2015) stated that learners, through the use of digital technologies, can discuss and engage in continuous problem-solving activities that involve:

- Selecting pertinent online information to extend or clarify concepts, definitions or examples of problems related to themes in study. Thus, learners need to develop ways to contrast, analyse and eventually synthetize important information to extend their knowledge and problem-solving strategies.
- Looking for different ways to solve problems and to openly discuss advantages and mathematical qualities associated with problem-solving approaches, in particular the importance of discussing the extent to which problem-solving methods can be applied to others' problems.
- Using different tools throughout the whole problem-solving process to make sense of problem statements and to represent, explore, solve and communicate results. Tools include both multiple purpose to share and communicate results and mathematical action tools to construct and dynamic models of problems.
- Reporting and discussing partial results or problem solutions that might appear during the solution process of tasks and share them with other students.
- Looking for creative or novel ways to solve and extend initial problems.

4.3 Methodological Components and Procedures

The nature of this study is qualitative since our interest is to characterize ways of reasoning that high school teachers exhibit in a technology-enhanced problem-solving environment. In particular, the aim is to document the extent to which the participants transit from empirical reasoning to the use of geometric and formal arguments to support mathematical conjectures. Eight high school teachers who have completed a Bachelor of Science in Mathematics participated in a problem-solving course during one semester. The course was part of master's degree program in mathematics education, and the participants met twice a week in sessions of 3 h each. An important goal of the course was to work on mathematical tasks via the use of digital tools throughout all problem-solving phases (understanding, plan design and implementation, finding and applying several solution methods and extending initial results).

The content of the course was structured around the concept of function as a central theme. The activities included problems or tasks that were structured around what Cooney, Beckmann, and Lloyd (2010) call the five big ideas that are essential during the study of functions: the function concept, covariation and rate of change, family of functions and the role in modelling, combining and transforming functions and multiple representations of functions. According to the authors, these big ideas are essential for teachers and students to develop or construct a robust and comprehensive understanding of functions and to solve related problems.

The sessions involved a brief introduction, provided by the course coordinator, regarding the task statement and its goal (including a task work sheet designed in advanced). The participants worked on the task individually and in pairs, and later they were asked to present their work to the group (plenary sessions).

It is important to mention that some of the participants had some experiences in the use of some digital tools and throughout the sessions, they were encouraged and guided to use multiple purpose technologies (communication technologies, Keynote or PowerPoint, YouTube) to share and follow up the work done during the sessions. To represent and explore the tasks, the participants mainly used a Dynamic Geometry System (GeoGebra) and the Wolfram Alpha computational knowledge engine (www.wolframalpha.com). Thus, the use of technologies became important not only to represent and explore the problems but also to keep an on-going discussion among the participants after the sessions. Data gathered came from individual reports, GeoGebra files and videos of plenary discussions of two sessions.

Two main intertwined phases were relevant during the design and development of the study, one in which the research team chose the problems and examined them in detail to identify different ways to represent, explore and solve them. This phase provided important information for the research team to structure the tasks and to generate potential questions to orient the participants' approaches to solve those (Sacristan et al., 2010). Based on the task analysis, an implementation guided for each task was designed. The second phase involved the actual implementation of the tasks. Here, the participants worked on the tasks individually and as group.

In the first phase, the research team worked on and discussed each task previous to its implementation. The goal was to identify different ways to represent and explore the problem in order to design and implement a guide to orient the participants' work. The guide includes ways to introduce the participants in the use of the tool, in this case the appropriation of a Dynamic Geometry System (GeoGebra). The idea is that the participants could think of the tasks as learning platforms to engage in continuous mathematical reflections and discussions associated with different ways to approach and extend the tasks. Specially, it was important to show that routine tasks could be transformed into a series of problem-solving activities that foster mathematical reflection (Santos-Trigo & Camacho-Machín, 2009; Santos-Trigo & Moreno-Armella, 2016).

We summarize the initial discussion of a task that was implemented during the problem-solving sessions. The goal was to represent and explore the tasks via the use of a Dynamic Geometry System. In this chapter, we focus on a task that involves analysing properties and relationships of two linear functions and its product.

4.4 Presentation of Results and Discussion

We chose a task to analyse and discuss ways in which the participants relied on the GeoGebra affordances to make sense, represent and explore the task. That is, the chosen task illustrates how the participants engaged in the use of a dynamic geometry system during problem phases that involve understanding and exploring the problem statement, the identification of relationships, looking for argument to support conjectures and searching for extensions and connections of the initial task. At the outset, we summarize main ideas that the research team discussed previously to the implementation of the task.

The Task (Functions) Based on a given linear function *f*, find another linear function *g*, such as both linear functions are tangent to the graph of its product function $h = f \bullet g$.

The task is an adjusted version of a problem used by Wilson and Barnes, and it can be found in jwilson.coe.uga.edu/Texts.Folder/tangent/f(x).g(x)%3Dh(x).html.

During the discussion of this task, the research team focused on identifying potential questions to guide the participants during the implementation of the task. Can you give an example of a linear function and then look for a function that fulfils the condition? Thus, if $f(x) = x + (1/2)$, then the goal is to find a function *g* such as both *f* and *g* are tangent to the graph of $h = f \bullet g$. That is, to make sense of the problem statement, it becomes important to think of a particular case in order to identify properties and relationships associated with the involved objects.

What does it mean that two linear functions are each tangent to its product? Can you sketch some examples? Is it always possible to find those functions holding the condition? What properties or how the product of two linear functions behave? With the use of GeoGebra, it is possible to explore a family of linear functions by changing, through sliders, the slope and the y-interception point. The idea is that the discussion of these types of question leads the participants to examine other examples and observe how they behave graphically.

Figure 4.1 shows some examples in which for a given function $f(x)$ it was possible to find the corresponding function $g(x)$, by changing the parameters slope and *y*-intersect (through the sliders) and observe the graphic representation of $h(x)$.

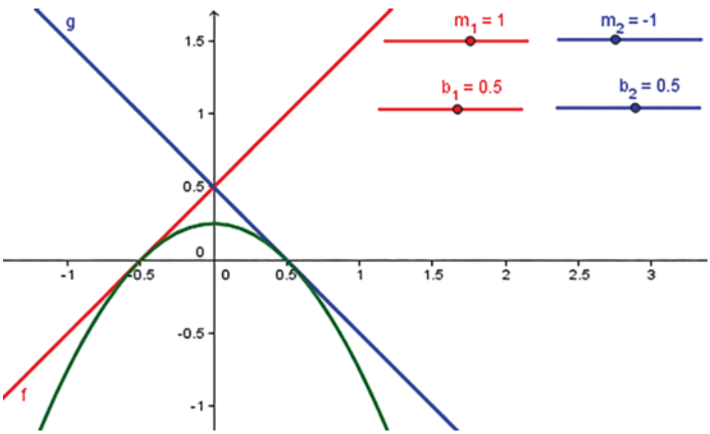


Fig. 4.1 Finding function *g*

Fig. 4.2 Examples of some solutions

$f(x) = -x + 0.5$	and	$g(x) = x + 0.5$
$f(x) = -2x + 2$	and	$g(x) = 2x - 1$
$f(x) = x + 2$	and	$g(x) = -x - 1$
$f(x) = -2x + 4$	and	$g(x) = 2x - 3$
$f(x) = -x - 2$	and	$g(x) = x + 3$

This empirical approach provides useful information regarding some possible relations among main parameters of the involved functions.

Are there some patterns, relationships or properties that relate the linear functions and its product? Based on the information provided in Fig. 4.2, it might be observed that:

(i) The product of two linear functions (assuming that the coefficient of the linear terms is not zero) always generates a quadratic function (what type of function does the product of two linear functions generate?).
(ii) The sign of slopes of the linear functions provides information regarding the product behaviour. That is, if both signs are positive or negative, the product function will open upward; otherwise it would open downward. What information does the product of the slopes of two linear functions provide regarding the graph of the product of those linear functions?
(iii) The possible candidates, for the linear functions, whose product is tangent to both lines, need to have slopes with opposite signs.

Comment The team discussed the task and identified possible routes to implement it. In particular, it was decided to orient the teachers' interaction with the task toward the use of the tool to explore properties associated with the behaviour of main parameters (slope and y-intersection) and the related product. Based on this analysis of the task, we intended to orient the participants to rely on the use of GeoGebra as a means to analyse changes in the parameters of the linear functions and the graphs of the product of those functions.

The Task Implementation Some of the participants read the task statement and began drawing a possible sketch of the linear function and its product. Figure 4.3 showed a graph that one of the participants initially proposed and that later was used to discuss the extent to which it fulfilled the tasks conditions. This discussion led the participants to identify properties of the slopes of the linear functions and their relations to the quadratic function generated by the product. It was important to observe that when the participants began to explore the properties of the graph, they relied on the use of the tool to examine examples which could match what they had initially proposed as possible graph.

Table 4.1 shows initial ideas that the participants expressed in their individual work with the task. Indeed, the example found by Sandra led them to recognize that it was possible to find the linear functions both tangent to the product.

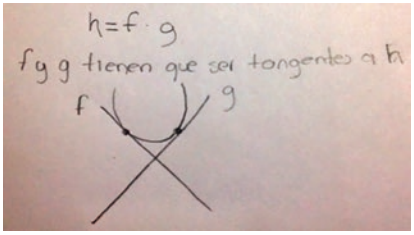


Fig. 4.3 Initial sketch of the problem

Table 4.1 The participants' initial ideas and sense-making of the task (exploration phase using GeoGebra)

<table>
<tr><th colspan="4">What initial ideas did participants show in their individual approach to solve the task?</th></tr>
<tr><th>Properties of the function product h (quadratic, roots, concavity, vertex)</th><th>Intersections between f and h</th><th>Intersections between g and h</th><th>Is there a situation where the graphs of the linear functions are tangent to the graph of function h?</th></tr>
<tr><td rowspan="3">h intersects x-axis when $f(x) = 0$ and $g(x) = 0$
h opens up when the both signs of the slopes are positive or negative (same sign)
h opens down when the signs of the slopes of the linear functions are opposite
The vertex of the parabola lies on the perpendicular bisector of the segment that joins the two roots of h. (All the participants)</td><td>There might be either one or two intersection points. One of them is obtained when $f(x) = 0$ (Isabel, Laura and Paola)</td><td>Similarly, there might be either one or two intersection points. One of them is obtained when $g(x) = 0$ (Isabel, Laura and Paola)</td><td>It might not be possible (Isabel, Gabriela, Laura and Paola)</td></tr>
<tr><td rowspan="2">There might be either one or two intersection points. One of them is x_1 verifying $f(x_1) = 0$ and the other point x_2 must verify $g(x_2) = 1$ (Ana, Gabriela, Sandra, Andrés and Gerardo)</td><td rowspan="2">There might be either one or two intersection points. One of them is x_1 verifying $g(x_1) = 0$ and the other point x_2 must verify $f(x_2) = 1$ (Ana, Gabriela, Sandra, Andrés and Gerardo)</td><td>It seems that we can draw the case where only one line is tangent to the product (Ana, Andrés and Gerardo)</td></tr>
<tr><td>Yes, it is possible. She found an example (Sandra)</td></tr>
</table>

Comment The participants' initial individual work focused on trying to make sense of the task in order to sketch or visualize the tangent condition of the two linear functions to its product. One of the participants drew two intersecting lines and a parabola (Fig. 4.4) without realizing that the sketch did not match the representation he had proposed. Others even mentioned that it was not possible to fulfil the task conditions. However, when these initial contributions were presented to the group, they were analysed and helped the group examine important properties associated with the product of two linear functions. In particular, the example that one of the participants (Sandra) found (via the use of GeoGebra) was important for the

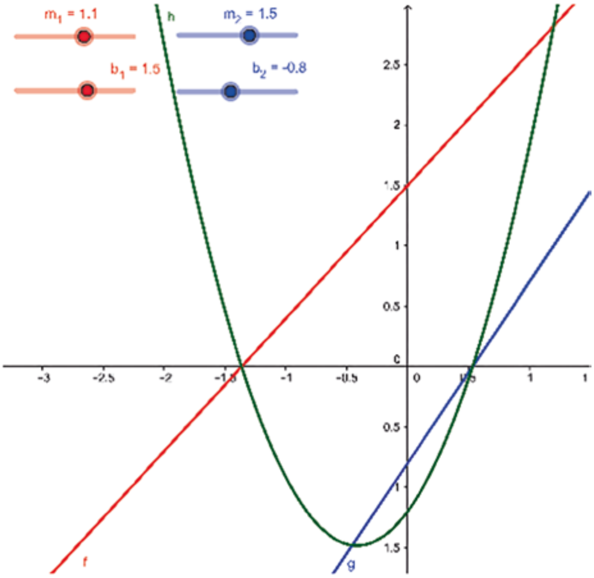


Fig. 4.4 Dynamic representation

participants to focus on what properties the linear functions and the product must hold to fulfil the task conditions.

At this stage, all the participants relied on GeoGebra to find examples of two linear functions each of which is tangent to the graph of the product of the functions. Figure 4.5 shows an example of some graph representations that they explored to examine relations between the linear function and the quadratic by focusing on the root of the functions, the axis of symmetry and the parabola vertex.

The use of sliders to examine the behaviour of the linear function and the product became efficient when they fixed one of the linear functions and looked for the second function by varying the corresponding parameters. Based on this strategy, they found more examples fulfilling the conditions:

$f(x) = -x$	$g(x) = x + 1$	$h(x) = -x^2 - 1$
$f(x) = -2x + 3$	$g(x) = 2x - 2$	$h(x) = -4x^2 + 10x - 6$
$f(x) = -x - 1$	$g(x) = x + 2$	$h(x) = -x^2 - 3x - 2$
$f(x) = 4x - 7$	$g(x) = -4x + 8$	$h(x) = -16x^2 + 60x - 56$
$f(x) = -2x + 6$	$g(x) = 2x - 5$	$h(x) = -4x^2 + 22 - 30$

At this stage, all participants recognized that it was possible to find examples, by changing the corresponding parameters via the use of sliders, that graphically satisfied the task conditions. Indeed, they recognized that the examples they found fulfilled the properties that appear in Table 4.1. And, they also formulated the following conjectures:

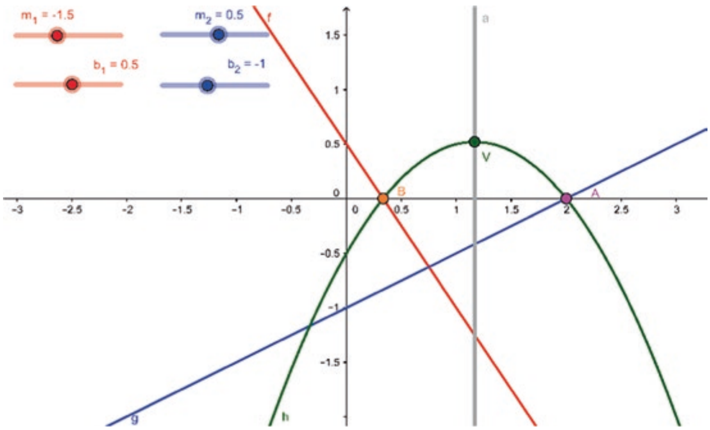


Fig. 4.5 Exploring properties and relations

1. For two linear functions $f(x) = ax + b$ and $g(x) = cx + d$ to hold that the graph of the product $f(x) \bullet g(x)$ is tangent to both f and g, then $a = -c$ (same slope with opposite sign) and $b + d = 1$.
2. The function f intersects the quadratic function h ($h = f \bullet g$) at one or two points, and one of them (in case that there are two) $B(x_1, f(x_1))$ also holds that $g(x_1) = 1$. This is because at the intersection point, it is true that $f(x_1) = h(x_1) = f(x_1) \bullet g(x_1)$ and therefore $g(x_1) = 1$. Likewise, there is an intersection point $C(x_2, g(x_2))$ between g and h such that $f(x_2) = 1$. And the case in which a linear function became tangent to the parabola, then intersection points between the product function and the linear functions coincide (Fig. 4.6).
3. The case in which both lines are tangent to the parabola, then the parabola vertex and the intersection point of both lines (point H) are on the axis of symmetry of the parabola and the point H, intersection of f and g, is located at the intersection point of the axis of symmetry and line $y = 0.5$ (Fig. 4.7). Furthermore, the intersection of function product and the axis of symmetry is the vertex (V) of the parabola.

Table 4.2 summarizes the set of conjectures that the participants formulated as a result of exploring empirically properties of the functions that fulfil the task conditions.

Comment At the beginning, some of the participants used four "sliders" to represent and get families of line by changing corresponding slopes or the y-intersecting points; however, they eventually realized that it was more efficient strategy to fix one linear function and to find the second function by changing the slope and y-intersection point (via the sliders) of particular function to get task conditions.

At this stage, the participants recognized that all conjectures were formulated based on examining graphically and empirically the behaviour of the linear functions and its product. Here, one of the participants posed this question: *Could we draw the graph of a linear function g if we know the graph of function f and then sketch the graph of their product?*

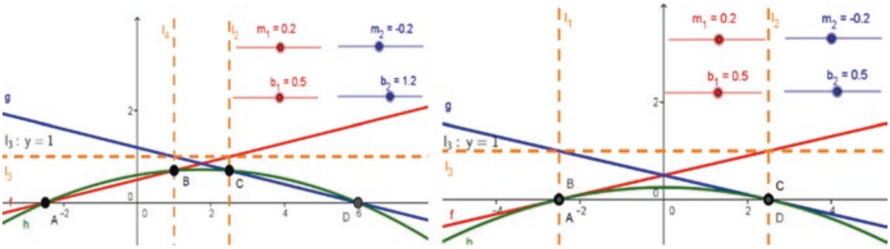


Fig. 4.6 The intersection points of the linear functions and the product function coincide when the lines are tangent to the product function

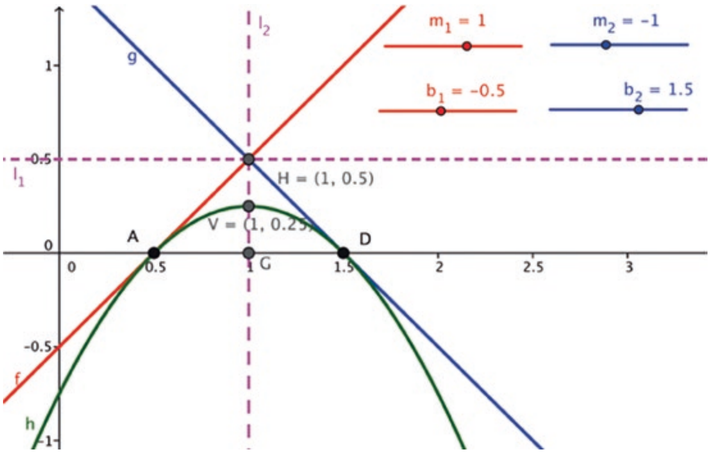


Fig. 4.7 Identifying the axis of symmetry of the intersection of *f* and *g*

Table 4.2 Conjecturing the previous results

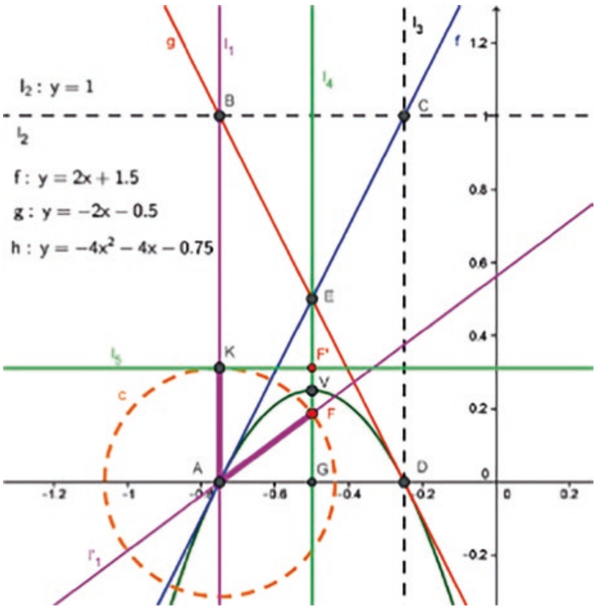
What conjectures did the participants formulate when they explored the task with GeoGebra?		
Slopes of the linear function, the independent terms and the graph of the product	Roots of the linear functions and the product function	Lines intersection
The slopes of the linear function need to be the same, but with opposite signs. And the sum of the independent terms must be 1 (All the participants)	Line *f* and parabola *h* get intersected at the root of *f* and when *g* has the value of 1. Similarly, line *g* and parabola *h* get intersected at the root of *g* and when *f* has the value of 1. In addition, when both *f and g* are tangent to *h* both intersection points coincide (Ana and Sandra)	The intersection point of the lines *f and g* is on the line $y = 1/2$ (Andrés and Gerardo)

There were several ways to find the graph of g from a given f and then the graph of product of f and g. A pair of participants followed up the properties and relations that they empirically had previously identified to graph the asked functions (the linear and the product functions). Figure 4.8 shows the graph they achieved based on the graph of $y = 2x + 1.5$.

(i) Figure 4.8 shows the graph of the given function $f(x) = 2x + 1.5$. The x-coordinate of Point A is the root of f, and they drew a perpendicular l_1 to x-axis from point A and line l_2 ($y = 1$). B is the intersection point between l_2 and l_1(B belongs to the graph of g), and point C is the intersection between line l_2 and f. Then, they drew a perpendicular line l_3 to l_2 that passes through point C and located point D that is the intersection point of l_3 and the x-axis (D also is a point of the graph of g). They drew the graph of g as the line that passes through points B and D. Point E is the intersection of graphs of f and g.

(ii) To sketch the function product $h = f \bullet g$, they drew the perpendicular bisector of segment AB (this is the axis of symmetry of h). This axis of symmetry intersects the x-axis at point G and the middle point of segment EG is the vertex of the parabola. At this stage, the teachers asked: Can we find the focus and directrix of the parabola? Another participant proposed to reflect line l_1 over line f (f is tangent to the parabola) to get l_1', and the intersection of l_1' and the axis of symmetry is the focus of the parabola (F). Then they drew the symmetric point of F (F') with respect to V and drew a perpendicular l_5 to the axis of symmetry that passes through point F' (this is the directrix). Based on this information, the function product h was drawn directly.

Fig. 4.8 Sketching g and h based on the graph of f



Comment The participants relied on the tool's affordances to identify a set of mathematical relations involved in the graphic representation of the functions. At this stage, they even posed questions that involved graphic relations (perpendicular bisector, axis of symmetry, etc.) to examine graphically the behaviour of the linear functions and their product. This approach led the participant to transit form empirical to algebraic approaches to the task.

Supporting the Conjectures through Formal Arguments The participants recognized that the conjectures emerged from visualizing the behaviour of objects when moving particular parameters and there was a need to justify or support those conjectures. Santos-Trigo, Moreno-Armella, and Camacho-Machín (2016) documented that the teachers relied on the tools' affordances to think of the task or problem in terms of mathematical properties. In this context, the participants found several properties and mathematical relations, through the use of GeoGebra, that later became important in constructing algebraic arguments to validate their conjectures.

Table 4.3 shows a summary of the arguments that the participants provided to support the conjectures.

(a) The signs and values of the slopes. Three participants (Sandra, Andres and Gerardo) considered the general case, that is, $f(x) = m_1x + b_1$ whose root ($f(x_1) = 0$) is $x_1 = -b_1 / m_1$ and $g(x) = m_2x + b_2$, which takes the value of 1 ($g(x_2) = 1$) when $x_2 = \left(1 - b_2\right) / m_2$. At the tangent point, both f and h must coincide, that is, $x_1 = x_2$; therefore, $m_2 / m_1 = \left(1 - b_2\right) / -b_1 \ldots \left(1\right)$. Similarly, for function g, $g(x_3) = 0$ when $x_3 = -b_2 / m_2$, and $f(x_4) = 1$ for $x_4 = \left(1 - b_1\right) / m_1$. Then, the graph of g is tangent to the product function when $x_3 = x_4; m_2 / m_1 = -b_2 / \left(1 - b_1\right) \ldots \left(2\right)$. By solving (1) and (2), they got that $\left(1 - b_2\right) / -b_1 = -b_2 / \left(1 - b_1\right)$, that is, $b_1 = 1 - b_2$. That is, they proved the pattern they found regarding that sum of the y-intersection of both linear functions must be 1. Then by replacing $b_1 = 1 - b_2$ in (1), they proved that $m_1 = -m_2$.

(b) The derivative of the function product and the slopes of the linear functions. Another approach followed by other participants involved expressing algebraically the linear functions and found its product. That is, $h(x) = f(x) \bullet g(x) = (mx + a)(nx + b) = mnx^2 + (mb + na)x + ab$. They focused on the values of the slopes at the tangent point of the linear functions and the derivative of the product function (h), that is, $h'(x) = 2mnx + mb + na$. They affirmed that at the tangency point the derivative of h is equal to the slope of each linear function. Then they found: $-m + mb - na = 0 \ldots (3)$ and $-n - mb + an = 0 \ldots (4)$, that is, $-m(-2b + 1) = n(2a - 1)$. Then, they introduced the condition that $-m = n$, to find that $b + a = 1$.

(c) A rectangle and the axis of symmetry of the parabola. Three participants observed that points A, B, C and D formed a rectangle with one side AD (distance between the roots) and the other side AB of one unit length. Then the diagonals of the rectangle $ABCD$ coincide with the linear functions. Therefore, the centre of this rectangle lies on the intersection point of $y = 0.5$ and the axis of symmetry of the parabola (Fig. 4.9).

Table 4.3 Looking for arguments to support main conjectures

How did the participants validate their conjectures?		
Slopes, intersection points and graphs	Relations between the linear functions and the quadratic function (product)	The lines and the diagonals of a rectangle
1. Getting the derivative of function *h* and applying the condition that it must be equal to the slope of each linear function. (Isabel, Gabriela, Laura, Paola and Sandra) 2. Finding the intersection points of *f and h* and *g and h* to make them coincide for the tangent condition. (Andrés and Gerardo) 3. Finding two points of the graph of *g* to get an algebraic expression for it (Ana)	*f and h*: Both get intersected when $f(x) = 0$ since $h(x) = f(x) \bullet g(x) = 0 \bullet g(x) = 0$ They also get intersected when $g(x) = 1$. This is because $h(x) = f(x) \bullet g(x) = f(x) \bullet 1 = f(x)$ *g and h*: They get intersected when $g(x) = 0$ since $h(x) = f(x) \bullet g(x) = f(x) \bullet 0 = 0$ They also get intersected when $f(x) = 1$, since $h(x) = f(x) \bullet g(x) = 1 \bullet g(x) = g(x)$ (All participants)	The lines determine the diagonals of a rectangle whose base is the segment between the roots of the linear functions (and quadratic) and a height of one unit. Therefore, the intersection point of the linear functions is the centre of the rectangle and is located on the line $y = 1/2$ (Sandra, Andrés and Gerardo)

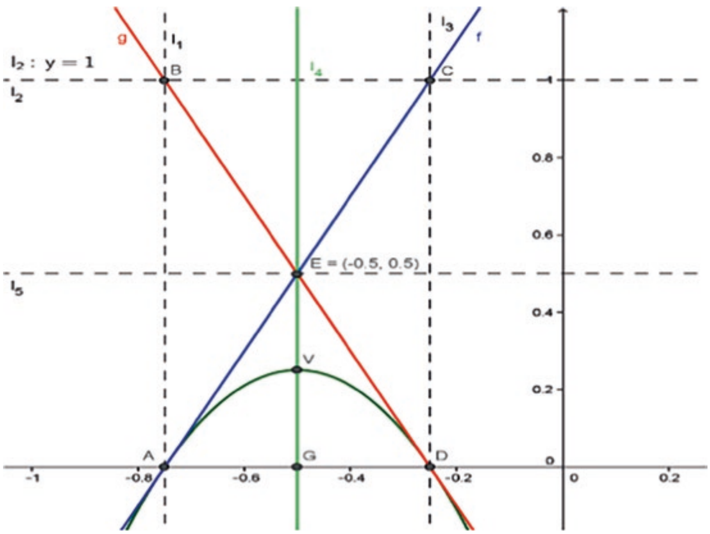


Fig. 4.9 Justifying that lines intersections lies on $y = 0.5$

4.5 Discussion of Results

It is important to reflect on the extent to which the participants relied on GeoGebra affordances to reason about concepts and to engage in making sense of activities that led them to explore, to formulate conjectures and to solve mathematical problems. In the first episode that involved understanding key ideas around the task, the participants' individual contributions included some incomplete or contradictory representations (Fig. 4.1) of the task that later were examined within the group. Here, the use of GeoGebra became important not only to reject some initial sketches of the task but also to explain what the slopes of the lines produce to the behaviour of the product. Thus, the tools affordances helped the participants to transit from intuitive sketches of the task to those that consider the geometric meaning of the involved parameters (slopes, parabola concavity, etc.).

Taking a particular example for *f* became an important strategy to explore a family of possible candidates for function *g* based on observing graphically both the behaviour of the functions and the product. In addition, the use of sliders for varying parameters of linear functions became a problem-solving strategy to generate data that later were used to detect patterns or conjectures and properties of the functions and the product. That is**,** the participants relied on GeoGebra's affordances to make sense of the task statement that led them to think of task parameters (slope and y-intersect) in terms of sliders. So, they reasoned about the problem via a dynamic representation that provided useful information to explore possible patterns and relations between the linear functions and its product. At this stage, there were evidences that the participants used the tool as a means to formulate conjectures and to identify properties of a family of linear functions that satisfies the task conditions (subjects' tool appropriation). For instance, during the exploration, they detected invariances such as convergence of roots of the involved functions, opposite signs of slopes of the linear functions, location of the lines intersections, etc. Furthermore, they were able to identify and visualize a set of conjectures.

In the second episode, they relied on the tool affordances to construct a graph solution of the task. Indeed, with the use of the tool, they empirically validated some of the relations and conjectures that they had identified during the exploration of the task. That is, by considering a graph of a linear function, they were able to construct graphically the second linear function and the product function. Although at this stage they were convinced that the conjectures guided them to solve the task geometrically, they also recognized that they had important information to present algebraic arguments to validate the conjectures. Furthermore, the use of the tool became crucial to reconcile algebraic approaches and visual or geometric arguments to support the emerging conjectures. Indeed, all the participants recognized that the construction of dynamic models of the tasks represented an opportunity for them to make sense of the task and to explore relations among parameters of objects that later were important to think of a solution plan. In addition, dynamic explorations not only provided useful information to support conjectures but also to connect particular cases with the analysis of a family of objects.

4.6 Closing Remarks

In general terms, there is evidence that the participants relied on GeoGebra affordances to work on all problem-solving phases that involve making sense of the task, recovering from wrong or inconsistent initial approaches and identifying relations and conjectures to design and implement diverse ways to solve the task and to extend the initial task statement. Santos-Trigo and Moreno-Armella (2016) pointed out that the use of the tool often helps problem solvers realize whether their initial ideas are consistent or can be geometrically achieved. For instance, an initial sketch proposed by one of the participants (Fig. 4.3) showed that it was drawn based on his initial intuition; but later it became a vehicle to introduce the use of GeoGebra to explore, based on particular examples, whether that sketch could be achieved.

In terms of ways of reasoning that the participants engaged while solving the task, it was clear that the use of the sliders helped them to explore dynamically the parameters' behaviour associated with the task. Thus, the controlled movement of those parameters led them to propose a set of conjectures and properties that later they support with mathematical arguments. Indeed, the list of properties and conjectures that they found empirically led them to explore whether, from a given graph of a linear function, it was possible to find the graph of the second function and the product fulfilling the task conditions. That is, in general terms, the geometric approach to find the functions became a way to validate the set of conjectures that they had identified empirically.

In addition, when they looked for formal arguments to support the conjectures, the use of the tools was crucial to identify properties that later were used to validate their results. In this context, the use of the tool became important to reconcile and transit from empirical evidences to geometric and algebraic arguments to support results. That is, the use of GeoGebra not only offered a set of affordances for the participants to test the pertinence of their initial ideas but also to examine a family of examples in order to detect patterns or possible relationships between the involved parameters. Thus, the participants exhibited several problem-solving strategies such as examining special cases, analysing family of functions by changing the slope and the y-intersection point, collecting data and using a table to organize them, looking for patterns, finding conjectures and providing different types of arguments to support conjectures.

Working on the tasks became important for the participants to reflect on their understanding of concepts that appear during the solution process. For instance, Figure 4.3 shows an initial wrong representation of the graphic representation of the required functions that later was analysed and changed as a result of exploring some examples through the use of the tool. That is, the use of a dynamic geometry system became important to relate examples provided algebraically to the graphic representation of their product. This type of exploration led the participants to reflect on ways to introduce the use of the tool in their actual teaching practices.

In general terms, the inquiring approach that framed the participants' approaches to the tasks helped conceptualize the tasks as a point of departure to engage in mathematical reflections and as an opportunity to look for mathematical relations and ways to support it through visual, empirical and eventually formal arguments. In addition, all the participants recognized that the technology affordances not only helped them construct and explore dynamic models of the tasks but also to reflect on ways to implement the tasks in actual practice.

Acknowledgement This report is part of a wider project that deals with teachers and students' use of digital technologies in extending both mathematics and didactic knowledge. We acknowledge the support received from research projects with reference numbers Conacyt-168543 (México) and EDU2015-65270-R (National I+D+I Plan of MEC-Spain) during the development of this research.

References

Churchill, D., Fox, B., & King, M. (2016). Framework for designing mobile learning environments. In D. Churchill, J. Lu, T. K. F. Chiu, & B. Fox (Eds.), *Mobile learning design, lectures notes in educational technologies* (pp. 22–52). New York: Springer.

Cooney, T., Beckmann, S., & Lloyd, G. (2010). *Developing essential understanding of functions for teaching mathematics in Grades 9–12*. Reston, VA: National Council of Teacher of Mathematics.

Dick, T. P., & Hollebrands, K. F. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making*. Reston, VA: National Council of Teacher of Mathematics.

Hoyles, C., & Noss, R. (2008). Next steps in implementing Kaput's research programme. *Educational Studies in Mathematics, 68*, 85–97.

Kereluik, K., Mishra, P., Fahnoe, C., & Terry, L. (2013). What knowledge is of most worth: Teacher knowledge for 21st century learning. *Journal of Digital Learning in Teacher Education, 29*(4), 127–140.

Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM Mathematics Education, 43*, 325–336.

Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problem solving in mathematics education*, *ICME-13 topical surveys*. Cham: Springer.

Mishra, P., & Koehler, M. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teacher College Record, 108*(6), 1017–1024.

National Council of Teacher of Mathematics. (2009). *Focus in high school mathematics: Reasoning and sense making*. Reston, VA: National Council of Teacher of Mathematics.

Sacristan, A. I., Calder, N., Rojano, T., Santos-Trigo, M., Friedlander, A., & Meissner, H. (2010). The influence and shaping of digital technologies on the learning -and learning trajectories- of mathematical concepts. In C. Hoyles & J. Lagrange (Eds.), *Mathematics education and technology rethinking the terrain. The 17th ICMI study* (pp. 179–226). New York: Springer.

Santos-Trigo, M. (2014). Problem solving in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 496–501). New York: Springer.

Santos-Trigo, M., & Camacho-Machín, M. (2009). Towards the construction of a framework to deal with routine problems to foster mathematical inquiry. *Primus, 19*(3), 260–279.

Santos-Trigo, M., & Moreno-Armella, L. (2016). The use of digital technologies to frame and foster learners' problem solvers' experiences. In P. Felmer et al. (Eds.), *Posing and solving mathematical problems*, *Research in Mathematics Education* (pp. 189–207). Cham: Springer. https://doi.org/10.1007/978-3-319-28023-3_12

Santos-Trigo, M., Moreno-Armella, L., & Camacho-Machín, M. (2016). Problem solving and the use of digital technologies within the mathematical working space framework. *ZDM Mathematics Education, 48*, 827–842. https://doi.org/10.1007/s11858-016-0757-0

Santos-Trigo, M., Reyes-Martínez, I., & Aguilar-Magallón, D. (2015). The use of digital technologies in extending mathematical problem solving reasoning. In L. Uden et al. (Eds.), *Learning technologies for education in cloud*, *Communications in computer and information science* (pp. 298–309). New York: Springer.

Santos-Trigo, M., Reyes-Martinez, I., & Aguilar-Magallón, D. (2016). Digital technologies and a modeling approach to learn mathematics and develop problem solving competencies. In L. Uden et al. (Eds.), *Learning technologies for education in cloud*, *Communications in computer and information science* (pp. 193–206). New York: Springer.

Santos-Trigo, M., & Reyes-Rodríguez, A. (2016). The use of digital technology in finding multiple paths to solve and extend an equilateral triangle task. *International Journal of Mathematical Education in Science and Technology, 47*(1), 58–81. https://doi.org/10.1080/0020739X.2015.1049228

Schoenfeld, A. H. (2009). An essay review of the NCTM high school curriculum project. *Mathematics Teacher, 103*(3), 168–171.

Schoenfeld, A. H. (2011). *How we think. A theory of goal-oriented decision making and its educational application*. New York: Routledge.

Schoenfeld, A. H. (2012). Toward professional development for teachers grounded in a theory of decision making. *ZDM Mathematics Education, 43*, 457–469. https://doi.org/10.1007/s1158-011-0307-8

Schoenfeld, A. H. (2015). How we think: A theory of human decision-making, with a focus on teaching. In S. J. Cho (Ed.), *The proceedings of the 12th international congress on mathematical education* (pp. 229–243). New York: Springer. https://doi.org/10.1007/978-3-319-12688-3_16

Trouche, L. (2016). Didactics of mathematics: Concepts, roots, interactions and dynamics from France. In J. Monaghan et al. (Eds.), *Tools and mathematics*, *athematics education library* (pp. 219–256). New York: Springer.

Chapter 5
The Interactive Whiteboard and the Development of Dialogic Interaction in the Context of Problem-Solving

Ana Paula Canavarro and Carla Sofia Pereira Reis

5.1 Introduction

The interactive whiteboard was introduced in Portugal on a large scale about a decade ago. It was the last technological tool introduced in schools, and nowadays, it is available in most of them (Torres, 2008). Its use opens a wide range of possibilities for work in the classroom, allowing the teacher to use digital resources, created by himself/herself or available on the Internet, taking advantage of the audiovisual use for experimentation and simulation of various situations, contributing to the promotion of interaction between the teacher and pupils and among themselves (Beauchamp & Kennewell, 2008; British Educational Communications and Technology Agency [BECTA], 2004).

However, in some contexts, there is a tendency to set the interactive whiteboard to expository teaching service: the whiteboard is used for purposes of projection, or, in other cases, some of its graphic capabilities are used to enhance the teacher's content presentation to a listening-only class.

So, the use of the interactive whiteboard does not necessarily transform the class dynamics nor does it improve the interactions among its participants, but it is in this particular topic that important advantages for the teaching and learning process are achieved (Mercer, Hennessy, & Warwick, 2010; Tanner, Jones, Kennewell, & Beauchamp, 2005; Warwick, Hennessy, & Mercer, 2011). The interactive whiteboard is a tool that can be used at the service of a dialogic approach to students' learning, based on truly interactive classroom environments (Reis, 2014).

A. P. Canavarro (✉)
Universidade de Évora, Évora, Portugal
e-mail: apc@uevora.pt

C. S. P. Reis
Agrupamento de Escolas de Amareleja, Amareleja, Portugal

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_5

In a dialogical approach to learning, the key idea is the construction of shared knowledge that is established in the class derived from student and teacher dialogue engagement (Mercer et al., 2010; Ruthven, Hofmann, & Mercer, 2011; Stein, Engle, Smith, & Hughes, 2008). This perspective on learning is especially well articulated with an inquiry-based mathematics teaching (Ponte, 2005; Stein et al., 2008; Warwick et al., 2011). In this model, the teacher's role is a key factor in the coordination of collective discussion on the student's work, resulting from challenging tasks and from the joint reflection produced, pointing to the construction of the mathematical knowledge that is being institutionalized in the class (Canavarro, Oliveira, & Menezes, 2012; Ruthven et al., 2011).

However, exploratory teaching practices are difficult to conduct while putting several challenges to the teacher (Canavarro, 2011). Special care must be taken in the choice of tasks to promote student work. The tasks must enable the confrontation of ideas, strategies and results, both during the work of the students in small groups and during the collective discussion with the whole class. Problematic tasks, for its challenging nature (Ponte, 2005), emerge as especially appropriate and valuable in the development of dialogical interaction in mathematics classes.

Another difficulty has to do with the role of the teacher. In the inquiry-based teaching of mathematics, he/she has to perform a multitude of simultaneous actions, imbued with two purposes: to manage the class as a whole (time, resources, interactions) and to promote students' mathematics learning (reasoning, solving, communicating, concluding) (Canavarro et al., 2012). To better face this complexity, it is important that the teacher can easily mobilize resources for supporting the achievement of these actions in an effective way, in particular in what concerns the facilitation of communication between different participants in class.

Being the interactive whiteboard a powerful resource to promote the interaction, it is of special importance and interest to understand how it can enhance the development of dialogical interaction in the classroom.

This paper focuses on the identification and analysis of the contributions that the interactive whiteboard can provide for the development of dialogical interaction in the context of problem-solving, by identifying the features of the board that can be used to support this interaction.

It is our belief that this study may contribute to deeper the knowledge necessary to support teacher's transformation of class dynamics, taking advantage of the use of the interactive whiteboard for helping teachers to better deal with the complexity of mathematics inquiry-based teaching practices.

5.2 Theoretical Perspectives

5.2.1 The Interactive Whiteboard in the Teaching and Learning of Mathematics

An interactive whiteboard is an educational technology associated with a computer, a projector and a specific monitoring device (pen or other), which allows designing digital resources in an interactive surface. Through this surface, it is possible to write and control software programs, without computer usage.

The software associated with the interactive whiteboard offers a wide range of functions that allow revolutionizing both teaching and learning. Among the main features are the possibility to write on the screen; drag and drop objects; hide and reveal; enhance the written or other objects by placing a transparent colour on them; rotate, enlarge or decrease the objects; use multiple pages without having to delete information; store materials during the desired time and retrieve them; give sound or visual feedback when a given object is touched; and use, at the same time, specific programs of the various subject areas ([BECTA], 2004). In the case of mathematics, it stands out the dynamic geometry programs, calculation programs and interactive applets, which can be integrated on the interactive whiteboard.

According to Glover, Miller, and Averis (2005), the demo tasks and the problem resolutions profit from the use of the item dragging and hidden answer revelation techniques, while working with graphics is facilitated by the use of colours and shadings. The object manipulation or rotation and the use of specific software allow students to better understand the concepts (Moss et al., 2007). Torres (2008) suggests that a student who is given the opportunity to interact with the interactive whiteboard can present "a reasoning or a different strategy to approach a problem, using a spreadsheet, dynamic geometry software, an applet or other" (p. 43), taking advantage of the interactivity provided by the board.

For students, the use of the interactive whiteboard has several advantages (British Educational Communications and Technology Agency [BECTA], 2003): it allows them greater attention and focus on the class, since students do not need to spend all the time taking notes in their personal notebooks; it enhances their participation and cooperation in the activities and, consequently, the development of personal and social skills; it increases their motivation to participate by the use of more dynamic approaches – the use of games, colours, images, the Internet and software; and it extends the possibilities of understanding more complex concepts.

Some studies point to the fact that the use of interactive whiteboard contributes to increase the students' motivation and their involvement in the learning process (Higgins et al., 2005; Lerman & Zevenbergen, 2007; Levy, 2002). According to BECTA (2004), the greatest source of students' motivation is the opportunity to physically interact with the board, together with the increased possibility of interaction and discussion. Higgins et al. (2005) also highlight that what makes the difference in the use of the interactive whiteboard are aspects related to the interactions established in the classroom, which can increase considerably. Levy (2002) emphasizes that this technology has a positive impact on the presentation of information and teaching resources, in the explanation of concepts and ideas and in the students' interaction and activities.

There are also references to contributions of the interactive whiteboard for the teaching, namely, at the management of the class by the teacher. Glover and Miller (2001) emphasize the visual contact that the teacher can keep with the students while exposing the subject and controlling the computer through the board, allowing him to be more attentive to the class events. Studies developed by Ball (2003) and Lerman and Zevenbergen (2007) showed that the use of interactive whiteboard contributes to increase the rhythm of the class.

Many advantages arise from the use of the interactive framework from the teachers' point of view. BECTA (2003) considers that this tool allows the possibilities of further integrating of ICT; registering of notes throughout the class, saving notes and printing or sending them to the students by email; sharing of materials with colleagues; reusing and reducing the work of lesson preparation; and adapting resources created to different age levels and curricular areas. However, according to Greiffenhagen (2000), cited by Lerman and Zevenbergen (2007), the effectiveness of the interactive whiteboard is only fully achieved if it is regularly used.

One key idea comes from this review: the interactive whiteboard is indeed a multipurpose tool for the teacher, but it is actually more than that, and its added value (for instance, in immediately switching between several pages when needed) is an essential point to be brought to the fore.

5.2.2 Dialogic Interaction and the Use of Interactive Whiteboard

Tanner et al. (2005) argue that a dialogue-based teaching methodology requires a powerful interaction with students, which is related to the degree of teachers' technology control, to the type of task presented to students, to the degree of questioning the students and to the culture established in the class to create moments of presentation and sharing of ideas.

This is a sustainable approach in a perspective of inquiry-based teaching, according to which the construction of knowledge is made together, as a result of the dialogic interaction between teacher and students (Mercer et al., 2010). In this process, students play an active role, being encouraged to expose their ideas and explain their reasoning, where particular importance is given to the orchestration of the discussions by the teacher.

Mercer et al. (2010) recognized that adopting a dialogical pedagogy is not easy. It is a demanding practice for teachers, both in preparation of the lessons and during the lessons, while the interactions are occurring, being the teachers' orchestration of the collective discussions especially complex (Stein et al., 2008).

According to Ruthven et al. (2011), teacher interventions should guide the course of the discussion, supporting students in articulating mathematical thinking and helping them relating the concepts with other examples, principles or existing tools. According to Ball (2003), the relationship between teacher and the student in the process of knowledge construction can take advantage of the use of the interactive whiteboard that allows the teacher to focus more on the students' answers, staring them directly, without having to worry about the computer mouse or keyboard to explain or rework procedures.

The student's role is relevant when he/she gets the possibility to interact with the teacher and with other colleagues handling directly the interactive whiteboard, which is a knowledge construction facilitator aspect, although it is not necessary for the student to handle the interactive whiteboard in order to achieve a good level of interaction (Reis, 2014). The teacher is the key element in the management and orchestration of the interactions among students and between them, and the interactive whiteboard and the dynamics of the class strongly depend on the teacher (Stein et al., 2008).

Studies developed by Mercer et al. (2010) revealed some of the specific features of the interactive whiteboard that can be used in the classroom to support the dialogic interaction. They highlight the sharing of ideas and its registration as notes, which can be easily modified, stored or reviewed; the use of options to hide and reveal that allow students to focus their attention in specific parts of one document; and the multimodality, which concerns the possibility of integration of many resources that are readily available in order to be effectively used.

Large group discussions may be supported by the use of audio, visual and written functions, which stimulate the organization and development of the students reasoning, encouraging them to introduce and test their own ideas. The different ideas can be recorded, analysed, stored and revisited, allowing the construction of meanings based both on dialogue and on the interactions between students themselves and between them and the teacher (Reis, 2014).

Another aspect to consider in promoting dialogic interaction is the nature of the mathematical task or question proposed to the students (Beauchamp & Kennewell, 2008). Smith and Higgins (2006) stress the value of open questions defining them as questions that invite a range of acceptable answers. As they say (p. 486), open questions "which invite pupils to use talk to explore understandings, to speculate, hypothesize, reason and evaluate, and to consider a range of possible answers (i.e., to give an open response), are more valuable in helping pupils construct and reconstruct knowledge and understanding". When responding to open tasks, students tend to reveal their knowledge, express their doubts, speculate, set hypotheses and consider a number of possible answers. These tasks, which include the problems, seem to be those who best provide the construction and reconstruction of knowledge. During the discussion of a problem, the stimulation of oral communication and of various types of interactions (teacher-student, student-student and student-class) must always be present in the teacher attitude, as well as the promotion of sharing and confrontation of ideas, resolution strategies and mathematical procedures (NCTM, 2000). In contrast, closed tasks, although being widely used by teachers, elicit short and factual responses of low-level cognitive demand and do not promote student's active participation in the construction of knowledge. In this way, the development of dialogical interaction in the classroom can be more easily and deeply achieved through the use of open tasks, particularly through solving problems (Reis, 2014). However, again, the role of the teacher and the type of communication that he/she promotes in class are essential for modelling the students' learning opportunities (NCTM, 2000), in particular the way in which teachers react to pupils' responses to questions (Smith & Higgins, 2006).

In order to promote the dialogue in the classroom, the dynamics based on practices of inquiry-based teaching and learning are also reinforced (Canavarro et al., 2012). These dynamics, combined with the solution of challenging tasks, potentiate the moments of discussion and debate, both in small groups and in large group (Canavarro, 2011). According to this perspective, the lessons are structured in four phases, which include the launching of the task to the students, solving the task in small groups, large group discussion of solution strategies and finally the systematization of the assembled mathematical learning.

It is interesting to note that when a class is using the interactive whiteboard, it is common to distinguish four phases (Kennewell & Beauchamp, 2007). The first phase of the class is characterized by the realization of lecture theme review, which is conducted by the teacher, and the attention of the class is focused on the interactive whiteboard. The second phase includes the introduction of concepts and the development of skills, involving students in animations, in observations registration and other forms of physical interaction with the interactive whiteboard. The third phase involves the development of students' work in groups and usually does not require the use of the interactive whiteboard. In the last phase, the interactive whiteboard plays the major role to review the central aspects of the learning objectives and the main questions posed by students, mainly led by the teacher.

The class model of inquiry-based teaching and the class pattern with the use of an interactive whiteboard do not match exactly, but both have in common something essential: the emphasis on exploring the interaction of students in collective moments of discussion or appropriation of knowledge.

It must be stated that to conduct this kind of practice, it is extremely important that the teacher owns resources that enables him/her to expedite the students' communication, taking advantage of previously prepared and structuring materials of class activity, getting good-quality written records of students' mathematical productions that may be compared and confronted at any time and enabling opportunities for joint reasoning, discussion and conclusions, including diverse tone voices and ideas from students (Stein et al., 2008).

The interactive whiteboard, given its characteristics, can provide to the teacher the streamline of the various actions that he/she must perform, driven by two simultaneous concerns: on one hand, the managing of the classroom and, on the other, the promotion of the mathematical learning (Canavarro et al., 2012).

In brief, we underline the idea that the interactive whiteboard is a versatile tool with many features that can be used for the promotion of the students' learning. Among these features, some are concerning the written record, others relating to the integration of multiple useful tools to support the mathematical reasoning (multimodality) and others relating to the interactivity provided. The process of teaching and learning will be enriched if both teacher and students can take advantage of this interactivity, particularly in conducting discussions involving the whole class to facilitate a dialogical construction of mathematical knowledge. This kind of practice is of great complexity, because the teacher has to undertake a number of actions that have to do with two distinct but interrelated purposes: manage the class as a whole and promote the learning of the students.

5.3 Methodology

5.3.1 Fundamental Options

The research reported here is based on a case study with a group of students from 8th grade of schooling from a basic school located in a city in an interior region of Portugal. This study arises in the context of a teaching experience held in 15 regular lessons of mathematics, during the second and third school terms. This experience consisted in a sequence of six tasks to be explored in class with the use of the interactive whiteboard and involved both the authors of this article as researchers and the mathematics teacher of the class.

The mathematics teacher has a positive attitude towards the use of ICT in mathematics teaching. She regularly uses the interactive whiteboard with the students but only in what concerns its use as a board for projection. The use of more sophisticated potentialities of the interactive whiteboard is very scarce and happens only in the teaching of specific topics like symmetry, despite the teacher had some courses in the use of technology. For the teacher, this project is an opportunity for her to learn how to use the interactive whiteboard as a tool for promoting interaction and development of the mathematical communication of their students.

The planning of the lessons involved several sessions of joint work between one of the researchers and the class teacher in a collaborative context. In these meetings, the tasks were selected and adapted in order to be implemented in the classroom, and the lesson development was anticipated, including the preparation of resources to use on the interactive whiteboard.

All the tasks were problem situations in nature, but they were diverse: they focused on different curricular topics and had different purposes, from the learning of new mathematical contents to the development of mathematical processes such as problem-solving and communication, providing students with different learning experiences (NCTM, 2000).

These lessons followed an inquiry-based teaching model, stressing the possibility of establishing interactions and class discussions aiming to the collective construction of mathematical knowledge. Although not being a recurrent practice of the teacher, she was able to create a classroom environment in the spirit of dialogical interaction. In general, the structure of the lessons followed four phases: (i) task presentation by the teacher to the students; (ii) task solution by the students organized in small groups; (iii) discussion of task solutions, in large group, with the participation of all the students and the teacher; and (iv) systematization of mathematical learning, conducted by the teacher with the participation of the students.

In this chapter, we focus on two of the problems. Its selection followed two main criteria: they have quite different learning objectives, allowing to observe and to analyse diverse scenarios of the use of the interactive whiteboard in the development of the lesson. Furthermore, they correspond to different moments of the implementation of the sequence, one at the beginning and the other by the end.

The first problem, “Introduction to the systems of two equations in two unknowns”, was inscribed in the curricular topic “Systems of two equations in two unknowns” and corresponded to the first task of the sequence. The teacher’s aim was to introduce the concept of a system of two linear equations in two unknowns, from the interpretation of the problem posed by means of two equations related one to the other. Based on initial teacher guided discussion in which students tried to solve the problem by trial and error, they have come to the conclusion that they did not have sufficient knowledge to solve the problem properly. The method of substitution for solving systems was then presented to the class by the teacher.

The second problem, “The cylindrical candles”, focused on the curricular topic “Geometric solids” and corresponded to the fifth task of the sequence. This problem had the purpose of consolidation of contents taught in previous lessons and, simultaneously, was an opportunity for solving a problem that allows different strategies to be used, requiring the organization of a sequence of steps until the final answer is achieved, fostering students’ creativity and critical sense.

5.3.2 Data Collection and Analysis

The data collection was based on observation and documental analysis (Creswell, 2012). The observation took place on all the lessons of the teaching experience, which were video recorded with proper authorization. The moments involving the large group discussion and the concluding remarks were integrally transcribed. The documental analysis was based on the researcher field notes of the lessons and also on the digital files relative to the use of the interactive whiteboard, as the flipchart relating to the solutions of the problems and other files (for instance, GeoGebra files produced in the context of a task not focused in this paper).

The collected data was analysed with the purpose to understand which interactive whiteboard features were used in the lesson development and which contributions resulted from there for the promotion of dialogical interaction development in class.

The data analysis was processed in two stages. In the first phase, a preliminary analysis of data collected from various sources was conducted, with the objective of identifying the categories of relevant analysis towards the interest of the present study, combined with the literature review. In a second phase, data analysis was performed according to the defined categories.

To analyse the features of the interactive whiteboard, three main categories have been considered. One of them is on the written recording functions, including, for example, the possibility to change the dimensions of the writing and to use colours for highlights, as well as the possibility of recording files and recover them. A second category concerns multimodality that includes the possibility of using different tools integrated in one screen, as it is the case of the calculator. Finally, we consider the category of interactivity, including the possibility of people interaction, students or teacher, with the screen, with immediate feedback, allowing, for example, experimentation and simulation of various situations without any other software or applets.

Regarding the contributions to the promotion of dialogical interaction, we consider two categories: on one hand, the contributions concerning whole-class global management (time, materials or resources to support communication or to promote students' reasoning, resources to manage students' participation, students' attentive attitude, etc.) and, on the other hand, the contributions concerning the promotion of students' learning mediated by dialogic interaction (possibility of considering various contributions, clarification of class questions, collective construction and record of a collective product institutionalized by the class, collective experimentation for obtaining joint conclusions, etc.).

5.4 Presentation of Results

5.4.1 Introduction to the Systems of Two Equations in Two Unknowns

This problem had two specific purposes: introducing the concept of a system of two equations in two unknowns to the students and presenting the substitution method for solving linear systems.

The exploration of this problem took place in two consecutive lessons. In the first one, the teacher presented the problem to the students and promoted a discussion concerning the obtaining of solutions, interacting with the class, so as to show them how to solve it by an efficient method. At the second lesson, the focus was pointed in the systematization of students learning regarding the substitution method for solving systems.

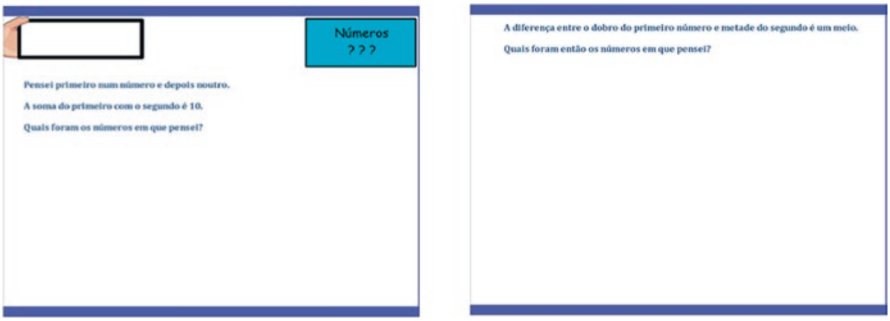


Fig. 5.1 Presentation of the problem *Introduction to the systems of two equations in two unknowns*

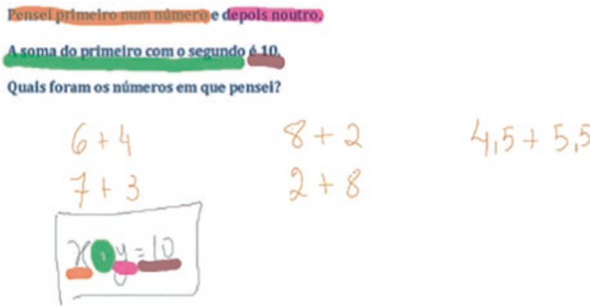


Fig. 5.2 The use of colour transparent markers in the first equation of the problem

During the presentation phase, the teacher used the interactive whiteboard using a previously prepared flipchart with the statement of the problem (Fig. 5.1) [translation from the first screen: "First I thought of a number and then of another one; adding both numbers, I got 10. What are these numbers?"; translation from the second screen: "The difference between the double of the first number and the half of the second one is one half. What are the numbers?"].

The students were informed that the solution of the problem was hidden under the mysterious blue rectangle (Fig. 5.1) [translation: Numbers???], thus motivating their curiosity. The students felt challenged and excited and tried to guess the solutions of the problem, interacting with the teacher that kept on pushing them:

> T: (…) I could think of a quantity of numbers. I've heard some decimal numbers... Negative numbers, could also be. But I can tell you that none of these values are the ones hidden there!

Students offered several possible hypotheses, and the teacher recorded them in colour on the screen (Fig. 5.2), valuing their contributions.

The writing of the problem in mathematical language was the result of the interaction among all. After the two equations were obtained, the teacher, in response to students' questions about the order of the variables, reviewed the writing of equations by taking advantage of the possibilities of using transparent coloured markers:

T: So, let's see. If I have here … Let's see … What did I say? I thought first of a number [highlights that phrase with a transparent orange marker]. This number, which I don't know what it is… You don't know what is the number I thought of! Do you represent it by what?

S1: *x* and *y*.

T: No, no. The first number that I thought of!

S1: *x*.

T: We represent it by *x* [teacher highlights the *x* with the transparent orange marker]. Then… let's change the colour. Then, I thought of another number [highlights that phrase with a transparent pink marker]. This number is represented by what?

S1: By *y*.

T: By *y* [highlights the *y* with the pink marker].

[…]

T: I finally say "it's ten" [changes the colour of the marker to brown and highlights that part with a brown-coloured marker]...

S3: It should be underlined the $x + y$.

T: Yes, you have some reason! What he's saying is that I should have put the $x + y$, this whole part, in green colour. You are not far from being right. It's true, because I'm adding the two, right? But, when I speak of the sum of the first with the second, I mean the sum... What is the symbol that will represent, mathematically, this sum? Is the "plus" sign, right? When I say it is 10, where am I going to get this "is 10" [points to the equation]?

S4: It is the result.

T: The "is" means is equal to ten [highlights "=10" with the brown marker]…

The colour highlights in the interactive board simplified the identification of the variables and the comprehension of their relationship (Figs. 5.2 and 5.3).

The next phase of the lesson consisted on the joint solution of the problem, since the students still lacked mathematical tools to its autonomous solution. The teacher guided the discussion in class, based on teacher-student dialogue, which occurred more frequently, and student-student, in a more occasional basis.

During the explanation of the problem solution method, the teacher felt the need of having more screen space so that it was possible to conclude the solution of the

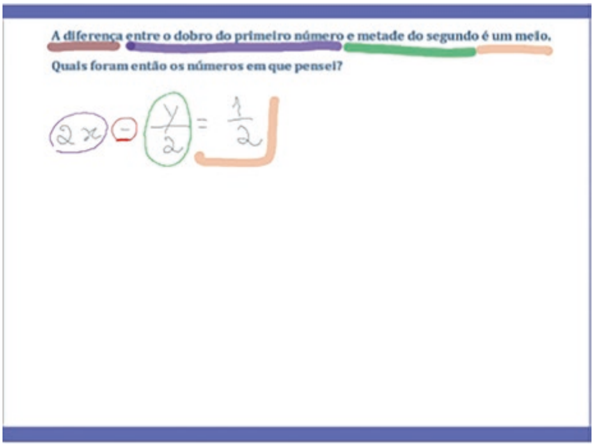


Fig. 5.3 The use of colour transparent markers in the second equation of the problem

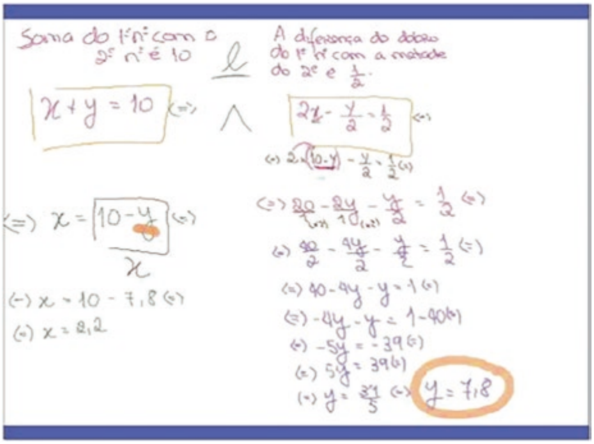


Fig. 5.4 Solution of the problem. *Introduction to the systems of two equations in two unknowns*

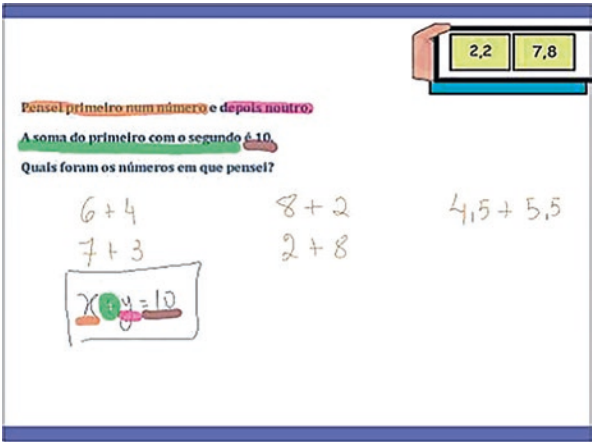


Fig. 5.5 The use of magic ink to reveal hidden objects in the problem. *Introduction to the systems of two equations in two unknowns*

system on the same page of the board. So, she used the functionality of the interactive whiteboard that allows to increase/decrease objects, keeping a good readability (Fig. 5.4) [translation of the sentence above the conjunction: "The sum of the numbers is equal to 10 and the difference between the double of the first number and the half of the second one is one half"]. This also enabled students to observe the complete sequence of steps performed and to have a global perspective of the method – this would only be possible when using a traditional board if prepared in advance and not in a dynamical way as it is possible with the interactive whiteboard.

The use of coloured pens contributed to stress the substitutions involved on the method, enhancing its understanding by the students.

After the solutions were obtained, the teacher revisited the flipchart first page and, using the magic ink, revealed the hidden numbers (Fig. 5.5), provoking a satisfaction and contentment reaction in the students.

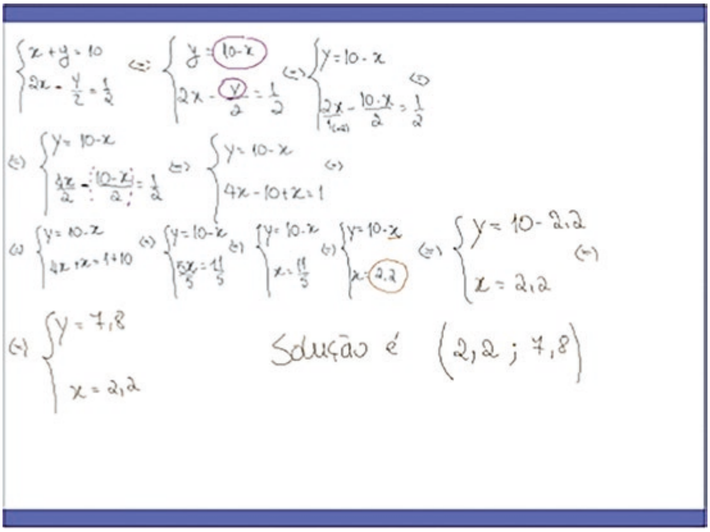


Fig. 5.6 Second solution process of the problem. *Introduction to the systems of two equations in two unknowns*

The teacher reviewed the steps of the system solution with the verbal participation of the students, who continued to express their interest and collaboration. They also suggested a new idea: to solve the first equation for y and verify if there was no difference in the final result. The teacher accepted the idea, again making use of the possibility of performing the problem solution in a single sheet of the interactive board (Fig. 5.6).

It was also important to be able to show the students the two different uses of the same method, which was easily done by the teacher recovering of the previous pages of the interactive whiteboard – this also would not be possible with a traditional board.

Throughout the lesson, it was notorious the teacher's concerns to focus all students on the interactive whiteboard where everything was recorded for later use. This allowed her to not have breaks in lesson rhythm that could arise if, for example, she had to wait for students to record the information on their notebook.

> T: You don't even have to write it right now. You will write some things on your notebook at a proper time. Now, I want your attention here. Can it be?

At the end of lesson, the teacher saved the pdf file containing all the pages written on interactive whiteboard and sent it to students by email.

In the beginning of the next lesson, the teacher proceeded to a synthesis of the substitution method for solving two linear systems, for better systemization of this solution method. For doing this, she recapped the various steps using the file obtained from the previous lesson. The use of this file allowed her to fast recover with accuracy all the previous work, improving the lesson rhythm and the management of time.

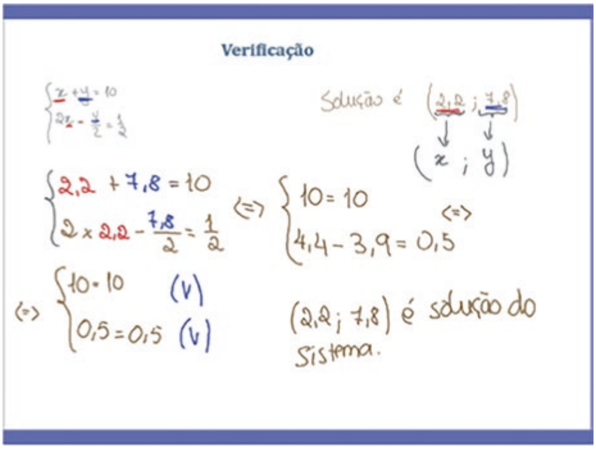


Fig. 5.7 Colour markers in the verification procedure of the solutions of the problem. *Introduction to the systems of two equations in two unknowns*

T: What was done? All we have here is what we did in the last lesson. Let's only review it…

The method of solution of the system was reviewed, and the teacher proceeded to the verification of the solutions, having again used the coloured pens for better identification (Fig. 5.7).

With this approach, the teacher managed to introduce new mathematical content, not in an expository way but in interaction with the students, based on the dialogue that was effectively streamlined by the use of the interactive whiteboard. Although there was no physical interaction of the students with the interactive whiteboard, the ideas presented by the students were discussed, integrated and recorded on the board, and the final product resulted from the dialogic interaction of the class, although fairly conducted by the teacher.

The analysis of the main contributions arising from the use of interactive whiteboard in the exploration of this problem shows its importance.

Both during the presentation of the problem, as throughout its exploration, the use of the interactive whiteboard enabled keeping the students on the same focus and contributed to increasing their motivation and interest. The fact that this board provides an enlarged good image of the content being explained allowed intensifying the involvement of students in the task, preventing them to divert their attention. The joint exploration of the problem on the board contributed to the promotion of interaction between the students, who have shown great willingness to participate.

In particular, during the discussion and class solution of the problem, the process of writing and noting of the answers given by the students allowed the class to achieve a built-together and cumulative product, which is authored by all.

In the solution of this problem, the use of colour allowed the highlight of specific parts, increased the visual stimuli and facilitated the understanding of the problem in general, as well as the specific procedures followed in its solution. As already mentioned, the annotations made on the interactive whiteboard allowed the good and clean visualization in a single page of the whole process of solving the system of equations, providing a better perception of the sequence of all steps to perform through the method.

The use of magic ink to reveal the solution to the problem has encouraged students who expressed satisfaction and contentment reactions.

In the synthesis phase, the reuse of the saved file with the work of the previous lesson enabled the rigorous and fast recap of the ideas of the class and to build on forward, stressing the feeling of collective authoring. It also contributed to a more effective time management.

5.4.2 *The Cylindrical Candles*

The problem "The cylindrical candles" had a double objective: on one hand, it invites students to solve a problem and, on the other, allows the application of mathematical contents taught in previous lessons on areas and volumes of geometric solids.

The work with this problem occurred in two lessons. The first included the presentation and solution phases of the problem and part of the discussion. The second lesson was devoted to conclude the discussion and the final phase of systematization of mathematical learning involved in the problem solution.

In the first phase, students were given a paper sheet with the written problem. The interactive whiteboard was used in addition to provide the problem to the class (Fig. 5.8) [translation: Task – Joana has two cylindrical moulds for candles, mould A

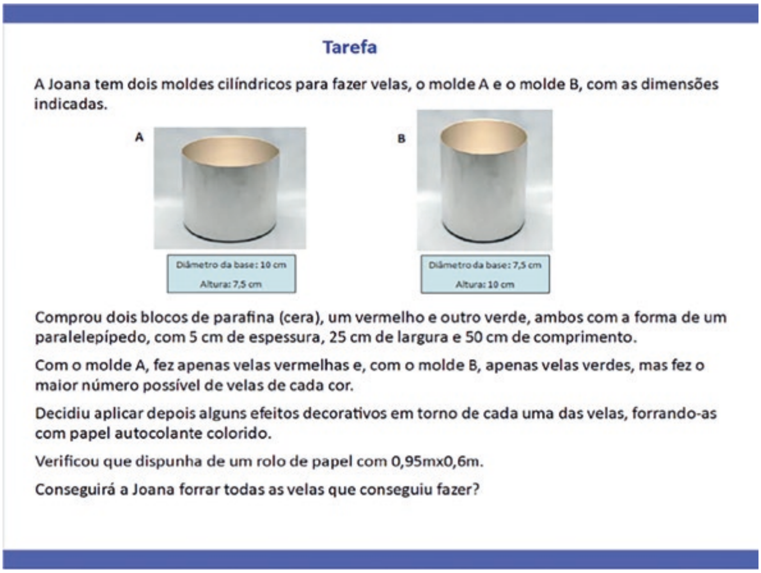


Fig. 5.8 Presentation of the problem. *The cylindrical candles*

and mould B, with different dimensions. Mould A: diameter of the base, 10 cm; height, 7.5 cm; Mould B – diameter of the base, 7.5 cm; height, 10 cm. She bought two blocs of paraffin, one red and one green, both with the shape of a parallelepiped, with 5 cm of height, 25 cm of width and 50 cm of length. With mould A, she did only red candles; with mould B, she did only green candles. In each case, she did the biggest possible number of candles. After, she decided to decorate the candles, rolling up them with decorative self-adhesive paper. The paper roll had dimensions of 0.95 m by 0.60 m. Will Joana have enough paper to roll up all the candles she did?]

During the second phase of the lesson, students sought to solve the problem organized in small groups. The teacher encouraged the autonomy of the groups, pushing the students to explain their ideas and giving them some hints when needed to progress on their reasoning.

The discussion phase of the strategies of solving the problem on the interactive whiteboard started without all the groups had completed the full solution of the problem. The teacher justified this option in order to use the remaining time of the lesson for collective discussion in order to reach a shared solution of the class:

> S1: Teacher, give us a little bit more of time… We haven't finished yet.
>
> T: I know, but everyone thought about the problem. Some groups already have almost everything done and now all together let's see if we can reach some conclusion. John [points to the Group 1], you will present! Move on to next page [of the interactive white board], pick the pen and start.

Having realized that the strategies used by the groups were very similar, the teacher decided to ask each group to contribute to the discussion with a small part of the solution of the problem.

A student of the first group explained on the interactive whiteboard his reasoning to determine the volume of each cylindrical mould, the amount of wax available and the number of candles that it was possible to do, recording all calculations they made.

A question regarding the result correctness of mould volume A created the necessity of verifying the calculations. The presenting student then used the integrated calculator on the interactive whiteboard (Fig. 5.9), clarifying the whole class at once about the correctness of the values.

Fig. 5.9 Use of the interactive whiteboard calculator in the solution of the problem. *The cylindrical candles*

In order to more clarification of the solution process, the information presented on the board was organized using coloured pens. The group of students highlighted in red the calculations relating to the mould A and in green the calculations relating to the mould B (Fig. 5.10).

During this presentation, some doubts arose concerning the default-like approach to the units of the number of red candles. Some students indicated that they could make only 10 candles, while others considered that it was possible to make 11. This was an opportunity for collective discussion about the significance of this approximation. The teacher let the interaction flow among students valuing the dynamics of the class and concluded the discussion by validating the argument of one of the students:

T: Could we fill 11 moulds with that amount of wax?
S1: No. But the 61 [0.61] must be used too.
T: Nobody says that…
[…]
S10: She used the most possible, but didn't use all of it.
P: Exactly. Say it again! Listen to what John said, I liked the phrase.
S10: She used as much wax as possible.
T: She doesn't have to use it all. She has to use as much as she can.

After the presentation of the first group, the next students continued to solve the problem on the same page of the flipchart. To do so, they had to reduce the space occupied by previous records.

This group started by the area determination of the coloured sticker paper. To this end, it was necessary to refer back to the statement of the problem to clarify the reductions made to the values of the paper length measures. So, the group switched quickly between the two flipchart pages, revisiting the statement of the problem. The group also used coloured pens to highlight the value of the paper area available to quilt the candles (Fig. 5.11).

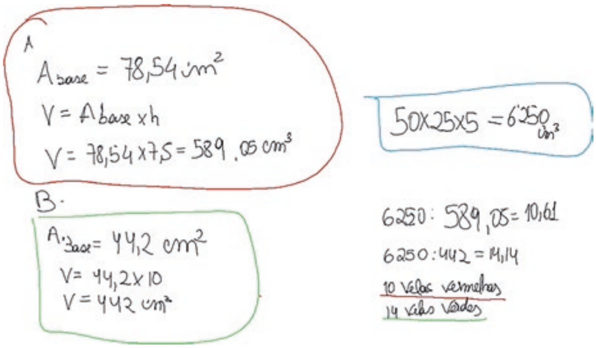


Fig. 5.10 Use of colour ink markers in the organization of the problem resolution

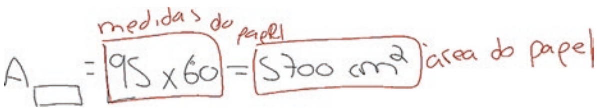


Fig. 5.11 Utilization of colour markers in the available paper area calculations to quilt the candles

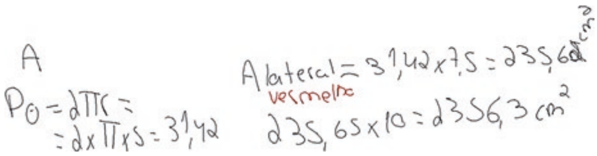


Fig. 5.12 Area determination occupied by the red candles

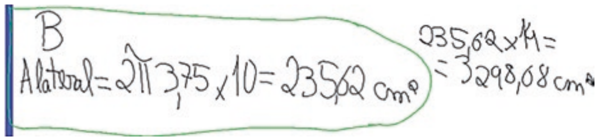


Fig. 5.13 Determination of the area occupied by the green candles

The students of this group calculated also the lateral area of each of the red candles, originated by the mould A, and also the consequent area occupied by those ten red candles (Fig. 5.12).

During the second group intervention on the interactive whiteboard, a low level of interaction between students was noticed, probably because there were no other resolutions to confront relating to this part of the solution.

When the lesson came to the end, the teacher asked students to complete the problem solution at home and saved the pdf file with all the written records.

The next lesson started with the state of the solution made so far, taking advantage of the visualization of the pdf file from the previous lesson.

From there, a third group was called to the board and determined the area occupied by the candles obtained from mould B, annotating the calculations in the same page and highlighting them in green colour (Fig. 5.13).

The discussion of the next step of the solution (whether if it was possible to cover all the candles with the available paper) led to the oral presentation of two distinct strategies, one addressing separately the candles of different colours and another that considered them together, based on the sum of both of the areas.

T: Any ideas? Say it, Joana.

S5 (Group 2): We need to divide 5700 by the result [refers to the area value of the 10 red candles] [...]

T: Tell me, John.

S2 (Group 1): I think we should add the 2356.3 to the 3298.68 and see if it was lower or higher than the value from above [refers to the area of available paper].

[…]

S1: The moulds are not equal…

S7: One is greater than the other.

S2: Okay! But if the 5700 is enough for the two [refers to the values of the areas: 2356.3 and 3298.68] it is a sign that it works.

S3: Oh, yes! I see your point.

T: Let's see. I have a sheet of paper. I want to cover; I could even have different things. Now I have here this piece of paper, let's imagine, and now I want to cut here a rectangle

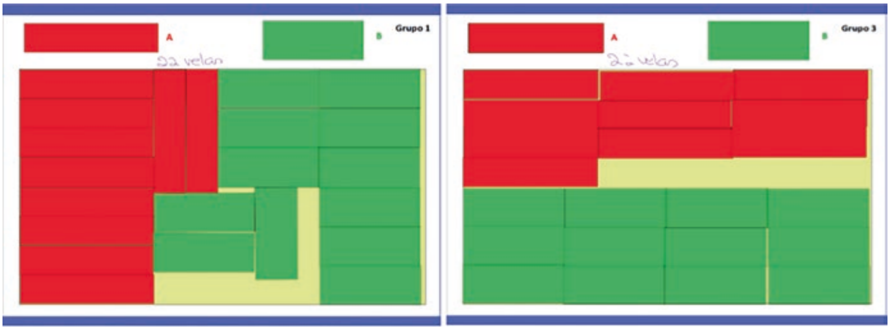


Fig. 5.14 Two simulations for cover the paper with the red mould A and green mould B

and quilt a little thing with a rectangle. So what do I have to check? If the areas fit here or not, right?

S5: But if we divide the area of the paper by the area of the model, we know how many candles we can cover.

T: Do you agree with Joana?

After some discussion, the conclusion was collectively withdrawn, involving the participation of several students, and was recorded on the interactive whiteboard by a student of another group.

T: Can Joana quilt every candle or not?

Several students: Yes.

T: Why?

S4: The value [of the area of all the candles] is lower than the other [refers to the area of paper available].

T: So we have more paper to quilt from than area to quilt. Right?

Several students: Yes.

T: So, John, go to the board and skip the page. Write the conclusion. And what was the conclusion that we took from this problem?

S13: That Joana can quilt....

[…]

S10: Joana can cover all the candles.

T: Then write it!

After this phase, a surprise to the class was reserved. Anticipating the possible difficulties of the students to understand effectively that the paper sticker was not enough to cover all the candles, the teacher introduced them to a new flipchart previously prepared, in which it was possible to simulate the situation of the problem by making use of the interactive whiteboard potential. The simulation was about the distribution of candle moulds through the colourful sticker paper by inserting and manipulating rectangles (Fig. 5.14). This promoted students' interaction with the board, some of them had the possibility of trying. Besides, this simulation allows the use of iconic representations, favouring the overall understanding of the situation and revealing important aspects for the process solution.

In general, the motivation of students increased significantly at this stage, having been retrieved a stronger degree of interactivity. Everyone wanted to make a trial on the interactive whiteboard, and the teacher was essential to control the student's contributions.

Figure 5.14 shows two of the attempts made by the students to cover the yellow rectangle simulating the sticker sheet of paper with red (mould A) and green (mould B) rectangles.

At this stage, the class gained a new life. All the students wanted to participate and gave lots of suggestions to those who were using the interactive whiteboard to manipulate the rectangles. Some commented: "Actually, I think it's not going to fit..."

During this discussion, previous pages were revisited to confirm how many rectangles had been managed to fit on each trial. Many students convinced themselves that it was not possible to cover all the candles and were debating out loud their opinions.

S4: There [points to the interactive whiteboard] is already a bit wasted.

S2: These spaces will always be remaining.

S10: So if this area is larger than the lateral area of the candles… How couldn't it work?!

S2: It's not going to happen! Look at the spaces that are not being used [points to the interactive whiteboard].

S4: The difference is so tiny and because there are some empty spaces left, it's not going to be enough!

S13: There will always be spaces that will not be used.

The teacher moved on to the systematization phase of the lesson when she evaluated that most students were already convinced that it was not possible to cover all the candles, because of the waste of paper. At this stage, the purpose of the problem was recalled; the various pages of the flipchart were reviewed, allowing a better understanding of the solving process. The main conclusions were then synthesized, and the final answer to the problem was recorded on the interactive whiteboard.

T: With this information we have here, do you think we can cover it or not?

[…]

S4: The spaces will not be all covered… It is not going to be enough! There is always paper waste.

[…]

S11: Teacher, with these little spaces, couldn't we put it all together and put them in a single piece?

T: Well, that's the issue here. Bernardo had already said it… If I want to quilt a candle with "patches", even little ones, then, in that case, would it work or not?

Several students: Yes, it can work.

S4: Enough to quilt yes, but it's not a practical task.

T: Then it works.

S13: By the calculations, yes...

T: Exactly! We've been able to confirm by calculations that it will be enough paper. But if you don't use "patches", do you think it fits or not?

Several students: No.

T: So, Bernardo, come here to the board and write down the conclusion.

The analysis of the main contributions arising from the use of the interactive whiteboard in the exploration of this problem shows its importance in two phases of the lesson.

During the discussion phase, the interactive whiteboard was effectively used by students, and no sort of handling embarrassment was detected. The use of the interactive whiteboard calculator has confirmed, in front of the class, the results obtained by the students, concentrating their attention again on the same focus. The use of coloured pens allowed to highlighting parts of the resolution of the problem, providing a better organization of the records made on the board.

Reducing the size of the written records allowed visualizing all the steps of the solution in one page, allowing students visual contact with the whole process.

The possibility of retrieval and visualization of pages previously written contributed for students understanding of the complete solving strategy and for teacher making of summaries and clarification of questions posed by the students.

The distribution "of moulds of candles" on "the coloured sticker paper", by the insertion and manipulation of objects on a page previously prepared for this purpose, took advantage of exploring interactivity associated with the board. The simulation allowed to confirm/disconfirm the first solution found by the class, which was incorrect. In a fast and effective way, students convinced themselves easily that the sticker paper was not sufficient to cover all the candles if full pieces of paper were selected. Their ideas and reasonings were supported or rejected in a great interactivity environment. The strategy of using specific features of the interactive whiteboard to explore this situation was quite rewarding, enabling the agile use of iconic representations. Without the use of the interactive whiteboard, the discussion of these strategies would have been too time-consuming and probably not implemented if a pencil and paper approach was required.

All groups contributed to the collective construction of a product common to the class, which was then made available by email to students. This pdf file was a supplementary support to the records that students made in their daily notebook.

5.5 Conclusions and Final Remarks

Table 5.1 summarizes the main features of the interactive whiteboard used in the exploration of the two problems focused on Sect. 5.4, as well as the contributions resulting from the use of these functionalities for promoting a dialogic interaction in class.

5.5.1 Features of the Interactive Whiteboard Used to Support Dialogic Interaction in Solving Problems

The interactive whiteboard was used by the teacher and also by the students although the last ones had used it only on the second problem, for which they had their own solutions to present. All of the students used it without apparent technical difficulties and took advantages of several of its functionalities.

Table 5.1 Synthesis of the main features of the interactive whiteboard used in the exploration of the two problems and contributions for promoting dialogic interaction

Phases of the lesson	Features used of the interactive whiteboard	Contributions to promote dialogic interaction
Presentation of the problem	Access to a digital resource previously prepared Inclusion of visual special effects (magic ink)	To concentrate students' attention on the same focus, simultaneously To increase students' motivation and willingness for participation
Discussion of strategies for solving the problem	Written record with the use of different colours and kind of pens Decreasing/increasing of written records and objects Effect of hide/show objects Use integrated of calculator Use of iconic representations that can be manipulated interactively Save of pdf files produced in previous lessons Recovery and visualization of pages previously saved	To concentrate students attention on the same focus, simultaneously To increase students' motivation and willingness for participation To obtain organized and well-presented written records To compare easily results and strategies presented by students or teacher To clarify doubts using specific tools To support the interactive experimentation and simulation To support the development of ideas, reasoning and communication of students To increase the interactions among students and teacher and studentsTo build collectively and cumulatively a common class product To save time and increase the rhythm of the lesson To retrieve work done in previous lessons for review, with completeness, accuracy and speed
Systematization of mathematical learning	Recovery and visualization of pages previously saved Written record using different colours and kind of pens	To retrieve work done in previous lessons for review, with completeness, accuracy and speed To build a collective product resulting from class dialogue conducted by the teacher To provide the relevant information recorded to students for posterior use

The analysis of Table 5.1, organized by the lesson phases where the interactive whiteboard was used, points to several conclusions.

In the presentation phase, the interactive whiteboard was used as a screen projection of problem statements, through access to digital resources (flipcharts) previously prepared and, in one case, making use of the functionality of the magic ink that hides or reveals objects.

In the phase of discussion of the strategies of solution of the problems, several features of the interactive whiteboard were used: the recording and annotation of responses and suggestions from students, the use of coloured pens for highlights and organization, the insertion and manipulation of objects and the use of integrated calculator.

In the phase of systematizations of mathematics learnings was used the retrieval and display of previously written pages and used the possibilities of written record in previously saved pages.

Thus, we observed that it was used a wide variety of functionalities. Some of them are related with the written record of quality; others are related to the possibility of interactivity and multimodality. This conclusion is consistent with early studies, in particular from Glover et al. (2005) and Mercer et al. (2010), which relate to many aspects of this study. Moss et al. (2007) put great emphasis on the use of colour and manipulation and resizing objects, two features also widely used in this study.

However, the different features do not seem to be used with the same frequency and intensity in all phases of the lesson in the context of solving problems, being less used in the phases of solution of the problem by the students and of presentation of the problem to the class.

The discussion phase of solutions of the problem that took place in plenary with the class made use of a large number of features of the interactive whiteboard. It was at this phase that the features of multimodality and interactivity proved to be particularly useful. This conclusion is not surprising if we assume that it is in this phase that different ideas of students emerge, as well as questions to clarify and hypotheses to test. The interactive whiteboard had here a fundamental role, in particular in the problem of "The cylindrical candles", where the simulation assisted by iconic representations students could manipulate came with great importance for deeper students understanding of the problem.

On the other side, it is important to note that at the systematization phase, which occurred at the second lesson devoted to the work with each one of the problems, the teacher always resorted to features on the written record, retrieving the files constructed and recorded. This allowed her to quickly, completely and accurately recapitulate the work previously done, reducing the difficulties that arise in time management and in obtaining the attention and concentration of students when the conclusion of the task discussion and synthesis does not occur in the class in which it was initiated (Canavarro, 2011).

5.5.2 Contributions of Interactive Whiteboard Use to Facilitate and Expedite the Dialogic Interaction in the Context of Problem Solving

The interactive whiteboard provided several contributions in promoting dialogic interaction, which was observed both in the lesson where the teacher introduced new knowledge to students and in the lesson in which they solved the problem autonomously. These contributions were revealed in the different phases of the class considered in Table 5.1.

In the problem presentation phase, it allowed to concentrate the students on the same focus, being the interactive whiteboard the spatial reference to look at and the screen where the action unfolded, involving technological innovation. This aroused students' interest and motivation, in particular in the problem where special visual effects were used, stimulating students' curiosity to find the solutions of the problem.

In the discussion phase of the problems, many contributions were taken from the use of the interactive whiteboard. On the one hand, some of the features contributed to increase motivation and interest of the student, to record more organized information and to enhance and highlight colours to specific parts of the solutions of tasks by providing conditions for the construction of a written product with quality.

Other features contributed to support the development of ideas, reasoning and student's communication. We highlight the interaction achieved from the use of simulation of the moulds of candles, which allowed also to exploring effectively new situations. It was a powerful contribution for supporting teacher actions in what concerns promoting students learning.

At the discussion phase and also at the systematization phase, we stress the possibility to easily retrieve the work already carried out in previous lessons, namely, when the discussion of solutions took more than one lesson. Also the effect of increasing the rhythm of the lesson and saving of time was obtained in this phase. Both are powerful contributions for supporting teacher actions in what concerns managing the class as a whole.

All these contributions allow a progressive construction of a product of the class, promoting in the students a feeling of authorship – the collective product evolves in the class and integrates the ideas of those who participate.

Taken together, these contributions have supported quite significantly the development of an inquiry-based teaching of mathematics. They favoured students' tracking of the work done by the class and recorded on the interactive whiteboard and fostered students' attention and willingness of participation in the discussion, interacting with the teacher, with colleagues and with the interactive whiteboard. Thus, this study confirms the results of other investigations (Higgins et al., 2005; Lerman & Zevenbergen, 2007; Levy, 2002), stressing those that enhance the interactive whiteboard as an extra motivating factor for students, increasing their involvement and interactions (BECTA, 2003; Higgins et al., 2005).

Analysing the diverse contributions, we can say that many of them support the actions that the teacher should perform for an efficient conduction of maths lesson. On one hand, the functionalities related to written record of quality were crucial for the management of time and resources. On the other hand, functionalities linked to multimodality and interactivity, they provide conditions that facilitate teacher actions regarding to the promotion of mathematical apprenticeship by the students.

5.5.3 Final Remarks

The interactive whiteboard is a very versatile tool. Its features support increased interaction even in lessons where students do not really manipulate the board by themselves or in lessons where they do not solve tasks for sharing their own solutions with the class. This study showed that the interactive whiteboard is worth full in the context of a class discussion orchestrated by the teacher, in order to present new contents. This scenario can be very rich as a stimulus for the promotion of interactions among all participants – the interaction can be dialogic depending more of teacher intentions and actions than of the technological resource being used in classroom.

But the interactive whiteboard is not a neutral device and brings many contributions to promote dialogic interaction. Of particular interest is its use in the collective discussion of problem solutions, where multimodality and interactivity are simultaneously stimulus and support for promoting student mathematical learning. The features concerning quality written records are valuable in any phase of the lesson, but its use facilitates teacher management of whole class, namely, in the management of lesson time.

This study is a first experience for this teacher and students but still revealed interesting results. A more regular use of the interactive whiteboard would create better conditions for its enhancement, in relation to the exploitation of more open tasks to make space to a deepening of the dialogue. Warwick et al. (2011) point out that teachers who adopt an interactive pedagogy, involving students in their learning through dialogue, are those who are better able to integrate the use of the interactive whiteboard in the service of their educational goals.

References

Ball, B. (2003). Teaching and learning mathematics with an interactive whiteboard. *Association of Teachers of Mathematics (ATM).* [online] [accessed on: 14/01/2013]. Available at: http://smartertechteaching.com/wp-content/uploads/2011/11/research-article.pdf

Beauchamp, G., & Kennewell, S. (2008). The influence of ICT on the interactivity of teaching. *Educational Information Technology, 13*, 305–315. [online] [accessed on: 10/08/2011]. https://doi.org/10.1007/s10639-008-9071-y

BECTA - British Educational Communications and Technology Agency. (2003). *What the research says about interactive whiteboards*. Coventry: British Educational Communications and Technology Agency. [online] [accessed on: 27/12/2010]. Available at: http://partners.becta.org.uk/pagedocuments/research/wtrswhiteboards.pdf

BECTA - British Educational Communications and Technology Agency. (2004). *Embedding ICT @ secondary: use of interactive whiteboards in mathematics*. Coventry: British Educational Communications and Technology Agency. [online] [accessed on: 31/01/2012]. Available at: http://www.drp.nl/digibord//scripts/getfile.php?id=302

Canavarro, A. P. (2011). Ensino exploratório da Matemática: práticas e desafios. *Educação e Matemática, 115*, 11–17.

Canavarro, A., Oliveira, H., & Menezes, L. (2012). Práticas de ensino exploratório da matemática: o caso de Célia. In L. Santos (Ed.), *Investigação em Educação Matemática 2012: Práticas de ensino da Matemática* (pp. 255–266). Portalegre: SPIEM.

Creswell, J. W. (2012). *Educational research*. London: Sage Publications.

Glover, D., & Miller, D. (2001). Running with technology: The pedagogic impact of the large-scale introduction of interactive whiteboards in one secondary school. *Journal of Information Technology for Teacher Education, 10*(3), 257–278. [online] [accessed on: 15/12/2010]. Available at: http://www.informaworld.com/smpp/content~db=all~content=a739086631

Glover, D., Miller, D., & Averis, D. (2005). *Developing pedagogic skills for the use of the interactive whiteboard in mathematics*. Staffordshire: Keele University. [online] [accessed on: 15/12/2010]. Available at: http://www.keele.ac.uk/media/keeleuniversity/fachumsocsci/sclpppp/education/interactivewhiteboard/BERA%20Paper%20Sep%202005.pdf

Higgins, S., Falzon, C., Hall, I., Moseley, D., Smith, F., Smith, H., et al. (2005). *Embedding ICT in the literacy and numeracy strategies*. Newcastle upon Tyne: University of Newcastle upon Tyne. [online] [accessed on: 31/01/2012]. Available at: http://dro.dur.ac.uk/1899/1/1899.pdf?DDD29+ded4ss

Kennewell, S., & Beauchamp, G. (2007). The features of interactive whiteboards and their influence on learning. *Learning, Media and Technology, 32*(3), 227–241. [online] [accessed on: 12/09/2011]. https://doi.org/10.1080/17439880701511073

Lerman, S., & Zevenbergen, R. (2007). Interactive whiteboards as mediating tools for teaching mathematics: Rhetoric or reality? In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 169–176). Seoul: PME. [online] [accessed on: 10/01/2012]. Available at: http://www.emis.de/proceedings/PME31/3/169.pdf

Levy, P. (2002). *Interactive Whiteboards in learning and teaching in two Sheffield schools: a developmental study*. Sheffield: Department of Information Studies, University of Sheffield. [online] [accessed on: 10/01/2012]. Available at: http://dis.shef.ac.uk/eirg/projects/wboards.htm#top

Mercer, N., Hennessy, S., & Warwick, P. (2010). Using interactive whiteboards to orchestrate classroom dialogue. *Technology, Pedagogy and Education, 19*(2), 195–209. [online] [accessed on 06/02/2012]. https://doi.org/10.1080/1475939X.2010.491230

Moss, G., Jewitt, C., Levaãiç, R., Armstrong, V., Cardini, A., & Castle, F. (2007). *The interactive whiteboards, pedagogy and pupil performance evaluation: an evaluation of the schools whiteboard expansion (SWE) project: London Challenge*. London: Institute of Education. [online] [accessed on: 27/12/2010]. Available at: http://www.pgce.soton.ac.uk/ict/NewPGCE/pdfs%20IWBs/The%20interactive%20whiteboard,%20pedagogy%20and%20pupil%20performance%20evaluation.pdf

NCTM-National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Ponte, J. (2005). Gestão curricular em Matemática. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11–34). Lisboa: APM.

Reis, C. (2014). *O quadro interativo e o desenvolvimento da interação dialógica na aula de Matemática* (Dissertação de Mestrado). Universidade de Évora.

Ruthven, K., Hofmann, R., & Mercer, N. (2011). A dialogic approach to plenary problem synthesis. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81–88). Ankara: PME.

Smith, H., & Higgins, S. (2006). Opening classroom interaction: The importance of feedback. *Cambridge Journal of Education, 36*(4), 485–502. [online] [accessed on: 12/09/2011]. https://doi.org/10.1080/03057640601048357

Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313–340.

Tanner, H., Jones, S., Kennewell, S., & Beauchamp, G. (2005). Interactive whole class teaching and interactive whiteboards. *Proceedings of the Conference of Mathematics Education Research*

Group of Australasia - 28 (Melbourne), 2, 720–727. [online] [accessed on: 10/01/2012]. Available at: http://www.merga.net.au/documents/RP832005.pdf

Torres, J. (2008). Quadros Interactivos (QI): a última tecnologia a chegar à escola. *Educação e Matemática, 97*, 42–43.

Warwick, P., Hennessy, S., & Mercer, N. (2011). Promoting teacher and school development through co-enquiry: Developing interactive whiteboard use in a 'dialogic classroom'. *Teachers and Teaching, 17*(3), 303–324. [online] [accessed on: 06/02/2012]. https://doi.org/10.1080/13540602.2011.554704

Chapter 6
Learning Scenarios with Robots Leading to Problem-Solving and Mathematics Learning

Elsa Fernandes, Paula Lopes, and Sónia Martins

6.1 Introduction

Children are growing up in a world where technology surrounds them and encompasses most of the aspects of their lives. This has an impact on a new generation of learners who are entering our educational institutions having technology as an integral part of their everyday life. Those learners are imbued with particular forms of talk, cultural history and social relations. All this transforms the way they learn. Thus, creating technological learning environments in the classroom seems to be important and needed. But it challenges the traditional process of teaching and learning.

We have therefore decided to challenge the teaching and learning of mathematics and informatics within the project DROIDE II – Robots in Mathematics and Informatics Education – that aimed to understand how the use of robots as mediating artefacts of learning contributes to meaning production and to the learning of mathematics and informatics topics and contents, by exploring possible articulations between both domains. This research problem, herein defined in general manner, was dissected into two research questions aimed at guiding our research work. (1) How do students learn mathematics and/or informatics, when robots are mediators of learning? (a) What is the shared repertoire that students build in those practices and what are its particular features? (b) Which kinds of contradictions arise from the use of robots on those learning environments? (c) How do such contradictions lead to new forms of activity? (d) How can the work with robots contribute to the development of mathematics and informatics competences? (2) What contributions to the learning of mathematics and/or informatics emerge from the participation in social digital environments? (a) How do young people make explicit and

E. Fernandes (✉) · P. Lopes · S. Martins
Faculdade de Ciências Exatas e da Engenharia, Universidade da Madeira, Madeira, Portugal
e-mail: elsaf@staff.uma.pt

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_6

communicate their ways of doing and thinking on those environments? (b) How do young people critically but constructively participate on those environments? (c) How are young people aware of their own responsibility and initiative that their participation entails?

DROIDE II has adopted a strategy that brings the theoretical field and the empirical field of research into dialogue, in the course of four phases: (1) creating problems in mathematics and informatics education to be solved with robots, (2) creating learning scenarios using robots in which the created problems will be included, (3) analysing the practice of the learners when solving problems using robots and (4) developing a set of guidelines about using robots as an educational tool.

We assumed that robots would be used to discuss mathematical concepts and to solve mathematical problems, but also that students using robots would be dealing and generate mathematical concepts and problems. We conceptualise learning as a social phenomenon, and our theoretical background is based on three approaches to learning: learning as participation (Lave & Wenger, 1991), learning as transformation (Engeström, 2001) and learning as dialogic action (Alrø & Skovsmose, 2004).

Due to the nature of our research problem, the phenomenon under study – learning – and our positioning towards learning, we designed a qualitative research by adopting an interpretive approach (Savenye & Robinson, 2004).

Data collection for analyses and evidence production was performed through (a) the researchers' participation in learning environments, in which robots had a fundamental role (namely, school contexts – mathematics classroom, in grades 1–9, and informatics, in grades 10–12; project area (a cross-disciplinary area taking place in middle school) – and other non-curricular and virtual learning environments), and (b) informal interviews to the participants. Field work was planned to be across a 6-month period and took place through the immersion of subgroups of researchers in the field. Work sessions as well as the participants' interviews were audio and/or video recorded.

In this chapter, we intend to explore and discuss the role of robots as mediating artefacts of mathematics learning and its effect on the reconstruction of the students' foregrounds regarding the learning of mathematics. We will explore the role of robots in the learning of mathematics at a micro-level of analysis, by bringing students' perspectives forward, and also at a macro-level of analysis, by bringing sociocultural and political aspects that frame students' lives and can sometimes hinder mathematics learning, using insights coming from the empirical field in two of the learning scenarios of the project DROIDE II: 'Making a Movie' and 'Robot Race'.

As the design and implementation of learning scenarios had a fundamental role in this project, in the following section we will discuss the concept of learning scenarios and arguments for their use. Afterwards we will briefly present the six learning scenarios designed in the project DROIDE II. Two of them will be further and deeper described as they will be used later in the chapter as a support to discuss learning with robots.

6.2 Learning Scenarios

Learning scenarios constitute resources that people use to modify or transform their previous ideas about something (Matos, 2013). These are not projections or plans for future actions, but they contain structural elements that shape people's learning trajectories.

Our positioning about learning led us to the concept of learning scenarios because this resource proved to be a good way of addressing and articulating several components of learning situations, by imagining different settings constituted by different actors.

6.2.1 What Are Learning Scenarios and Why to Use It?

During times of rapid change or complexity, existing ways of thinking are often based on rationales that are no longer valid and that prevent us from seeing new relationships. Scenarios are useful tools when complexity and uncertainty are high. Complexity and uncertainty are typical of many situations and, for that reason, anticipating changes through creative processes, such as formulating scenarios, becomes helpful.

The concept of learning scenario is a prospective concept used when we intend to introduce changes in a certain context (Carroll, 1999). A learning scenario tells a story of how various elements might interact under certain conditions:

> Scenarios are stories of what might be. Unlike projections, scenarios do not necessarily portray what we expect the future to actually look like. Instead scenarios aim to stimulate creative ways of thinking that help people break out of established ways of looking at situations and planning their action. (Wollenberg, Edmunds, & Buck, 2000, p. 2).

Learning scenarios are stories of persons in activity and because people are in action, they learn. The fundamental is in the action and interaction between people and not in the resources used. A learning scenario is a hypothetical situation of teaching and learning composed of a set of essential elements: a setting in which learning occurs (in which people are included); the knowledge domain in which the scenario may be situated (including multi- and transdisciplinary domains); the roles played by different agents, shaped by their goals; the story establishing the conditions for the development of the scenario, including sequences of events and creating a coordinated structure that constitute an activity. The learning scenario should also predict an outcome and/or products.

Learning scenarios should not be something that teachers design for the students. They are joint constructions of the different actors involved – such as teachers, students, researchers, etc. They should be configured in a dynamic process of creation, experimentation and reflection and conceived as something 'under construction' since they should be modified while they are being implemented, according to the teachers' and students' reflections and their evaluation of the work being done, in trying to meet the needs of the different actors involved.

In project DROIDE II, learning scenarios were a powerful way of enabling researchers, teachers and students to jointly engage in creative learning contexts. According to our positioning towards learning – as a social phenomenon – learning scenarios were used as a tool for engaging different sets of actors, understanding their motives and preferences, encouraging the use of different kinds of tools and determining equitable arrangements among actors where abilities, preferences and power relations differ. Our experience and also our learning through the project development led us to gradually design learning scenarios that were less and less structured and also with increasing complexity and involving more interdisciplinarity.

6.2.2 Scenarios Designed and Implemented in the Project DROIDE II[1]

A possible trip This was the first learning scenario we created and possibly the one with the most tight and straightforward academic structure. We created a worksheet about the 'notion of function', which aimed for students in seventh grade to understand, learn and define the concept of function, by working with robots as they followed the questions of the worksheet. The innovation was just the inclusion of the robots for students to think about the mathematical concepts involved.

The worksheet proposed that students think about two robot trips represented by two graphics provided. First, we intended that the students analyse both graphics and make a description of the robot trip having the starting point as a reference. After, we intended that students would programme the robot to perform the trips, if possible. One of the graphics represented a relation between time and distance that was a function and the other represented a relation that was not a function.

Students also solved a worksheet about proportionality as a function. Through the comparison of the velocities of two robots, they redefined their notion of proportionality.

Introducing robots in the school mathematics class exposed a dynamic link between the work with those artefacts and the way students thought about the notion of function (Fernandes, 2012).

Making a movie We designed a learning scenario involving two primary school classes working together with robots. The learning scenario was developed in two moments. In a first moment, students built robots and defined their physical and emotional features. Their creations become characters in a play-story written by all of them. After writing the story, students programmed their robots in order to

[1] More information about each learning scenario can be found at http://www.cee.uma.pt/droide2/cenarios/index.htm and at http://www.cee.uma.pt/droide2/ebook

perform their roles in the play; in a second moment, students, teachers and researchers decided to produce a film, using the story written as a storyline. Students established new tasks to produce the film, and they created teams to accomplish those tasks. Each student chose which team(s) they wanted to work in (Martins, 2013a).

The use of robots in this learning scenario contributed to the emergence of mathematical and other concepts. Robots were a powerful tool for students to perceive, use, expand on and talk about mathematical concepts. Despite the contributions that robots brought to this learning scenario, we cannot disregard the working methodology encouraged. Our positioning towards learning – as participation in social practices – led to a methodology with characteristics that potentiate the learning scenario (Martins & Fernandes, 2015).

Robot race To work with NXT robots, students from an eighth-grade class received assembling kits and had the opportunity to build, in groups, a car out of Lego bricks, by following instructions. Each working group also created a prototype of a racing route. In the whole-class group, the students chose the racing route to be used and then built it in real size. Afterwards, the robot races were held. Each working group collected data and worked on them in order to decide how to elect the winning robot. With the data collected from the races, each group made a statistical study where conclusions were provided and generalisations were established. The statistical content matter had not been studied before the implementation of this learning scenario; rather, those emerged from the robot race development (Lopes, 2012). Having provided a relevant context – Robot Race – was important to promote knowledge and awareness, among students, about the relevance of statistical data. With this project, students have become more able to solve problems and to understand, to interpret, to analyse, to relate, to compare and to summarise data, thus developing statistical competence and citizenship attitudes (Lopes & Fernandes, 2015).

Virtual droide Three groups comprised of three or four students (each one from a different part of Portugal – Lisbon, Porto and Funchal) that did not know each other before the scenario implementation built a robot (collaboratively) that could solve eight problems proposed by the research team. Each group had a tutor (an informatics teacher) working with them on weekly meetings. Each student had a robot, but we imposed the condition that they had to build a robot that could solve all the problems. This condition was central for youngsters to work collaboratively (Santos, 2012).

Robot guide dog for the blind girl Ema was a blind girl. The informatics teacher proposed, to their pupils of grade 12, in a cross-disciplinary area called project area, to build a robot guide dog for the blind girl. Pupils were organised in working groups. Jointly, they defined four important tasks in order to make the project: (1) to search for the features of a guide dog in order to define robots' functions, (2) to decide how the robot has to be in order to execute predefined functions, (3) to programme the robot and (4) to create an arena in which they can test the robot (Abrantes & Matos, 2012).

A journey to the centre of the Earth The activities were developed with students of ninth grade working in groups of four or five elements. During the classes, there were moments of discussion in large and small groups, and in order to assess the goals' achievement, we monitored discussions in the working groups as well as those that occurred between students of different groups.

First, students watched the movie trailer *The Core* in which they were challenged to 'save the planet Earth'. The challenge appears written as subtitle. Then they built a ship prototype, to reach the centre of the Earth in order to detonate a bomb of great intensity in the core to reactivate it. Several bombs with less intensity should be dropped close to the paths that were open during the journey. Initially the working groups had to build the ship prototype (Phase 1) and then test the programming (Phase 2) to become familiar with the programming environment. During the classes, different groups were working on different phases of scenario's implementation, according to their own rhythms.

The teachers helped the students when they were testing the prototype programming by setting small challenges. Using a small two-dimensional scheme of the centre of the Earth, students projected a round journey to the Earth's core. At the end, students prepared a written report requested by 'NASA'. With this report, we intended that students could achieve the meaning of trigonometric ratios as relationships between the lengths of the sides in similar right triangles (Fernandes, 2013c).

6.3 Our Positioning Towards Learning

For thousands of years, children have learned the lessons of their community by participating with parents and others in important activities to their lives and the lives of their families. Learning occurred while the children were trying to be part of the ongoing activities of their families – be they agriculture, weaving, fishing, trade, understanding of spiritual narratives, treatment of diseases or discussion of moral principles (Rogoff, Turkanis, & Bartlett, 2001). The children were in the same scenario that adults were and realised the importance of learning the skills needed for survival. From this point of view, learning is conceived as part of identity construction. It is connected and embedded with the social and cultural setting in which it occurs. Slowly and steadily the idea of school and the schools themselves as institutions appeared. Today, for many, it is impossible to imagine that children can learn without being taught (in the traditional sense) and also to imagine instruction that is organised in other ways than what is common in schools (Fernandes, 2004).

Associated with this idea, learning is seen as a process in which the individual moves from one phase of not understanding to one stage of understanding a topic or subject; and it is also assumed that everybody learns in the same way. This conception of learning determines a style and an educational process. 'A number of authors have challenged that view and proposed new forms of understanding how people learn. In particular, authors from a situated learning perspective claim that learning is intimately linked with participation in social practices' (Matos, 2010, p. 42).

Whatever the social theory we adopted to think about learning, it has always an underlying way of looking at the person, the social world and the relationship between the person and the world. In project DROIDE II, (i) the person is seen as the agent in the world; (ii) we assume the relation between the agent and the world as a dialectic one (Fernandes & Santos, 2013):

> People, even when looked at in their individuality, are considered in relation to the social practices in which they act. As participants in social practices, they participate in the social and institutional world, which is inherently collective. Subjects, practices, and the social world (in which culture and knowledge are embedded, but also artifacts, meanings, and rules) are perceived as constituting one another and are therefore codependent. (p. 2).

The way learning is conceptualised in the context of this project connects to three metaphors that complement each other – participation, transformation and dialogic action – as discussed in the project LEARN (Matos, 2010).

First, we take learning as participation, recognising it as an integrative part of generative social practices and considering it 'located in the social co-participation processes and not in the head/mind of the people' (Santos, 2004, p. 43). Knowledge, identities and communities are constructed and reified in social practices being that people, who are participating in them, learn. But because, in social practices, there are people in transformation (learning), then collectives, communities (of practice) and organisations transform themselves, or 'learn' (Fernandes & Santos, 2013).

Furthermore, we take learning as transformation, which is inherently linked to the idea of activity, and therefore we may speak of 'learning activity'. But we can only understand the meaning and significance that learning activity takes when we consider it as framed by a system (the activity system) that represents the relationships established between the subjects and the social world (Engeström, 2001).

Finally, we envision learning as dialogic action, which implies participating in dialogues with a purpose that usually stems from motives that lead people to participate in a certain practice, even if the practices in which people participate have not been organised so as to meet those motives. However, the ways in which people participate in social practices hold a strong relationship with the motives and dispositions (those being the resources of the intentions-in-learning (Alrø & Skovsmose, 2004)). Learning involves being in a dialectic process that demands, from the learner, intentionality, reflection and critique, that is, acting dialogically with the world (artefacts, meanings, etc.) and with others.

6.4 Artefacts of Mediation on the Learning of Mathematics

Every theory of learning that assumes the situated nature of learning embodies the notion of participation. Participating does not only refer to events of local engagement in certain forms of activities or with certain types of people but to a wider process of being an active participant on the practices of social communities

(Wenger, 1998). Participation makes us not only what we are but also who we are and the way we look and interpret what we do. It also shapes the communities in which we participate; in fact, our ability or lack of it to shape our communities of practice is an important aspect of our participating experience.

Participation in a social practice implies constant negotiation. To negotiate a shared enterprise entails responsibility from the parts involved. These relations include what matters and what does not, what is important and what is not, what to do and what not to do, what parts need attention and what to ignore, what to say and what not to say, what should be justified and what to presume justified and what to show and what to conceal. It also include to understand when actions and artefacts are good enough and when they still need improvement or refining (Wenger, 1998), given that 'their usefulness is not revealed in the characteristic identified independently of the use in the practices where they are put in action'. (Santos & Matos, 2008, p. 201).

Understanding learning, especially mathematics learning, involves understanding how the use of artefacts in mathematics classes 'impacts on the learners' conceptions of mathematical objects encountered through the use of such artefacts' (Jones, 2000, p. 1). Learning mathematics is a process of people becoming more capable of participating, and it is also a social practice which encompasses the relations between people, knowing and artefacts. Boaler and Greeno (2000) consider knowing and understanding mathematics as aspects resulting from participation on social practices, in particular, those in which the individuals engage themselves on sense-making and solving problems using mathematics representations, concepts and methods as resources. Throughout this process, multiple moments of negotiation take place. Those moments of negotiation that take place in mathematics classes shape the practice of school mathematics, affecting participants and their way of participating.

To talk about participation also requires talking about reification. Wenger (1998) uses the term reification more generally to talk about the process of shaping our experience by producing objects that congeal this experience into 'thingness'. In doing that, we create focus points around which the negotiation of meaning becomes organised. According to Matos (2010, p. 48):

> [t]he key issue is that negotiation of meaning occurs in the convergence of the two: reification calls for transformation of experience, for its commodification; reification produces projections of meaning in the social world giving them a dimension of existence frequently perceived as independent.

Writing down a law or producing a tool is a similar process. A certain understanding is given a form. 'This form then becomes a focus for the negotiation of meaning, as people use the law to argue a point, use the procedure to know what to do, or use the tool to perform an action' (Wenger, 1998, p. 59). The reification process is central in any practice. In every practice, there are 'abstractions, tools, symbols, stories, terms and concepts that reify something of this practice in a congealed form' (Wenger, 1998, p. 59). What usually happens in mathematics classes is that the concepts and procedures are presented to students in a way that reification it is not tangible for them. They do not participate in the process of giving meaning and significance to those concepts and procedures.

With the term reification, Wenger (1998) intends to cover a wide variety of processes that include making, designing, representing, naming, encoding and describing, as well as perceiving, interpreting, using, reusing, decoding and recasting. Reification shapes our experience. Having a tool to perform an activity changes the nature of the activity. The word processor reifies our view of the activity of writing but also changes the way we position ourselves in relation to writing, in the sense that we pay attention to different aspects of those we pay attention when we write by hand. Incorporating the word processor as a mediational instrument in the activity of writing had the power to change the activity. Furthermore, the artefact of mediation is also transformed during the activity and carries in it a particular culture, that is, the traces of a historical activity. This transformation does not have to be physical; it may be conceptual too, which is a consequence of the way the subject acts with the artefact (Martins, 2016).

6.4.1 Making a Movie

For the purpose of illustrating and discussing the role of robots as mediating artefacts in learning as participation, we will bring the learning scenario – 'Making a Movie', from Martins (2013b).

This learning scenario was developed with two primary school classes – grade 2 and grade 3 – from the same school. Students from both classes worked together throughout all the learning scenario implementation.

At the beginning, the research team presented to both teachers a draft of the learning scenario to be implemented. That initial draft was discussed and modified several times according to ideas presented by the teachers and students. In this process, students have made options, which were very important for them and for the success of the project. The learning scenario was constructed by the research team, by the teachers from both classes and by their students. Parents have been informed about the project and driven by the enthusiasm of their children, which in turn, fomented enthusiasm in the parents. Between some of the working sessions, the teachers have often contacted the researchers to report on students' opinions and expectations.

In this learning scenario, students, teachers and researchers had a main purpose – Making a Movie with robots – and to achieve it, many actions were developed. In those actions all the participants chose what was important to do, what physical and conceptual artefacts were to be used and also the way each one of the actors could actively participate in a meaningful and effective way. The decisions and strategies were jointly discussed.

This learning scenario was implemented in a less conventional setting, and we tried to make a bridge between the work on the project with robots and the work that teachers and students had developed in regular classes. The learning scenario was developed in two time frames: the first, between May and June 2011 (students in grade 2 and grade 3), and the second, between April and July 2012 (students in grade 3 and grade 4). The scenario's activities followed a project work methodology. In this project, students worked with the Lego robots RCX and NXT. In both RCX and NXT models, the programming environment is a very intuitive icon-based drag-and-drop programming language, designed for an easy introduction to programming. By choosing programme blocks that work with the motors and make the sensors react to inputs, students simply build up their programme, block by block, and they could create programmes that range from simple to complex. The students and the teachers had never worked with robots before.

In the beginning of the project, students played a game to choose the working teams. Those teams were composed of students from both classes. At different stages of the scenario's implementation, those teams have changed because students were involved in different actions, and they chose what they wanted to do and with whom they wanted to work. The teachers had to support students in their work, and the researchers sought to support students and teachers alike in taking advantage of situations that could contribute to the emergence of mathematical concepts. Based on that intention, researchers assumed a questioning attitude towards students' work in their practice with robots.

In the first phase of the scenario's implementation, students had to construct different robots (e.g. a dog, a spider, a bug, a football player) and define their physical and emotional features (e.g. whether the robot was strong or weak, whether it was sad or happy). Their creations would become characters in a play-story written by all of them. After writing the story, the students had to programme their robots so that they performed their roles in the play. The initial goal was to accomplish those tasks and make the robot characters for the play. The play was not produced in this first phase of the scenario's implementation.

In the second phase of the scenario's implementation, students, teachers and researchers decided to produce a film using the story written as its storyline. Students established new tasks to produce the film, and they created teams to accomplish those tasks. Each student chose in which team(s) she/he wanted to work.

Two teams were created to programme the robots, one team for each of the robot models. The voices team was constituted by ten students who gave voices to the ten constructed robots. This team recorded the voices using the Microsoft Audio Recorder and chose the film's soundtrack. The film was edited using the Windows Live Movie Maker by the editing team. Some students were responsible for the filming, and others were in charge of the lights. Based on the story previously written, the direction team wrote the script for the film, and this team was also responsible for making the communication between all the involved teams.

According to the created storyboard, students from both classes decided to build the physical 'scenarios' needed for filming the movie within the arts classes.

All students decided that the grade 3 class would construct the 'floor' in which filming would be made and the grade 4 class would be responsible for the 'vertical' scenarios. In order to accomplish those tasks, students explained to the arts teachers from both classes the way the robots would perform the scenes in the film, the way they would move on the filming scenario and which environments would be necessary to create (a restaurant, a park, a castle, etc.). A researcher followed and participated in the arts classes.

In the second moment of the scenario's implementation, when students were filming, they wanted to paint routes to define the robots' trajectories. This intention emerged when grade 3 students were working in the arts classes, creating the 'floor' for the filming.

As it was found to be interesting that, in different moments of the filming, the streets' position could change, the researcher suggested the use of black cardboard strips representing the streets in which the robots would move. Thus, the streets' position could easily be changed, which obtained the agreement of the students and arts teachers.

The researcher prepared the black cardboard strips, and by observing how students positioned the strips on the floor, she considered that this should be a good moment to bring the relative position of lines into discussion. Hence, it opened the space, in this learning scenario, to the negotiation of the mathematical meanings related to this topic. The researcher knew that the grade 3 students had not yet studied the relative position of straight lines in their regular classes and that grade 4 students had already studied it. Supported by our theoretical framework, we believed that using the cardboard strips to discuss robots' trajectories would be fruitful to grade 3 students for negotiating the mathematical meaning of the straight lines' positions and to grade 4 students to renegotiate it in this specific practice. This turned out to be another good opportunity to reinforce the bridge between what students were learning in their regular classes and the learning that was occurring with the scenario's implementation.

6.4.2 Robots as Mediating Artefacts of the Concepts of Relative Position of Two Lines

We will next analyse how the mathematical meanings were negotiated (Wenger, 1998) in the practice developed within the learning scenario's implementation and how the robot, as a mediating artefact of learning (Engeström, 1987, 1999a, 2001; Vygotsky, 1978), shaped the way students appropriated those concepts.

The transcription below will support the discussion on the role of the robot on the negotiation of mathematical meaning.

> Res: The way that you want to put the streets on the 'floor' seems to be related to what you want the robots to do. Let's imagine, for instance, that I have two robots in distinct streets and that I do not want them to meet each other… How would you place the streets, in this case?
>
> Ine_3: We could place it like this. [The student placed the black cardboard strips on the floor, representing two parallel lines]
>
> Res: Very well. Do you think that our situation is solved?
>
> Mat_3: The robots, moving like that, in those streets, they will never find each other…
>
> Fran_4: Yes, in oblique streets those robots will never find each other…
>
> Res: In oblique streets? What do you mean by that?

Fran_4: Oblique streets are streets that never met [Putting his arms in parallel position].

Mar_4: Those streets are parallel.

Fran_4: I'm confused again. I did it correctly a few days ago, and I'm already confused again...

Mar_4: The oblique streets are others.

Res: And how are they?

Fran_4: Oblique are two lines that intersect, and all angles between the lines are 90°.

Res: Do the angles measure 90°? [Students started whispering without ideas to offer.] Yes or no? [Students continued whispering]

Fran_4: I'm confused all over again.

Grade 3 Teacher: My students haven't learnt the angles yet… [Talking to the researcher]

Res: But without talking about the angles, we can simply use a cardboard square to support our discussion. [The researcher uses a cardboard square and asks the students.] What can we say about these sides of the square? [Pointing at two parallel sides]

Several Students: Parallel.

Res: And the other two? [Pointing to perpendicular sides of the square]

Bea_3: These are not parallel. If the robots were there, they will surely find each other.

Jes_3: But the robots could both be moving on the streets in the perpendicular position, and they might miss each other. Possibly, one was moving faster than the other.

[The deliberation continued, and students concluded that perpendicular lines are oblique lines that have 'perfect corners' in their intersection.]

In analysing the previous discussion, we can see that there was a strong connection between the way the streets (representing lines) were placed on the floor and the way the students claimed that robots would be programmed to travel across them. The meaning of parallel lines was an outcome negotiated from the fact that students did not want certain robots to meet each other. Students concluded that the best way to ensure that two robots would not find each other was to put them moving in parallel streets. The robot, although only conceptually present (through the streets), was the mediator in the negotiation of this meaning.

As we can see in the transcription above, the discussion about the lines' position (parallel, oblique and perpendicular) was supported by the use of robots. By analysing its trajectories, grade 3 students expanded their knowledge about positions of lines and started using a mathematical vocabulary that was subsequently shared by all students that were working on the project. On the other hand, students from grade 4 needed to make those contents clear to others, which allowed that such specific topics also became apparent to themselves (Martins & Fernandes, 2015).

Students also realised that it would be possible for two robots moving in two intersecting streets (lines) not meeting each other. This question was raised by 'Jes': 'But the robots could both be moving on the streets in the perpendicular position and they might miss each other. Possibly, one was moving faster than the other'. Jes' finding is probably not completely disconnected from his actions taken in the robots' programming team. The knowledge that 'Jes' had about the effect of the robots' programming in its movement was brought by him to support his argument. In robots' programming, students used blocks to define robots' actions such as moving forwards or backwards for a specific period of time, in seconds. They also established the power of the robots' motors to produce those movements. By doing that, they were programming robots to move themselves in different velocities. The students of the programming teams were responsible for programming all the robots. Those students therefore knew how robots could move and interact with each other.

Despite the fact that the focus of this article is not on the robots' programming, we must highlight that the students' actions in that domain was a very productive field for the negotiation of the meaning of mathematical concepts such as positioning, orientation, duration, trajectories, direction and movement. Dealing with robots was a powerful opportunity for students to engage in solving mathematical problems in a very specific context. By programming robots, students learn to predict how a robot would move from one place to another by establishing time and directions for that action. By manipulating robots, students had several experiences that gradually allowed them to negotiate the meaning of time and space. Robots transformed the students' conceptions about those mathematical contents. Robots were physically present, and they were a tool for students to perceive and to expand on those concepts. But, as stated above, robots were a structural resource for students' learning even when they were not physically present. The robot has transformed the way students acted, reasoned and communicated using mathematical contents.

Based on the proposal by Engeström (1999b), the mediating nature of mathematics learning in this activity can be presented in a triangular form (Fig. 6.1).

The robot, while being a mediating artefact, is much more than something being used between the student and the mathematical content (the position of lines and/or temporal and spatial notions). In fact, as the episode emphasises, in that interaction the mathematical contents appear, in a unique and concrete way, in the actions developed by the students with those artefacts. In this sense, the data led us to claim that when students used robots in this learning scenario to think about mathematical contents and properties, their activity was mediated in a particular way by those artefacts (Wertsch, 1991). The artefacts reflect joint stories of learning and the member's engagement in the practice (Wenger, 1998). In this learning scenario, robots represent reifying elements that reflect, in a particular way, the students, the teachers and the researcher's sense of belonging (Martins & Fernandes, 2012) and their unique stories of participation in that practice.

The robot was also a mediating artefact between the several activities in the students' school practice. That being said, many concepts (mathematical or others) developed in the classroom activity of all the classes (language, mathematics, natural sciences, technology, arts, etc.) were brought to the project activity 'Making a Movie'. Similarly, concepts and ideas from the project were taken to the classroom. This was not completely disconnected from the teachers' and researcher's intentionality, but it also happened because of the students' high engagement with the practice that became integrated in the project.

6.5 Students' Foregrounds Regarding the Learning of Mathematics

In the previous section, we have drawn on two social theories of learning to analyse the researched phenomena, by adopting the perspective of the learners at a micro-level, which emerged from their participation in a practice, and by assuming the perspective of the school community at a macro-level of analysis. But learning can

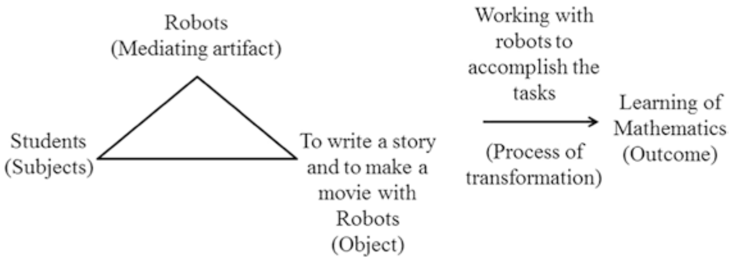


Fig. 6.1 Robot as mediating artefact on the learning of mathematics

also be addressed by relating the social, cultural, economic and political aspects that frame students' lives, at another macro-level of analysis, with the perspective of the students about learning, at a micro-level of analysis.

We thus bring into play the concept of *foreground* (Alrø & Skovsmose, 2004; Skovsmose, 1994, 2011, 2012) in order to analyse the way students approach learning. According to Skovsmose (1994), every person has a background and a foreground. Background can be interpreted as 'that socially constructed network of relationships and meanings which belong to the history of the person' (Skovsmose, 1994, p. 179).

Usually when we talk about the intentions of a person, we relate them with the person's background, but this is not the only source of intentions. The person's foreground is also an important source. The foreground is seen as 'the opportunities which the social, political, economic and cultural situation provides for the person' (Skovsmose, 2011, p. 22). However, two people having similar foregrounds can act out their possibilities in quite different forms. Thus, the foreground includes both collective and individual features. 'Foregrounds include experiences and interpretations, which are elaborated through interaction and communication' (Skovsmose, 2011, p. 23).

One can see the foreground of a person as a complex combination of two sets of features. On one hand, 'a foreground is formed through the possibilities, tendencies, propensities, obstructions, barriers, hindrances, et cetera, which his or her context provides for a person' (Skovsmose, 2012, p. 2). One can therefore see the foreground as structured through social, economic, political and cultural parameters; however, the foreground is not uniformly determined by this. The foreground is also formed through the person's experiences and interpretations of the above mentioned parameters.

The notion of *intentionality* is related to the notion of foreground. If we look for the motives for an action, it is important to consider the person's foreground (Skovsmose, 2011). Learning is an action which includes intentions and motives. When we want to investigate learning phenomena, we have to consider the intentionality of the learners.

Can the learning that happens at school be considered in terms of action? This draws our attention to the students' intentionality, to their foregrounds and to their motives (or lack of it) for learning (Skovsmose, 2011, p. 26):

> When learning is seen as action we can interpret different learning phenomena – students' engagement (or lack thereof) and their achievements (or lack thereof) with reference to their foregrounds. In particular, a ruined foreground can obstruct bringing intentions into the learning process.

For there have to be action and therefore learning, it is important that students experience open and challenging learning situations so that their intentionality to learn 'is born' in a natural way. Closed situations can completely block the emergence of intentionality and end up with no need of action. Situations that can bring out intentionality to learn may not be related to the students' backgrounds. They may be connected 'with student's possibilities in future life, not the objective possibilities but the possibilities as the student perceives them' (Skovsmose, 1994, p. 182).

According to Skovsmose (2011), this aspect is very important given that the meaning of an action is related to the intentionality included in that action, which in turn relates to the foreground of the person in action. The meaning of a classroom activity is constructed by the students, and that construction depends on what students can perceive as their possibilities, and it is related with their foreground and intentions. In view of that, there is a close relationship between meaning, intentionality and foreground.

To learn implies to live meaningful experiences, which can be either relationships between what is taking place in the classroom and the students' backgrounds or their daily life experiences. However, 'experience of meaningfulness has much to do with experienced relationship between activities in the classroom and students' foregrounds' (Skovsmose, 2011, p. 93). The student produces meaning in a learning situation when he or she feels like he or she is learning something that is socially valued. Students also produce meaning if they understand that their contributions are valued within a situation of cooperation (with colleagues or with the teacher).

The dialogic action is a search for and a questioning on shared perspectives in an attempt to produce meaning. Dialogic action means acting in cooperation.

Learning implies action, which can be a dialogic action in terms of producing means to read the world and also to transform it. But to learn implies also reflection about that action. Dialogic action is an interaction which provides (a visible) basis for critical learning (Alrø & Skovsmose, 2004).

Alrø and Skovsmose (2004) conceptualise critical mathematics learning as being grounded in dialogic actions, and they define mathemacy as the competence that critical mathematical learning represents. That is, mathemacy is not only the ability to calculate and to use mathematical techniques but a competence associated with reflecting and acting in a world strongly structured by mathematical models. Skovsmose (2011) points to mathemacy as the competence of dealing with mathematical notions, applying these concepts in different contexts and reflecting on those applications. Frankestein (1998) advocates the need of mathematics to be worked, lived, taught and learned in the real context and dealing with problems as they are presented to us on a daily basis. In real life, we have to deal with many problems presented to us in an unorganised way. The problem-solving of traditional curriculum isolates and simplifies particular aspects of reality in order to provide students the training of techniques.

6.5.1 Robot Race

For the purpose of illustrating and discussing the role of students' foregrounds on the learning of mathematics, we will bring the learning scenario – 'Robot Race' from Lopes (2013).

One year prior to the scenarios' implementation (school year 2010–2011), the students worked with robots in the learning scenario 'a possible trip' that we described above (Fernandes, 2012). In the following school year (in grade 8), they asked the mathematics teacher to work with robots again. The mathematics teacher contacted the research team, and together they designed a learning scenario. We proposed students to organise a robot race. With this learning scenario, students had their first experience with NXT robots along with its programming environment. During nine classes of 90 minutes each, in the last term of the school year, these students learned mathematics using robots.

With the purpose of students learning to programme the robots to hold the races, three challenges were launched: (i) to programme the robot to run around four tables arranged in pairs (forming a rectangle); (ii) to hold races in a straight line from side to side of the classroom (the robot should stop when it detects a wall (ultrasonic sensor)); and (iii) to programme, taking that into account, the robot so that it would have to start the race upon the starting signal (sound sensor), to follow a black line (light sensor) and to stop 15 cm before the end of the line (ultrasonic sensor). Each working group created a prototype of a race route with provided parts, so that two robots could race simultaneously. Each two robots should have the same chance to win. The race route should fit inside the classroom.

In the whole-class group, students selected the race route to be used, and then they built it with the real dimensions. They decided that all robots would run under the same conditions, that is, each one would run twice against each opponent and once in each line of the race route.

They started making some training in the race route in order to improve their programming and thus to increase the chance of winning the races. Each working group collected data from the races performed and analysed it in order to determine the winning robot. Each group made a statistical study – within the scope of an inquiry process – whereby conclusions were provided and generalisations were established. Statistical contents emerged from students' actions within that activity (Lopes & Fernandes, 2012).

6.5.2 Racing to Reconstruct Foregrounds Regarding Mathematics

These students were from a poor neighbourhood close to their school. They had a poor school performance, and many of them had low grades in mathematics. Their parents without any constraints accepted this fact. The cultural roots of these students, as well as the way they and their parents interpreted their past experiences regarding school mathematics, led them to have a low participation and a highly disinterested positioning in mathematics classes, including a high rate of absenteeism (Lopes, 2016). We can say that most of these students had a ruined foreground in relation to school mathematics, that is, most of them had only experienced desolated possibilities in relation to the learning of mathematics. The students' foregrounds did not allow them to see what possibilities school (and particularly mathematics) could bring to their lives since, from the perception that they have about their possibilities of future life, they considered that having good grades in mathematics was not necessary to finish school. Thus, the intentionality to learn mathematics was not easily activated by most of these students. Ten of the fourteen students received bad grades in maths in the second term of the school year 2011–2012, and some of them had never had a good grade in mathematics tests.

So, what changed from the implementation of the learning scenario – 'Robot Race'?

The fact that students' request to work again with robots has been accepted by their mathematics teacher seems to have been the starting point for the reconstruction of the students' foregrounds related to mathematics. Realising that their interests were valued by the teacher opened a space for the emergence of students' intentions-in-learning.

From the moment the 'big idea' of the learning scenario was presented – to hold races with robots – the way students acted has started changing. Working in the mathematics classes with robots that were built by them also triggered the students' learning intentions. The emotional relationship created with the robot they built to run and to win the races helped students to find motives to engage in school mathematics practice and in the learning of mathematics (Fernandes, 2013b).

The openness of the learning situation also created a fruitful environment to the emergence of the intentionality of the students. Robots as well as the challenge proposed to the students that had embedded a playful tone and the idea of gamification (Gee, 2008) also contributed to the reconstruction of students' foregrounds regarding mathematics. The intentions-in-learning emerged, and learning as action arose naturally in mathematics classes.

After students have learned to programme with the aim of performing the three proposed challenges, they held races, and thereafter they searched for mathematical arguments for their own robot to become the winner, if possible.

The intention, action and reflection were closely connected with the practical activities, and its product was some physical element (a robot, a programme that makes what is expected and a race route that fits in the classroom). But, in the moment of finding mathematical arguments to choose the winning robot, the product was no longer a physical element; students had to create strategies and justifications to convince themselves and others that their perspective was a valid one. The transcription below supports the discussion of students' conceptions about what means to solve a problem in mathematics class.

Res: What are you doing?

R: We have finished. We have already found the winner!

N: It is *Jagunço*.

Res: And how did you find out?

D: We calculated the average.

Res: Can you explain, please?

R: We calculated the average time of the races of each robot, and we assigned the robots' classification.

D: The robot that has the best average is the first one; the robot that has the second best average is the second one and so on...

N: We are in the third position.

Res: Can you find a strategy that makes your robot become the winner?

D and R: Not worthy, teacher. We'll never be the winner.

At this moment, students did not know that using different criteria would define a different winner. According to their conceptions about mathematics and about solving problems in the mathematics class, finding a solution for a problem is to solve the problem. So, the problem was solved: *Jagunço* was the winner.

Through questioning, the researcher tries to understand the students' perspectives and to explore them, leading students to develop or abandon them, in order to construct a shared understanding of mathematical concepts. Furthermore, with this positioning, the researcher led students to reflect about their work and to recognise the mathematical ideas they used and also helped them to reconstruct the idea of what can be solving problems in the mathematics class. Through an inquiry process, the researcher fostered dialogue and moderated discussions in the small and large groups. The last question of the researcher 'Can you find a strategy …?' was formulated to challenge students, to open up new perspectives and also to keep them involved in the mathematical activity. Thus, the researcher is making visible other possibilities of robots' classification. However, the students did not still understand what to do. Then, the researcher suggested, in the form of a hypothetical question:

Res: What if you analyse the times of the races, in order to find out who has the best of all racing times? (The researcher knew that the best racing time belonged to this group.)

Students understood the suggestion proposed by the researcher, and the zooming in was activated. Zooming in indicates a quest for shared perspectives. The entire dialogue presented in the episode shows the search for a shared perspective between the researcher and the students. It represents action, since students started analysing the data from the races, again.

After some minutes analysing the data, the students called the researcher and said:

D and R: Using this criteria we win.

Res: Can you explain it, please?

N: We have the best time of all races… if we use only the best time of each robot…

D, N and R: We win.

Res: Well done. Try to organise the information in order to present the classification of each robot…

The researcher moved away from the group to let the students work autonomously.

Through questioning, the researcher helped the students to see that their robot could also be the winner. This fact made students' intentions for learning to emerge. They were able to find mathematical arguments to show that their robot could also be the winner. Thus, the researcher managed to establish a relationship between the classroom activity and students' foregrounds regarding mathematics, helping them to reconstruct it and leading them to live meaningful experiences in maths classes.

So, the students create the following table (Table 6.1).

Then, the researcher asked them to explain the table.

D: First, we chose only the best time of each robot… in this way we win.

Res: Well done. We can say that you have chosen the 'minimum time' of each robot.

N: Yes… minimum time. Do you think we should write, in the table, minimum time instead of best time?

Res: Yes, I do… to use statistical language.

D: And what should we put instead of 'worst time'?

Res: What term do you think we should put?

N: Maximum time?

Res: Yes. Very good. The lowest time is called minimum time, and the greatest time is called maximum time. These values correspond to the extreme values of the times of the races, of each set of data. How did you choose the second and the third robot position?

D: To be the second, we chose 'the best worst time'.

Res: Can you explain it, please?

N: As the two robots have had the same 'best time', that is, the same 'minimum time', we decided that the one that had the lower... hum... hum... 'maximum time' would win.

Res: Ok. Have you used the sample range?

D: Sample range?

Res: Yes. You decided that, if there is a draw, the robot that has the smallest difference between the 'maximum time' and 'minimum time' wins, that is, the robot which has the lower range.

Throughout the above episode, the students have calculated and have used mathematical techniques but have also reflected upon the data collected in the race in order to build argumentation to justify that their robot had been the winner. Besides that, students solved problems as they appear in real life and not in a structured way as they usually appear in a traditional mathematics class.

The episode also shows evidence that the students gave meaning to the mathematical (statistical) contents by relating them to the races.

All groups, except one, were able to find out mathematical arguments to make their robot be the winner. Each group used a statistical measure (average, median, or minimum time) as an argument for their robot to be the winner of the races.

Table 6.1 Table created by the DNR group for the classification of robots

	Best time (s)	Worst time (s)	Classification
DNR	26.54	29.75	1st
Jagunço	27.05	27.57	2nd
Vinagre	27.05	30.44	3rd
X – 5	28.32	33.52	4th

The group that failed to make their robot to be the winner also sought mathematical arguments. But the data from their robot did not allow them to win. However, they were also analysing data within an inquiry process.

At the presentation of the conclusions and generalisations of their statistical analysis, students showed the criteria for a robot to be the winner using the statistical concepts in a correct way and explaining their meaning in the races, therefore showing that they learned statistics.

For the researcher, it was clear that students had learned mathematics, but the teacher felt the need to evaluate students also through a written test. In this test all the students got good grades.

Mathematics learning occurred within an inquiry process, by reflecting on mathematics and with mathematics. We may say that students have been developing their mathemacy competence (Skovsmose, 2011) once they have been dealing with mathematical concepts, applying them and reflecting on them. The lived experience opened the field of possibilities provided to these students and contributed to an ongoing construction and reconstruction of their foregrounds (Skovsmose, 2011) regarding mathematics. Most of the students became actively involved in mathematics classes and finished the middle school with good grades in mathematics.

6.6 Conclusions

The design of the learning scenarios in DROIDE II project was supported by theoretical assumptions that framed the research project, namely, our positioning towards learning as a situated phenomenon that embodies the notion of participation.

The creation and implementation of learning scenarios with robots revealed itself to be a powerful artefact (Engeström, 2001) to enhance mathematics learning (Fernandes, 2013a). In fact, learning scenarios, usually designed by teachers, are often presented to students as a reified product (Wenger, 1998) without explanation of the reasons underlying their creation. Therefore, students are not expected to participate in the negotiating of the fundamental aspects of the practice, leaving it meaningless to most of them. The fact that the students were cocreators of the learning scenarios in our research project was decisive for their involvement in the practice, because this aspect presupposes that the students' intentionality is part of the process once they are allowed to participate in the negotiation of the several activities that constitute the practice. From this point of view, learning was also sought as action (Skovsmose, 2011), once students' engagement was a central key in the process.

The openness of the situation proposed to the students and also the interdisciplinary nature that was potentiated from working around big themes – 'Making a Movie' and 'Robot Race' – were also important aspects of the learning scenarios. The thematic proposed, which in the project DROIDE II we call the 'big idea', led students to find motives to engage in the practice in which they were coactors. From the work held by students, teachers and researchers to achieve the 'big idea', learning mathematics (but not only that) emerged in an ongoing process. There was intentionality of teachers and researchers to bring out the mathematical contents. Nevertheless, it was not only the mathematical contents that governed the creation of the scenarios (Fernandes, 2013d). The 'big idea' is what Wenger (1998) also calls the joint enterprise. The joint enterprise is not dictated by someone who creates the scenario (and teaches) and is accepted by others who learn. It is jointly negotiated by the participants. For this reason, it was important that the research team created a 'sketch' of a scenario and opened the space so that the other actors could participate in their construction (Fernandes, 2013d). During the implementation of the learning scenarios, the initial intentions of the different actors were considered but also those that emerged along the way, with a view to achieving the objectives initially established and which were renegotiated throughout the process (Martins & Fernandes, 2015).

The working methodology adopted, including in it the researchers' positioning and the questioning in both learning scenarios, was important because of how it emphasised the cooperation between students and between students and teachers, opening spaces for negotiation and sharing of mathematical meanings but also for negotiation of ways of acting and thinking in the mathematics class.

The creation of learning scenarios where challenges are posed (which entails the resolution of various mathematical and other problems) leads students to deal with knowledge from different areas and is a powerful artefact for mediating learning. Nevertheless, it requires a very different positioning of the teacher (in this case, the researchers had also this role), particularly in relation to the mastery of knowledge in their subject area, but also in other areas. The fact that the project team was pluridisciplinar was very useful. However, to think about the empirical field in a dialectical relationship with the theoretical framework was very fruitful to the decision-making.

When we are trying to understand learning, namely, mathematics learning, as a social phenomenon, it is important to pay attention to the social and cultural context in which it occurs. The analysis of the relationship between students and between students and teachers is a rich approach for understanding how the learning takes place. Equally important, the relationship between students and the tools they are using to learn is also central.

The artefacts people use in a certain activity cannot be constructed and conceptualised outside that activity. In fact, 'artefacts should not be considered by themselves' (Santos & Matos, 2008, p. 183); they can only be understood within their own history. The robots constructed by students in the researched learning scenarios are productions from developed practices in the implementation of those scenarios. For someone external to those two practices these robots would not be more than just 'simple' robots. For the students involved, these robots reify a history of participation in a practice and may represent an idealised character or a robot that allowed them to compete and eventually win a race. The robot reifies those practices as well as the mathematical ideas and concepts students used to create those characters or to win the race.

These robots transformed the way students negotiated the meaning of mathematical knowledge. As shown above, the position of straight lines, space and time were notions that were negotiated based on robots' trajectories. In addition, statistics was a tool to make a robot become the winner of a race. Mathematics was, together with the robots, a tool to act in the learning scenarios. Robots reified the experience of learning mathematics, and certain mathematical concepts 'were born' in association with the robot (Fernandes, 2012). Mathematics emerged embedded in the work with robots, and as a result of that, students reconstructed the meaning of learning mathematics.

The robots had also a significant role over the reconstruction of students' foregrounds, which are in themselves dynamic entities (Skovsmose, 2011). In the learning scenario 'Robot Race', the inclusion of the robots in the mathematics classroom, within the 'big idea' of the learning scenario, challenged the students' positioning towards school mathematics learning.

'There are no regulations and simple guidelines for establishing meaningful education and for anticipating students' intentionality' (Skovsmose, 2011, p. 30). But we can argue that school, and in particular mathematics education, have to create different learning scenarios, enriched with technologies, in order to open the field of possibilities for students to reconstruct the meaning of learning mathematics and of the usefulness of mathematics in problem-solving. And, for the students with ruined foregrounds in relation to mathematics (and to school), it may open the range of possibilities for them to reconstruct it, thus providing them with new opportunities in life.

Acknowledgement The work reported in this chapter was developed under the support of the project Technology Enhanced Learning @ Future Teacher Education Lab supported by FCT under the contract number PTDC/MHC-CED/0588/2014.

References

Abrantes, P., & Matos, J. F. (2012). Using activity theory to understand how using robots can foster learning of basic concepts of programming. In L. Gómez Chova, A. López Martínez, & I. Candel Torres (Eds.), *2012 Proceedings of the 6th International Technology, Education and Development Conference* (pp. 299–305). Spain: Valencia.

Alrø, H., & Skovsmose, O. (2004). *Dialogue and learning in mathematics education: Intention, reflection, critique*. Dordrecht: Kluwer Academic Publishers.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematical worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171–200). Westport, CT: Ablex.

Carroll, J. M. (1999). Five reasons for scenario-based design. In *Proceedings of the 32nd Hawaii International Conference on System Sciences,* Hawaii. Retrieved 10 Jan 2015, from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.106.5310&rep=rep1&type=pdf

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit.

Engeström, Y. (1999a). Activity theory and individual and social transformation. In Y. Engeström, R. Miettien, & R.-L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19–38). Cambridge: Cambridge University Press.

Engeström, Y. (1999b). Innovative learning in work teams: Analysing cycles of knowledges creation in practice. In Y. Engeström, R. Miettinen, & R.-L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 377–404). Cambridge: Cambridge University Press.

Engeström, Y. (2001). Expansive learning at work: Toward an activity-theoretical conceptualization. *Journal of Education and Work, 14*(1), 133–156.

Fernandes, E. (2004). Aprender Matemática para Viver e Trabalhar no Nosso Mundo. [Learning mathematics to live and work in our world]. PhD thesis. Universidade de Lisboa. http://cee.uma.pt/people/faculty/elsa.fernandes/artigos/Tese%20EMdSF.pdf

Fernandes, E. (2012). 'Robots can't be at two places at the same time': Material agency in mathematics class. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 227–234). Taipei: PME.

Fernandes, E. (Ed.). (2013a). Aprender matemática e informática com robots. Funchal: Universidade da Madeira. Retrieved 10 Jan 2015, from www.cee.uma.pt/droide2/ebook/index.html

Fernandes, E. (2013b). The emergence of agency in a mathematics class with robots. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education (CERME 8)* (pp. 1725–1734). Ankara, Turkey: Middle East Technical University and ERME.

Fernandes, E. (2013c). Trigonometria - uma viagem ao centro da terra. In E. Fernandes (Ed.), *Aprender matemática e informática com robots* (pp. 96–102). Funchal: Universidade da Madeira.

Fernandes, E. (2013d). O que aprendemos sobre aprender com robots. In E. Fernandes (Ed.), *Aprender matemática e informática com robots* (pp. 248–259). Funchal: Universidade da Madeira.

Fernandes, E., & Santos, M. (2013). A visão da aprendizagem adotada no projeto DROIDE II. In E. Fernandes (Ed.), *Aprender matemática e informática com robots* (pp. 1–24). Funchal: Universidade da Madeira.

Frankestein, M. (1998). Reading the world with math: Goals for a critical mathematical literacy curriculum. In E. Lee, D. Menkart, & M. Okazawa-Rey (Eds.), *Beyond heroes and holidays: A practical guide to K-12 anti-racist, multicultural education and staff development* (pp. 306–313). Washington, DC: Network of Educators on the Americas.

Gee, J. P. (2008). Learning and games. In K. Salen (Ed.), *The ecology of games: Connecting youth, games, and learning* (pp. 21–40). Cambridge, MA: The MIT Press.

Jones, K. (2000, March). The mediation of mathematical learning though the use of pedagogical tools: A sociocultural analysis. Invited paper presented at the conference on Social Constructivism, Socioculturalism, and Social Practice Theory: Relevance and Rationalisations in Mathematics Education, Norway.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Lopes, P. C. (2012). Racing with robots. In L. Gómez Chova, A. López Martínez, & I. Candel Torres (Eds.), *INTED2012 Proceedings of 6th International Technology, Education and Development Conference* (pp. 6217–6221). Valencia: IATED.

Lopes, P. C. (2013). Uma corrida com robots. In E. Fernandes (Ed.), *Aprender matemática e informática com robots* (pp. 86–95). Funchal: Universidade da Madeira.

Lopes, P. C. (2016). *Aprender matemática com recurso a tecnologias: Robots na sala de aula* (PhD thesis). Universidade da Madeira.

Lopes, P. C., & Fernandes, E. (2012). Participation in a school mathematics practice with robots: Racing with robots. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 250). Taipei: PME.

Lopes, P. C., & Fernandes, E. (2015). Robots to learn statistics and citizenship. In M. F. Costae & B. V. Dorrío (Eds.), *Hands-on science network* (pp. 70–78). Braga: Universidade do Minho.

Martins, S. (2013a). 'Regime of competence' in a school practice with robots. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society* *for Research in Mathematics Education (CERME 8)* (pp. 1774–1783). Ankara/Turkey: Middle East Technical University and ERME.

Martins, S. (2013b). Da escrita de uma história à produção de um filme. In E. Fernandes (Ed.), *Aprender matemática e informática com robots* (pp. 114–142). Funchal: Universidade da Madeira.

Martins, S. (2016). *Aprendizagem de tópicos e conceitos matemáticos no 1.° Ciclo do Ensino Básico: Uma história com robots* (PhD thesis). Universidade da Madeira.

Martins, S., & Fernandes, E. (2012). We perceive two minutes to be a fast achievement while for robots this presents a lifetime: Analyzing mathematics learning from a situated perspective. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, p. 302). Taipei: PME.

Martins, S., & Fernandes, E. (2015). Hands-on mathematics with lego robots. In M. F. Costae & B. V. Dorrío (Eds.), *Hands-on science network* (pp. 161–165). Braga: Universidade do Minho.

Matos, J. F. (2010). Towards a learning framework in mathematics: Taking participation and transformation as key concepts. In M. Pinto & T. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 41–59). Belo Horizonte, Brazil: PME.

Matos, J. F. (2013). Cenários de Aprendizagem como recursos estruturantes da ação em educação. In E. Fernandes (Ed.), *Aprender matemática e informática com robots* (pp. 47–54). Funchal: Universidade da Madeira.

Rogoff, B., Turkanis, C., & Bartlett, L. (2001). *Learning together – Children and adults in a school community*. Oxford: Oxford University Press.

Santos, E. A. (2012). Robots as learning mediators in a virtual community of practice. In L. Gómez Chova, A. López Martínez, & I. Candel Torres (Eds.), *INTED2012 Proceedings of 6th International Technology, Education and Development Conference* (pp. 6222–6225). Valencia: IATED.

Santos, M. P. (2004). *Encontros e Esperas com os ardinas de Cabo Verde: Aprendizagem e participação numa prática social* (PhD thesis). Universidade de Lisboa. Retrieved 20 July 2013 from: https://sites.google.com/site/madalenapintosantos/doutoramento

Santos, M., & Matos, J. (2008). The role of artefacts in mathematical thinking: A situated learning perspective. In A. Watson & P. Winbourne (Eds.), *New directions for situated cognition in mathematics education*, *Mathematics education library* (Vol. 45, pp. 179–204). Boston: Springer.

Savenye, W. C., & Robinson, R. S. (2004). Qualitative research issues and methods: An introduction for educational technologists. In D. H. Jonassen (Ed.), *Handbook of research on educational communications and technology* (2nd ed., pp. 1045–1071). Mahwah, NJ: Lawrence Erlbaum.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematical education*. Dordrecht: Kluwer Academic Publishers.

Skovsmose, O. (2011). *An invitation to critical mathematics education*. Rotterdam: Sense Publishers.

Skovsmose, O. (2012). Students' foregrounds: Hope, despair, uncertainty. *Pythagoras, 33*(2), 1–8. https://doi.org/10.4102/pythagoras.v33i2.162

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wenger, E. (1998). *Communities of practice – Learning, meaning and identity*. Cambridge: Cambridge University Press.

Wertsch, J. (1991). *Voices of the mind: A socio-cultural approach to mediated action*. Hertfordshire: Harvester Wheatsheaf.

Wollenberg, E., Edmunds, D., & Buck, L. (2000). *Anticipating change: Scenarios as a tool for adaptive forest management. A guide*. Bogor: SMT Grafika Desa Putera.

Chapter 7
Developing the Mathematical Eye Through Problem-Solving in a Dynamic Geometry Environment

Maria Alessandra Mariotti and Anna Baccaglini-Frank

7.1 Introduction

Images play different roles, but they play a key role in human thinking and, generally speaking, in human mental activities. It is a familiar experience to have images accompanying our thoughts either in fantasying or in trying to solve a difficult mathematical problem. And it is exactly in solving problems that very often we look for help from images, for instance, by sketching a drawing on a sheet of paper; this is what Pólya (1957) wrote in his book *How to Solve It*, opening the discussion on figures in geometry:

> Figures are not only the object of geometric problems but also an important help for all sorts of problems in which there is nothing geometric at the outset. Thus, we have two good reasons to consider the role of figures in solving problems. (p. 103)

Starting with the seminal work of Alan Bishop (1980, 1983), studies on visualization have been developed (Presmeg, 2006) recently focusing on specific aspects of the relationship between images and mathematical thinking. The advent of digital technologies has opened a new direction of investigation on how specific digital environments might affect conceptualization processes and problem-solving in mathematics (Arcavi & Hadas, 2002).

A first fundamental result from the studies on visualization is about reflecting upon the use of a varied and vague set of terms commonly used in the current language both for referring to the internal and the external context – such as visualization,

M. A. Mariotti
Università di Siena, Siena, Italy
e-mail: mariotti21@unisi.it

A. Baccaglini-Frank (✉)
Università di Pisa, Pisa, Italy
e-mail: anna.baccaglinifrank@unipi.it

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_7

visual thinking, mental images, drawing, figures and schema – in order to establish a shared terminology.

According to Presmeg (2006), visualization can be characterized in the following way:

> […] visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (Presmeg, 1997). This characterization is broad enough to include two aspects of spatial thinking elaborated by Bishop (1983), namely, interpreting figural information (IFI) and visual processing (VP). (p. 206)

As Bishop (1989) clearly pointed out, there seems to be a contrast between positive aspects and pitfalls related to visualization. On the one hand, starting from the original work of Krutetskii (1976), mainly based on experts' experience/reports, authors claimed the value and the power of visualization; on the other hand, the first studies concerning students' behaviour have highlighted difficulties related to visualization. Among these are the following (Presmeg, 1986):

> Especially if it is vague, imagery which is not coupled with rigorous analytical thought processes may be unhelpful. (p. 45)

Our aim is that of discussing and clarifying some aspects related to processes of visualization in problem-solving in Euclidean geometry. We claim that specific visualization skills (the *mathematical eye*) that are necessary for solving geometrical problems are actively involved in problem-solving processes that take place in a dynamic geometry environment (DGE). In the following sections, we start discussing and clarifying the meaning of the expression *mathematical eye*. Building on previous studies and on the current literature, we introduce specific cognitive constructs, which we call *visual skills*, involved in the elaboration of visual stimuli and, which are, consequently, fundamental in solving geometrical tasks. As we illustrate how different visual skills are involved in the solution processes of geometrical problems, we will see how a DGE can afford the mobilization of the same visual skills. Therefore, we claim that not only problem-solving activities can be designed with the aim of fostering the student's development of specific visual skills but also that acting within a DGE might strengthen the didactic potential of geometrical problem-solving activities and eventually affect the development of the mathematical eye.

7.2 Selected Skills Involved in Geometrical Problem Solving Using the "Mathematical Eye"

According to the notion of *figural concept* (Fischbein, 1993; Mariotti, 1995), geometrical reasoning consists of a dialectic between figural and conceptual components, so that the solution of a geometrical problem results from a coherent interaction between such components. Indeed, geometrical problem-solving is based on elaborations of images, both external representations (drawings on the paper or drawings on the screen) and internal representations (mental images). This is why we are interested in studying processes involved in treating images and, specifically, in describing and explaining how spatial properties of images are noticed, identified and interpreted geometrically, and eventually linked together logically in a conditional statement.

For instance, how can it happen that, looking at a scribble on a piece of paper (see Fig. 7.1), the observer thinks of "a square"? Or, similarly, looking at a moving image on the screen, the observer suddenly exclaims, "it is a parallelogram!"?

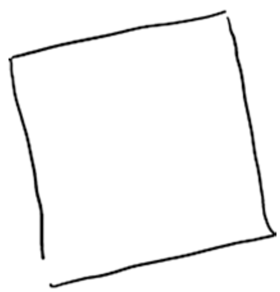
Though these experiences may be considered common to any student who has learned a bit of geometry, other more sophisticated experiences are common for expert mathematicians, thanks to a high level of competence in the treatment of images that supports problem solving in geometry; we can call such competence the *mathematical eye*.

In the following, we will be assuming that the process of perception occurs thanks to specific mental schemes that allow us to interpret visual stimuli, that is, to transform them into a coherent *perceived* image. The internal elaboration that occurs in our brains can be modelled through these mental schemes that we will be talking about in terms of *skills* because of their important role in problem solving and, specifically, in the process of geometrical problem solving. In this section, our intent is not to be exhaustive in describing all skills that allow the mathematical eye to function; instead, we introduce a selection of specific skills, and we identify and describe their roles in processes of problem solving.

The following skills are elaborated based on the theory of figural concepts (Fischbein, 1993), Duval's theory on cognition in geometry (Duval, 1994, 1998) and on the literature in cognitive psychology on visual-spatial abilities (e.g. Cornoldi & Vecchi, 2004). In the following we will refer to these skills as *visual-geometrical skills*, considering them as a possible characterization of what we call the "mathematical eye".

- Identification: immediate identification of a geometrical property of a figure on the plane or in space, with a goal in mind; this skill echoes Duval's perceptual apprehension and the visual-spatial abilities of visual organization and (planned) visual scan applied to the context of geometry.

Fig. 7.1 Scribble that could be recognized as a "square"



- Reconstruction: reconstruction of a figure from parts that are not correctly organized in space or that are not visible. This skill echoes an aspect of Duval's operative apprehension (1995a, 1995b) and the reconstructive visual ability (Cornoldi & Vecchi, 2004).
- Construction: construction of a representation of a figure, taking into account the use of tools and the construction sequence. This skill echoes Duval's sequential apprehension and the visual-spatial ability referred to as reconstructive visual ability.
- Part-whole awareness: abstract a part of the figure and consider it separately from the rest; this skill echoes Duval's attention to relevant subconfigurations and is described by Hadamard (1944, p. 80) as the ability to "abstract some special part of the diagram and consider it apart from the rest".
- Manipulation: manipulating a figure to transform it. This skill echoes the visual-spatial ability of image manipulation and aspects of Duval's operative apprehension.
- Theoretical control: mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry; this skill allows to "see" a figure in relation to verbal statements describing its geometrical properties. Moreover, it allows the solver, in Hadamard's words, to achieve a "simultaneous view of all elements of the argument, to hold them together, to make a whole of them in short", and "understanding the […] proof" (1944, p. 77). Consistently, we will also speak of advanced theoretical control intending to mean how the mathematical eye can bring an expert to automatically make use of sophisticated (for a high school student) mathematical notions to "see" aspects of a figure that are invisible to a less trained eye. Examples would be seeing a configuration "modulus similarity or translation" or declaring certain points "inessential" in the manipulation of a figure. This skill echoes aspects of Duval's discursive apprehension, and operative apprehension, and Fischbein's conceptual component of a figure.

There are two more visual-geometrical skills, which, for their specific role in geometrical reasoning, will be more widely described in the following sections: geometric prediction and crystallization.

7.2.1 *The Skill of Geometric Prediction*

In solving a geometrical problem, reasoning is heavily guided by the goals the solver has in mind: for example, identifying geometrical properties of a figure and classifying it is a different process from identifying properties that need to remain invariant as a manipulation of the image is performed. Indeed, a process that seems rather frequent in geometric reasoning is to mentally manipulate a figure and imagine how it will change given certain constraints, that is, maintaining certain properties invariant. Such process can be carried out through the use of the various skills listed above, but it is so common for experts to use it as a skill in its own right that we will call it *geometric prediction*[1]. With geometric prediction, we intend the identification of particular properties or configurations of a new figure, arising from a manipulation process. This process does not seem to be precisely described in the psychological literature; however it appears to be coherent with respect to the notions of anticipatory image (Piaget & Inhelder, 1966) and anticipatory schemes (Neisser, 2014), which suggest an ability to make predictions, orienting both perception and imagination, in the presence of a specific goal.

7.2.2 The Skill of Crystallization

Dynamism seems to be a component of experts' reasoning. Indeed, the skills of manipulation and of geometric prediction involve "movement" of the figure. If this movement is imagined by the solver, it may appear in different forms. For example, some may imagine a continuous deformation of the figure and others a "generic" figure that has at the same time infinite realizations that the solver can "move across" selecting the most useful ones. The use of movement (of any type) involves a temporal dimension in this kind of reasoning, which has been well documented in the literature, especially in reference to the use of a DGE during geometrical explorations. These were initially developed and used by mathematicians during their processes of problem solving, thanks to the possibility they offer to "externalize the set of relations defining a figure" (see, e.g., Laborde & Laborde, 1992; Laborde & Straesser, 1990).

Indeed, Laborde, speaking of a specific DGE, Cabri-Géomètre, states:

> The nature of the graphical experiment is entirely new because it entails movement. The movement produced by the drag mode is the way of externalising the set of relations defining a figure. The novelty here is that the variability inherent in a figure is expressed in graphical means of representation and not only in language. A further dimension is added to the graphical space as a medium of geometry: the movement. (1993, p. 56)

In this context, geometrical properties are interpreted as invariants (Laborde, 2005):

> A geometric property is an *invariant* satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions. (p. 22; emphasis added)

Therefore, the identification of invariants is also an important skill, constituting the mathematical eye.

Moreover, as the solver produces conjectures as part of the solution process, s/he can find him-/herself in the need of crystallizing an experimental situation by eliminating the temporal dimension to move to conditional statements. Research has shown that this is not always a spontaneous process. Some research has been conducted on processes of generation of conditionality (PGC) (Boero, Garuti & Lemut, 1999; Boero, Garuti & Mariotti, 1996). We will touch upon this again briefly in the analyses.

[1] This construct has also been used in a recent study by Miragliotta, Baccaglini-Frank and Tomasi (2017) and is being used in the doctoral work of Miaragliotta (Miragliotta and Baccaglini-Frank 2017, 2018).

7.3 Skills Involved in Problem-Solving Processes: Analyses of Two Problems

In this section, we give two examples of problems and provide *a priori* analyses of how the mathematical eye, described through the previous visual-geometrical skills, might guide possible solution processes. The first problem is rather classical, typically found in Italian geometry textbooks (and indeed it is a translation of one such problem), for the fact that it explicitly states what is to be proved, given the described construction. The second is an "open problem", less typical in the curriculum, but with the potential of fostering development of the mathematical eye, because of the skills it is necessary to use to solve it. Moreover, it lends itself quite naturally to be explored within a DGE, which can support the development of such skills thanks to its affordance of specific tools that will be analysed in a later section.

In the following analyses of this section, we imagine working without the support of a DGE.

7.3.1 Visual-Geometrical Skills Used in Solving a "Prove That" Problem

Problem 1: Finding triangles with equal area
Given a triangle ABC and the midpoint D of side BC, consider a point E on segment BD and construct line AE and the parallel to AE through D. This line intersects AC in F. Construct segments AD and EF; these segments intersect at H. Prove that triangles EHD and FHA have the same area.

What are key geometrical problem-solving skills that come into play?

The solver will probably first use the construction skill to produce a representation of the figure described in the problem, and s/he might wonder whether triangles FHA and EHD are congruent in general. Without further questioning, s/he may simply assume this and attempt to prove their congruence by trying to apply the triangle congruence criteria (if the triangles can be proved to be congruent, then of course their areas will be congruent).

However, if the image that is first realized following the construction steps looks like our Fig. 7.2, this (incorrect) assumption may not be made.

The solution process is guided by the aim of finding a relationship between the two triangles that is not their congruence. The process will start with the identification of properties of the figure and in particular of the property "AE parallel to FD". Such property may be identified through a manipulation of the figure: the solver may imagine E moving along BD, and as the figure changes, through geometric prediction, s/he can notice properties of the figure as those that remain unvaried

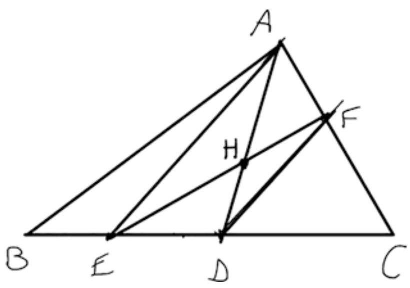


Fig. 7.2 Possible figure obtained by accomplishing the construction described in Problem 1

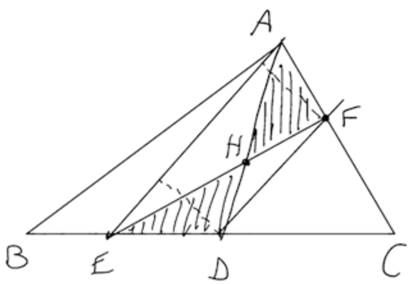


Fig. 7.3 Possible outcome of a (mental) manipulation of the constructed figure for Problem 1 by changing the position of E on BD

throughout the manipulation. Though a changing figure might be imagined, the dynamism is then eliminated through crystallization of the figure into a "generic" configuration. In other words, the mathematical eye will interpret a specific configuration as representing all the characteristic properties and nothing else; in terms of figural concepts, the figural and the conceptual components are fully integrated, and the drawing is properly controlled (Fig. 7.3).

For example, while the property "EHD congruent to AFH" is not always true, the property "AE parallel to FD" is (indeed it is one of the construction properties), as well as the (derived) property "triangles AFE and ADE have congruent heights". Notice that to "see" congruent heights, the solver needs to use the skill of reconstruction, as no segments corresponding to the heights are included in the construction given in the problem. These properties are key in solving the problem, because triangles AFE and ADE, having the same base AE and congruent heights, must have the same area. Identification of these properties is guided by theoretical control over the figure. Indeed, the following theorem needs to be recalled: "Given a triangle ABC, all triangles with the same base AB and the third vertex on a line parallel to AB through C have the same area". Such theorem could be recalled in a dynamic form, for example, with the third vertex "moving" along the parallel line through C.

However, the solution of the problem is possible only through awareness of a specific part-whole relationship, that is, each of the two triangles AFE and ADE can be seen as made of a common part AEH and one of the two triangles AHF and EHD. From such awareness the solver can deduce the equivalence of the two triangles AHF and EHD, as it was to be proved.

There are different reasons why the solver could be led to perceiving the key properties, “common base AE” and “equal heights” of AHF and EHD. The following two reasons we consider come from somewhat opposite directions of reasoning, but they include the same geometrical ingredients and similar activity of the mathematical eye.

1. In an attempt to identify significant (for the problem) properties starting from the configuration, the solver can manipulate the figure, imagining E to vary on segment BD or the triangle ABC to vary. Thanks to geometric prediction, this can lead to noticing that there are very few “special” properties in the figure, so it may be rather straightforward to identify the parallelism of AE and FD (a construction property), seen as constancy of the distance between the two lines and of any other consequent properties; this may be promoted also through theoretical control over the figure.
2. On the other hand, the solver can think about the conclusion (“area EHD = area FAH”) and reason about these surfaces, thinking about how to decompose them or see them as part of greater surfaces, thanks to part-whole awareness (as described above at the end of the first process outlined). In this case, triangles AFE and ADE may be seen as made of the “parts” AHE (common), EHD and FAH; the solver may attempt to search for an argumentation leading to the property “area AFE = area ADE” and in doing so identify their property of sharing a base and having congruent heights (that s/he “sees” thanks to the skill of reconstruction).

At this point, through part-whole awareness, AE can be seen as the common base of two triangles with the same height, and thanks to the same skill, these triangles can then be decomposed into two parts each, of which one (AHE) is common. S/he can conclude that the areas of AFH and HED are equal because these triangles are parts of congruent triangles to which a common region (AHE) is subtracted.

The first form of reasoning seems to be heavily based on identification of relevant (to the problem) properties, thanks to manipulation and geometric prediction, guided by theoretical control over the figure.

The second form of reasoning seems to depend much more on the solver’s skills of recognizing the geometrical rather than the algebraic nature of the area searched for: instead of focusing on the product of two numbers, focusing on the product of two segments (one side and the height relative to it).

7.3.2 Visual-Geometrical Skills in Solving an "Open Problem"

Problem 2: Finding possible types of quadrilaterals a constructed figure can become

Let A, M and K be three points, and construct B as the symmetric point of A with respect to M, and construct C as the symmetric point of A with respect to K. Construct the parallel line l to BC through A and the perpendicular line r to l through C. Let D be the intersection of l and r. Make conjectures about which types of quadrilaterals can ABCD be (Fig. 7.4).

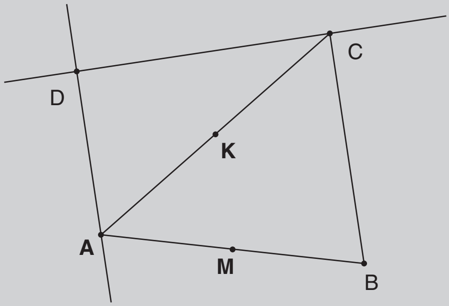


Fig. 7.4 Figure obtained following the construction steps in Problem 2

One way to proceed in solving Problem 2 is to construct a figure and proceed to identify properties of the quadrilateral ABCD. The variable points are A, M and K. Properties can be identified by manipulating mentally the figure, for example, by thinking of moving points A, M or K. The properties constructed will definitely be invariant (the parallelism of AD and BC, the perpendicularity of BC and CD, AM = MB, AK = KC), but also properties that are logical consequences of these in the theory of Euclidean geometry can be identified during any manipulation and process of geometrical prediction (e.g. CD perpendicular to DA, KM parallel to CB, KM = 1/2 CB). Of course, to identify such properties, it is also necessary to have good theoretical control over the figure.

Through manipulation, an expert mathematician may imagine inducing movement on ABCD to explore possible configurations, realizing, as a colleague once reported, that M and K act as a sort of "handles" for moving B and C. The expert might quickly realize this because of the fixed (by construction) relationships between A, M and C (AM/MC = 1/2) and A, K and B (AK/KB = 1/2). Indeed, experts interviewed on this problem have answered that "no matter what fixed ratios" there were between these sets of points, "M and K are inessential" in the manipulation of the quadrilateral. This seems to be an automatic process (the expert says it is "natural and immediate"), characteristic of the mathematical eye of an expert. This automatized process has surely been reached through years of experience in “looking at” and “working with/talking about” mathematical objects. In other words, the trained mathematical eye, thanks to advanced theoretical control acquired over the years, guides the exploration process and the choices of features of the figure that are “relevant”.

Once these properties have been identified, the solver can advance a conjecture such as “ABCD is (always) a right trapezoid”, which can easily be proved using the theorems that allowed him/her to infer “CD perpendicular to DA” from the construction properties. The expert may also make a claim like “The types of quadrilaterals that can be obtained depend only on the choice of K”, arguing that a type of quadrilateral is determined “modulo similarity and rotation”, so A and M can be thought of as fixed. Again, the mathematical eye must be quite experienced and well-trained to be able to see “modulus similarity and rotation”.

What other types of quadrilaterals can ABCD be? Using his/her knowledge of how quadrilaterals can be theoretically classified, the solver can list other possible types of quadrilaterals to test, as subtypes of right trapezoids with a second pair of parallel sides (rectangles, including squares). The solver can perform manipulations of the figure, guided by theoretical control and by part-whole awareness through which specific subconfigurations can be searched for and identified.

At this point, in a possible attempt to construct a figure with the properties listed in the problem plus additional properties that will guarantee the figure’s belonging to a subtype of quadrilateral, the solver can search for conditions under which the subtype may be identified, possibly using geometric prediction.

Conceiving the necessity of the angle at M to be right in order for ABCD to be a rectangle may come from noticing, thanks to (possibly advanced) theoretical control over the figure: (1) triangle AMK is similar to ABC and (2) a right trapezoid becomes a rectangle if either of its non-right angles become right (this is a sufficient condition).

A conjecture might be advanced such as “If AMK is a right triangle (at M), then ABCD is a rectangle”.

7.4 How Development of the Mathematical Eye Can Be Fostered Through Problem Solving in a DGE

In this section, we analyse possible solution processes of the two problems introduced above when these are solved within a DGE. From the analyses, we will show how the tools in the DGE can support the solver’s solution process, either supporting the use or compensating for the weakness of certain visual-geometrical skills and eventually developing the mathematical eye. These analyses will lead into our hypotheses on how the development of certain geometrical exploration modalities can be fostered within a DGE through appropriately designed activities.

7.4.1 Analysis of a Solution Process of the "Prove That" Problem Within a DGE

Within a DGE, the first step is generally to realize a construction that incorporates all the properties given in the problem. So the solver will need to use his/her skills of construction, determining which commands to use from the menus in the DGE to incorporate properly each property into the figure described in the problem's hypothesis: in this case D as the midpoint of BC, E as a mobile point "attached to" segment BD and DF as the parallel through D to AE. In order to guide processes of identification and part-whole awareness, the solver may also decide to highlight the surfaces of the two triangles AFH and EHD (Fig. 7.5). In doing this, the solver is supported by the DGE that carries out the constructions correctly and precisely; this is something that is not guaranteed in a hand-made drawing.

The solver might now try to identify properties of the triangles AHF and EHD, as crystallized invariants as s/he manipulates the figure. However, unlike in the previous cases analysed without the DGE, here the solver can physically drag points of the figure; so the DGE becomes responsible for much of the theoretical control over the figure and compensates the solver's geometric prediction ability, showing the result of each manipulation instantly (e.g. see Figs. 7.6 and 7.7).

These manipulations may allow the solver to notice, for example, that the property "triangles AFH and EHD are congruent" is not true in general; indeed, some results of the manipulations (dragging) can be identified as counter examples to such a property.

Through such (physical) manipulations, the solver can search for invariant properties, identifying them through a process of crystallization in which dynamism is eliminated and a particular configuration (product of the crystallization of an invariant) becomes the figural component of a figural concept – an identified property. This may occur even for invariants that were defined by the steps of the construction (we have evidence of this in various students' protocols); in this case, for example, it could occur for the property "AE parallel to FD".

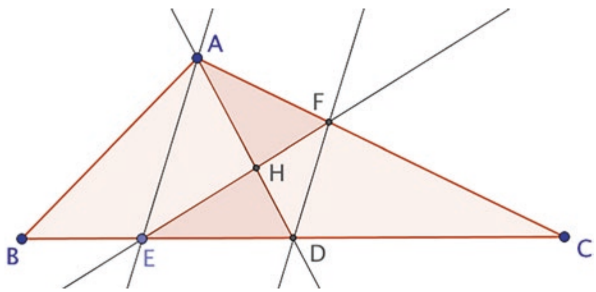


Fig. 7.5 Possible figure obtained by accomplishing the construction described in Problem 1 in a DGE

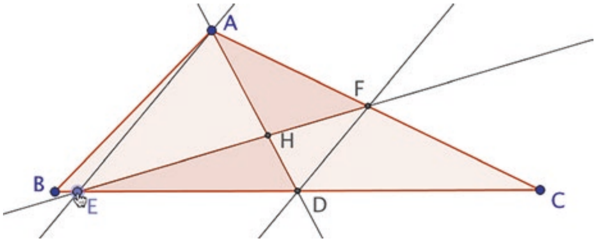


Fig. 7.6 Possible outcome of a manipulation through dragging of the constructed figure for Problem 1 by dragging E along BD

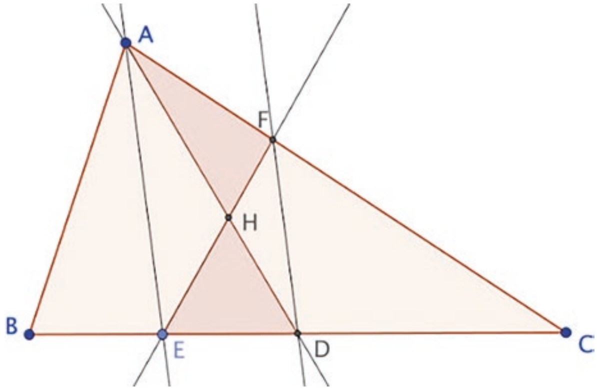


Fig. 7.7 Possible outcome of a manipulation through dragging of the constructed figure for Problem 1 by dragging vertex A of the triangle ABC

Now, the solver might refer to his/her theoretical knowledge and ask him-/herself what can be inferred from segments being parallel, and this could lead to the identification of congruent segments, perpendicular to the parallel lines, seen as an indication of the lines always having a "same distance". This way, through reconstruction, two segments could be seen in the figure and actually constructed in the DGE. Since points A, E, F and D are visible in the figure, it is likely that the solver will choose to construct the segments through two of these points and, possibly, from two points on the same line because of the types of images s/he might be familiar with (Fig. 7.8).

This choice – which may happen without a conscious decision being made by the solver – can guide the part-whole awareness and, thanks to theoretical knowledge, help noticing triangles with a same base and congruent heights and thus identifying properties such as "triangles AFE and ADE have congruent heights", "triangles AFE and ADE have base AE in common" or (if the distances from A and E to FD were drawn) "triangles DFA and DFE have congruent heights" and "triangles DFA and DFE have base FD in common". Moreover, the solver can identify the theorem relating the two properties noticed: "Given a triangle ABC, all triangles with the same base AB and the third vertex on a line parallel to AB through C have the same area". Of course, as described in the preceding solution process for Problem 1, much of the whole solution process may be guided by the solver's identifying a proper configuration of such a theorem within the figure on the screen.

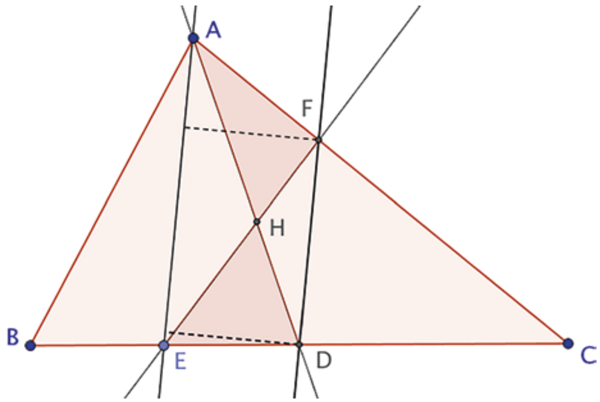


Fig. 7.8 A DGE figure in which segments showing the parallel lines AE and FD having the "same distance" are shown

Finally, as in the previous solution process, the solution can be reached through awareness of a specific part-whole relationship, that is, both of the two triangles AFE and ADE can be seen as made up of a common part AEH and of one of the triangles AHF and EHD. From such awareness, the solver can deduce the equivalence of the two triangles AHF and EHD, as it was to be proved.

Compared to the solution process described previously, we highlight how the skill of geometric prediction plays a much more minor role in the context of the DGE because the software exercises on the figure the theoretical control that the solver would have to otherwise exercise him-/herself. Instead, in the DGE, the solution processes typically involve crystallization, a skill that seems less dominant in geometrical problem-solving with paper and pencil. As a matter of fact, in a DGE, the solution might be achieved, eliminating dynamicity and grasping the invariance of a relation between properties, in other words recognizing the occurrence of a theorem.

7.4.2 Analysis of a Solution Process of the "Open Problem" Within a DGE

The solver can start by constructing a dynamic figure incorporating the properties described in the problem, in this case: C symmetric to A with respect to K, B symmetric to A with respect to M, DA parallel to CB and CD perpendicular to DA (Fig. 7.9).

The solver can manipulate the figure dragging A, M or K and see (on the screen) the effect of the dragging of any of these points. In doing this s/he can identify the property "ABCD is a right trapezoid", as a crystallized invariant emerging from all the configurations that appear. The identification of this property may not require the reconstruction and part-whole awareness skills (to realize that only the angle ADC is right by construction, while angle DCB is right as a consequence of the construction properties), since the quadrilateral ABCD may be perceived as a dynamic whole. A first conjecture like "ABCD is always a trapezoid" may be put forth.

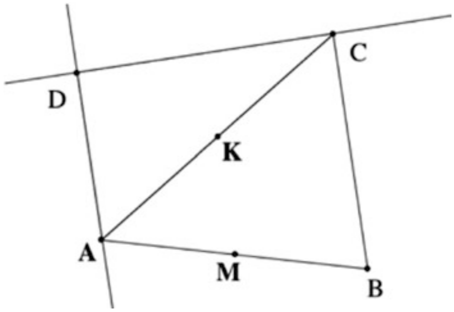


Fig. 7.9 Possible figure constructed for Problem 2 within a DGE

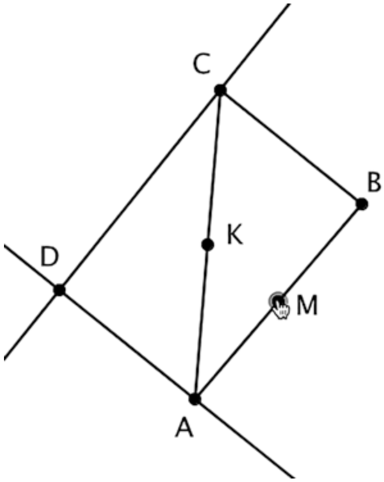


Fig. 7.10 The figure is manipulated by dragging M as the solver searches for "good positions" in which ABCD is a rectangle

Then the solver may ask him-/herself what other types of quadrilateral ABCD can become, theoretically controlling the possibilities by recognizing that the quadrilateral can only become subtypes of a right trapezoid, that is, rectangles or squares.

The exploration (and solution process and products) changes based on which points the solver drags (we are assuming to be in a DGE in which only one point at a time can be dragged). For example, manipulating the figure by dragging M, the solver can search for "good positions", positions in which the property "ABCD rectangle" can be *identified* (Fig. 7.10).

The solver can also search for regularity in the movement of a certain point s/he decides to drag. For example, if K is dragged, and the solver is trying to figure out "when ABCD is a rectangle", s/he will have to drag K along a line, and not just any line, the line through M perpendicular to AM. This line (in this case actually the geometrical locus of the points M such that ABCD is a rectangle) can be discovered without exercising much theoretical control over the figure but instead other skills, including eye-hand coordination and crystallization transforming the regular "straight" movement of K into a geometrical property, such as "K belongs to the perpendicular to AM through M". Such process can be fostered through a tool offered by the DGE that is the trace mark activated on the dragged point (Fig. 7.11).

This particular way of dragging while trying to maintain a desired property has been referred to as maintaining dragging (Baccaglini-Frank & Mariotti, 2010), and we consider it a potential support for mobilizing specific visual skills for the problem-solving in a DGE.

If A is dragged, a similar regularity in its induced movement can be crystallized and another property identified: "A belongs to the line through M and perpendicular to MK". Carrying out the maintaining dragging and identifying that property may stimulate the skill of geometric prediction as, for example, when a path is marked by the trace. Now, the simultaneous presence of two identified properties can be crystallized into the conditional statement (a conjecture): "if A belongs to the line through M and perpendicular to MK, then ABCD is a rectangle".

Finally, if M is the dragged point, the solver can crystallize and identify the property "M belongs to the circle with diameter AK", and the simultaneous presence of two identified properties can be crystallized into the conditional statement (a conjecture): "If M belongs to the circle with diameter AK, then ABCD is a rectangle". In our research, we have witnessed the possibility that solvers, before expressing their conjectures, linking the identified properties, decide to construct the property that will become the premise of their conditional statement in order to obtain a figure in which the whole conjecture, if correct, can be identified and possibly crystallized (Fig. 7.12).

Therefore, based on the point dragged, the solver may make a number of different conjectures that vary in the condition added to the construction to obtain a rectangle and, possibly, in the terminology used to express the conjecture itself. The conditions may be the following: "K belongs to the line perpendicular to AM through M", "A belongs to the line through M perpendicular to MK", "M belongs to the circle with diameter AK" and "AMK is a right triangle (in M)". Based on the results of our research, we can definitely conclude that the last condition was the

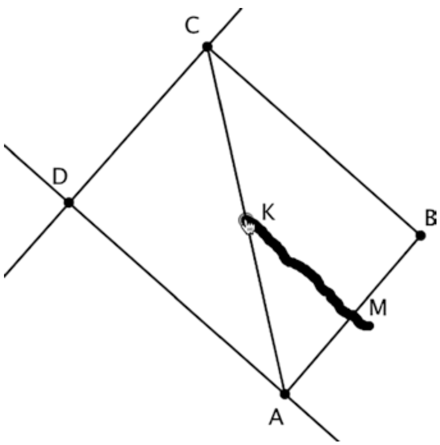


Fig. 7.11 What is shown on the screen in a DGE when the trace mark is active on the dragged point D as the solver tries to maintain the property ABCD rectangle

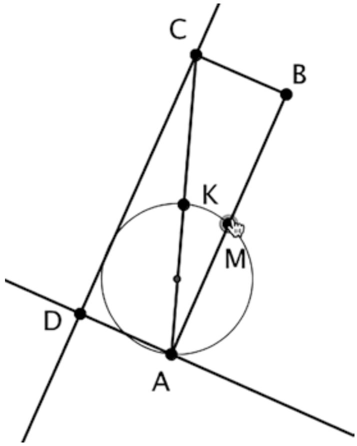


Fig. 7.12 M can be linked to the circle with diameter AK so that the figure embeds properties "M belongs to the circle with diameter AK" and, as a consequence, "ABCD is a rectangle"

least common in students who used maintaining dragging to reach their conjectures in this problem (for further details, see Baccaglini-Frank, 2010a).

The conjecture may not contain only "static", crystallized words; indeed, if dynamic explorations are carried out, our studies (and many others) have shown that students use expressions like "A moves on a line", "ABCD is a quadrilateral when [a certain condition is verified]", "in order for ABCD to remain [a rectangle], the point has to move [in a certain constrained way]" or even "when I move the point [on some path], then ABCD stays a rectangle".

Speaking about the visual-geometrical skills involved in reaching such conjectures, we wish to briefly comment on how crystallization may contribute to the solver's reaching a conjecture as a product of the problem-solving process in a problem like the one described. The skill we have described as crystallization and used in the analyses has important roots in the literature, especially regarding studies of students' "processes of generation of conditionality (PGC)" (Boero et al., 1996, 1999).

In particular, processes like the one we have described echo processes described by Boero and his colleagues, such as the following PGC.

> The conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, that will be subsequently detached from it and then "crystallize" from a logic point of view ("If…, then…"). (Boero et al., 1996, p. 121)

Studying processes of conjecture generation in open problem situations in a DGE, we found that PGCs, as described by Boero and colleagues, seem to be present during the processes analysed (Baccaglini-Frank, 2010a) and also that a new element comes into play: continuity that is induced by the specific kind of motions that occurs in the DGE. Indeed, the examples provided for the PGCs in literature have mostly been of a "discrete" nature, the use of dragging in the processes of conjecture generation attributes a new "continuous" nature to the processes. We note that although dynamism seems to provide support for the processes of conjecture generation like the ones described, making it more “natural”, it may turn into an obstacle as far as the aim to formulate conjectures within the “static” theory of Euclidean geometry is concerned, where it becomes necessary to “eliminate” time.

Going back to the analysis of the open problem solved within a DGE, a final consideration can be made in how the role of the mathematical eye can contribute. While for some students, the three premises in the conjectures (1. “K belongs to the line perpendicular to AM through M then ABCD, 2. “A belongs to the line through M perpendicular to MK”, 3. “M belongs to the circle with diameter AK”) have nothing to do with one another, the mathematical eye, thanks to advanced theoretical control, will allow to condense them into a condition that expresses a general relationship between A, M and K, such as “AMK is a right triangle (at M)” or “the angle AMK is right”. Moreover, as we described in the first analysis of this problem, the expert will also exercise advanced theoretical control to reduce the exploration to a minimum, for example, deeming the dragging of A and M as “inessential” as the configurations can be uniquely determined “modulus similarity and rotation”.

Similar to what we concluded in the other case, here, too, for the open problem solved in a DGE, manipulation can be carried out through dragging, offloading from the solver much of his/her theoretical control over the figure. Moreover, we highlight how the skill of geometric prediction plays a much more minor role; however it may be stimulated by the solver’s use of a tool such as maintaining dragging, as we have described. The use of such tool can be fostered through open problem activities like asking for the production of conjectures, like in the case discussed in the example. In the problem-solving process in a DGE, the crystallization skill seems to emerge much more than in a paper and pencil environment, where, in fact, frequently there seem to be no traces of it at all.

7.5 How Geometric Problem Solving in a DGE Can Foster the Development of Skills Pertaining to the Mathematical Eye

The analyses presented above have highlighted how certain tools offered within the DGE seem to induce the use of particular skills, allowing a less expert solver to take part in explorations in which his/her experience resembles that of an expert, thus fostering the development of his/her mathematical eye. For example, while the expert mentally manipulates the figure and performs geometric prediction, the less expert solver can drag and change the configuration, identifying properties (and relationships between properties) through the crystallization of invariants. In doing this, s/he enriches with dynamism (which cannot be achieved through static images drawn on paper) his/her figural components of the geometric concepts involved; the dynamism represents variability and generality (for the expert). So, we argue that through geometric problem solving in a DGE, it is possible to enhance particular skills supporting development of the mathematical eye. The use of some skills, such as crystallization, and other skills associated to the use of specific dragging modalities like maintaining dragging seems to be mostly present within a DGE, but we have argued that these can be seen as preparatory and supporting for other skills comprising the mathematical eye.

In this section, we take this argument a step further and show a case of how solvers' expert use of skills associated to maintaining dragging actually fostered the development of skills pertaining to the mathematical eye that the solvers were able to use without support from the DGE. For this purpose, we will introduce a third problem, similar to Problem 2, highlighting design aspects that seem to foster the development of desired skills, and then present excerpts from the students' exploration in which we found evidence of the students' strengthened geometric problem-solving skills.

Problem 3
Construct the quadrilateral ABCD following these steps. Construct: a point P and a line r through P, the perpendicular line to r through P, C on the perpendicular line, a point A symmetric to C with respect to P, a point D on the side of r containing A, the circle with centre C and radius CP, point B as the second intersection between the circle and the line through P and D. Formulate conjectures about the possible types of quadrilaterals it can become describing all the ways you can obtain a particular type of quadrilateral.

The design of this problem is similar to that of Problem 2: the task asked of the solver is an open-ended one, in which explorations of the figure are promoted through dragging. The processes of problem-solving induced by these kinds of problems involve the generating of conjectures as an outcome of various kinds of manipulation of the figure. Moreover, our studies have suggested that a request in which the solver is asked to describe "all the ways" in which a certain configuration may be visually verified can foster the use of certain dragging modalities such as maintaining dragging, assuming that this modality is familiar to the solvers (e.g. Baccaglini-Frank, 2010a, 2010b; Baccaglini-Frank & Mariotti, 2010).

Coming to the specific construction proposed in Problem 3, when solving the problem in a DGE, the figure (see Fig. 7.13) can be acted upon by dragging points C, P or D. We will concentrate on dragging D. Among the properties that can be identified, there are the properties described in the steps of the construction.

If the solver tries to obtain the configuration "ABCD parallelogram" through maintaining dragging, new invariants can be crystallized into geometrical properties (e.g. "D lies on a circle C_{AP} with centre in A and radius AP").

As in the analysis of a solution process for Problem 2, here, too, identification of these new properties during dragging can be supported by the use of the trace mark, a functionality in most DGEs. The properties that appear to be visually verified simultaneously as the figure is acted upon through dragging may be crystallized into

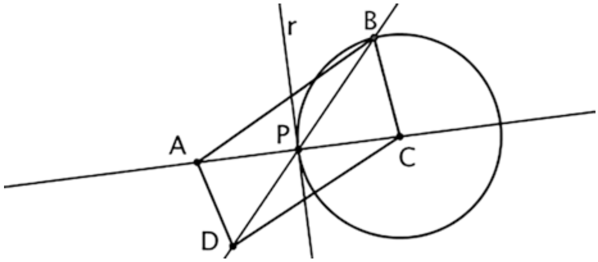
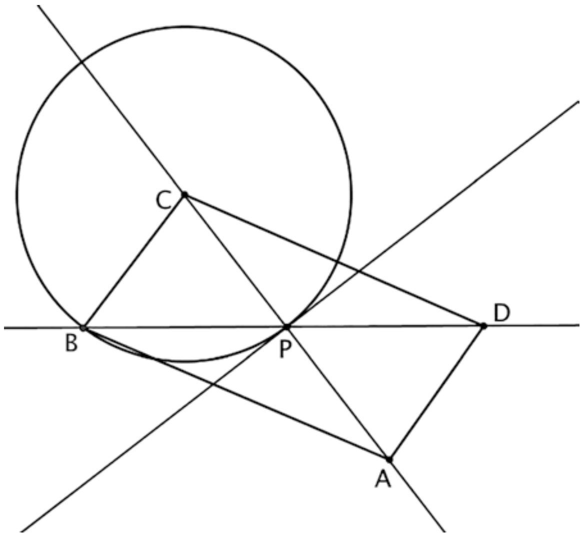


Fig. 7.13 A possible result of the construction in the situation described above

Fig. 7.14 Construction of the figure in Problem 3 with which the students are working



a conditional statement, a conjecture (e.g. "If D belongs to C_{AP}, then ABCD is a parallelogram"), as the outcome of the problem-solving process. Indeed, our research has shown that many solvers who decide to use maintaining dragging perceive a relationship between properties that are simultaneously visually verified on a dynamic figure they are manipulating (e.g. Baccaglini-Frank & Mariotti, 2010; Leung, Baccaglini-Frank & Mariotti, 2013; Mariotti, 2014)[2] (Fig. 7.14).

However, their attempt fails, as shown in the following excerpt[3].

[2] The process is described in further detail by Baccaglini-Frank and Mariotti (Baccaglini-Frank, 2010a, 2010b; Baccaglini-Frank & Mariotti, 2011).

[3] Adapted from Baccaglini-Frank and Mariotti (2010, pp. 238, 240, 241) with permission from International Journal of Computers for Mathematical Learning, copyright 2010, by Springer

What is said [and done]	*Comments*
Gianni: And now what are we doing? Oh yes, for the parallelogram? **Francesco**: Yes [as he drags D with the trace activated] yes, we are trying to see when it remains a parallelogram. Gianni: Yes, okay the usual circle comes out. **Francesco**: Wait, because here...Oh dear! Where is it going? […] so, maybe it's not necessarily the case that D is on a circle so that ABCD is a parallelogram. Because you see, if we then do a kind of circle starting from here, like this, it's good, it's good, it's good, it's good [he drags along a circle he imagines], and then here… see, if I go more or less along a circle that seemed good, instead it's no good…so when is it any good?	Francesco and Gianni seem to have conceived a geometric prediction for the path along which point D should be dragged. This prediction does not seem to fit with the shape of the trace mark appearing on the screen as Francesco performs maintaining dragging. This leads the failure of the students' use of maintaining dragging as a physical tool, so they abandon it.

Suddenly Gianni expresses the result of a geometric prediction he has carried out mentally, thanks to theoretical control he exercises on the figure.

What is said [and done]	*Comments*
Gianni: Eh, since this is a chord, it's a chord right? We have to, it means that this has to be an equal chord of another circle, in my opinion with centre in A. Because I think if you do, like, a circle with centre. **Francesco**: A, you say… Gianni: Symmetric with respect to this one, you have to make it with centre A. […] Gianni: With centre A and radius AP. I, I think… **Francesco**: Let's move D. More or less… Gianni: It looks right doesn't it? **Francesco**: Yes. Gianni: Maybe we found it! (Fig. 7.15)	Gianni, who was not dragging, carries out a geometric prediction. The students proceed to construct the newly conceived circle along which Gianni has imagined D to move. The identified relationship "if D belongs to the symmetric circle then ABCD is a parallelogram" is constructed in the DGE.

The students seem quite satisfied and formulate the following conjecture explicitly: "If D belongs to the circle with centre in A and radius AP, then ABCD is a parallelogram".

What happened to maintaining dragging here? The students continue the exploration mentally as if they were dragging. Gianni seems to be using the maintaining dragging tool mentally (Baccaglini-Frank, 2010a, 2010b). Therefore, the conjecturing process relies entirely on his theoretical control over the figure.

In this case, geometric prediction, paired with theoretical control, plays a key role in the process of problem solving. We conjecture that this way of thinking was fostered by Gianni's extensive experience using maintaining dragging as a physical tool that strengthened his theoretical control and geometric prediction skills to the extent that he was able to take-upon himself to manipulate the figure, controlling its conceptual components and carry out an accurate geometric prediction. In sum-

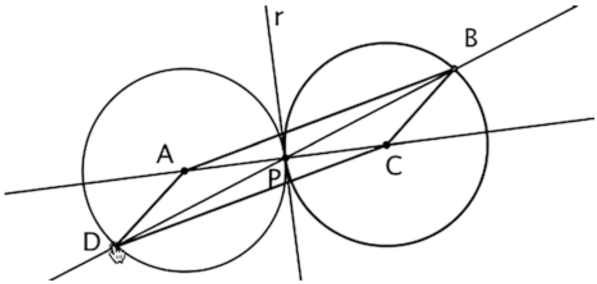


Fig. 7.15 Francesco drags D along the newly constructed circle

mary, looking at the transcript, we can infer that this prediction was possible for him because the combination of visual-geometrical skills comprising his mathematical eye allowed him to "see" the circle with centre in A as the image of the circle in the original construction through reflectional symmetry across *r*.

This case suggests that appropriation of dragging modalities such as maintaining dragging is possible and can lead to strengthening particular visual-geometrical skills and, therefore, to development of the mathematical eye. Moreover, specific analyses of the case have also suggested that use of maintaining dragging as a psychological tool (Vygotski, 1978) can bring continuity between the conjecturing phase and the proving phase (if students are later asked to prove their conjectures), because many theoretical notions become explicit to the solver who can then control them at will (Baccaglini-Frank & Antonini, 2016).

7.6 Concluding Remarks

The aim of this chapter was to discuss some aspects related to visualization processes in problem solving. We started with the idea of the necessity of a mathematical eye to successfully perform geometrical problem solving. In order to clearly describe what the mathematical eye could mean, we introduced a selection of specific skills and elaborated on some theoretical constructs, previously introduced in the literature concerning cognitive aspects of visualization; we called these visual-geometric skills.

Our analyses showed how the problem-solving process can be described through the combination of different visual-geometrical skills that are necessary in the problem-solving process. This raises the educational problem of how it might be possible to improve such skills and eventually develop a mathematical eye.

The general hypothesis presented in this chapter concerns the educational role played by moving into a dynamic geometry environment, that is, proposing and solving a problem with the support of a DGE.

The analyses developed in Sects. 7.3 and 7.4 show how the same visual-geometrical skills are actively involved in the solving processes but also how the functionalities of the DGE can support the emergence and consolidation of these skills. For example, in Sect. 7.4 we have shown how specific DGE skills associated to a way of dragging points, maintaining dragging, can support the development of the skill of geometric prediction.

Moreover, repeated experiences mobilizing the skill of identification and the skill of crystallization seem to contribute to the strengthening of the solver's theoretical control; these visual-geometrical skills guide the mathematical eye during problem-solving processes. In particular, the visual-geometrical skill of theoretical control can be at the basis of the elaboration of a theorem.

Indeed, let us imagine how this could happen if a student who is elaborating the theorem "Given a triangle ABC, all triangles with the same base AB and the third vertex on a line parallel to AB through C have the same area." used the first problem we analysed. The student could be introduced to the theorem in a dynamic form, for example, exploring a figure in which a triangle ABC is constructed as a segment (AB) and a third vertex C attached to a line parallel to AB. As C is moved on the parallel line, the invariant area of the triangle may be identified. Such dynamic configuration, with the invariant relationship between C being on the parallel line and the constant area of ABC, can be crystallized by the student into a statement such as "any triangle with the third vertex C on a line parallel to side AB has a given area" or "any two triangles with a common base and congruent heights have the same area", by eliminating the dynamism in favour of awareness of the being generic of the configuration. Once the process of crystallization is complete, the student will have probably also strengthened his/her theoretical control, since a new fragment of theory is now at his/her disposal. According to the notion of figural concept, the new fragment of theory will not only have a conceptual component – possibly expressed in a verbal text – but also a figural component encompassed in a crystallized configuration. That means that such a theorem, if needed, may be also recalled in a dynamic form, for example, with the third vertex "moving" along the parallel line through C.

The analyses presented and these final considerations support our claim that problem-solving activities can be designed with the aim of fostering students' development of specific visual skills, which can, in turn, contribute to the development of a mathematical eye.

References

Arcavi, A., & Hadas, N. (2002). Computer mediated learning: An example of an approach. In F. Hitt (Ed.), *Representations and mathematical visualization*. Mexico D.F.: Cinvestav – IPN.

Baccaglini-Frank, A. (2010a). *Conjecturing in Dynamic Geometry: A Model for Conjecture-generation through Maintaining Dragging*. Doctoral dissertation, University of New Hampshire, Durham, NH, USA.

Baccaglini-Frank, A. (2010b). The maintaining dragging scheme and the notion of instrumented abduction. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the PMENA* (Vol. VI, pp. 607–615). Columbus, OH: The Ohio State University.

Baccaglini-Frank, A., & Antonini, S. (2016). From conjecture generation by maintaining dragging to proof. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th conference of the International Group for the Psychology of mathematics education* (Vol. 2, pp. 43–50). Szeged, Hungary: PME.

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning, 15*(3), 225–253.

Baccaglini-Frank, A., & Mariotti, M. A. (2011). Conjecture-generation through dragging and abduction in dynamic geometry. In A. Méndez-Vilas (Ed.), *Education in a technological world: Communicating current and emerging research and technological efforts* (pp. 100–107). Badajoz, Spain: Formatex.

Boero, P., Garuti, R., & Lemut, E. (1999). About the generation of conditionality. In O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of the International Group for the Psychology of mathematics education* (Vol. 2, pp. 137–144). Haifa, Israel: Technion.

Boero, P., Garuti, R., & Mariotti, M. A. (1996). Some dynamic mental process underlying producing and proving conjectures. In L. Puig (Ed.), *Proceedings of 20th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 121–128). Valencia, Spain: Universitat de València.

Bishop, A. J. (1980). Spatial abilities and mathematics education: A review. *Educational Studies in Mathematics, 11*, 257–269.

Bishop, A. J. (1983). Space and geometry. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 175–203). New York, USA: Academic Press.

Bishop, A. (1989). Review of research in visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7–16.

Cornoldi, C., & Vecchi, T. (2004). *Visuo-spatial working memory and individual differences*. Hove, UK: Psychology Press.

Duval, R. (1994). Les différents fonctionnements d'une figure dans une démarche géométrique. *Repères-IREM, 17*, 121–138.

Duval, R. (1995a). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (Vol. 138 of the series NATO ASI Series, pp. 142–157). Berlin, Heidelberg: Springer.

Duval, R. (1995b). *Sémiosis et noésis*. Berne, Switzerland: Peter Lang.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 37–52). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–162.

Hadamard, J. (1944). *The psychology of mathematical invention*. New York: Dover Publications.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago, USA: University of Chicago Press.

Laborde, C., & Laborde, J. M. (1992). Problem solving in geometry: From microworlds to intelligent computer environments. In J. P. Ponte, J. F. Matos, J. M. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and new information technologies* (pp. 177–192). Berlin, Germany: Springer.

Laborde, J. M., & Straesser, R. (1990). Cabri-Géomètre: A microworld of geometry for guided discovery learning. *Zentralblatt für Didaktik der Mathematik, 22*(5), 171–177.

Laborde, C. (2005). Proceedings of the 10th Asian technology conference in mathematics. In *Robust and soft constructions* (pp. 22–35). Korea: National University of Education.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Mariotti, M. A. (2014). Transforming images in a DGS: The semiotic potential of the dragging tool for introducing the notion of conditional statement. In S. Rezat et al. (Eds.), *Transformation – a fundamental idea of mathematics education* (pp. 155–172). New York: Springer. https://doi.org/10.1007/978-1-4614-3489-4_8

Miragliotta, E., Baccaglini-Frank, A., & Tomasi, L. (2017). Apprendimento della geometria e abilità visuo-spaziali: un possibile quadro teorico e un'esperienza didattica. *L'Insegnamento della Matematica e delle Scienze Integrate, 40B*(3), 339–360.

Mariotti, M. A. (1995). In R. Sutherland & J. Mason Eds (Eds.), *Images and concepts in geometrical reasoning. Exploiting mental imagery with computers in mathematics education* (pp. 97–116).

Miragliotta, E., & Baccaglini-Frank, A. (2017). Visuo-spatial abilities and geometry: a first proposal of a theoretical framework for interpreting processes of visualization. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the tenth congress of the European society for research in mathematics education (CERME10, February 1–5, 2017)* (pp. 3952–3959). Dublin, Ireland: DCU Institute of Education and ERME.

Miragliotta, E., & Baccaglini-Frank, A. (2018). You see (only) what you predict: the power of geometric prediction. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 387–394). Umeå, Sweden: PME.

Neisser, U. (2014). *Cognitive psychology. Psychology press classical editions*. New York: Taylor and Francis.

Piaget, J., & Inhelder, B. (1966). *L'image mentale chez l'enfant*. Presses Universitaires de France.

Pólya, G. (1957). *How to Solve It*. Garden City, NY: Doubleday.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Rotterdam: Sense Publishers.

Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6*(3), 42–46.

Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematics learning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 267–279). Mahwah, NJ, USA: Lawrence Erlbaum.

Vygotski, L. S. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Chapter 8
Reflecting on Digital Technology in Mathematical Problem-Solving Situations

Arthur B. Powell

8.1 Introduction

The preceding six chapters concern the first tripartite theme of this book: the role of digital technology in mathematical problem-solving situations. The chapters represent research pursued in four different Europe countries, one in France, one in Italy, three in Portugal, and one in Spain. Of the six chapters, five are qualitative empirical investigations, and one is a theoretical study for which later empirical studies can further elucidate and refine the proposed theory.

Collectively, these chapters broaden research in mathematics education on problem-solving with digital technologies as they investigate theoretical, epistemological, and pedagogical uses of a wide range of technologies, both hardware types and software applications, as well as in formal and informal educational settings. The research reported in the chapters are notable since they manifest a certain state of affairs: digital technology in mathematics education has moved from novelty, needing exploration to determine appropriateness, to essential, requiring further understanding of the specific ways it shapes instruction and learning. This is not to say that in actual classrooms instructional uses of digital technologies abound.

Notwithstanding, a statistical analysis of the use of digital technologies for mathematics teaching and learning, the significance of the chapters is their theoretical framing, their methodological approaches, and their findings. The goal of this reaction chapter is not to rehearse what is already well-presented in the chapters themselves but rather to reflect on a few common themes. To do it, we first review the research problem that motivates each chapter.

A. B. Powell (✉)
Department of Urban Education, Rutgers University-Newark, Newark, NJ, USA
e-mail: powellab@newark.rutgers.edu

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_8

8.2 Research Problems

As this section of the book concerns roles that technology plays to broaden research on mathematical problem-solving, the six chapters highlight a diversity of investigations into how technology shapes instruction and learning. It is particularly interesting, therefore, to compare the six conceptual issues or problems about which these studies report. Each study's research problem represents the researchers' insight into a condition of incomplete knowledge or understanding reflected in the extant mathematics education literature. Each set of authors specified explicitly or implicitly the research problem that motivated their study. Table 8.1 lists what I surmised to be the research problem that each chapter addresses.

These research problems coexist with a reality of digital technologies in mathematics education. Outside of schools, digital technologies are evermore part of the quotidian cultural and social environment of both teachers and students. Nevertheless, in schools, these technologies have yet to permeate mathematics instructional and learning spaces. Some of the chapters echoed this state of affairs as they introduced their research problem.

Investigations such as those in Chaps. 2–7 represent the next stage in the use of technological tools. Though teaching with technology is recommended (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 7), meta-analytic and large-scale studies indicate that teaching with technology do not guarantee positive learning outcomes (Cheung & Slavin, 2013;

Table 8.1 Research problems posited in each of Chaps. 2–7

Research problem	
Chapter 2	Need to understand levels of sophistication and robustness of representational and conceptual expressiveness that young competitors (10- to 14-year-olds) evidence when they use digital tools to shape and support their reasoning and solutions to mathematical problems
Chapter 3	Need to understand, along with Duval's semantic congruence and Rabardel's instrumental genesis, insights derived from that didactical framework—Mathematical working spaces (MWS)—Inquire into the cognitive and epistemological aspects of solving of probability problems with digital technology
Chapter 4	Need to "analyse and characterize ways in which high school teachers rely on a dynamic geometry system's affordances to represent and reason about mathematical tasks in a problem solving environment" (p. 83)
Chapter 5	Need to understand how interactive whiteboards can be used to develop dialogical interactions as students problem-solve and what features of the boards can be used to support these interactions
Chapter 6	Need to "understand how the use of robots as mediating artefacts of learning contributes to meaning production and to the learning of mathematics and informatics topics and contents, by exploring possible articulations between both domains" (p. 129)
Chapter 7	Need to understand how different visual-geometrical skills are involved in geometrical problem-solving and how DGEs support the mobilization of these visual-geometrical skills

Higgins, Xiao, & Katsipataki, 2012; Kaput & Thompson, 1994; Li & Ma, 2010; Rakes, Valentine, McGatha, & Ronau, 2010; Wenglinsky, 1998). The research problems that the chapters address are investigations to understand how appropriate digital technologies shape mathematics learning in formal and informal settings as well as mathematics teachers' content and pedagogical knowledge.

Collectively, another theme evident among how researchers approached their research problems is that with digital technologies, the software applications provide feedback to learners as or after they manipulate dynamic objects. Several chapters indicate that, in turn, this affects the user's interaction with the software. The environment reacts to the users' actions through engineered infrastructure that responds according, for instance, in the case of dynamic geometry environments (DGEs), to the theory of geometry. This reaction can inform the learners' actions and shape their thinking.

8.3 Responses to Research Problems: Seeing Mathematically as an Educated Capability

The research problems motivated the investigations reported in the six chapters of this section of the book. Table 8.2 attempts to summarize the empirical and theoretical responses. From the empirical and theoretical responses to research problems in each of Chaps. 2–7, a common theme emerges. The theme concerns how digital technology shapes learners' discourse about mathematical objects and reflects in their discourse what learners learn to see. The use of digital technologies, whether computers, whiteboard, or robots or whether GeoGebra, WolframAlpha computational knowledge engine, or Excel, enables learners to develop new ways to act mathematically and new mathematical capabilities. These technology-based actions and associated capabilities gear learners to develop new ways of seeing.

These technology-enhanced new ways of seeing educate what, in Chap. 7, Mariotti and Baccaglini-Frank term, with reference to geometry, the "mathematical eye". Their term refers to theorized "mental schemes that allow us to interpret visual stimuli, that is, to transform them into a coherent *perceived* image" (p. 155). They identify eight schemes, which they call "visual-geometrical skills" and provide a priori exemplification of how they enable geometrical problem-solving. Each of these visual-geometrical skills as elements of a mathematical eye enables both seeing and acting in powerful ways.

In a similar fashion, as we discover in Chaps. 2–6, learners develop powerful ways of seeing and acting mathematically in the context of technology-enhanced problem-solving in informatics, geometry, probability and statistics, analytic geometry, and algebra. Moreover, the investigation of solutions produced by youngsters who participate in the mathematical competitions SUB12® and SUB14® provide evidence of different levels of sophistication in how with technology the youngsters develop, represent, and communicate their solutions of mathematical

Table 8.2 Empirical and theoretical responses to research problems in each of Chaps. 2–7

Research problem	
Chapter 2	A gradual scale can be discerned among the conceptual models, characterized by details, variations, and specificities that are directly involved in understanding of the mathematical task and in developing an approach to the solution. "[T] o-think-with-GeoGebra or to-think-with-excel means to incorporate the possibilities of mathematical 'expressiveness' conveyed by the digital tools, which invite the student to develop mathematical thinking. Solving-and-expressing by means of digital tools is therefore something that comes as a result of introducing a mathematical point of view to address a situation combined with recognizing a mathematical point of view in the tool to deal with the same situation" (p. 39–40)
Chapter 3	"In tasks dealing with simulation, the semantic congruence between the real situation and the model implemented in the technological device should be taken explicitly into account in teaching. Moreover, in such problems the process which has to be put into play by students is complex, since they have to deal with several domains and shift from one to another: Reality, probability and statistics. The student's personal MWS [mathematical working spaces] is constantly changing, first through a modelling process – often implicit – then through a simulation on spreadsheet or calculator and finally through a return back to the model. In this process the computer acts as a versatile help, since it is used in turn as random generator, calculator, logical tool, plotter, both to calculate (and then it produces *results*) and to 'show' theoretical results (and then it produces *conjectures*), which may not be so clear for students" (p. 76)
Chapter 4	"The inquiring approach that framed the participants' approaches to the tasks helped conceptualize the tasks as a point of departure to engage in math-ematical reflections and as an opportunity to look for mathematical relations and ways to support it through visual, empirical and eventually formal arguments. In addition, all the participants recognized that the technology affordances not only helped them construct and explore dynamic models of the tasks but also to reflect on ways to implement the tasks in actual practice" (p. 98–99)
Chapter 5	Given a four-phase, inquiry-based teaching model: (1) teacher presents task; (2) students solve the task in small groups; (3) all students and the teacher discuss task solutions; and (4) the teacher with the participation of the students systematize mathematical learning, for phases 1, 3, and 4, specific features of the whiteboard promote dialogic interactions among students and among the teacher and students
Chapter 6	Considering learning as a situated, participatory phenomenon, robots are mediating artefacts that can enhance mathematics learning when students are co-creators to meaning production and to the learning of mathematics and informatics topics and contents, by exploring possible articulations between both domains
Chapter 7	Grounded in the literature and a priori analyses, postulate eight visual-geometrical skills that contribute to the development of a mathematical eye and enable geometrical problem-solving. Moreover, learners' mobilization of their visual-geometrical skills is afforded by DGEs

problem situations. These levels can be thought of as different levels of sophistication with their ways of seeing and acting exemplified in their solutions.

These levels of sophistication represent differing and evolving mathematical capabilities, power, or ability to solve mathematical problems. From the reports in each chapter, using digital technological tools, the capabilities that learners develop are to discern critical features of problem situations. Those features are variant or invariant as in the movement of dynamic geometric constructions using dynamic geometry environments as well as delimiting parts from wholes as the cylindrical candles problem in Chap. 5 or the stabilization of relative frequencies analysed in Chap. 3. The use of digital tools and specially designed task enables learners to develop their capability of seeing mathematically. This is a way of seeing something as defined by the aspects discerned, the critical feature of what is seen (Marton et al., 2004; Runesson, 2005; Watson & Mason, 2006).

8.4 What Is to Be Done?

Much is left to do. Each chapter implicitly or explicitly mentions further areas that their findings suggest need to be investigated. To complement those suggested areas of research, it seems worthwhile to indicate a research domain not mentioned. It concerns what has come to be called group cognition (Stahl, 2008, 2016) and its manifestation and support using collective Web 2.0 technologies.

The emergence of new collaborative, digital technologies occasions opportunities to rethink school learning and classroom practice. These technologies permit learners working collectively to construct, visualize, conjecture, and manipulate mathematical objects and relations such as dependencies (Christou, Mousoulides, & Pittalis, 2004). Furthermore, the new technologies' collaborative affordances support social argumentation, justification, and deductive reasoning (Öner, 2008; Silverman, 2011). This potential for knowledge building enables learners to engage jointly in ways that parallel the real world online, collaborative work of mathematicians, including Fields Medal recipients (Alagic & Alagic, 2013). To realize the educational potential of these emergent collaborative technologies in mathematics classrooms requires teacher professional development.

Models of teacher professional development, whether face-to-face, hybrid, or virtual, tend to extract teachers from classroom practice, narrate to them new information, engage them in activities, and guide them on changing their practice to implement the new knowledge. However, these models often neither change teacher practice nor improve student outcomes (Gulamhussein, 2013). Other models may be more effective. Recent surveys of teachers found "teachers in strong collaborative environments see significant benefits in their day-to-day work" (Bill & Melinda Gates Foundation, 2014, p. 5). Effective professional development engages teachers in "learning how to put knowledge into practice through engagement *in* practice within a community of practitioners" (Schlager & Fusco, 2004, p. 132). As Brown and Duguid (2000) emphasize, "*practice* is an effective teacher and the *community of practice* an ideal learning environment" (p. 127, as quoted in Schlager & Fusco, 2004, emphasis added).

References

Alagic, G., & Alagic, M. (2013). Collaborative mathematics learning in online environments. In D. Martinovic, V. Freiman, & Z. Karadag (Eds.), *Visual mathematics and cyberlearning* (Vol. 1, pp. 23–48). New York: Springer.

Bill & Melinda Gates Foundation. (2014). Teachers know best: Teachers' view on professional development. Retrieved from http://k12education.gatesfoundation.org/resource/teachers-know-best-teachers-views-on-professional-development/

Brown, J. S., & Duguid, P. (2000). *The social life of information*. Cambridge, MA: Harvard Business School.

Cheung, A. C., & Slavin, R. E. (2013). The effectiveness of educational technology applications for enhancing mathematics achievement in K-12 classrooms: A meta-analysis. *Educational Research Review, 9*, 88–113.

Christou, C., Mousoulides, N., & Pittalis, M. (2004). Proofs through exploration in dynamic geometry environment. *International Journal of Science and Mathematics Education, 2*(3), 339–352.

Gulamhussein, A. (2013). Teaching the teachers: Effective professional development in an era of high stakes accountability. Retrieved from http://www.centerforpubliceducation.org/Main-Menu/Staffingstudents/Teaching-the-Teachers-Effective-Professional-Development-in-an-Era-of-High-Stakes-Accountability/Teaching-the-Teachers-Full-Report.pdf

Higgins, S., Xiao, Z., & Katsipataki, M. (2012). *The impact of digital technology on learning: A summary for the education endowment foundation*. Durham, UK: Education Endowment Foundation and Durham University.

Kaput, J. J., & Thompson, P. W. (1994). Technology in mathematics education research: The first 25 years in the JRME. *Journal for Research in Mathematics Education, 25*(6), 676–684.

Li, Q., & Ma, X. (2010). A meta-analysis of the effects of computer technology on school students' mathematics learning. *Educational Psychology Review, 22*(3), 215–243.

Marton, F., Tsui, A. B. M., Chik, P. P. M., Ko, P. K., Lo, M. L., Mok, I. A. C., et al. (2004). *Classroom discourse and the space of learning*. Mahwah, NJ: Lawrence Erlbaum Associates.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

Öner, D. (2008). Supporting students' participation in authentic proof activities in computer supported collaborative learning (CSCL) environments. *Computer-Supported Collaborative Learning, 3*(3), 343–359.

Rakes, C. R., Valentine, J. C., McGatha, M. B., & Ronau, R. N. (2010). Methods of instructional improvement in algebra a systematic review and meta-analysis. *Review of Educational Research, 80*(3), 372–400.

Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education, 35*(1), 69–87. https://doi.org/10.1080/0305764042000332506

Schlager, M. S., & Fusco, J. (2004). Teacher professional development, technology, and communities of practice: Are we putting the cart before the horse? In S. A. Barab, R. Kling, & J. H. Gray (Eds.), *Designing for virtual communities in the service of learning* (pp. 129–158). New York: Cambridge University Press.

Silverman, J. (2011). Supporting the development of mathematical knowledge for teaching through online asynchronous collaboration. *The Journal of Computers in Mathematics and Science Teaching, 30*, 61–78.

Stahl, G. (2008). Social practices of group cognition in virtual math teams. In S. Ludvigsen, A. Lund, & R. Säljö (Eds.), *Learning in social practices: ICT and new artifacts—Transformation of social and cultural practices*. New York: Routledge.

Stahl, G. (2016). *Constructing dynamic triangles together: The development of mathematical group cognition*. New York: Cambridge University.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking & Learning: An International Journal, 8*(2), 91–111.

Wenglinsky, H. (1998). Does it compute? The relationship between educational technology and student achievement in mathematics. Retrieved from http://www.ets.org/Media/Research/pdf/PICTECHNOLOG.pdf

Part II
Creativity in Mathematical Problem Solving

Chapter 9
Mathematical Problem Solving Beyond School: A Tool for Highlighting Creativity in Children's Solutions

Susana Carreira and Nuno Amaral

9.1 Mathematics Competitions as a Research Field

Today, more and more activities seek to attract students to the world of mathematics through projects that excite their curiosity and propose them good mathematical challenges. Worldwide many organizations are promoting mathematical enrichment extracurricular activities, along with several well-known international contests and competitions. Those projects represent an important contribution to mathematics education and particularly to a large number of promising students insofar as it offers a complement to the work developed in the classroom (Koichu & Andzans, 2009). The awareness of such importance has been nurturing a growing interest in knowing, as accurately as possible, the impact and benefits of competitions related to mathematics education, in particular those that take place beyond the school context.

Previous studies have shown that mathematical problem-solving competitions contribute to increase the mathematical skills of the individuals who take part in them (Freiman & Applebaum, 2009). At the same time, they can positively influence the motivation of young people and create contexts for discovery, exploration and communication of new ideas, thus providing a rich learning context (Freiman & Lirette-Pitre, 2008). Competitions may actually be considered as a source of curriculum enrichment as they strengthen the abilities required for solving problems while developing an increasing appreciation for mathematics as a discipline

S. Carreira (✉)
Universidade do Algarve, Faro, and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal
e-mail: scarrei@ualg.pt

N. Amaral
Agrupamento de Escolas Cardoso Lopes, Amadora, and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_9

(Freiman, 2009). They have also a primary role in the realization of the students' intellectual and creative potential and offer opportunities for their mathematical empowerment in ways that are compatible with their ages (Koichu & Andzans, 2009). To some extent, they challenge the traditional classroom by letting students learn independently or together, leading them to be persistent and making them aware of thinking processes, strategies and learning media (Freiman & Lirette-Pitre, 2008). In general, competitions emphasize that mathematics is not just about definitions, theorems, formulas, etc. but it is full of processes including experimenting, reasoning, conjecturing, refuting and justifying.

The current study, in theoretical and empirical terms, was developed under the research project *Mathematical Problem Solving: Perspectives on an Interactive Web-Based Competition (SUB12 and SUB14)* more commonly referred to as the Problem@Web Project. The general goal of the project was to expand the current knowledge on mathematical problem solving taking place outside the classroom, namely, in the context of mathematical competitions based on the Internet. One of the project's research foci was the study of mathematical creativity manifested in young students' solutions to mathematics problems in the course of their participation in web-based competitions. One of the issues that soon caught our attention was the need to develop tools that would allow us to assess the creativity of a solution to a challenging problem. In this chapter, we propose a tool, which is both descriptive and numerical, and show how it was used to highlight the creativity of the solutions produced by the students in a mathematics competition.

The empirical field is the mathematical problem-solving competition SUB12 promoted by the University of Algarve and covering the southern region of Portugal. This competition takes place through the Internet, over a period of 6 months, and addresses students attending fifth and sixth grades (10–12 years old). Every 2 weeks, a new problem is proposed in the competition website; the problems are varied and cover different mathematical domains. It is the kind of problems where students are supposed to be challenged and think for themselves, where diverse strategies and approaches are possible and where they may use various forms of representation in the process of finding the solution.

In this competition, it is explicitly required that students present and explain the whole process leading to the solution, as clearly as possible. Moreover, the competition allows participants time and freedom to create inventive solutions. Those are key factors for the development of creativity, which are sometimes lacking in the classroom due to the pressure of curricular targets or other constraints of the school system. It is therefore an environment with obvious potential for the development of mathematical creativity and independent thinking, where students can display their talent each time they solve problems in unexpected and innovative ways.

In taking this competition as a research setting, our aim was to devise a tool for characterizing the mathematical creativity in the solutions submitted through email by the students participating in the SUB12. In this particular context, and bearing in mind the very young age of the students and the moderate challenge of the problems, mathematical creativity is often revealed in unusual and remarkable products, whether by the insights of the solvers or by the ways they find to express the solution.

Our search for a tool that would account for the creativity involved in this mathematical problem-solving contest has led us to develop an analytical framework to assess students' creative solutions. Its main purpose is to track observable indicators of creativity in solutions to mathematical problems. It is no more than a reconfiguration of a traditional psychometric conception of evaluating the creativity of a product, by suggesting three basic dimensions to that effect. In this sense, it is a tentative formulation that we have been working on and revising over time, both by refining the descriptors of such dimensions and by deepening their meaning (cf. Amaral, 2016; Amaral & Carreira, 2012, 2017). The evolutionary construction of an analytical tool is seen as inherent to the process of researching a complex and theoretically dense phenomenon, such as that of mathematical creativity.

9.2 Mathematical Creativity and Problem Solving

Acquired knowledge is certainly an important condition for the emergence of creativity, but in no way can it be said to be enough (Sternberg, 2007). Mastery of a given content may be crucial in interpreting, in processing and integrating information and in establishing connections between different concepts (Kattou, Kontoyianni, Pitta-Pantazi & Christou, 2013). But in addition to the mastery of content knowledge, an inclination to innovate, to work intensely on ideas and to look for new, different and unusual possibilities is also vital. The creative talent is often associated with the ability to produce original, surprising and interesting ideas that go beyond convergent thinking (Milgram & Hong, 2009).

Silver (1997) has pointed out a close connection between problem solving and the genuine activity of mathematicians, as far as to suggest that problem solving could be the best mirror of mathematical creativity. In discussing the signs of creativity in mathematics, Silver refers to the novelty of the problem-posing or problem-solving approach, the shifts taking place during the problem-solving process and the number of different paths explored and solutions achieved. Those are broadly the kind of parameters involved in the classical tests designed to measure creativity. In fact, the dimensions of *originality*, *fluency* and *flexibility* have been extensively used in developing tools to determine creativity (Mann, 2005; Silver, 1997; Yuan & Sriraman, 2011). One of the most renowned tools that are based on those three variables is the Torrance Test of Creative Thinking (TTCT).

The recognition of originality is decisive in determining creativity, but this relevant dimension should not be narrowly viewed. It has been recurrently argued by several authors that creativity is a two-part concept: it simultaneously embraces originality and effectiveness (Aldous, 2007; Runco & Jaeger, 2012; Selter, 2009). According to Aldous (2007), a simplified definition of creativity would be the production of something new and effective. Both qualities – novelty and usefulness – are jointly crucial to yield an understanding of creativity; something that is new and original but does not present any usefulness, adequacy or suitability loses its value as a creative product. Moreover, Aldous (2007) points out that the production of effective novelty has to be seen in relation to the subject who creates it. If an individual produces an independent and effective solution to a problem, though not a new discovery in some domain, it should be accepted as a manifestation of creativity.

This means that creativity must not be restricted to eminent discoveries and instead should include more modest and everyday achievements like those that arise with school students solving problems that are new to them. In such a setting, creative products could be those that support the understanding of mathematical relations and the discovery of hidden ones or that may lead to unpredicted results (Leikin, 2009). In a sense, we are considering the inclusive creativity – the little-c instead of the Big-C creativity – that allows seeing the diversity in the apparent uniformity (Beghetto & Kaufman, 2009).

> Traditionally, creativity scholars have focused on creative outcomes classified either as Big-C (eminent) or little-c (everyday) creativity. Big-C creativity focuses on clear cut examples of great creative expression (e.g., Pythagoras's theorems, Dickenson's poetry, Mozart's compositions). In contrast, little-c creativity — focuses more on the creativity of everyday life, accessible to most anyone (Runco & Richards, 1998). One example of everyday creativity might be the creative way you arrange plants and flowers in your garden — an arrangement that receives praise from friends and family. (Beghetto & Kaufman, 2009, p. 40)

Fluency is one of the characteristics of the creative process and refers to the number of ideas generated in response to a query and to the flow of associations between different ideas. To become an indicator of creativity, it is essential that fluency comes along with originality and flexibility, in the sense that generating a lot of ideas does not guarantee any creative result. So, this indicator of creativity may not be as important and decisive as originality and flexibility (Guerra, 2007), but it has a deeply rooted tradition in the tests of divergent thinking that are very much associated with the quantification of creativity (Plucker & Renzulli, 1999). To test fluency, individuals are asked to produce various responses to visual or verbal prompts, and scores are given to the number of different ideas produced. However, as Plucker and Renzulli (1999) refer, there are risks into overemphasizing ideational fluency, especially given that it may be detrimental to originality (the rareness of ideas). According to the authors, the ability to generate numerous ideas is just one of the aspects of the creative process, but to overstate its importance may dissipate the fundamental role of problem solving in creativity, which substantially draws on convergent thinking. Therefore, as they conclude, the role of fluency in the assessment of creativity may be more complex than it is often admitted.

Flexibility is a feature of creativity that is related to the apparent changes in the approaches taken in producing a response to a request. It has to do with the ability to consider a variety of options and think carefully about each alternative before choosing one. Cognitive flexibility stands out through the ability to combine ideas in solving problems (Vale, 2012). Therefore, flexibility is an aptitude that enables individuals to cope with constant changes (Runco, 2004).

The relation between problem solving and creativity is not straightforward. Runco (2004), for example, states that creativity underlines problem solving and problem finding. He notices that creativity in problem solving may be related to avoiding fixation and routine practices and over-relying on established knowledge.

Some authors have also argued that creative problem solving is a matter of how one defines the problem (Runco & Sakamoto, 1999). In fact, in the literature on mathematics problem solving, the well-known distinction between routine and nonroutine problem solving brings the key point of the need to develop some novel and insightful approach to arrive at a solution.

The creative application of mathematical knowledge in solving problems involves the activation of mathematical rules and procedures in non-standard combinations (Pehkonen, 1997; Silver, 1997). Thus, the manifestation of mathematical creativity and its consequent materialization in solutions to problems may arise from the combination of convergent thinking and divergent thinking, since this intersection involves a wide variety of ideas, some of which are useful in problem solving (Pehkonen, 1997). Convergent and divergent thinking are therefore both essential in problem-solving experience. While convergent thinking emphasizes the reproduction and adaptation of existing knowledge to new situations, divergent thinking involves the production of new ideas in reaction to the challenge at hand.

The act of solving a problem, which involves mobilizing knowledge inherent to a certain domain, as well as isolating the structural elements of the situation, may be seen as a creative act. Good mathematical problems are, by nature, challenging situations that encourage individuals to structure thinking in order to generate links between given and absent elements, based on their previous experience (Freiman, 2006).

In the case of mathematical problem solving, ideational fluency may be paralleled to the use of various concepts, procedures and mathematical results and the ability to coherently structure several steps to solve a problem. Thus, the problem presents a request to which the solver responds with a number of appropriate, useful and meaningful mathematical ideas that must be articulated in the search for a solution. In this context, knowledge activation is seen as a necessary condition for successful problem solving, and it entails the ability to implement strategies and procedures with efficiency, precision and flexibility (NCTM, 2014). Therefore, applying mathematical content knowledge is more than retrieving memorized facts and procedures or performing algorithms; it refers to the readiness of a mathematical knowledge base (NCTM, 2014; Russel, 2000).

In the same way, as it has been strongly emphasized in the research on mathematical problem solving, the ability to use and adapt multiple representations and to switch between representations is part of a cognitive flexibility that contributes to success in achieving a solution (Heinze, Star & Verschaffel, 2009). Good problem solvers generally develop very rich representations while solving problems. In fact, the flexibility of representation in mathematics is a characteristic of the solvers who choose appropriate and strategic representational forms, according to the task proposed (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009). Each form of representation expresses elements of the underlying reasoning, in line with the choice of a particular strategy or approach (Preston & Garner, 2003; Steele, 2008). So, ideational flexibility seems to have also parallels with the ability of creating and using powerful mathematical representations, even because the representational means highlight the developed reasoning and its expression. In postulating the inseparability between solving the problem and expressing the solution to others

(Carreira, 2015; Carreira, Jones, Amado, Jacinto, & Nobre, 2016), challenging problems should encourage both mathematical reasoning (including adapting and applying concepts, procedures and strategies) and mathematical communication (including the choice and use of representational means to support and express mathematical thinking).

9.3 A Tool for the Analysis of Creative Mathematical Solutions

The present study reconsiders the traditional psychometric model underlying the measuring of creativity and divergent thinking, namely, by acknowledging the well-established components of creativity. However, we intend otherwise to characterize the mathematical creativity of the products submitted by children participating in the mathematics competition SUB12. The focus is placed on the creative product, and the assessment of creativity is based on a consensual approach supported by an analytical tool aimed to assist in examining the solutions and interpreting their creative quality and variability.

As stated before, we will not be looking for exceptional mathematical talent; we aim, on the contrary, for the little-c creativity: the mathematical creativity in the solutions of children solving moderately difficult problems in effective ways.

The problems proposed in the competition can be solved in different ways and do not require the application of specific curriculum contents, giving opportunity and freedom for students to find their own resolution processes and even to invent their own strategies. To describe creative solutions, we will look at the dimension of *originality* as being related to the ability to generate unique resolutions within a given sample, taking into account the context of the competition and the level of proficiency which is attainable by the participants. In other words, originality reflects the ability to produce unusual ideas and unusual ways to solve problems (Gomez, 2007; Sriraman, 2008). Another dimension to be considered is the *knowledge-based activation*, which relates to the use of concepts, procedures and mathematical notions, and also to the way of imagining and performing steps to solve the problem (Fig. 9.1). The activation of mathematical knowledge thus concerns the ability to use knowledge in a clear and effective way, such as in devising strategies, as well as linking together ideas, concepts and procedures. Creative mathematical thinking is here connected to proficiency in mathematical concepts and procedures, especially when applied in novel ways to solve nonroutine tasks (Aizikovitsh-Udi, 2013).

The third dimension that the model postulates is the *representational means activation*. This dimension is linked to the ability to select, combine, create and use representations, according to the characteristics of the problem, supporting the reasoning followed and the expression of mathematical thinking (Heinze et al., 2009). Thus the flexible use of representational means involves conveying the reasoning that leads to a solution, the choice of strategic representations and the communication and expression of the resolution in clear and accurate ways (Fig. 9.2). The idea

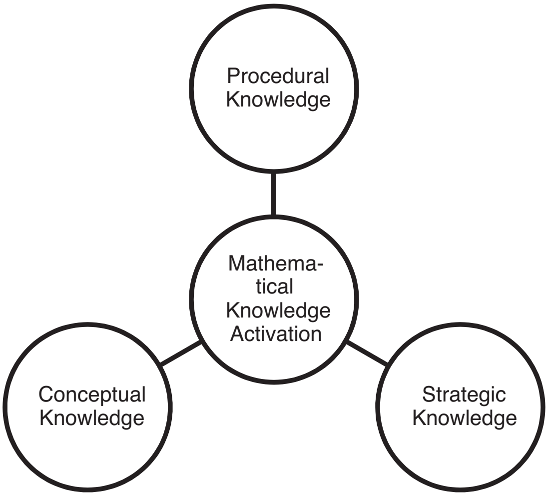


Fig. 9.1 Elements of mathematical knowledge activation in creative mathematical problem solving

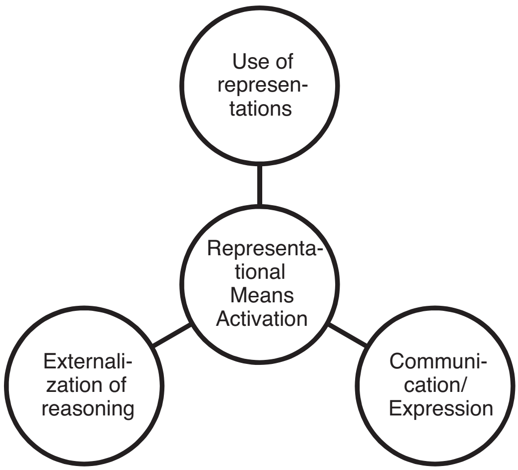


Fig. 9.2 Elements of representational means activation in creative mathematical problem solving

of representational flexibility has to do with the fact that the same mathematical concept can be viewed from different perspectives and represented in different ways, but it also means that representations are forms of expressing and communicating reasoning (Andresen, 2007; Gagatsis & Elia, 2004).

In short, the concept of creativity will be operationalized by the combination of the three dimensions: (i) originality, (ii) activation of mathematical knowledge and (iii) activation of representational means. In this system, originality plays a central role in ascribing creativity to a resolution.

Bearing in mind the three dimensions and their most relevant elements, the analytical tool shown in Table 9.1 offers a set of coded indicators and descriptors that summarize the essential elements attributed to each of the three dimensions of creativity, in a total of 15 descriptors (O1 to O5; K1 to K5; R1 to R5).

Table 9.1 The analytical tool for describing the mathematical creativity of the resolutions

Indicators (code)	Descriptors (code)
Originality (O)	1) effective and unusual resolution (O1) 2) uncommon and significant ideas (O2) 3) striking and effective strategy (O3) 4) clear and uncommon reasoning (O4) 5) distinctive and accurate form of communication (O5)
Mathematical knowledge activation (K)	1) relevant mathematical content according to givens and goal (K1) 2) insightful and efficient use of mathematical content (K2) 3) procedural knowledge used with skill and efficiency (K3) 4) conceptual knowledge expressed in formal or informal ways (K4) 5) display of the structuring stages of the resolution (K5)
Representational means activation (R)	1) use of appropriate representations (R1) 2) different representations cleverly interconnected (R2) 3) presence of personal and distinctive forms of representation (R3) 4) use of strategic representations for solving the problem (R4) 5) representations supporting the communication and the reasoning (R5)

The above tool, in short the "O.K.R" model, is intended as a practical tool for highlighting evidence of mathematical creativity in students' solutions to mathematical problems in the specific context of the competition SUB12. Even though it has been tested and applied to several empirical data (Amaral, 2016; Amaral & Carreira, 2012) and has undergone successive refinements and revisions, it remains in an evolutionary state as further investigations and results are being assembled.

9.4 The Methodological Procedure

As part of a larger study on the creative participants' solutions to nonroutine problems, within the Problem@Web Project, here we will focus on a set of 10 selected resolutions (within a population of 177) to problem #7 posed in the 2012/2013 edition of the SUB12 competition, which is transcribed below:

"*All locked*"
In a drawer we have 20 locks and 20 keys. Each key opens exactly one lock, but we do not know which key corresponds to each lock. To associate each key to the corresponding lock, we have to proceed by trial. Suppose that one trial means trying a key in a lock. In the worst-case scenario, what is the number of trials that we need to make to associate each key to the respective lock?
Do not forget to explain your problem-solving process.

The selection of the ten resolutions of the problem "All locked" was subject to the following criteria: its uniqueness, the mathematical knowledge revealed, the strategies developed and the forms of representation used. They were the result of filtering the most original ones from the complete set of resolutions (n = 177, 98 from fifth graders and 79 from sixth graders), in line with the adopted concept of creativity in which originality is a decisive component, albeit not enough for the characterization of a creative product. In the assessment of the originality dimension, each resolution was evaluated within the population of the resolutions that were submitted to the competition. The two authors were responsible, through consensual assessment, for the analysis of all the solutions that led to the selection of the 10 final ones; there was a complete agreement on the application of the established criteria. The small number of selected resolutions is intentional, given the main purpose of highlighting, in a qualitative way and in a sample of productions, the nature of mathematical creativity in problem solving, according to the indicators of originality (within a target group), mathematical knowledge activation (which refers to concepts, procedures and strategies) and representational means activated (which includes the suitability and relevance of the representations used).

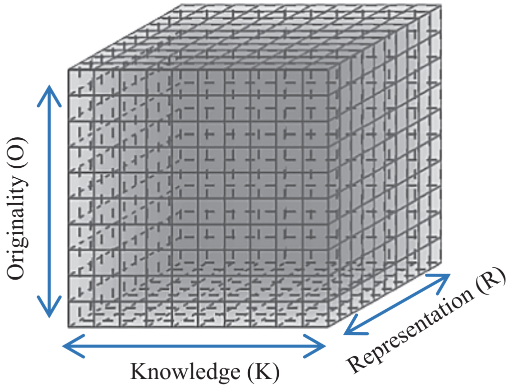
Content analysis is a method of data analysis supported by systematic procedures of content description intended to determine the presence of certain expressions or concepts in the discourse of participants and/or in documents produced by them. Through it, the researcher interprets the meaning of the concepts and relationships between them, making inferences from the description of the data (Bardin, 2004). It is generally based on systematic encodings that help to understand the phenomena beyond the common sense (Moraes, 1999).

The thematic content analysis adopted in this study requires at least three steps (Bardin, 2004). At first, a pre-analysis is performed to prepare the actions to be taken in the analysis itself. This involves defining the corpus of analysis (choice and organization of documents), the formulation of the goals and the development of indicators that support the final interpretation. In this study the analysis of the students' resolutions will be supported by the application of the above analytical tool consisting of three indicators and several descriptors specifically created for identifying the mathematical creativity of submitted solutions in the particular context of the SUB12 competition.

The next phase is dedicated to the exploration of the material, which consists of the systematic transformation of the raw data to allow the description of relevant evidence. In this step, the operations of coding, classification and categorization are performed. For each resolution, the tool is used firstly to detect evidence of originality and subsequently to identify traces of knowledge activation and representational means used, by briefly describing how the encoding was obtained based on the descriptors.

Since the analysis is based on a comparison of resolutions in the sample, we have set the symbols plus (+), minus (−) and zero (0) for labelling a greater or lesser presence of descriptors or its total absence in each of the selected solutions. To allow a more clear and concise picture of the codification of the mathematical creativity, the coded descriptors for each resolution are compiled in tables.

Fig. 9.3 The 3-D graph of the indicators of mathematical creativity in problem solutions



After the encoded description of each resolution, a schematic summary is created, using a three-dimensional graph (Fig. 9.3) representing the size of the three dimensions: originality (O), mathematical knowledge activation (K) and representational means activation (R). The number represented in each axis of the graph aims to give an impression of the breadth of that same dimension. That number is obtained by converting the symbolic codes assigned to each of the 15 descriptors, plus (+), minus (−) and zero (0), to the numerical values 1, 1/2 and 0, respectively. In this way, the size of a dimension is the sum of the values given to the descriptors that compose it, therefore ranging between 0 and 5, with half-unit increments. At the end of the analysis of the resolutions, a snapshot of the type of mathematical creativity arises from the tables of encoded descriptors and the three-dimensional graphs.

The third, and final stage, comprises the interpretation of the results obtained in all the collected material, and a comparative analysis aimed at highlighting the similar and different aspects.

9.5 The Analysis of the Solutions Using the Analytical Tool

The situation described in the problem "All locked" involves the discovery of a number pattern, drawing on inductive reasoning, since it is necessary to test, one by one, each key in all the locks to find the lock that matches it. The problem refers to an extreme situation (the worst-case scenario) where a key-lock-pair is found only on the last attempt – when all the locks, except one, have been tested and rejected. Inductive reasoning may suggest an ordering of the keys, k_1, k_2, …, k_{20}, and padlocks, p_1, p_2, …, p_{20}, and a strategy that sets out all the tests on the key k_1 (19 padlocks), the key k_2 (18 padlocks), etc., bearing in mind that only the last padlock matches the key that is being tested. Thus, one has to find the number of tests that will be performed with each key (or each lock) to arrive at the total number of trials. The result is given by the sum of the first 19 natural numbers. In solving the problem, several ideas will be relevant to approach the situation from a mathematical perspective: (i) find a systematic method of counting the failed attempts; (ii) consider simpler situations (fewer keys and padlocks); (iii) make trials with some particular cases and look for a pattern; (iv) focus on an iterative process and look for a rule; and (v) create forms of pairing, such as a double-entry table.

In the set of selected resolutions, these ideas or approaches are present in various ways, different from case to case, which makes them unique and distinct yet sharing common points.

9.5.1 *Coding of the Solution S1*

The solution S1 (Fig. 9.4) is distinct from the others in the way it emphasizes an organized process of counting the failed attempts and in making use of a double-entry table to make the counting systematic, comprehensive and categorical.

1. I have read and understood.
2. Givens: 20 locks and 20 keys. Each key opens a lock.
3. Question: In the worst-case scenario, which is the number of attempts that we need to do to associate each key to the respective lock?
4. Count: 190
5. Answer: 190 attempts.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	X
2	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	X	X
3	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	X	X	X
4	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	X	X	X	X
5	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	X	X	X	X	X
6	86	87	88	89	90	91	92	93	94	95	96	97	98	99	X	X	X	X	X	X
7	100	101	102	103	104	105	106	107	108	109	110	111	112	X	X	X	X	X	X	X
8	113	114	115	116	117	118	119	120	121	122	123	124	X	X	X	X	X	X	X	X
9	125	126	127	128	129	130	131	132	133	134	135	X	X	X	X	X	X	X	X	X
10	136	137	138	139	140	141	142	143	144	145	X	X	X	X	X	X	X	X	X	X
11	146	147	148	149	150	151	152	153	154	X	X	X	X	X	X	X	X	X	X	X
12	155	156	157	158	159	160	161	162	X	X	X	X	X	X	X	X	X	X	X	X
13	163	164	165	166	167	168	169	X	X	X	X	X	X	X	X	X	X	X	X	X
14	170	171	172	173	174	175	X	X	X	X	X	X	X	X	X	X	X	X	X	X
15	176	177	178	179	180	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
16	181	182	183	184	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
17	185	186	187	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
18	188	189	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
19	190	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
20	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Fig. 9.4 Solution S1 from a sixth grader

9.5.1.1 Originality Description

The solution includes two main elements: a small text explaining successive steps of the solving process and a double-entry table with 20 columns and 20 rows, each of them labelled with the numbers from 1 to 20. The entries of the table are shadowed in grey and apparently serve as a continuous counting register of the failed trials, while the symbols X seem to represent the cases that are not to be considered in the counting.

In general, the resolution can be characterized by a median originality, according to the coding of its five descriptors. This is an effective yet unusual resolution, considering that a full counting of the attempts, one by one, is not a very expeditious approach. In this case, the effectiveness means having reached the desired result even if using mostly endurance (O1+). It is not obvious any uncommon mathematical idea in the approach (O2 absent); in any case, the strategy of using the consecutive numbering of cells in a double-entry table strikes for its simplicity and the fact that it clearly works (O3+). As for the reasoning set out in the search for the solution, it is clearly based on finding a reliable and organized method of counting, which is materialized in the table; some lack of clarity is felt as the meaning of the "X" in the non-numbered entries is not explicit and can only be guessed. On the other hand, the lack of labels on the table rows and columns seems to indicate that it is indifferent (keys and padlocks are commutable) and it only matters the recording of the pairs that are checked and do not work (O4−). In what concerns the process of communicating the way to achieve the solution, it is a proposal that though revealing the essential elements does not offer an explanation of the entries of the table and of the way it was conceived to provide a counting method; therefore accuracy in the communication is not a strong quality (O5−). In short, the reasoning that led to the solution and the method of communicating and explaining the process somehow restrict the originality.

9.5.1.2 Knowledge Description

The most relevant mathematical knowledge to achieve the solution appears to be the understanding of the given conditions and the implementation of a comprehensive and systematic counting method of the failed attempts (K1+).

There is no significant use of mathematical content or procedural knowledge skilfully executed, namely, the possibility of using addition to get the total of trials; on the contrary, there is a long counting process of the trials, one by one (K2, K3 absent). The use of a double-entry table does reflect the notion that there is a combination process between keys and padlocks, and this is a useful form of translation of a cross product of two sets (K4+). Finally, the structure of the successive stages of the resolution seems evident although the stage dedicated to count is not properly documented and the process of filling the double-entry table is not perfectly clear (K5−).

In short, in terms of knowledge, both conceptual and procedural, there is little proficiency in approaching the situation, without this invalidating an organized way to structure the search for the solution.

9.5.1.3 Representational Means Description

In the domain of representation, it stands out the double-entry table and its completion with successive numbers starting at 1 and ending at 190, indicating all the failed attempts and their total. There is also a brief description of the process that basically reproduces the problem conditions and points to a counting procedure. The connection between the two elements is very weak, and the table meaning must be decoded by the reader. The double-entry table although not sufficiently explained is an appropriate representation to the purposes of the counting procedure (R1+). There are no other mathematical representations presented as the basic strategy stands on doing a thoroughgoing counting (R2 absent). In any case, the representation found for this purpose has a personal touch in supporting a process which is naive and somewhat trivial but which requires no technical expertise and turns out to be reliable (R3+). The use of a double-entry table is a strategic choice in developing a systematic representation of checking each pair of key lock, but, in addition, it serves as a means to count, bypassing the need for any calculations or other procedures (R4+). Finally, the representation clearly supports the reasoning, i.e. the counting procedure, but its communicational ability is not fully harnessed in expressing the solving process (R5−).

In short, there is the resort to a strategic representation that supports a counting process, but that representation is handled poorly, and it shows some opacity.

9.5.1.4 Mathematical Creativity Description

According to the coding of the three indicators of mathematical creativity, below is a descriptive summary of the solution S1 (Table 9.2).

Table 9.2 Coded descriptors of creativity in the solution S1

	Indicators/descriptors														
	Originality (O)					Knowledge (K)					Representation (R)				
Solutions	O1	O2	O3	O4	O5	K1	K2	K3	K4	K5	R1	R2	R3	R4	R5
S1	1	0	1	1/2	1/2	1	0	0	1	1/2	1	0	1	1	1/2

9.5.2 *Coding of the Solutions S2 and S3*

The two solutions (Figs. 9.5 and 9.6) reveal approaches with many similar aspects. Both rely on a double-entry table to record the tests done with the various locks and keys, by organizing the entries in the form of successes or failures. The use of colours is common to both tabular representations. Also in each case, the pairs that are correct associations are highlighted in the diagonal. The main distinctions occur in terms of detail and also in the explanation attached to the double-entry table.

I considered that the best case would be to get the right lock - key pairs at the first attempt (20). On the contrary, in the WORST case the number of at tempts **to open** all the locks is [[(20×20)-20]/2]+20 = 210 attempts (this is assuming that each key that "does not fit" in a lock will not be used again in the same wrong lock). However if the requirement is "**to associate** each key to the respective lock" then it is just 210 attempts MINUS 20 hits = 190. (For instance, for the CAD1 it is required 19 attempts because even without doing the 20th test it is known that the remaining key is the right one!)

Note: CAD is short for CADEADO (Lock); CH is short for CHAVE (Key).

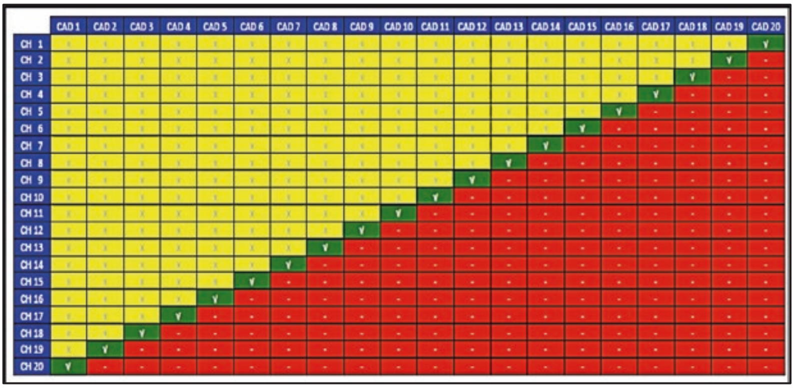


Fig. 9.5 Solution S2 from a sixth grader

Sum of trials: 19 + 18 + 17 + 16 + 15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = **190**

Or: using the average number of trials for the 20 keys, which is 9.5, we have 9.5×20 = **190** (this is to confirm).

This is the solution if we assume that we don't check that each key fits in the last lock available. If we had to check that each key works in the last lock, then it would be another 20 trials, therefore 210 trials.

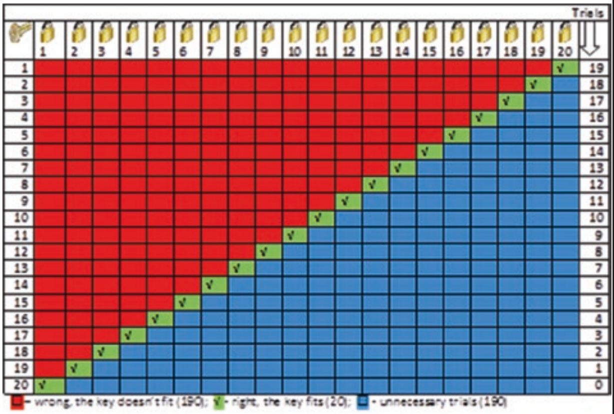


Fig. 9.6 Solution S3 from a fifth grader

9.5.2.1 Originality Description

In both cases we are dealing with effective resolutions, but obviously not unusual due to the strong similarity between the two (and also with others in the bulk of answers submitted), in particular as regard to the tabular representation that each includes (O1− is assigned to both). Any of the two contains very interesting and different ideas. The solution S2 suggests an approach to compute the number of cells with crosses as a kind of "finding an area", i.e. subtracting the diagonal to the total of cells and dividing the remaining by two. The solution S3 goes towards the successive addition of values because the table includes a column displaying the record of failed attempts for each verified key (O2+ is assigned to both).

Any of the strategies proves to be very clear and effective. However, the first of the solutions has a distinctive element in the computation method of the failed attempts, which captures the spatial arrangement of the cells in the table. Therefore, the originality of the strategy is perceived as more striking in the solution S2 (O3+) than in the solution S3 (O3−). There is also evidence, in both cases, of a correct reasoning and accuracy in the justifications. However, the solution S3 presents a more unusual approach, with a "geometric" flavour. Therefore, some differentiation is made between S2 (O4+) and S3 (O4−). As to the clarity and distinctiveness of communication, both cases are noticeable by the ability to generate verbal and non-verbal elements that allow us to understand the solving process and its expression (O5+ assigned to both). These are two resolutions with a high degree of originality that take good advantage of the tabular representation, with a small lead of the solution S2 due to the curious form of calculation of the failed attempts.

9.5.2.2 Knowledge Description

Both resolutions reveal the use of valid and useful mathematical content to go from the givens to the goal. The number of keys and padlocks and their representation, in clearly labelled rows and columns in a table, are a suitable way to get a systematic and organized record of successes and failures. The calculation of the number of failed attempts is mathematically well founded in both cases (K1+ assigned to both). Still, there is a clear presence of a mathematical process related to the sum of the terms of a finite arithmetic progression in the solution S3 (K2+), something that is not recognized in the solution S2 (K2−). At the level of the efficient use of procedural knowledge, both solutions show great skill and clarity in every detail (K3+ assigned to both). The same happens with the expressed conceptual knowledge (K4+ assigned to both), although revealing distinct conceptual models: the result as the sum of the average number of attempts $((0 + n - 1)/2) \times n$ or as given by the formula $(n^2 - n)/2$. It is also apparent a capacity to structure the resolution in both the solutions, although the solution S2 shows greater concern to verbalize an understanding of what is asked and what the answer means as opposed to other possible views of the situation (K5+) compared to the solution S3 that goes directly to the table construction and from there to the calculations (K5−).

9.5.2.3 Representational Means Description

It can easily be seen that the representational power in both resolutions is very high. The representations are appropriate: in particular the double-entry table and the translation into the arithmetic language are done cleverly, which leads to see the two resolutions as quite balanced, more verbal the first and more figurative the second one (R1+ and R2+ assigned to both). Any of the tables has a distinctive feature, though both rely heavily on the use of colours and a few icons, which makes them rather similar (R3− assigned to both).

The tabular representations are clearly strategic to solve the problem as well as the use of synthetic calculations that are well set out (R4+ assigned to both). Lastly the use of well-constructed representations is a fundamental basis for the development and expression of the mathematical reasoning in both the solutions (R5+ assigned to both). Thus, the ability to represent is a crucial point in these two solutions, contributing substantially to their level of representational activation.

9.5.2.4 Mathematical Creativity Description

In summary, we present a condensed description of the mathematical creativity of the two solutions in Table 9.3.

Table 9.3 Coded descriptors of creativity in the solutions S2 and S3

	Indicators/descriptors														
	Originality (O)					Knowledge (K)					Representation (R)				
Solutions	O1	O2	O3	O4	O5	K1	K2	K3	K4	K5	R1	R2	R3	R4	R5
S2	1/2	1	1	1	1	1	1/2	1	1	1	1	1	1/2	1	1
S3	1/2	1	1/2	1/2	1	1	1	1	1	1/2	1	1	1/2	1	1

9.5.3 Coding of the Solutions S4, S5, and S6

The three resolutions (Figs. 9.7, 9.8, and 9.9) have in common the fact that there is the search for a numerical pattern depending on the number of keys and locks to be taken, based on a recursive process that relates the next term of the sequence to its previous one. Thus, all the cases refer to the idea of starting with a small initial number of keys and locks and then successively going up a unit. At the end, the several terms of the sequence are added, and the answer is found.

1. We have read the problem…
2. We know that there are twenty keys and twenty locks. Each key opens only one lock…
3. And we have to know the number of trials in the worst case.
4. And we remembered a problem of handshakes that was very similar…
5. Then we started to think about... the handshakes between people… Each handshake corresponds to a trial of a key in a lock…
6. With 1 person there are no handshakes to give!
7. With a new person, we get 2 persons and there is 1 handshake;
8. If a 3rd person comes, that person gives two handshakes to the people who were there and who had already greeted. Total: 2 +1 handshakes.
9. And so on... When a person comes, that person gives a handshake to the people who had already greeted... and then we add it.
10. And now we represent this on a table.
11. Conclusion: at worst, with twenty keys and twenty locks, there are 190 trials.

One lock and one key	0 trials
Two locks and two keys	1 trial
Three locks and three keys	2+1=3 trials
Four locks and four keys	3+2+1=6 trials
Five locks and five keys	4+3+2+1= 10 trials
Six locks and six keys	5+4+3+2+1=15 trials
Seven locks and seven keys	6+5+4+3+2+1=21 trials
Eight locks and eight keys	7+6+5+4+3+2+1=28 trials
Nine locks and nine keys	8+7+6+5+4+3+2+1=36 trials
Ten locks and ten keys	9+8+7+6+5+4+3+2+1=45 trials
Eleven locks and eleven keys	10+9+8+7+6+5+4+3+2+1=55 trials
Twelve locks and twelve keys	11+10+9+8+7+6+5+4+3+2+1=66 trials
Thirteen locks and thirteen keys	12+11+10+9+8+7+6+5+4+3+2+1=78 trials
Fourteen locks and fourteen keys	13+12+11+10+9+8+7+6+5+4+3+2+1=91 trials
Fifteen locks and fifteen keys	14+13+12+11+10+9+8+7+6+5+4+3+2+1=105 trials
Sixteen locks and sixteen keys	15+14+13+12+11+10+9+8+7+6+5+4+3+2+1=120 trials
Seventeen locks and seventeen keys	16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1=136 trials
Eighteen locks and eighteen keys	17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1=153 trials
Nineteen locks and nineteen keys	18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1=171 trials
Twenty locks and twenty keys	19+18+17+16+15+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1=190 trials

Fig. 9.7 Solution S4 from a group of two fifth graders

9.5.3.1 Originality Description

From the standpoint of the effectiveness and the unique character, it is clear that the three resolutions are effective and convincing, but they share many commonalities. However, the solution S4 is distinguished by the insightful association to a related situation (the problem of the handshakes), and the solution S5 has the benefit of being the only one that does not thoroughly specifies all the terms of the sequence (O1+ assigned to both), so by comparison, this reduces the strength of the solution S6 (O1−). In terms of powerful and unusual ideas, it is clear that the solution S4 stands out for the connection to the handshakes problem and for the way that connection is understood and described. Likewise, the solution S5 lies on inductive reasoning and seeks to make a generalization leading to the sum of the terms of an

To find an easy way to discover the answer to the problem, I considered a few cases. I started with the easiest situation, as if I had initially 2 locks and 2 keys and then I extended the process by adding more locks and keys. For that, I made a "table"...
We have to realize that in order to get the next number of trials we have to add the new trials to the previous number of trials... So when we get to the number 20 (twenty locks, twenty keys), we just have to add all the trials.
I have a way of adding many sequential numbers that give large sums; I hope you understand the way that I use.
We have to do:
1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19=(1+19)/2×19=190.
Ans: At worst, 190 trials are made.

No. of locks or keys	No. of trials
2	1
3	2 + 1
4	3 + 2 + 1
5	4 + 3 + 2 + 1
20	19 + 18 + 17 ... + 1

Fig. 9.8 Solution S5 from a sixth grader

arithmetic progression (O2+ assigned to both). In the case of the solution S6, the relevant ideas are hardly evident (O2 absent). For an impressive and effective strategy, it appears that all the three resolutions follow an inductive process by increasing the number of keys and padlocks. Apparently, this strategy has a greater impact on the solution S4, because of the link to the handshakes lending meaning to the recursive model (O3+), than on the solutions S5 and S6 (O3−).

Concerning the reasoning revealed by the three resolutions, it seems clear that the solution S6 has several limitations in showing the way to get the table contents, that is, the justification of the calculations. We argue that this solution reveals the absence of this descriptor (O4 absent). When comparing the other two solutions, one realizes that the solution S4 is clearer in explaining the reasoning that leads to the calculations (O4+) than the solution S5 (O4−). Finally, with regard to the form of communication and expression of the solution, all the resolutions show precision and detail, although the solution S6 is less able to account for the entire process (O5−) than the solutions S4 and S5 (O5+).

9.5.3.2 Knowledge Description

Any of the resolutions demonstrates the use of relevant mathematical content to solve the problem, namely, the ability to formulate a numerical pattern that is recursively developed and to represent this pattern in a table for the calculation of the failed attempts as a function of the number of keys and padlocks (K1+ assigned to

1. I read the problem in order to understand it.
2. When I was reading, I took the following information:
-There are 20 locks and 20 keys that have been shuffled;
-Every single key corresponds to a lock;
3. Trying a key is the same as making an attempt;
4. I want to know: "At worst, what is the number of attempts we need to do to associate each key to the respective lock?".
5. To make it easier, I started working with a small number of keys and an equal number of padlocks.
6. Then I kept increasing the number of keys and padlocks and organized everything in a table.
7. The table is the following.
8. In short, the answer to the problem is: The number of attempts I need to do to make sure that each key finds its lock is 190 attempts.

Keys and locks	Trials	Sum of the trials
1	0	0
2	1	1
3	2+1	3
4	3+2+1	6
5	4+3+2+1	10
6	5+4+3+2+1	15
7	6+5+4+3+2+1	21
8	7+6+5+4+3+2+1	28
9	8+7+6+5+4+3+2+1	36
10	9+8+7+6+5+4+3+2+1	45
11	10+9+8+7+6+5+4+3+2+1	55
12	11+10+9+8+7+6+5+4+3+2+1	66
13	12+11+10+9+8+7+6+5+4+3+2+1	78
14	13+12+11+10+9+8+7+6+5+4+3+2+1	91
15	14+13+12+11+10+9+8+7+6+5+4+3+2+1	105
16	15+14+13+12+11+10+9+8+7+6+5+4+3+2+1	120
17	16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1	136
18	17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1	153
19	18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1	171
20	19+18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1	190

Fig. 9.9 Solution S6 from a fifth grader

all). The more efficient use of mathematical concepts emerges in the solutions S4 and S5 (K2+) either by the recursive process or by the generalization that leads to the terms of an arithmetic progression. In the case of the solution S6, the use of mathematics is less efficient (K2−).

Regarding the procedural knowledge, the S5 solution stands out because it is the only one capable of proposing a formula from a set of particular cases and advancing to the sum without specifying all the iterations of the process (K3+). By contrast, the solutions S4 and S6 proceed with the completion of each successive addition, in a redundant way (K3−). As for the conceptual knowledge involved in the problem, all resolutions show the mastery of essential ideas that lead to the construction of a numerical sequence and the discovery of a pattern that is used to get the answer (K4+ assigned to all). Finally, it is also clear, in all the solutions, an understanding of the steps that structure the resolution of the problem, in particular, the central role of the table created (K5+ assigned to all).

9.5.3.3 Representational Means Description

Moving to the indicator of representational means activation, we notice that the three resolutions show a high level of ability in the use of representations. All rely on appropriate representations, namely, tables and arithmetic language, to carry out the inductive reasoning that leads to the solution (R1+ assigned to all). Also the different types of representations are well connected, namely, in the solutions S4 and S5 (R2+), and only poorly interconnected in the solution S6 since the recursive formula displayed in the successive rows is not linked to any other elements of the resolution (R2 absent). In any case, those are solutions in which the presence of distinctive representations is visible, both regarding the use of tables and the use of verbal and mathematical language (R3+ assigned to all). Likewise, the use of tables to express the ordered sequence with the increasing number of keys and locks is a strategic choice in all cases (R4+ assigned to all). Finally, the case of the solution S6 is one where the tabular representation is not totally effective as a way of communicating thinking (R5−), unlike what happens in the solutions S4 and S5 (R5+ assigned to both).

9.5.3.4 Mathematical Creativity Description

The three solutions S4, S5 and S6 that were coded above by means of the adopted descriptors are now described in terms of their mathematical creativity (Table 9.4).

Table 9.4 Coded descriptors of creativity in the solutions S4, S5 and S6

	Indicators/descriptors														
	Originality (O)					Knowledge (K)					Representation (R)				
Solutions	O1	O2	O3	O4	O5	K1	K2	K3	K4	K5	R1	R2	R3	R4	R5
S4	1	1	1	1	1	1	1	1/2	1	1	1	1	1	1	1
S5	1	1	1/2	1/2	1	1	1	1	1	1	1	1	1	1	1
S6	1/2	0	1/2	0	1/2	1	1/2	1/2	1	1	1	0	1	1	1/2

9.5.4 Coding of the Solutions S7, S8, S9, and S10

It is common to the four solutions (Figs. 9.10, 9.11, 9.12, and 9.13) the fact that all of them conceive ways of portraying the actual situation of trying the keys in the locks, one after the other. In a certain sense, these solutions offer a narrative of the

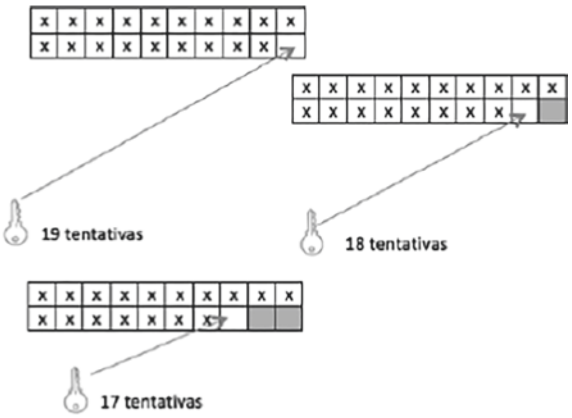
I saw that it would always decrease one unit. Since there were 20 locks, it means to decrease 1 unit 20 times. 19+18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1+0 = 190 attempts.	

Fig. 9.10 Solution S7 from a sixth grader

problem situation and in it the process to achieve the solution, by doing that in a more pictorial way or in a more verbal one.

9.5.4.1 Originality Description

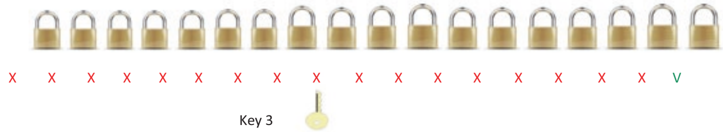
The solution S7 shows in a schematic way a brief simulation of the situation for the first three keys and then generalizes the rule for the remaining ones. Likewise, the solution S8 uses pictures and thoroughly conveys the idea that the number of trials is decreasing by one for each new key. And the solution S9 presents a remarkable symmetric table showing in iconic terms how each new lock is successively hit in the last attempt. The solution S10 just explains in a very clear and remarkably simple way what happens in the process, and it also uses a table to present the ordered keys and the respective number of failed attempts. It is apparent that all the solutions are effective and unusual in their one way (O1+ assigned to all). The same may be said about the presence of relevant and original ideas to express the problem situation (O2+ assigned to all).

As to the strategy used, it is possible to see that the solution S7 is the most economical one, and it easily provides the basis to move towards the general rule. Therefore the solution S7 is distinguished in this point (O3+) by comparison with the other three (O3− assigned to the other solutions). In what concerns the disclosure of solid and clear reasoning, the solutions S7 and S10 demonstrate more clearly and simply the inductive reasoning that leads to the rule for the number of failed attempts (O4+ assigned to both). In fact, the solutions S8 and S9 tend to be more expensive in terms of developing pictorial representations to support the reasoning (O4− assigned to both). Finally, the solutions S7 and S10 are the most effective and accurate in terms of the communication process. They easily tell how the thinking is conducted and reinforce it with the help of simple representations (O5+ assigned

With the key 1, in the worst case, I only get a hit on the 20th lock, so I make **19 attempts**

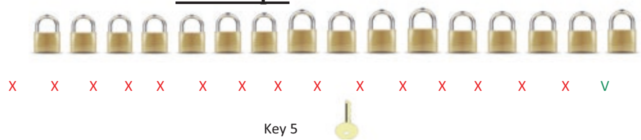
Now I only have 19 locks and 19 keys and I make new trials, and in the worst case I only get a hit on the 19th lock – I make **18 attempts**

I keep making attempts, now with 18 keys and 18 locks and in the worst case I get a hit in the 18th lock – I make **17 attempts**



Now I only have 17 locks and 17 keys and I make new trials and in the worst case I get a hit on the 17th lock _ I make **16 attempts**

Now I only have 16 locks and 16 keys and I try again and in the worst case I get a hit on the 16th lock – I make **15 attempts**



Each time I use a new key, since it is always in the worst possible case, I hit the last lock (that does not count as a trial for it is certain that it's the key to that lock), so I still have to try with the remaining 15 keys:

Key 6 – 14 trials	Key 11 – 9 trials	Key 16 – 4 trials
Key 7 – 13 trials	Key 12 – 8 trials	Key 17 – 3 trials
Key 8 – 12 trials	Key 13 – 7 trials	Key 18 – 2 trials
Key 9 – 11 trials	Key 14 – 6 trials	Key 19 – 1 trials
Key 10 – 10 trials	Key 15 – 5 trials	Key 20 – 0 trials

Then I added the number of trials:
19+18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1+0 = 190 trials.
ANSWER: At worst, the minimum number of trials I have to do to associate each key to the respective lock is 190 trials.

Fig. 9.11 Solution S8 from a fifth grader

In the worst case, the number of attempts will be 190. I made this chart to help explain my reasoning. I concluded that the number of attempts in each row is equal to the number of locks to open, minus one.
In the first row there are nineteen attempts, because after nineteen attempts, all unsuccessful, the twentieth is no longer an attempt, it is a certainty, because now we are sure that this key belongs to the lock, since the others did not. In the second row there are eighteen attempts, since the eighteenth attempt is the last one before the lock that was already opened. They are eighteen because the lock that was previously opened no longer counts, since we already know its corresponding key. And so on, until we have just two locks and one failed attempt.

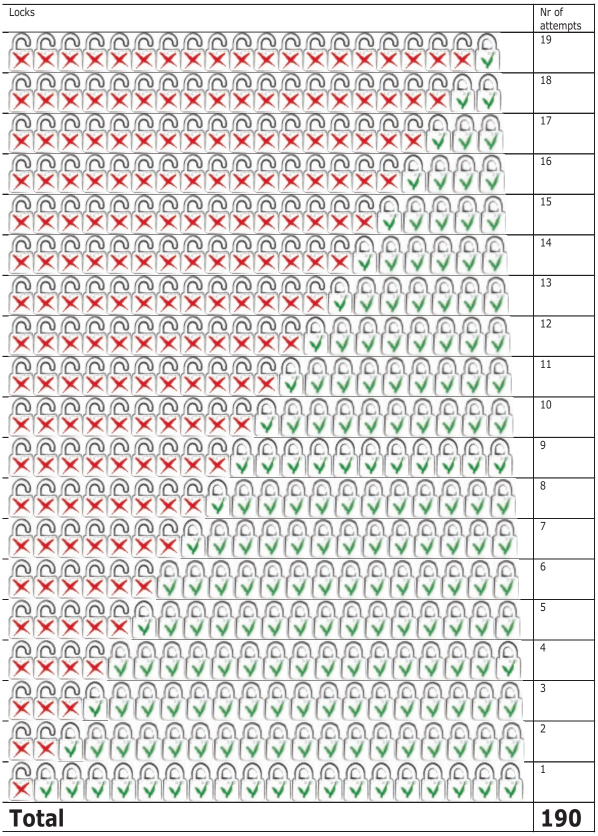


Fig. 9.12 Solution S9 from a fifth grader

to both). The other solutions although interesting and effective in the communicational facet reveal a more intricate style in the expression of the solution (O5–assigned to the others).

9.5.4.2 Knowledge Description

The four resolutions reveal considerable mobilization of mathematical concepts related to formulating numerical sequences, both through recursive reasoning and as a relationship between the number of attempts and the number of keys or padlocks.

Initially, I have 20 keys for 20 locks.
I will number the keys from 1 to 20.
For the key #1 I have 20 locks, of which 19 are wrong and only one is right.
So at worst I make 19 attempts for the key #1, and after those attempts I associate it to the correct lock.
Now the key #2 has only 19 locks available, so I make 18 attempts to get to the right lock, and so on, as shown in this table.
Answer:
To associate all the keys to their locks thereby the attempts needed are 190 in total (1+2+3+4+5+....+ 18+19).

keys	attempts	locks
1ª	19	20
2ª	18	19
3ª	17	18
4ª	16	17
5ª	15	16
6ª	14	15
7ª	13	14
8ª	12	13
9ª	11	12
10ª	10	11
11ª	9	10
12ª	8	9
13ª	7	8
14ª	6	7
15ª	5	6
16ª	4	5
17ª	3	4
18ª	2	3
19ª	1	2
20ª	0	1

Fig. 9.13 Solution S10 from a fifth grader

In this sense, it seems legitimate to consider that all resolutions exhibit relevant mathematical knowledge that take into account the givens and the goal (K1+ assigned to all). Likewise, this knowledge is used efficiently and insightfully (K2+ assigned to all). In what concerns the application of specific techniques or procedures, it is noted that all these resolutions make use of the simple addition of the successive values of the sequence (terms of progression). For this reason, a major level of procedural skill is not detected (K3− assigned to all). In contrast, the conceptual models developed in the resolutions are adequate and clearly lead to obtaining an arithmetic progression whose terms are essential for calculating the desired result (K4+ assigned to all). Finally, all the resolutions show how the approach taken is organized in a number of steps leading to the final calculation that provides the answer (K5+ assigned to all).

9.5.4.3 Representational Means Description

At the level of representational means activated, there seems to be more pronounced distinctions between the four resolutions. In any case, it is clear that all of them resort to multiple and appropriate representations (R1+ assigned to all). Besides, all the resolutions show the interconnections of different representations in the narrative that expresses the solving process (R2+ assigned to all). No doubt the personal and distinctive touch of the representations used is also a trademark of any of the four resolutions. However, the solutions S7 and S8 propose illustrative schemes that have many similarities, thus reducing their differentiation (R3−) compared with the solutions S9 and S10 (R3+).

At the level of the strategic power of the representations used, we highlight the solution S7 for the use of an instructive diagram that illustrates a small number of tests but suggests its extension for the following cases (R4+). Likewise, the iconic representation involving a table of unpaired and paired padlocks proposed in the solution S9 has a very important role in the visualization of the rule and the apparent symmetry between the mismatches and hits (R4+). In the solutions S8 and S10, the forms of representation used (diagram and covariation table) are important but have a somewhat redundant character in that they thoroughly display the calculation of all the terms of the sequence (R4− assigned to both). Finally, the various types of representations chosen and used in these solutions play a fundamental role in supporting the communication of the reasoning (R5+ assigned to all).

9.5.4.4 Mathematical Creativity Description

For a summary of the mathematical creativity identified in the solutions S7, S8, S9, and S10, Table 9.5 presents the relative magnitudes of the three indicators based on the coding of the solutions.

Table 9.5 Coded descriptors of creativity in the solutions S7, S8, S9, and S10

	Indicators/descriptors														
	Originality (O)					Knowledge (K)					Representation (R)				
Solutions	O1	O2	O3	O4	O5	K1	K2	K3	K4	K5	R1	R2	R3	R4	R5
S7	1	1	1	1	1	1	1	1/2	1	1	1	1	1/2	1	1
S8	1	1	1/2	1/2	1/2	1	1	1/2	1	1	1	1	1/2	1/2	1
S9	1	1	1/2	1/2	1/2	1	1	1/2	1	1	1	1	1	1	1
S10	1	1	1/2	1	1	1	1	1/2	1	1	1	1	1	1/2	1

9.6 Discussion and Conclusions

Resuming the aim of this study, it was intended to obtain a descriptive view of the mathematical creativity in problem solving, drawing on an analytical tool that seeks to guide the identification of features pertaining to the three dimensions: originality, knowledge activation and representational means activation. Given the results of the data analysis, the analytical tool reveals potential for describing, in a qualitative and comparative way, the creativity of the solutions in the context of the SUB12 competition.

In conclusion, the three-dimensional representations presented in Fig. 9.14 show some diversity in the creativity manifested in the ten selected resolutions, which indicates that the mathematical creative product assumes a heterogeneous character in an inclusive competition. Indeed, this is a competition that aims to include a wide range of young students with different levels of aptitude and achievement in school

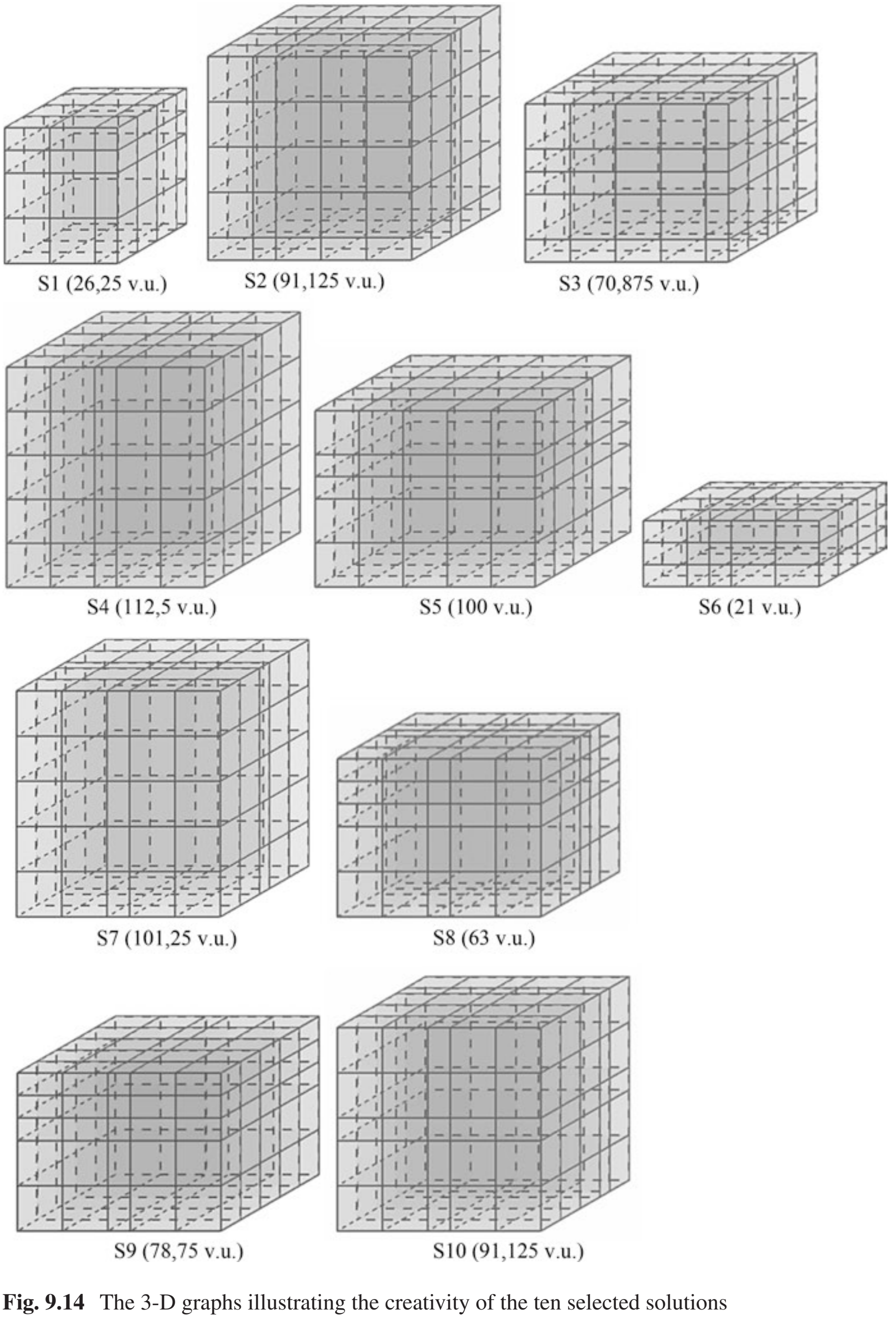


Fig. 9.14 The 3-D graphs illustrating the creativity of the ten selected solutions

mathematics. From observing the 3-D representations, one realizes that the boxes have different shapes and that their largeness is quite discernible in some cases, such as the cases of the solutions S1 and S6, but not immediately graspable in other cases. One possibility of making that distinction more evident is to consider the solid volume of the box, which is why this information is visible in the legend of each graph in Fig. 9.14.

The values of the volumes reveal that the top three solutions in terms of mathematical creativity are, in descending order, the solutions S4, S7, and S5. It is noteworthy that such volumes are the closest to the maximum volume (125 volume units) and therefore to the largest cube. This tells us how important the convergence between the three indicators of creativity is. In other words, mathematical creativity is based on the simultaneous contribution of the three dimensions considered. None of them is irrelevant and negligible, and none can alone explain the mathematical creativity of the solutions. Thus, the creativity cannot be reduced to the originality, because elements of novelty are not enough to qualify a creative product. On the contrary, it is clear from the results that the originality of a resolution is not detached from the activation of mathematical knowledge or the representational means. The latter are important indicators related to the interpretation and understanding of the problem as well as to the creation and representation of interesting, effective and relatively unusual mathematical approaches to the problem situation.

The mini-c creativity expressed in this set of resolutions is characterized by being associated with unique products, which acquire meaning through the mathematical knowledge deployed and the usefulness of the involved representational resources, since originality alone does not ensure ingenious results (Runco & Jaeger, 2012).

In this problem, the originality was supported by the ability to create interpretive tables of the situation, as a major component of several strategies, together with the ability to organize and explore data through them, which was combined with other numerical, iconic and verbal representations. There were cases in which the originality came together with the ability of inferring numerical patterns from other types of representations, including schematic drawings of the process of trying each key in a set of locks. In general, higher levels of originality tend to appear in interesting approaches in which the representational systems carry genuine authorship (Preston & Garner, 2003).

The knowledge activation is a critical component of creativity that has to do with the mathematical proficiency in problem solving, favouring the application of methods, strategies and procedures efficiently and accurately (NCTM, 2014). The way this indicator highlights the nature of mathematical creativity relates to the activation and use of concepts and potentially useful mathematical procedures, strategically combined and structured, as well as with the ability to clearly express mathematical reasoning.

As several researchers have argued, prior knowledge influences the understanding of the problems and the choice of solving strategies. In fact, prior knowledge and previous experience are all that a solver has to design a strategy to attack a problem (Mayer, 2006; Sternberg, 2007). When individuals can take advantage of prior knowledge, they do not need to spend cognitive effort in an attempt to reconstruct necessary procedures during the course of a resolution. For example, if someone has automated procedures and arithmetic concepts, then that person can focus on more critical issues, including the preparation of a plan, from which a creative solution may arise (Mayer, 2006). If students have a good repertoire of concepts and strategies, they can use it to build creative solutions; for example, through thinking by analogy, the resolution of a new problem may remind another problem that is then used as a reference to develop a method for solving the new one (Mayer, 2006). A good example of this is the solution S4 that makes an insightful connection to the famous problem of the handshakes, showing how the mathematical knowledge base is linked to a high capacity to represent and to great novelty.

The results show that the majority of the resolutions entail proficiency levels that contribute to the creativity of those solutions. Either way, it is important to refer the case of the solution S1, in which the lower mathematical knowledge activation has an impact on the overall creativity, whereas the solution S6 despite a higher level of knowledge activation has little originality which impoverishes the overall creativity. In short, a high proficiency in the domain of mathematics is not all you need to achieve a highly creative solution. Instead, knowledge activation must be in line with the two other indicators in order to be an effective component of mathematical creativity (Guerra, 2007), in particular to ensure a suitable and interesting application of mathematical knowledge.

Representational ability has also apparent consequences in the level of creativity achieved in the field of problem solving. In view of the proposed problem, the activation of representational means is expressed by the adequacy and articulation between mathematical representations, which allow representing the problem conditions in a mathematical way and support a strategy to solve the problem. The representational ability is revealed in the way the representations were operationalized in each case, were tuned and adapted to the goal of the problem and played a role in expressing and revealing the reasoning clearly. Representational flexibility is one of the important skills in mathematical problem solving which translates into imaginative and inventive capacity (Ainsworth, 1999; Benko & Maher, 2006). The various representations chosen by the solvers often showed a strategic character, in the sense that they have shaped an approach: a double-entry table to register a process of cross product, or a function table relating the number of keys and padlocks to the number of attempts in supporting inductive thinking, or pictorial schemes that seek to depict the process of trying out the keys in suggesting a pattern or a rule for calculating the total of attempts. In many cases, the representations were themselves invested with originality or led to original ideas.

Therefore, it should be noted that in general, the selected solutions showed a high level of representational flexibility. This salient features leads to stress the apparent centrality of this dimension for mathematical creativity in problem solving. In particular, it seems plausible that the freedom to design and build personal approaches based on the more readily available representations allowed students to generate significant ideas and make sense of the problem (Benko & Maher, 2006).

In a final comment, it must be said that the description of mathematical creativity of the solutions is the result of applying the "O.K.R" model where the definition of categories is admittedly loose and not rigid and whose implementation involves a subjective interpretation of the data, far from being a neutral procedure (Guerra, 2006). Thus, the description of the 10 solutions is not expected to be exact nor it assumes the purpose of a creativity measurement. However, despite different possible interpretations, the indicators have a reasonable analytical power; in highlighting the mathematical creativity of the solutions, it suggests that it is not merely a question of novelty but rather depends on the activation of mathematical knowledge or even more likely on the activation of representational means.

References

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education, 33*, 131–152.

Aizikovitsh-Udi, E. (2013). Creativity and critical thinking in solving non-routine problems among talented students. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of European Research in Mathematics Education (CERME 8)* (pp. 1270–1271). Ankara, Turkey: European Society for Research in Mathematics Education.

Aldous, C. R. (2007). Creativity, problem solving and innovative science: Insights from history, cognitive psychology and neuroscience. *International Education Journal, 8*(2), 176–186.

Amaral, N. (2016). *A criatividade matemática no contexto de uma competição de resolução de problemas*. Doctoral Thesis. Lisboa: Instituto de Educação, Universidade de Lisboa.

Amaral, N., & Carreira, S. (2012). An essay on students' creativity in problem solving beyond school – Proposing a framework of analysis. In *Pre-Proceedings of the 12th International Congress on Mathematical Education (ICME 12) – Topic Study 3* (pp. 1584–1593). Seoul, South Korea: ICMI.

Amaral, N., & Carreira, S. (2017). A Criatividade Matemática nas Respostas de Alunos Participantes de uma Competição de Resolução de Problemas. *Bolema, 31*(59), 880–906.

Andresen, M. (2007). Introduction of a new construct: The conceptual tool "flexibility". *The Montana Mathematics Enthusiast, 4*(2), 230–250.

Bardin, L. (2004). *Análise de Conteúdo* (3rd ed.). Lisboa: Edições 70.

Beghetto, R. A., & Kaufman, J. C. (2009). Do we all have multicreative potential? *ZDM – International Journal on Mathematics Education, 41*(1–2), 39–44.

Benko, P., & Maher, C. A. (2006). Students constructing representations for outcomes of experiments. In J. N. H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of mathematics education* (Vol. 2, pp. 137–143). Prague, Czech Republic: PME.

Carreira, S. (2015). Mathematical problem solving beyond school: Digital tools and students mathematical representations. In J. S. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 93–113). New York: Springer.

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematical problems with technology: The results and implications of the problem@web project*. New York: Springer.

Freiman, V. (2006). Problems to discover and to boost mathematical talent in early grades: A challenging situations approach. *The Mathematics Enthusiast*, *3*(1), 51–75.

Freiman, V. (2009). Mathematical e-nrichement: Problem-of-the-week model. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 367–381). Rotterdam: Sense Publishers.

Freiman, V., & Applebaum, M. (2009). Involving students in extra-curricular school mathematical activity: Virtual mathematical marathon case study. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd conference of the International Group for the Psychology of mathematics education* (Vol. 1, pp. 203–206). Thessaloniki, Greece: PME.

Freiman, V., & Lirette-Pitre, N. (2008). Building a virtual learning community of problem solvers: Example of CASMI community. *ZDM - International Journal on Mathematics Education, 41*, 245–256.

Gagatsis, A., & Elia, I. (2004). The effects of different modes of representation on mathematical problem solving. In M. Høines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of mathematics education* (Vol. 2, pp. 447–454). Bergen, Norway: PME.

Gomez, J. G. (2007). What do we know about creativity? *The Journal of Effective Teaching, 7*(1), 31–43.

Guerra, E. C. S. (2007). *Creatividad y desarrolo professional docente en matemáticas para la educación primária*. Doctoral Thesis. Barcelona: Universidad de Barcelona.

Guerra, I. C. (2006). *Pesquisa Qualitativa e Análise de Conteúdo – Sentidos e forma de uso*. Estoril, Portugal: Princípia Editora.

Heinze, A., Star, J. R., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM – International Journal on Mathematics Education, 41*(5), 535–540.

Kattou, M., Kontoyianni, K., Pitta-Pantazi, D., & Christou, C. (2013). Connecting mathematical creativity to mathematical ability. *ZDM – International Journal on Mathematics Education, 45*, 167–181.

Koichu, B., & Andzans, A. (2009). Mathematical creativity and giftedness in out-of-school activities. In R. A. Berman & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 286–307). Rotterdam: Sense Publishers.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 130–144). Rotterdam: Sense Publishers.

Mann, E. L. (2005). Mathematical creativity and school mathematics: *Indicators of mathematical creativity in middle school students*. Doctoral thesis. University of Connecticut, USA.

Mayer, R. E. (2006). The role of domain knowledge in creative problem solving. In J. C. Kaufman & J. Baer (Eds.), *Creativity and reason in cognitive development* (pp. 145–158). Cambridge, New Jersey: Cambridge University Press.

Milgram, R. M., & Hong, E. (2009). Talent loss in mathematics: Causes and solutions. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 150–163). Rotterdam: Sense Publishers.

Moraes, R. (1999). Análise de conteúdo. *Revista Educação, 22*(37), 7–32.

NCTM. (2014). *Procedural fluency in mathematics: A position of the National Council of Teachers of Mathematics*. [Article]. Available at http://www.nctm.org/about/content.aspx?id=42833

Nistal, A. A., Van Dooren, V., Clarebout, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: A critical review. *ZDM – International Journal on Mathematics Education, 41*(3), 627–636.

Pehkonen, E. (1997). The state-of-art in mathematical creativity. *ZDM – International Journal on Mathematics Education, 29*(3), 63–67.

Plucker, J., & Renzulli, J. (1999). Psychometric approaches to the study of human creativity. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 35–61). Cambridge: Cambridge University Press.

Preston, R., & Garner, A. S. (2003). Representation as a vehicle for solving and communicating. *Mathematics Teaching in the Middle School, 9*(1), 38–43.

Runco, M. A. (2004). Creativity. *Annual Review of Psychology, 55*, 657–687.

Runco, M. A., & Jaeger, G. J. (2012). The standard definition of creativity. *Creativity Research Journal, 24*, 92–96.

Runco, M. A., & Richards, R. (1998). *Eminent creativity, everyday creativity, and health*. Norwood, NJ: Ablex.

Runco, M. A., & Sakamoto, S. (1999). Experimental studies of creativity. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 62–92). Cambridge: Cambridge University Press.

Russell, S. J. (2000). Principles and standards: Developing computational fluency with whole numbers. *Teaching Children Mathematics, 7*(3), 154–158.

Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM – International Journal on Mathematics Education, 41*(5), 619–625.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM – International Journal on Mathematics Education, 29*(3), 75–80.

Sriraman, B. (2008). Are mathematical giftedness and mathematical creativity synonyms? A theoretical analysis of constructs. In B. Sriraman (Ed.), *Creativity giftedess, and talent development in mathematics* (pp. 85–112). Charlotte, NC: IAP.

Steele, D. (2008). Seventh-grade students' representations for pictorial growth and change problems. *ZDM – International Journal on Mathematics Education, 40*, 97–110.

Sternberg, R. (2007). Creativity as a habit. In A. Tan (Ed.), *Creativity: A handbook for teachers* (pp. 3–25). Singapore: World Scientific.

Vale, I. (2012). As Tarefas de Padrões na Aula de Matemática: Um Desafio para Professores e Alunos. *Interações, 8*(20), 181–207.

Yuan, X., & Sriraman, B. (2011). An exploratory study of relationships between students' creativity and mathematical problem-posing abilities. In B. Sriraman & K. H. Lee (Eds.), *The elements of creativity and giftedness in mathematics* (pp. 5–28). Rotterdam: Sense Publishers.

Chapter 10
Solving a Task with Infinitely Many Solutions: Convergent and Divergent Thinking in Mathematical Creativity

Michal Tabach and Esther Levenson

10.1 Introduction

The field of creativity in general, and mathematical creativity in particular, has become a growing arena for innovative research, as evidenced, for example, by two recently published special issues (Leikin & Pitta-Pantazi, 2013; Singer, Ellerton, & Cai, 2013). So far, these research efforts have not yielded a single acceptable definition of creative mathematical thinking in general or of creative mathematical thinking in the school context in particular (Mann, 2006). Nonetheless, researchers generally agree that problem-solving has the potential to elicit creative mathematical thinking (e.g. Silver, 1997). Specifically, "assignments in which a student is explicitly required to solve a mathematical problem in different ways" (Leikin, 2009, p. 133) may help elicit creative thinking. Indeed, many studies aimed at investigating creative mathematical thinking have used this explicit instruction as a tool for eliciting and evaluating creative mathematical thinking (e.g. Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 2013; Tabach, & Friedlander, 2013; Tsamir, Tirosh, Tabach, & Levenson, 2010).

Often, researchers use a task that has one final solution outcome and several strategies by which that outcome can be found. Even when one considers open-ended tasks, the number of solution outcomes is varied but finite. Take, for example, a typical open-ended task: Which of the following numbers is different from the others? If possible, try to find many possible cases or answers (Kwon, Park, & Park, 2006). In this case, each number can be considered to be unique. Hence the number of solution outcomes is equal to the number of numbers presented. More than one reason may exist as to why a particular number is unique, though again the number of variations is finite. What would happen if there were an infinite number of solution outcomes? Could such a problem also be used to elicit and evaluate mathematical creativity?

M. Tabach (✉) · E. Levenson
Tel-Aviv University, Tel Aviv, Israel
e-mail: TabachM@post.tau.ac.il

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_10

The current study examines an extreme case of a task with infinitely many mathematical solution outcomes, in which participants are explicitly asked to solve the problem in more than one way. Such a task was administered to adults with a variety of mathematical backgrounds and resources (Schoenfeld, 1985). The first aim of this study is to explore whether indeed such a task is a suitable vehicle for eliciting and evaluating creative thinking. Many times, creative thinking is associated with divergent thinking (e.g. Torrance, 1974). In turn, divergent thinking may be evaluated in terms of fluency, flexibility, and originality of outcomes (e.g. Leikin & Lev, 2007). On the other hand, problem-solving, including creative problem-solving, is also linked with convergent thinking (Lin & Cho, 2011). Taking into consideration that a task with an infinite amount of solution outcomes may lead to excessive divergent thinking, the second aim of this study is to investigate participants' solutions in terms of both divergent and convergent thinking (Guilford, 1973). Finally, the third aim of this study is to use the outcomes created by all participants to highlight several methodological questions regarding outcome assessment in terms of fluency, flexibility, and mathematical creativity.

10.2 Theoretical Background

10.2.1 Mathematical Problems as Relative Phenomena

The use of mathematical problems to elicit creative mathematical thinking necessitates being able to discern routine algorithmic tasks from problems. Routine algorithmic tasks are used to drill and practise acquired skills and techniques. Nevertheless, "a task in which the student is interested and engaged and for which he wishes to obtain a resolution; and for which the student does not have a readily accessible means by which to achieve that resolution" (Schoenfeld, 1985, pp. 87–88) constitutes a problem for the student. Note that the first part of Schoenfeld's definition refers to emotional aspects of interest and engagement. The definition does not exclude the possibility of being engaged with an algorithmic activity. The second part of the definition, however, distinguishes mathematical problems from routine algorithmic problems, indicating that the learner does not have a ready-made algorithm to tackle the task. In fact, being engaged with a problem involves searching for a strategy by which the problem can be solved.

This important second distinction in Schoenfeld's definition highlights another aspect of the task as a problem – its relative nature. That is, the same task may be a routine activity for one student and a problem for another. Moreover, for a task to become a problem, students need to become engaged with it. That is, their mathematical knowledge should be sufficiently developed to allow them to become engaged with the problem. In other words, for a task to become a mathematical problem, the problem-solver must be taken into consideration as well.

10.2.2 Creativity as a Relative Phenomenon

Creative mathematical thinking can be considered to be the ability to solve problems or to develop structural thinking while referring to the logical-deductive nature of a domain that has connections to mathematics (Ervynck, 1991). For professional mathematicians, creative mathematical thinking, like creativity in any other domain, involves "the ability to produce original work that significantly extends the body of knowledge" and that is expected to open up "avenues of new questions for other mathematicians" (Liljedahl & Sriraman, 2006, p. 18). This is akin to what Kaufman and Beghetto (2009) called Pro-C creativity or professional creativity. This type of creativity represents a professional-level expertise in a creative area that is both developmental and effortful. Taking this to a higher-level, Big-C creativity consists of clear-cut, eminent creative contributions that have a significant and lasting contribution to society.

Of course, one can hardly expect to describe the achievements of K-12 students in such terms. Little-c creativity underscores the important role that creativity plays in everyday life and points to the importance of identifying and nurturing creativity in everyday settings such as schools (Kaufman & Beghetto, 2009). According to Liljedahl and Sriraman (2006), school-level mathematical creativity includes unusual and/or insightful solutions to a given problem or viewing an old problem from a new angle, raising new questions and possibilities. As such, creativity in K-12 mathematics can be evaluated with respect to students' previous experiences and the performance of their peers (Leikin, 2009). In other words, like mathematical problems, creative mathematical thinking is relative to the participants.

Creativity may also be relevant to the task at hand. Silver (1997) refers to creative mathematical thinking in relation to problem-solving. He claims that creative mathematical thinking can be developed by working on relevant mathematical activities. In fact, he considers creative mathematical thinking to be relevant for all students. Silver (1997) argues that "mathematics educators can view creativity not as a domain of only a few exceptional individuals but rather as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (p. 79). The use of open-ended problems with different solution methods or solution outcomes may provide learners with valuable experience in analysing and creating new solutions on the basis of known solutions.

Solving problems in multiple ways is a widely recognized tool for developing connectedness of mathematical knowledge (NCTM, 2000; Polya, 1973; Schoenfeld, 1985). Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun (2005) emphasized that "different solutions can facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" (p. 288). Mathematical activities with the potential to promote creative mathematical thinking can be used to evoke, condense, and reorganize prior knowledge (Mumford, Baughman, Maher, Costanza, & Supinski, 1997). Specifically, open problems such as construction problems may elicit learning because (a) creative production requires learners to reactivate their prior knowledge and use it to derive the necessary parts relevant to the task; (b) as learners make and verify conjectures, they actively construct their knowledge; and (c) while evaluating their own conjectures and reorganizing their knowledge, learners also develop a general problem-solving strategy. Ervynck (1991) claims that creative mathematical thinking allows individuals to build a schema from which they can create an unlimited number of connections that can lead from one point in the schema to any other desirable target.

Finally, mathematical creativity, as other forms of creativity, may be related to the environment. Davies et al. (2012) describe the learning environment as extending "beyond the physical architecture of the space in which learning takes place … to encompass psychosocial and pedagogical features … [and includes] the influence of places and people outside of school" (Davies et al., 2012, p. 80). Thus, in addition to flexible use of physical space, an environment which encourages creativity provides a balance between structure and freedom, allowing participants to take risks with minimal pressure. The atmosphere of such an environment is of mutual respect for all participants that incorporates open dialogue. An open environment may also encourage participants to be open to new experiences, which in turn may be related to intrinsic motivation, engagement in creative processes, and finally to improved creative performance (Tan, Lau, Kung, & Kailsan, 2016).

10.2.3 Divergent and Convergent Thinking in Problem-Solving

Two thinking processes that play a major role in problem-solving have been identified as a basis for creative thinking (Guilford, 1973): convergent thinking and divergent thinking. Convergent thinking takes place when the solver logically strives to find a solution to a problem. In other words, the solver seeks to understand the logical connections among knowledge elements in the problem and to apply the relevant standard algorithm. Convergent thinking is usually not associated with creativity. Haylock (1987), for example, linked convergent thinking with narrow thinking that relies on routine processes.

Divergent thinking, on the other hand, is often thought of as the hallmark of creative thinking. It is often measured in terms of the fluency, flexibility, and originality of ideas produced. Silver (1997) related fluency to "the number of ideas generated in response to a prompt" (p. 76). Flexibility, according to Silver (1997), refers to "apparent shifts in approaches taken when generating responses to a prompt" (p. 76). Leikin (2009) evaluated flexibility by assessing if different solutions employ strategies based on different representations (e.g. algebraic and graphical representations), properties, or branches of mathematics. Flexibility may also be thought of as the opposite of fixation. In problem-solving, fixation is related to mental rigidity (Haylock, 1997). Overcoming fixation and breaking away from stereotypes are signs of flexible thinking.

Not all divergent thinking leads to creativity. Runco (1996) distinguished between divergent thinking at the service of creative thinking and divergent thinking as the production of random useless ideas. In his view, creativity is "manifested in the intentions and motivation to transform the objective world into original interpretations, coupled with the ability to decide when this is useful and when it is not" (p. 4). In other words, Runco adds the aspect of discretion. With regard to problem-solving, divergent thinking allows the solver to seek and find different solution paths. However, the intrapersonal evaluation of ideas is necessary in order to differentiate between useful and unproductive paths. In other words, divergent thinking, along with convergent thinking, may be necessary for creativity.

Another view of divergent and convergent thinking describes them each as two different but related creative processes: divergent-exploratory and convergent-integrative (Storme et al., 2015). Divergent thinking tasks generate many brief solutions to a problem, while integrative thinking tasks generate one elaborate solution to a problem. As opposed to divergent thinking tasks, convergent tasks usually call for the participant to elaborate on one single idea, without diverging from that idea. Several researchers consider these two types of thinking processes as involving different neurocognitive processes (e.g. Storme et al., 2015) and thus take each into account when investigating creativity.

10.2.4 Problem-Solving and Mathematical Creativity

Several studies have focused on problems that have more than one final solution or problems in which an explicit request was given to solve it in several ways. In the following we briefly report on some of these studies, organized according to participants' age group, starting with the youngest. The ability of 5-year-old children to handle a task for which there is more than one solution method and more than one correct outcome was evaluated in an interview setting (Tsamir et al., 2010). The results show that the children were able to find more than one solution and more than one strategy, including the solution involving two empty sets, which is known to be problematic even for mathematics teachers (e.g. Linchevsky & Vinner, 1998). Hayne and Tabach (2014) used problems embedded in a story context and a technological environment to examine whether a sequence of such tasks would develop the knowledge of 6-year-old children regarding equality and whether it would promote their creative mathematical thinking. The results show that indeed the children's answers were beyond the researchers' expectations, based on their early age.

Maher and Martino (1996) described a longitudinal study in which students engaged in a task of building different towers at a height of three cubes, when two colours were available. This task was given to the same students once a year for five consecutive years, from grade 1 and on. The results suggest that encouraging students to justify their solutions as they engage in such a task may foster the development of the notion of mathematical proof. A different longitudinal study approach was taken by Kattou et al. (2013). In an attempt to articulate and empirically test a theoretical model that predicts abilities in mathematics requiring creative potential, the researchers administered a problem-solving questionnaire to elementary school students from grades 4 to 6 to evaluate their creative mathematical thinking. The results showed that mathematical creativity is a predictor of mathematical ability. Similarly, problems with an explicit request to solve the problem in several ways were used as a tool in evaluating students' creativity levels. The study examined whether indeed becoming more knowledgeable in math, as expressed in advancing from grades 4 to 9, would be reflected in measures of creativity while engaging with the same problem (Tabach & Friedlander, 2013). Results show that the solution space of eighth grade students became more narrowed down, possibly because they had begun to study algebra. This tendency was no longer seen in the solution spaces of ninth graders.

In the high school context, Levav-Waynberg and Leikin (2009, 2012) used problems with an explicit request to solve the problems in several ways as a tool for evaluating mathematical thinking in geometry. These problems served a double role – didactic and diagnostic. Tenth graders learned geometry while proving theorems in multiple ways as a classroom norm. The problems were also used as a tool for evaluation, by comparing the individual and collective solution spaces. The authors claim that such problems can be used for multiple purposes: (1) "as a diagnostic research instrument to assess students' mathematical knowledge and creativity", (2) "to measure and follow the development of mathematical knowledge and creativity criteria", (3) "for comparative research", and (4) "as a research instrument for the verification of theoretical hypotheses such as the relations between mathematical knowledge and flexibility or between mathematical creativity and original reasoning" (2012, p. 330).

Problems with an explicit call to solve them in several ways were also used in professional development courses for prospective (Guberman & Leikin, 2013) and practising (Lev-Zamir, & Leikin, 2011, 2013) mathematics teachers. The problems play a dual role – in research and as didactic tools. In all the studies mentioned above, the number of strategies in using the problems and the number of correct outcomes for any given problem were finite. In the current study, in contrast, the task has infinitely many outcomes, as demonstrated in the methods section.

10.2.5 Evaluating Creative Mathematical Thinking

The application and assessment of routine procedures and algorithms are widely documented in the literature (see, e.g. Ayres, 2000; Demby, 1997; Kieran, 1981). In contrast, the essence and measurement of creative problem-solving are different. Krutetskii (1976) argued that problems with several solution methods allow for examining the flexibility of an individual's mathematical thinking by investigating moves from one mental operation to another.

Creative mathematical thinking can be analysed along several dimensions (Guilford, 1973; Torrance, 1974) with respect to students' abilities when offering strategies and solutions for solving a mathematical problem: fluency, flexibility, and originality. Fluency is the ability to provide a number of correct solutions for a given problem within a specific time limit. Fluency is related to an individual's active and available knowledge with respect to a problem. Flexibility is the ability to modify one's way of thinking regarding a given task. The solution strategies may derive from different mathematical domains or different mathematical principles. Originality is the ability to relate to a given problem in a novel way, producing solutions that are unexpected, unusual, or uncommon to the problem at hand. Some researchers see fluency, flexibility, and originality as manifestations of divergent thinking (Kwon et al. 2006; Levenson, 2011).

10.3 Methods

As stated above, the general aim of this study is to examine the potential of a mathematical task with infinitely many solutions to be used as a tool for evaluating creative mathematical thinking. The task was to draw as many different[1] polygons as possible with an area of 15 units in a period of 5 min. The task was taken from a previous study conducted with elementary school students (Swisa, 2015).

10.3.1 Settings

The task was administered to an international group of adults comprising 23 researchers in mathematics education who all attended one session at the Problem@ Web Conference. The researchers varied in terms of background, but all had at least a master's degree in mathematics or mathematics education, and many of them had a PhD in one of these domains. Each participant received one sheet of grid paper and was asked to answer anonymously. More sheets were available on demand. The time limit for the researchers to complete the task was 5 min, which is a relatively short period of time, and the aim was to consider initial thinking directions. Next we consider the expert solution space for this task.

10.3.2 Solution Spaces for the Task

As the task is about polygons, we begin with polygons with the least possible number of sides, namely, triangles. Our first candidate is a triangle with a base length of 10 units and an altitude length of 3 units. Next, we draw a line parallel to the base at a distance of 3 units; any point on this parallel line can be linked by two segments

[1] It was clear for participants in this study that "different" polygons referred to noncongruent polygons.

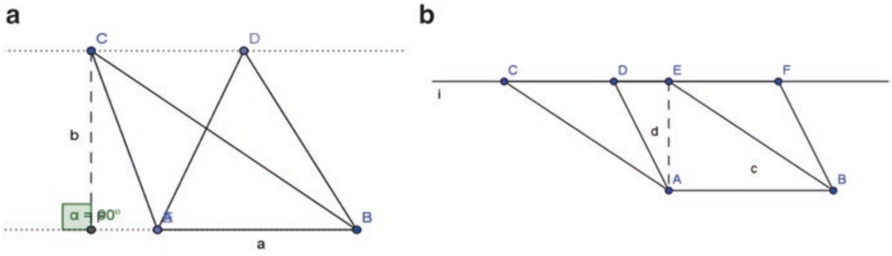


Fig. 10.1 (**a**) *a* ‖ CD, Triangles ABD and ABC share a base of length *a* and altitude of length *b*, such that $a \times b = 30$. (**b**) *c* ‖ *i*, Quadrilaterals ABEC and ABFD share one side of length *c* and the altitude *d*, such that $c \times d = 15$

to the endpoints of the base, creating infinitely many different triangles with an area of 15 square units. The cardinality of this set of triangles is ℵ (aleph). The same process of constructing a set of triangles can be carried out with any positive base length b and altitude length a, such that $a \times b = 30$ (see Fig. 10.1a for illustration). The cardinality of the set of all base lengths is also ℵ (aleph), and hence the cardinality of the set of all triangles satisfying the requirement (area of 15 square units) is $\aleph \times \aleph = \aleph$ (aleph × aleph = aleph).

We then turn to four-sided polygons, beginning with rectangles. Each positive side length c yields one rectangle of area 15, namely, one whose second side length is d such that $c \times d = 15$. Hence the cardinality of the set of all rectangles of area 15 is also ℵ (aleph). If we refer to quadrilaterals with one side length denoted by c and the altitude length for that side denoted by d, where c and d are positive real numbers such that $c \times d = 15$, we have one solution. By drawing a parallel line through the opposite side c, we can construct a set of quadrilaterals that satisfy the task conditions (see Fig. 10.1b). The cardinality of that set is also ℵ(aleph). Since we can choose a new value for c and repeat the whole process, we can create ℵ (aleph) sets of quadrilaterals satisfying the task conditions, and the cardinality of the set of all such sets of quadrilaterals is $\aleph \times \aleph = \aleph$ (aleph × aleph = aleph). Note that the set of rectangles is a subset of the set of quadrilaterals.

It can also be proven that the set of all n-gons (for any $n \geq 5$) has cardinality ℵ. From this it follows that the cardinality of the set of all polygons is ℵ (see Appendix). Note that the proof in the appendix can be generalized not only to any n-gons but also to any area.

10.3.3 Analysing Participants' Outcomes

In analysing participants' responses, we refer to their outcomes. We try to follow their line of thought as represented by the collection of their outcomes, and we look for manifestations of convergent and divergent thinking. In cases where systematic modifications were made, resulting in a set of modified outcomes, we categorized this as a case of convergent thinking. When a different representation was used or when no clear linkage could be noted between outcomes, we saw it as a case of divergent thinking.

We also referred to participants' outcomes in terms of fluency, flexibility, and originality. Recall that fluency is the ability to provide a number of correct solutions for a given problem. Hence, measuring fluency is a straightforward process: after identifying the outcomes that satisfy the task requirements, counting how many valid solutions a participant suggests is a measure of the said participant's fluency. Flexibility is a more subtle matter. Referring back to the definition, flexibility is the ability to modify one's way of thinking regarding a given task. The question to be asked, then, is what should be considered modification of a way of thinking. Solutions deriving from different mathematical domains, different mathematical principles, or a different mathematical representation may be indications of flexible thinking. Originality is the ability to relate to a given problem in a novel way, producing solutions that are unexpected, unusual, or uncommon to the problem at hand. Due to the nature of the task (i.e. an infinite number of solutions) and the limited number of participants, we do not refer to originality in our analysis.

10.4 Findings

Based upon initial observations of the participants' solutions, we divided the responses into 2 groups: 8 responses clearly indicate that infinitely many such polygons can be found, while the remaining 15 responses yielded between 1 and 18 polygons or an average of 8.3 polygons per participant. We report first on the responses of these 15 participants. After that, we consider the outcomes of the eight researchers (R1-R8) who explicitly referred to infinitely many polygons with an area of 15 units. To some extent we were also able to trace their solution process, as recorded on the grid paper. For each of them, we briefly describe their outcomes and offer our analysis.

The group of 15 participants used drawings, sometimes coupled with numerical notations. To illustrate one of the solutions we present R10's solution (see Fig. 10.2), which includes 9 polygons. We can see a 3×5 rectangle at the top left and next to it a 1×15 rectangle. Under these are all kinds of irregular polygons, all having sides that are parallel to the grid. All the polygons contain small dots, as if R10 was counting the square units to verify that indeed each polygon contains 15 square units. We cannot point to a systematic line of thinking. It looks as if each polygon is a new endeavour, exhibiting divergent thinking, with no attempt to further pursue any specific direction. Similarly, the other 14 participants drew sketches of polygons in a seemingly random manner, without any verbal comments, until the allotted time was over.

Next we discuss each of the responses of the participants from the other group in some detail.

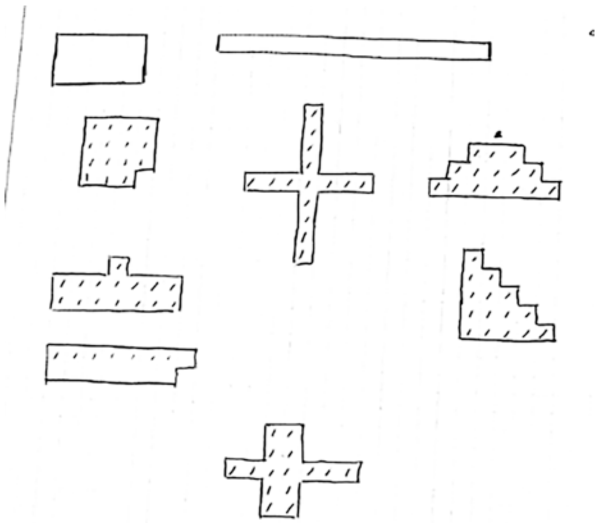


Fig. 10.2 R10's attempts at solving the task

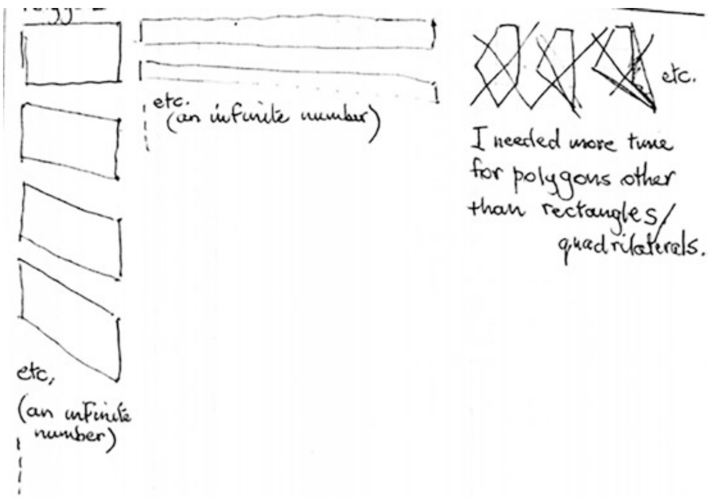


Fig. 10.3 R1's attempts at solving the task

R1 began by drawing a 3 × 5 rectangle. Under it she drew a quadrilateral with one pair of parallel sides of length 3, while the altitude to that side was of length 5. Below that she drew yet another quadrilateral, with more tilted sides, and then another one. At the bottom of this row, which clearly represents a sequence of quadrilaterals, she wrote "etc. (an infinite number)" and also "…". To the right of the first drawing of the 3 × 5 rectangle, she drew another rectangle, with side lengths of

1 × 15, below a quadrilateral with side lengths of 1 and altitude length of 15, and under it she wrote "etc. (an infinite number)" and "…". Next she drew three more polygons, crossed them out, and wrote below "I need more time for polygons other than rectangles/quadrilaterals" (see Fig. 10.3).

We can see that R1 began with a very basic polygon, which she was able to modify almost dynamically (in a static media), obtaining two sets of infinitely many solutions. While we can consider the first move from a rectangle to quadrilaterals in each sequence as evidence of some (low) level of flexibility, the changes alongside each set of quadrilaterals have a different flavour. They could be an illustration of convergent thinking, applying systematic modification to the same basic shape. Likewise, in moving from the first set of quadrilaterals to the second, R1 did not exhibit a moment of flexible thinking. Rather, this represented a second application of the same idea on a rectangle with different side lengths. The three crossed-out polygons on the far right, although they did not yield successful results, evidently point to an attempt to flexibly alter the working strategy applied to the two initial sets. In other words, this may have been the beginning of divergent thinking. Interestingly, finding the first and then the second infinite sets of polygons did not stop R1 from continuing to search.

R2's approach was totally different, although he too worked systematically. He started by drawing a 1 × 15 rectangle. Beneath it he drew an irregular polygon which he crossed out, and below that he drew a sequence of irregular-shaped polygons. The first looks like a 1 × 13 rectangle with the addition of 2 square units on top, marked as "2". The second looks the same, but this time the top part has 3 square units and the rectangle is 1 × 12 and marked as "3". The fourth already hints at the shape, and the last polygon in the sequence is marked as "7" and is fully drawn (see Fig. 10.4). Beneath this R2 began a new sequence of shapes, denoted by 1&2, 1&3, up to 1&6. In this second sequence, the shapes are arranged in three rows: the uppermost shape with 1 square unit, the middle with a changing number of square units that increase, and the lower row including the rest of the square units. Under this R2 started yet another sequence, denoted by 2&1, 3&1, … in which three rows were again drawn, though this time the upper row had a changing number of square units and the middle row consistently had 1 unit. Next R2 drew three more irregular polygons, each of which could be thought of as an element in yet another sequence. This time there were four rows, where the upper row had 1 square unit as a constant, the second had an increasing number of square units, the third again had a constant number of 3 square units, and the rest of the units were in the fourth and lowest row. Each element was also denoted with a code: 1-2-3-9, 1-3-3-8, and 1-4-3-7 (see Fig. 10.4, right side). Under this R2 wrote "…" to denote that there were more, and below that he wrote the infinity symbol, ∞. Below that was yet another polygon, this time a triangle whose base length was denoted by y and altitude length denoted by x, and under this R2 wrote: for every x, y: $x \bullet y = 30$.

An examination of the way R2 created the sequences indicates that the systematic thinking process is almost visual. In contrast to the dynamic continuum of modifications observed in R1's work, here the feeling is that more discrete modifications were made. The way R2 coupled a drawing with a numeric symbolic code for the

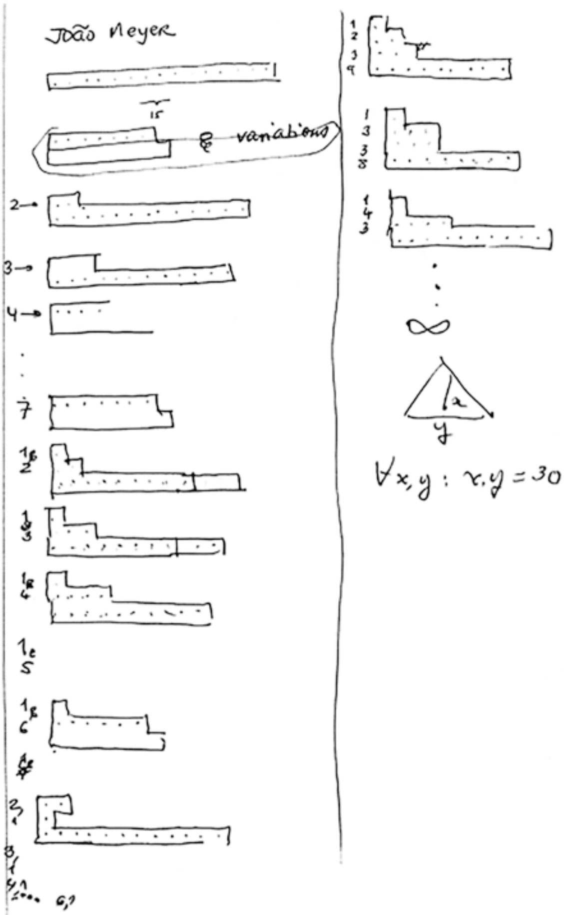


Fig. 10.4 R2's attempts at solving the task

irregular polygons highlights the discrete nature of his modifications. In fact, the infinity symbol at the end of the sequence is a bit puzzling. It is not clear that there are indeed infinitely many such sequences and elements within them. In any case, moving from one sequence to the next would not count as flexibility. Rather, the entire process seems to be more in line with convergent thinking, i.e. following the same line of thoughts systematically. Yet we can consider the last polygon drawn by R2, the triangle, and the accompanying symbolic notations, as a manifestation of divergent thinking that exhibits a different set of considerations. The change is apparent in the way it was expressed by using general algebraic symbolic notations. Note also that R2's conclusion that he can create infinitely many irregular polygons did not stop him from flexibly considering the task from an entirely new approach, which, based on the general notations he formulated, also ended up with infinitely many triangles.

R3 began like R1 by considering a 3 × 5 rectangle but continued in a different direction. She drew the 3 × 5 rectangle three times, each time with a different orientation on the paper, but next to it, she wrote "orientation doesn't matter". She drew yet another square-like rectangle and denoted its sides by $\sqrt{15}$. Next she wrote "numerous rectangles, possible others (trapezoids)". Next she drew a rectangle with side lengths 5 and 6 in such a way that two of its sides are dashed and its diagonal is marked and wrote "half of any rectangle (again orientation doesn't matter)" (see Fig. 10.5). Next R3 drew a five-sided polygon composed of a rectangle and a triangle with a shared side. Next to the triangle, she wrote "3 units" and next to the rectangle "12 units". Next to the polygon, R3 wrote: "5-sided (pentagon). Many variants of this area Δ + area rect. Orientation doesn't matter". R3 added: "possible for regular pentagon but I won't do that now due to time constraints" and "others too". Below that she wrote: "6-sided (hexagons). Regular (easier) area of equilateral Δ × 6" and below "irregular". Finally, she wrote: "out of time (didn't even look at concave options)".

As opposed to R1 and R2, R3 did not at any point explicitly state that in fact there are infinitely many solutions. Nevertheless, she did relate to "numerous rectangles possible", and hence we included her in this group. She systematically considered polygons with an increasing number of sides, and for each she provided an example

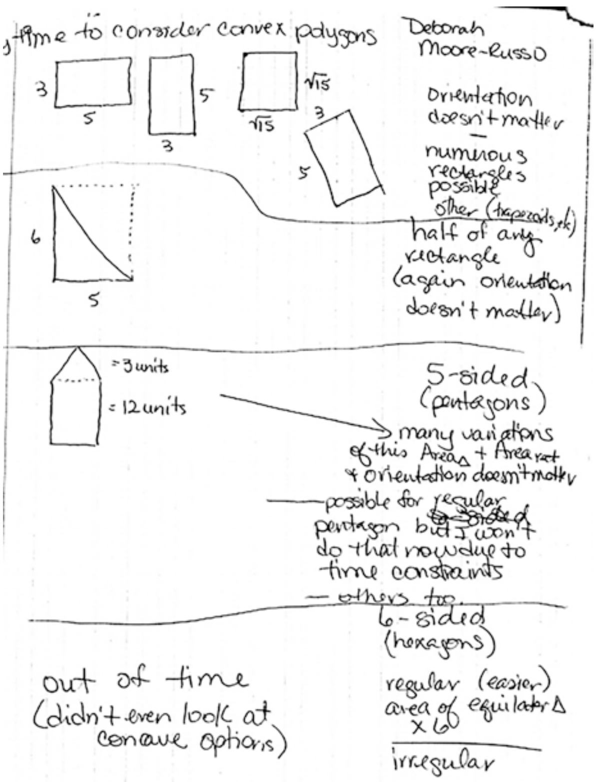


Fig. 10.5 R3's attempts at solving the task

or a way to build an example of polygons that would fit the task requirements. We can consider her move between polygon types as an example of convergent thinking, coupled with systematic considerations. Her last written comment about concave polygons may hint at applying divergent thinking.

As opposed to R1-R3, R4 did not begin by considering a particular polygon with 15 square units in area. Rather, he drew a triangle, and next to it, he wrote $a \bullet b = 30$ and then the ∞ sign, an arrow, and again the ∞ sign, this time with an exclamation mark. Below that he drew a rectangle. Next to it he wrote $a \bullet b = 15$ and, next the ∞ sign, an arrow and again the ∞ sign and an exclamation mark. Below he wrote "…" to denote continuation, and all this was marked with 1. Next he wrote 2, as if indicating a new line of thoughts. He wrote "if a, b integers, Δ ab = 30, $a = 30/b$; $a = 1$, $b = 1, 30$; $a = 2$, $b = 15$; $a = 3$, $b = 10$; 5 6; 6 5". Next he drew the first quadrant of a coordinate system; marked the axis with a and the other with b; marked 2 points on each axis, 1 and 30; and drew a curve (see Fig. 10.6). Just below the graph he wrote "4 Δ", as if counting how many triangles there are with base length and altitude length as integers. Under that, it seems he was making the same calculations for rectangles, concluding that there are two rectangles that meet the task requirements. Below he drew a pentagon, but he did not continue.

R4's initial considerations were general in nature, starting with the smallest polygon possible, a triangle, and immediately acknowledging the fact that there are infinitely many such triangles. The exclamation mark may hint at his surprise at this discovery. He next moved on to rectangles, came to the same conclusion, and then wrote "…". It is not completely clear how he was going to advance next

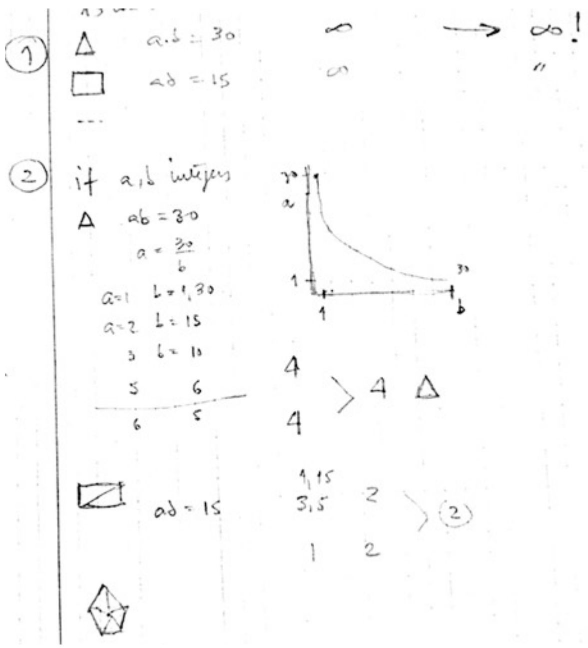


Fig. 10.6 R4's attempts at solving the task

along this line of thinking, but then he took a different approach, acknowledged by marking 1 next to what was done up to this point and 2 from then on. We can refer to this first line of thinking as a manifestation of systematic convergent thinking. The interesting thing is that now he constrained the solution space to include only integers. It is not clear why he did this. It could be the influence of the grid paper he was given. In any case, he found four pairs of triangle base lengths and altitude lengths (a, b) and probably due to symmetry considerations stopped there and did not continue to the reversed four pairs, concluding that there are four such triangles. However, here the symbolic notations hide the fact that a and b, where one represents the side length of a triangle and the second represents its altitude length, were not symmetrical, and in fact there are eight different triangles. The symmetry considerations were valid for the case of rectangles. The entire second line of thinking, marked by 2, is on its own another example of convergent thinking. Yet the transition from the first to the second line of thinking exemplifies flexibility. In conclusion, we note that both flexibility and convergent modifications can be seen in R4's solution process. The same phenomenon of continuing after obtaining infinitely many solutions was observed again.

R5's solutions are as follows: He drew a 3×5 rectangle but then wrote $15 \to 1 \times 15 \to \infty$. He then drew an arrow to a drawing of a parallelogram with side length 15 and altitude length 1 as an explanation of the infinity conclusion (see Fig. 10.7). Under this, he also wrote $3 \times 5 \to \infty$ with no further explanations. Next he wrote $\Delta \to 30 \to 1 \times 30 \to \infty$; below that, he listed: $2 \times 15 \to \infty$; $3 \times 10 \to \infty$; and finally $5 \times 6 \to \infty$.

In all, R5 found six different infinite sets. Two sets are based on parallelograms, and four are based on triangles. We saw the notion of modifying a specific parallelogram in R1's solution, but while R1 presented his solution in a visual manner by drawing the continuous dynamic changes, R5 used one drawing and did not elaborate on it. The drawing was self-explanatory for him. The generality of his solution was communicated by the use of symbolic algebraic notations. Furthermore, he did not provide any explanation for the four infinite sets of triangles. In terms of flexibility, we see a small modification from considering rectangles to considering triangles, and within each type of polygon, we see systematic convergent thinking based on natural numbers. Here again we see that finding one set of infinitely many solutions did not stop R5 from searching further for solutions.

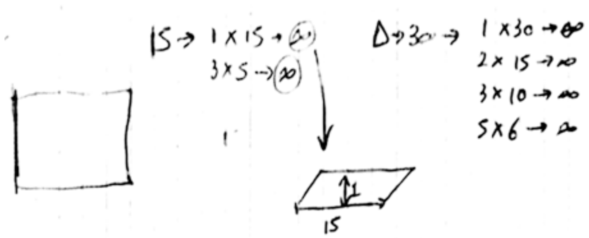


Fig. 10.7 R5's attempts at solving the task

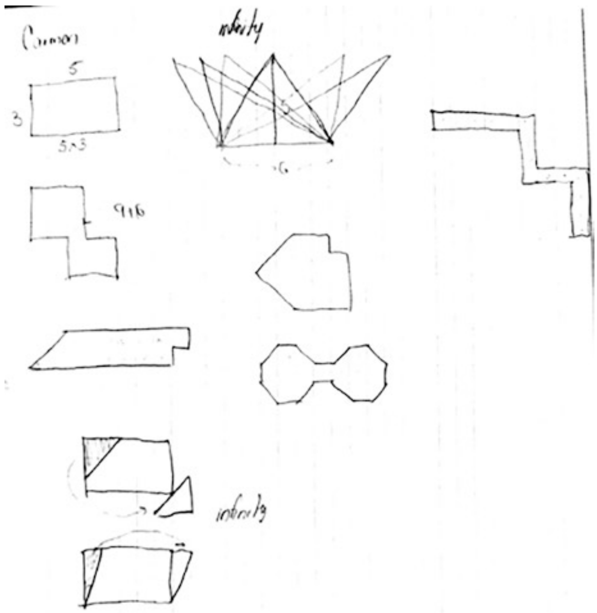


Fig. 10.8 R6's attempts at solving the task

Determining the solution path is less clear from R6's solution sheet. It seems that he first drew a 3 × 5 rectangle and perhaps several irregular polygons (see Fig. 10.8). At one point, he drew an equilateral triangle with a base length of 6 and an altitude length of 5. Yet at this point, he drew several other triangles, all using the same base length and the same altitude length, but not equal sides, and above it, he denoted "infinity". At the lower part of his answer sheet, he again drew two 3 × 5 rectangles: on each, he highlighted a triangular part which he then drew attached to the rectangle at some other point and with an arrow denoting that in fact the highlighted part had been moved around (see Fig. 10.8). Next to these two drawings, he again wrote "infinity".

Thus, R6 drew three sets of infinitely many solutions – one based on triangles, and the other two based on rectangles. It seems that the underlying reasoning for the triangle sets was different from the underlying reasoning on which the two rectangular sets were based. In this sense, we can say there is flexibility in R5's thinking. In between we see some irregular polygons. It seems that in R6's case, the changes were less systematic than the ones observed in other responses, as if his thinking was more divergent. Yet, the phenomenon of having one set of infinitely many solutions while still looking for other solutions is manifested here as well.

R7's solution began with a header – the word "triangles". Beneath that, he drew a triangle of base length 30 and altitude length 1. He drew a line parallel to the triangle's base and marked several points on that parallel line connecting them to the base sides, yielding many triangles. Below he wrote a new header – "rectangles" – though

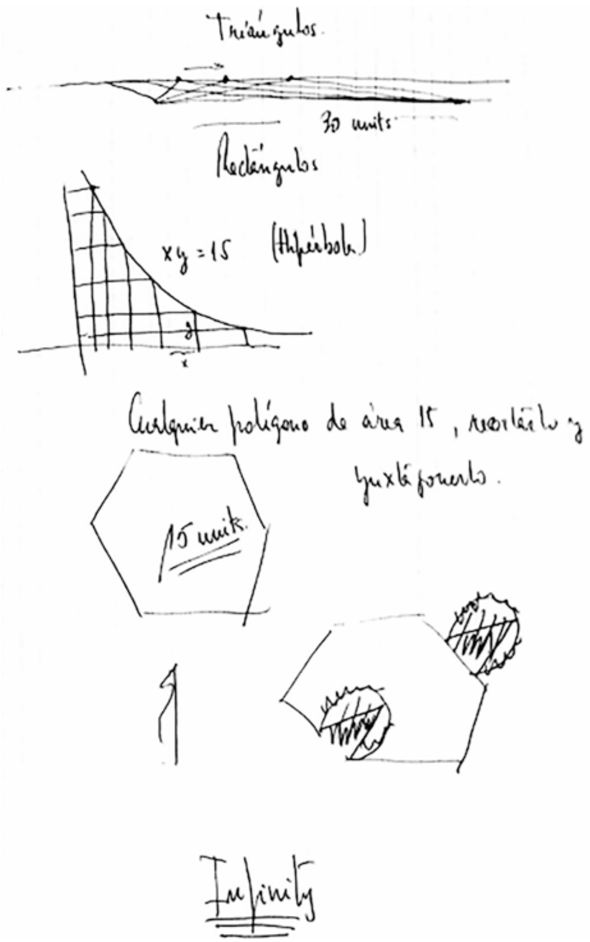


Fig. 10.9 R7's attempts at solving the task

here he did not draw even a single rectangle. Rather, he drew a graph of the curve xy = 15 (see Fig. 10.9). At the bottom of the page, he wrote the word "infinity" with a double underline.

A systematic move from considering triangles to considering rectangles can also be seen in R4's solution. However, while R4's reasoning concerning triangles seems to be based on symbolic considerations, R7 seems to be using a more visual form of thinking for the case of triangles. It also seems that R7 went on to consider rectangles in a general symbolic way, as also expressed graphically. So, while R7 referred to graphical representations in considering all rectangles that satisfy the task, R4 considered the graphical representations for discerning cases of solutions that were based on integers only. In any case, we can see flexibility in R7's thinking when moving from triangles to rectangles, as expressed in the different representations used for triangles [drawing and numerical] and for rectangles [algebraic symbols and graphical drawings]. Again we see a systematic approach that continues after the first set of infinitely many solutions.

R8 began by considering a 3 × 5 rectangle, modified to be a parallelogram with the same side length and altitude length, yielding a parallelogram with the same area. Next to this, he explained that $15 = b \times a$, and it seems that he came to the conclusion that there are infinitely many solutions. Next he drew a 1 × 15 rectangle, possibly with a similar set of considerations. After that, he moved on to triangles, denoting that $(a \times b)/2 = 15$, $a \times b = 30$. Next to this, he drew a triangle with a base length of 6 and an altitude length of 5, where a line parallel to the base is drawn through the vertex and many triangles are drawn with one shared base. He then wrote that there are an infinite number of triangles, though only one of them is equilateral. Next he drew a square, writing "if it's a square, there is only one, $l = \sqrt{15}$". He then drew an irregular polygon, based on a 3 × 5 rectangle, with 1 unit moved around, and then drew a second example of such a modification and indicated that it can be accomplished in other ways as well (see Fig. 10.10).

In R8's solution, we can again observe a systematic move. He started with basic polygons, modified each to see if more polygons suitable for the task could be obtained, and then moved on to the next polygon. We see this as a manifestation of convergent thinking. However, the last set of infinitely many irregular polygons may be a sign of flexible divergent thinking. Here also we can see that the first instance of infinitely many solutions did not prevent him from suggesting other alternatives in a combination of visual drawings and explanations accompanied by symbols.

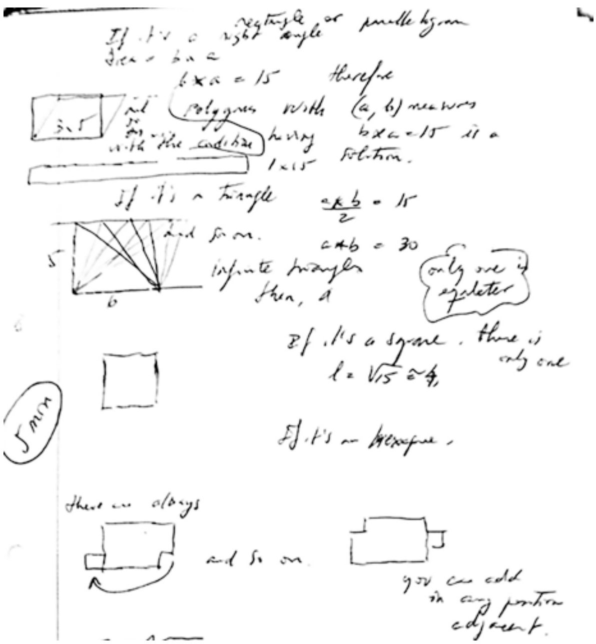


Fig. 10.10 R8's attempts at solving the task

10.5 Discussion

The first question posed at the outset of this research was: What is the potential of a mathematical task with infinitely many solutions to be used as a tool for evaluating creative mathematical thinking? The formulation of the task used in this study is not complex. When such a task is given to upper elementary school students, results indicate that with an ample amount of time, the students were able to find many solutions for it (Swisa, 2015). The participants in this study, 23 mathematics education researchers, were given a very limited amount of time, 5 min, to perform the task. As reported above, some of them noted that the limited time did not allow them to further explore the situation.

Several observations can be made on the basis of the findings reported above. We note that the task elicited two different sets of responses. One set may be characterized by drawing polygons with an area of 15 square units. It may be that the grid paper provided a means for recording the solutions, functioning as a "hidden" variable and constraining some of the participants to draw mainly polygons with sides parallel to the grid. It may also be that the instruction "to draw" polygons combined with the physical tool of the grid paper led the participants to consider concrete solutions, as opposed to abstract and perhaps logical lines of thinking. Considering that this group comprised two-thirds of the responses, we could count how many correct polygons were recorded in each response, yielding a fluency measure. We could distinguish between various polygons in an attempt to define which polygons would possibly point at a different line of thinking in order to discern a flexibility measure. Finally, we could see which polygons were unique and measure originality.

What stopped us from doing all the above, usually considered to be normative when evaluating creativity in such a research study, was the second set of responses. We see the production of the first two-thirds of the participants as a manifestation of what Runco (1996) characterized as divergent thinking. It exemplifies the production of random ideas. Like the example we showed in the results section, these participants did not seem to pursue any one direction which might lead to some coherent general statement. Instead, it seemed that each solution was a new endeavour, in a different direction. In other words, for us, these responses lack evidence of critical discretion. Hence, in line with Runco (1996), we say that the 15 participants who drew actual polygons exhibited noncritical divergent thinking without signs of convergent thinking.

The second group of responses, which represented about a third of the entire group, explicitly referred to the existence of infinitely many polygons that meet the task requirements. As reported in the findings section, each of the eight participants in this group had his or her own unique approach to the task. However, we can infer some general characteristics of their working process. One is that they usually pointed to more than one set of infinitely many polygons. This is surprising as one might think that finding a set of infinitely many polygons would cause the solver to stop looking for more solutions. However, all eight participants in the present study who found such an infinite set did not stop working. In fact, in most cases, they looked for other solutions based on a different thinking strategy, thus exhibiting flexible and divergent thinking. This is in line with Storme et al.'s (2015) theory that creative thinking consists of two different but related creative processes: divergent-exploratory and convergent-integrative (Storme et al., 2015). It may be that they interpreted the task instruction to "find as many as possible" to be, in fact, an instruction to find as many categories as possible.

A different commonality relates to the thinking process that may have led to finding an infinite set of polygons. In many cases, we noticed convergent thinking, where a sequence of slightly small modifications led the solver to conclude that an infinite number of similar solutions can be found. In fact, when considering the 15 responses that found a finite number of solutions, this type of thinking seems to be missing. That is, although in many of the solutions of both groups, starting with a regular 3 × 5 rectangle was the first step, for those who reached the conclusion of infinitely many polygons, convergent thinking followed, while the other two-thirds of the participants just moved on to consider some other shape.

While Guilford (1973) identified divergent thinking with creative mathematical thinking, it seems that we can point to convergent and divergent thinking as playing two complementary roles in creative mathematical thinking. On the one hand, the divergent thinking of the 15 participants led to relative fluency in producing results. On the other hand, each result was an isolated solution. In general, convergent thinking leads the solver to the one sought-after result. In this case, however, convergent thinking actually led to different sets in which each set contained an infinite number of solutions. Furthermore, after reaching a solution set with an infinite number of solutions, participants continued to search for an additional solution set. In other words, it seemed that the convergent thinking process also led to divergent thinking processes. This is a significant finding of this study. It makes sense that, for a problem to be solved, convergent thinking must eventually follow divergent thinking. However, to allow for creative mathematical problem-solving, this study shows how the combination of both processes is necessary and that following one line of thought to an eventual end may actually lead to a new direction and new solution path. Thus, we may conclude that a task with an infinite amount of solution outcomes can be suitable for eliciting and evaluating creativity, by taking into consideration both convergent and divergent thinking. It also raises methodological issues.

In line with most other studies which attempted to evaluate mathematical creativity, we also set out to assess the outcomes in terms of fluency, flexibility, and originality (although from the beginning we stated that originality would not be part of this study). In the end, this standard method of analysis was revised. Fluency was apparent for the most part in the 15 participants who drew a variety of polygons. Yet, what can be said about the participants who indicated that an infinite amount of solutions may be found? How can fluency be assessed in a task with an infinite amount of outcomes? Regarding flexibility, although we did not explicitly analyse the flexibility of the 15 participants, it may be said that flexibility was evidenced in the form of different-shaped polygons (e.g. rectangles and triangles). Among the eight participants who obtained an infinite number of solutions, a richer form of flexibility was evident. For these eight participants, flexibility took the form of using different representations (R1-R8). For some of them, this flexibility involved using both algebraic thinking and visual thinking (R2 & R4) or using algebraic expressions and verbal expressions (R7 & R2). When comparing this task with more common problems, it seems that the issue of flexibility is essentially similar to how it may be assessed in those other problems. What is perhaps different in this case is that for a task with an infinite number of solutions, a second degree of flexibility comes into play, as the solver seeks not only multiple isolated solutions but also multiple sets of solutions. Thus, the method of analysing and evaluation creativity may differ in a task with an infinite amount of outcomes. Finally, although in this study we did not analyse originality, how to analyse originality when solving this type of task is another methodological question.

This study raises several possible directions for future study. First is the time aspect. The stated reason for not studying originality was the short time limit given to solve the problem. Task time is a variable in need of further investigation. Further studies in which more participants are given more than 5 min to solve a problem are necessary in order to examine this complex aspect of mathematical creativity. A second research path is how the same problem may be solved by different groups of participants and what that might mean in terms of creativity. Was this task perceived as a problem in the way Schoenfeld (1985) defined it? On the one hand, this task did not pose a challenge to the participants of the Problem@Web Conference. Recall that this task was also given to elementary school students. Yet, this is not an algorithmic task – there was no ready-made algorithm to be applied. Indeed, the participants took different starting points to solve the problem. Also, it seems that the participants were engaged with the problem – how else can we explain that even after finding a set of infinitely many solutions some of them kept looking for another such set? Or the fact that some wrote that not enough time was given to consider other polygons? Could it be that the 5-min limit turned this task into a challenge? We need more experience with such problems to draw conclusions. Finally, we note that the special environment of the conference could have also affected the results. At the conference, participants solved the problem in an environment that was conducive to taking chances, fostered mutual respect for all, and, perhaps most importantly, encouraged participants to be open to new experiences. As educators, we ask ourselves how such an environment can be replicated in the classroom and, if so, how might that extend the possibilities for fostering mathematical creativity.

Appendix

Claim: The cardinality of the set of all polygons in the plane is $\aleph$. ($\aleph$ denotes the cardinality of the continuum.)

Lemma: For any given natural number $n \geq 3$, the cardinality of the set of all n-gons in the plane is $\aleph$.

The claim follows from the Lemma and the fact that $\aleph 0\times \aleph = \aleph$.

Proof of the Lemma:

The set of all n-gons in the plane is at most as large as the set of all n-tuples of points in the plane because different n-gons, when placed in the plane, determine different n-tuples of points. But the set of n-tuples of points in the plane is equivalent to the set of vectors in 2n-dimensional space (every point has two coordinates), which is of cardinality $(\aleph)^{2n}=\aleph$.

References

Ayres, P. (2000). An analysis of bracket expansion errors. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25–32). PME, Hiroshima, Japan.

Davies, D., Jindal-Snape, D., Collier, C., Digby, R., Hay, P., & Howe, A. (2012). Creative learning environments in education: A systematic literature review. *Thinking Skills and Creativity, 8*, 80–91.

Demby, A. (1997). Algebraic procedures used by 13- to 15-year-olds. *Educational Studies in Mathematics, 33*, 45–70.

Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 42–53). Alphen aan den Rijn, Netherlands: Kluwer.

Guilford, J. (1973). *The nature of human intelligence*. New York: McGraw-Hill Book Company.

Guberman, R., & Leikin, R. (2013). Interesting and difficult mathematical problems: Changing teachers' views by employing multiple-solution tasks. *Journal of Mathematics Teacher Education, 16*(1), 33–56.

Haylock, D. W. (1987). A framwork for assessing mathematical creativity in schoolchildren. *Educational Studies in Mathematics, 18*, 59–74.

Haylock, D. W. (1997). Recognising mathematical creativity in schoolchildren. *ZDM Mathematics Education, 29*(3), 68–74.

Hayne, N., & Tabach, M. (2014). When stories and multiple solution tasks meet in a technological environment: The case of equality. *Mediterranean Journal for Research in Mathematics Education, 12*(1–2), 23–37.

Kattou, M., Kontoyianni, K., Pitta-Pantazi, D., & Christou, C. (2013). Connecting mathematical creativity to mathematical ability. *ZDM, 45*(2), 167–181.

Kaufman, J. C., & Beghetto, R. A. (2009). Beyond big and little: The four c model of creativity. *Review of General Psychology, 13*(1), 1.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Krutetskii, V. A. (1976). The psychology of mathematical abilities in schoolchildren. In J. Teller (Trans), J. Kilpatrick & I. Wirszup (Eds.). Chicago: The University of Chicago Press.

Kwon, O. N., Park, J. S., & Park, J. H. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. *Asia Pacific Education Review, 7*(1), 51–61.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–135). Rotterdam, The Netherlands: Sense Publishers.

Leikin, R., & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st international conference for the psychology of mathematics education* (Vol. 3, pp. 161–168). Seoul, Korea: The Korea Society of Educational Studies in Mathematics.

Leikin, R., & Pitta-Pantazi, D. (2013). Creativity and mathematics education. *Special issue in ZDM – The International Journal on Mathematics Education, 45*(2), 159–332.

Levav-Waynberg, A., & Leikin, R. (2012). Using multiple solution tasks for the evaluation of students' problem-solving performance in geometry. *Canadian Journal of Science, Mathematics, and Technology Education, 12*(4), 311–333.

Levav-Waynberg, A., & Leikin, R. (2009). Multiple solutions for a problem: A tool for evaluation of mathematical thinking in geometry. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of sixth conference of European Research in Mathematics Education* (pp. 776–785). Lyon, France: Institut National de Recherche Pédagogique.

Levenson, E. (2011). Exploring collective mathematical creativity in elementary school. *Journal of Creative Behavior, 45*(3), 215–234.

Lev-Zamir, H., & Leikin, R. (2011). Creative mathematics teaching in the eye of the beholder: Focusing on teachers' conceptions. *Research in Mathematics Education, 13*(1), 17–32.

Lev-Zamir, H., & Leikin, R. (2013). Saying versus doing: Teachers' conceptions of creativity in elementary mathematics teaching. *ZDM, 45*(2), 295–308.

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For The Learning of Mathematics, 26*(1), 20–23.

Lin, C. Y., & Cho, S. (2011). Predicting creative problem-solving in math from a dynamic system model of creative problem solving ability. *Creativity Research Journal, 23*(3), 255–261.

Linchevsky, L., & Vinner, S. (1998). The naïve concept of sets in elementary teachers. In A. Borbás (Ed.), *Proceedings of the 12th international conference psychology of mathematics education* (Vol. 1, pp. 471–478). PME, Vezprem, Hungary.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of proof: A five year case study. *Journal for Research in Mathematics Education, 27*(2), 194–219.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30*(2), 236–262.

Mumford, M. D., Baughman, W. A., Maher, M. A., Costanza, D. P., & Supinski, E. V. (1997). Process-based measures of creative problem-solving skills: IV. Category combination. *Creativity Research Journal, 10*, 59–71.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Polya, G. (1973). *How to solve it. A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Runco, M. (1996). Personal creativity: Definition and developmental issues. *New Directions for Child Development, 1996*(72), 3–30.

Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM International Journal on Mathematics Education, 29*(3), 75–80.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Font Strawhun, B. T. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *The Journal of Mathematical Behavior, 24*, 287–301.

Singer, F. M., Ellerton, N., & Cai, J. (2013). Problem-posing research in mathematics education: New questions and directions. *Educational Studies in Mathematics, 83*(1), 1–7. https://doi.org/10.1007/s10649-013-9478-2

Storme, M., Lubart, T., Myszkowski, N., Cheung, P. C., Tong, T., & Lau, S. (2015). A cross-cultural study of task specificity in creativity. *Journal of Creative Behaviour*. https://doi.org/10.1002/jocb.123

Swisa, R. (2015). *Mathematical* Creativity *in both, a technological and non-technological* Environment*: student's work and* Teacher*'s actions*. Unpublished master thesis. Tel-Aviv: Tel-Aviv University.

Tabach, M., & Friedlander, A. (2013). School mathematics and creativity at the elementary and middle-grade levels: How are they related? *ZDM, 45*(2), 227–238.

Tan, C.-S., Lau, X.-S., Kung, Y.-T., & Kailsan, R. A. (2016). Openness to experience enhances creativity: The mediating role of intrinsic motivation and the creative process engagement. *Journal of Creative Behaviour*. https://doi.org/10.1002/jocb.170

Torrance, E. P. (1974). *Torrance tests of creative thinking*. Bensenville, IL: Scholastic Testing Service.

Tsamir, P., Tirosh, D., Tabach, M., & Levenson, E. (2010). Multiple solution methods and multiple outcomes—Is it a task for kindergarten children? *Educational Studies in Mathematics, 73*(3), 217–231.

Chapter 11
The Power of Seeing in Problem Solving and Creativity: An Issue Under Discussion

Isabel Vale, Teresa Pimentel, and Ana Barbosa

11.1 Introduction

The teaching of mathematics is a complex task, as there are several aspects that must be considered simultaneously, which requires that teachers have knowledge and deep understanding of the issues and mathematical content they expect to teach and also requires that teachers have a good understanding of how to teach, in order to be effective in the development of mathematics learning of all students.

Effective teaching of mathematics should engage students in the discussion and solution of tasks that promote reasoning and problem solving, skills that should be emphasized in mathematics classrooms at all levels. One way for teachers to meet this goal involves the selection of tasks that require students to apply their mathematical knowledge and involvement in higher-level thinking. These tasks must include the use of different representations and tools, and their flexible handling, allowing students to find multiple ways to approach a problem and various strategies (NCTM, 2015). Ever since 1980, the National Council of Teachers of Mathematics, in its Agenda for Action, recommended that problem solving should be the focus of school mathematics. Thus, problem solving has been considered an essential theme in the teaching and learning of mathematics given the existence of unanimity about its importance in the curriculum. Students' abilities in solving problems still require substantial improvement, especially when considering the nature and rapid evolution of today's world. However, due to the complexity of the process of problem solving, many of the issues related to greater efficiency in

I. Vale (✉) · A. Barbosa
School of Education of the Polytechnic Institute of Viana do Castelo,
Viana do Castelo, Portugal
e-mail: isabel.vale@ese.ipvc.pt; anabarbosa@ese.ipvc.pt

T. Pimentel
Escola Secundária de Santa Maria Maior de Viana do Castelo, Viana do Castelo, Portugal

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_11

teaching problem solving remain unanswered. What effective perspectives can be considered in the teaching of problem solving? May the performance of students in problem solving be affected by a specific type of approach?

We believe that problem solving must still remain as a central goal of mathematics learning in the twenty-first century; it will eventually be necessary, however, to rethink its approach in the classroom. This change results from the growing demand for professionals with higher-order skills. This change concurs with the objective that all students have access to an education that emphasizes creativity, innovation and problem solving (e.g. Pitta-Pantazi, Sophocleous, & Christou, 2013). A promising strategy may involve the enhancement of visualization; and this ability can be developed in students if it is promoted by the teaching practices. Although visual representations have been underrated for several decades, in the late twentieth century, there was a revival of the interest in visualization as a powerful tool in mathematical reasoning.

Recent research in the area of cognition, in particular in problem-solving processes, comes to the conclusion that, for certain kinds of tasks, the use of visual representations may have advantages over the use of other representations, therefore facilitating problem solving. Some authors (e.g. Campbell, Watson, & Collis, 1995; Dreyfus & Eisenberg, 1986; Presmeg, 2014; Zimmermann & Cunningham, 1991) suggest that for students to be mathematically competent and creative, they have to be able not only to solve traditional computational and logical problems but also to use visual images and intuitive skills at all stages of the developmental process. We emphasize the potential of visual solutions, especially because of the simplicity and creativity that such proposals may exhibit. In this sense, we begin by presenting an overview of some aspects associated with problem solving, creativity and visualization, grounded in reference authors and in some studies we carried out (e.g. Barbosa & Vale, 2014; Vale & Pimentel, 2011; Vale, Pimentel, Cabrita, Barbosa, & Fonseca, 2012), where we highlight the importance and power of the problem-solving strategy *seeing* and its links with creativity.

11.2 Revisiting Problem Solving

Some authors suggest that problem solving is the process of applying the previously acquired knowledge to new situations and that it may involve exploring, applying strategies, formulating and testing conjectures. This is a very absorbing activity, because the solver is challenged to think beyond the starting point, to think differently, to expand his/her thinking and, consequently, to reason mathematically. We think it is pertinent to start by defining problem, in order to delimit the concept. In general, it is consensual to consider that problems are situations for which a procedure or method to get to the solution is not known and whose solution requires critical thinking. Otherwise, if the situation can be solved using routine or standardized processes known by the individual that lead directly to the solution, the question is classified as an exercise. Thus, analysing if a given situation is or not a problem doesn't only depend on the task but also on the individual to whom it is proposed to. For example, the question *Calculate the product* 8 × 6 can have multiple interpretations, according to the level of knowledge of the individual who faces it: a specific fact, if the response is automatic and the solver makes use of memory; an exercise, if the solver mobilizes training or mechanization; or a problem, if it involves finding a path. It is usually recognized that problem solving depends on many factors such as an organized body of knowledge about the problem domain, techniques of representation and metacognitive processes (Kilpatrick, 1985). Other features may be the difficulties of understanding the statement and those that come from too much importance given to the calculations and quick answers from the earliest years of schooling.

When facing a real problem, in which we don't know the path to the solution, it is necessary to choose and use strategies according to each situation. Polya (1945) introduced the notion of heuristics, through his book *How to solve it*, a concept later referred by mathematics educators as strategy. Other authors have also proposed models for problem solving where one of the phases involves the search for strategies that enables the solver to attack the problem and get to the solution (e.g. Guzmán, 1990). Strategies have been recognized as useful as they can give clues about the path to follow and over the years they have been considered as abilities that students should develop. For these reasons a trend emerged advocating the explicit teaching of strategies and also defending the approach of the different stages of Polya's model. This perspective, called teaching *about* problem solving (Hatfield, 1978), considers that problem solving should be seen as a curricular content, taught in the same way as, for example, multiplication. However, it was found that this way of teaching did not produce the desired effect, that is, it did not contribute to make students better problem solvers. Strategies help to reflect and interpret problems, but there are authors (e.g. Cai & Lester, 2010) that consider they don't help students who don't know what to do, since they are too general, independent from mathematical subjects, and so it is not likely that they help to solve an algebra problem, for example. To overcome this difficulty, Schoenfeld (1985) recommends (a) developing with students a greater number of more specific strategies, more connected to certain categories of problems; (b) teaching metacognitive strategies so students learn to apply problem-solving strategies and the knowledge acquired in the right situations; and (c) studying ways to eliminate counterproductive beliefs of students and foster productive beliefs about mathematics, problem solving and their own personal skills. Strategies may be defined as tools which, most of the times, are identified with thinking processes and can be very useful in different phases of the problem-solving activity; moreover, mathematical knowledge and reasoning strategies must be learnt and used simultaneously and not separately (e.g. draw a picture or diagram, make an organized list, look for a pattern, trial and error, work backwards). We believe that the acquisition of a repertoire of feasible strategies – already known and applied – is a body of knowledge in action that (a) helps students to address the problem and to trace a path, (b) may be an alternative to the direct use of concepts that students do not have or are not immediately accessible and (c) frequently facilitates the interpretation of situations (Vale & Pimentel, 2004).

Problem solving can also be seen as the ultimate purpose of mathematics teaching, considered as a way of thinking. Routine procedures are just tools, and it is therefore necessary to teach students to think and prepare them to effectively solve problems. This is teaching *for* problem solving. In this perspective the teaching of concepts and procedures is considered basic and a prerequisite. Some authors criticize this approach, in considering that problem solving is thus seen as an independent and isolated subject, with a secondary importance in the development of mathematical ideas; they also argue that there is little evidence supporting that student's ability to solve problems improves when isolating the problem solving from the learning of concepts and mathematical processes (Cai & Lester, 2010; English, Lesh, & Fennwald, 2008).

Problem solving can also be considered as a way of instruction. We can teach in our mathematics classes using problem solving as a guideline to mathematical concepts, thus becoming the basis for teaching the various contents. This perspective is referred to as teaching *through* problem solving. Cai and Lester (2010) present problem solving as referring to mathematical tasks that have the potential to provide intellectual challenges that can increase the development and understanding of the students. In this sense, problem solving is an integral part of the learning of mathematics, not being considered as a separate topic in the curriculum but as a means for teaching mathematical concepts and skills.

While agreeing that problem solving is a key context for learning mathematics, we recognize that the definition of learning through problem solving seems to be excessive in certain circumstances. Even the authors who advocate teaching through problem solving consider that not all the proposed tasks have to be problematic (Cai & Lester, 2010). There are moments in the learning of mathematics to develop certain skills that involve training exercises. We also consider that this is a real situation and we know that teachers are confronted with it daily. Without some training it is not possible to acquire dexterity in routine procedures, necessary even to solve complex problems.

Thus, and after this brief incursion into different ways of looking at problem solving and its relations with the learning of mathematics, we propose the expression, perhaps more consistent with practice, of teaching *with* problem solving; in this view, we defend that problem solving should follow in parallel the curriculum and classroom practice, along with other more procedural tasks, developing an understanding of concepts and of mathematical structures and leading students to progressively acquire a list of useful and productive strategies in other approaches.

11.3 From Problem Solving to Creativity

In recent decades, problem solving has played an important role around the world as an organizing axis of the mathematics curriculum. However, it had not the expected impact on teachers' practices and, consequently, on students learning. Students' mathematics learning should include more than routine tasks; it should be enriched with challenging problem-solving tasks. According to Liljedahl and

Sriraman (2006), the ability of students to present new insights and/or solutions in mathematics is considered as an indicator of creativity. This is of great importance, not only for students but also for teachers, especially if these tasks lead to structural understanding of mathematical concepts and encourage fluency, flexibility and originality as essential components of creative thinking. Teaching that does not provide moments in which students are creative denies them any opportunity to develop their skills in mathematics but also to appreciate this subject.

Mathematical creativity is a very complex construct to define and measure (e.g. Haylock, 1997; Pitta-Pantazi et al., 2013). In an examination of the research about how to define mathematical creativity, we found that there is a lack of an accepted or used definition for mathematical creativity, since there are numerous ways to express it (e.g. Mann, 2006). It is argued that creativity begins with curiosity and engages students in exploration and experimentation tasks where they can manifest their imagination and originality; in general it is a notion that embraces a wide range of cognitive styles, categories of performance and many kinds of outcomes (Barbeau, 2009; Haylock, 1997). When we talk about creativity of young students, we do not mean the same creativity of famous people who made significant contributions to society (also known as creativity with big-C). Instead, we are concerned with the creativity of every day (with little-c) that can happen and is manifested in the classroom. Creativity is an essential part of mathematics learning (e.g. Mann, 2006; Pehkonen, 1997), but it is a process often neglected and considered impossible to pursue in mathematics classes; however, creativity is not a mysterious, unobservable process or an innate ability, not capable of being learnt. It is, instead, a range of abilities that can be taught and learnt by the student, so teachers must provide them a learning environment, where they propose adequate mathematical tasks in order to develop some of the mathematical creativity components. Research findings show that mathematical problem solving and problem posing are closely related to creativity (e.g. Pehkonen, 1997; Silver, 1997), leading to the understanding of structural mathematical concepts. Thus, teachers should provide tasks with multiple solutions that arouse curiosity and involvement and provide the flow of mathematical ideas, flexibility of thought and originality (Vale et al., 2012).

11.3.1 Insight and Intuition

To do mathematics is in fact to solve problems. And, in this activity, there is little difference between a mathematician and a mathematics student. Each one is following a path unknown to them. So, between the homework problem of a student and the invention of a mathematician, there is only a difference of cognitive degree (Silver, 1997). Invention is also required to the student. Here, we discuss some related concepts like invention, aha experience, intuition and creativity.

Mathematical invention has four stages (Hadamard, 1945): (a) initiation. In the initiation phase, there is deliberate and conscious work. The person engages voluntarily with the problem, using previous knowledge and experience; (b) incubation. This phase can last some minutes or some years. In this stage new ideas are created and discarded, and so imagination relies heavily on the contributions of the unconscious mind. The incubation phase begins when the solver, unable to obtain a solution, stops to work on the problem. But actually he is working at an unconscious level; (c) illumination. The illumination stage may then occur in the form of a solution, coming quickly when the person is not thinking about the problem and producing a feeling of joy; and (d) verification. In this phase there is a validation, testing and reducing the idea to an exact form.

Davis and Hersh (1981) refer to the illumination phase as a flash of insight. Some authors call this the Aha! experience. According to Liljedahl (2004), the Aha! experience is self-defining. At the moment of insight, in the flash of understanding when everything seems to make sense and the answer appears before you, that is, you know it. So the Aha! experience is one of the stages of the invention process, illumination, although it is strongly influenced by the previous stages of initiation and incubation.

No one can explain exactly what happens in the mind of the person who has a flash of this kind. It is known that this happens unexpectedly, often when the person is not thinking about it but doing something else. This process is described as a creative leap of the mind (Gardner, 1978), which *sees* the situation in a more simple way, thereby solving a problem that, using conventional methods, may prove to be difficult. This ability is developed by solving problems and getting used to look at them in an unconventional way.

Invention is a global process that includes intuition, imagination and creativity (Liljedahl, 2004).

The term intuition can be defined as a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident or immediate (Fischbein, 2002). A factor that contributes to the effect of immediacy is visualization, because a visual image delivers simultaneously most of the information related to a situation. It can also be described as plausible, convincing, incomplete, heuristic and holistic. Associating this phenomenon with teaching, Davis and Hersh (1981) state that we try to teach mathematical concepts not formally but intuitively – showing examples, solving problems and developing a reasoning technique – all expressions of having successfully internalized something: an intuitive mathematical idea. As a result of those experiences, something reminds in the student mind which is his/her representation of the mathematical concept. So intuition is not the direct perception of something existing internally or externally but the effect in the mind of certain experiences of activity, manipulation of concrete objects and later of recordings on paper or mental images. We have intuition because we have mental representations of the mathematical objects (Hersh, 1997). And we acquire those representations, not through the memorization of verbal formulas but through repeated experiences of manipulation of physical objects and then problem solving and self-discoveries. This way, intuition doesn't exist without knowledge of the subject. It definitely develops because if you have a false one, that intuition is modified (Burton, 1999). Intuition is slightly different from insight since the latter is a sort of flash, although they are both components for developing knowledge.

11.3.2 Problem Solving and Creativity

Polya (1945) states that creativity is an innate characteristic of individuals but teachers have the responsibility to stimulate mathematical creative thinking of students. This way, the environment plays a critical role in nurturing mathematical creativity if the teacher promotes an exploratory approach. Mainly, we believe that creativity involves the ability to challenge assumptions; break boundaries; recognize patterns; *see* things in new ways or have new eyes to old situations; have a chance to imagine, communicate, solve and invent problems; take risks; make mistakes; reflect; extend; and make new connections. Based on these assumptions, creativity plays a crucial role in problem solving. We argue that creative thinking can be learnt and developed if teachers propose to their students tasks that enable multiple solutions and involve the use of different representations and different properties of a mathematical concept (e.g. Leikin, 2009; Vale, Barbosa, & Pimentel, 2014).

To judge creative productions in mathematics, in particular creative solutions to problem solving and problem-posing tasks, researchers propose several criteria, but we can notice that there are some commonalities in the different attempts, especially in three main components/dimensions: fluency, flexibility and originality (e.g. Guilford, 1956; Leikin, 2009; Silver, 1997). *Fluency* is the ability to produce a large number of different solutions for the same task. This ability can be acquired by trying to achieve the greatest possible number of different ideas. Ideas come sometimes associated, and the more a person works on a topic, the more fluent he/she becomes. This skill is very important because the first step to problem solving or generating anything creative is having as many ideas as possible to choose from. Although generating many ideas does not mean that all are interesting, it is important because it is associated with flexibility and originality. Teachers' practices tend to make students seek only for one correct answer instead of considering multiple possibilities; as a consequence, students don't feel the need to present more than one solution. Silver (1997) states that the use of open or ill-structured tasks, during the teaching process, can encourage students to generate various solutions, which contributes to the development of fluency. Moreover, it has to be explicitly asked to students to solve the task in many different ways (Vale et al., 2014). *Flexibility* is the ability to think in different ways to produce a variety of different views on the same issue, and it is thus an important factor in solving problems. A flexible thinking, as it implies to view the problem from different perspectives, allows making associations between different areas of knowledge. Within mathematics education, Krutetskii (1976) considers reversibility, the ability to reverse a mental operation, as an aspect of the flexibility of thinking. It is associated with a change in the understanding and interpretation of tasks, a change of ideas and strategies when you are solving a problem to find other solutions or to choose the optimal solution. Silver (1997) argues that flexibility in problem solving and posing is identified by the number of different ways the student uses to solve, express or explain a problem. This means that the different solutions can be categorized by the different processes or contents used to achieve the solution. Flexibility requires a change in the way we look for situations, that is, the opposite of fixation (e.g. Haylock, 1997). What makes a solution creative is, as Presmeg (2014) suggests, the fact of being necessary to break the usual mental scheme, for example, which suggests the use of formulas when the word "area" is displayed, in order to search for a more fruitful and simple method of visual solution. It is flexibility that may lead to originality. *Originality* is the ability to think in an unusual way, producing new and unique ideas (e.g. Leikin, 2009; Silver, 1997), thinking outside the obvious or having a rare idea. Originality is the pinnacle of creativity. Often it is this skill that makes us finding a person to be "creative". This skill is the most difficult to develop but it can be reinforced. By definition, originality means producing ideas and products that have not existed before, but we judge it in relative terms. This means that a student that has never been exposed to an idea before and comes up with it on his/her own displays originality.

Another dimension of creativity, also considered in some situations, is *elaboration*. It refers to the number of details of an idea and is used to extend or improve a solution, that is, the ability that the student shows to have a careful thought about particular aspects of a problem or situation, changing one or more of its aspects, replacing, combining, adapting, changing, expanding, removing, rearranging or going back and then speculating on how this change would have a ripple effect on other aspects of the problem or situation. We do not consider it in our work, and one reason is related to the difficulty in determining different development levels of the solutions presented by the students in many of the tasks. Several authors (e.g. Leikin, 2009; Pitta-Pantazi et al., 2013; Silver, 1997) do not use this dimension either when analysing the mathematical creativity of students.

It is the combination of these dimensions that allows us to characterize, through the analysis of the students' productions to the proposed tasks, the creativity of students in school mathematics and help design tasks and strategies to use in the process of teaching and learning.

11.4 Visualization as a Different Mode of Thinking

Visualization has nowadays a crucial role in our society. Currently, the need to think and reason visually in problem solving is much stronger, and it can be a very important cognitive tool in the development of mathematical concepts and processes, including problem solving (Rivera, 2011). Before analysing and presenting some potentialities and constraints of visualization in the area of mathematics education, and their connections with problem solving and creativity, we will start by discussing the perspectives of different authors, in an attempt to delimit its meaning. Finally we discuss some implications for classroom practice.

11.4.1 Characterizing Visualization Towards Problem Solving

When reading about research in the area of visualization, many terms appear related to it, like visual reasoning, spatial thinking, visual images and visual representations, among others. Although it is important to clearly define a concept, the field of visualization is so wide that it is not reasonable to try to enclose it all (Gutiérrez, 1996). Professionals from diverse areas, like psychology or mathematics education, normally use different words or terminology. If we attentively analyse their definitions, we conclude that in some cases, different terms are used with the same meanings, and in other cases different meanings may have been developed for the same words. There is no general agreement about the terminology to be used in this field, but it is undeniable that visualization is considered relevant in a variety of areas of knowledge (e.g. Gutiérrez, 1996; Hershkowitz, Arcavi, & Bruckheimer, 2001; Zimmermann & Cunningham, 1991).

Presmeg (2006) presents an exhaustive synthesis of more than 150 studies about visualization in mathematics education, discussing several overlapping dimensions, like psychological, curricular, instructional, technological and semiotic. In her opinion visualization involves simultaneous acts of creating a spatial arrangement and constructing its image, stating that it includes processes of constructing and transforming both visual mental imagery and all of the inscriptions of spatial nature that may be implicated in doing mathematics. The term mathematical inscription refers to graphical representations (e.g. symbols, diagrams, drawings, figures) necessary in mathematical activity (Rivera, 2011). As for visual imagery, it is important to address first the view of Dreyfus (1995) that defines it as the use of "mental images with a strong visual component" (p. 3). This perspective broadens the possibilities in terms of the types of images we can consider. Presmeg (1986) proposes to define visual image as a "mental scheme depicting visual or spatial information" (p. 42), with or without requiring the presence of an object or other external representations. In this sense, this author established different categories for visual images: (1) concrete, pictorial images as being the pictures in the mind; (2) pattern images that are pure relationships stripped of concrete details; (3) memory images of formulas, when we can see in our minds a formula as it appeared in an external representation; (4) kinaesthetic images, reporting physical movement; and (5) dynamic images, when an image is transformed or moved in the mind. This categorization makes sense in the context of the definition of visualization presented by Presmeg, since it is wide enough in order to include different types of images, representing models, diagrams but also pictures in the mind.

Ben-Chaim, Lappan, and Houang (1989) argue that visualization involves the ability to interpret and understand information represented in the form of figures and the ability to conceptualize and translate abstract relations and information that is not displayed visually. Here it is also possible to distinguish two cases: interpretation of visual information and production of visual images based on non-visual information. The latter situation can also be identified by the words of Dreyfus and Eisenberg (1986) when they state that "many concepts and processes in school mathematics may be tied to visual representations, that is, visual models can be built which reflect (a large part of) the underlying mathematical structure" (p. 1). For the authors, visualization is associated with visual representations, considering that any mathematical concept can be translated by means of a diagram or a graphic. Arcavi (2003) states that visualization is no longer related to a merely illustrative ending, it is also recognized as a key component of reasoning, problem solving and proving. It can be considered as a tool of thinking in the sense that is fundamental in the process of mathematical discovery, involving components of creative thinking.

In the same line of reasoning, Zimmermann and Cunningham (1991) understand that visualization stands for "the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding" (p. 123). In the opinion of the authors, visualization is related to the student's ability to draw an appropriate diagram to represent a certain concept or solve a problem and to use the diagram to achieve understanding. In this sense, in mathematics, visualization is not an end in itself but a means towards an end, which is understanding. In the words of Hershkowitz et al. (1990), visual and spatial thinking and reasoning "generally refers to the ability to represent, transform, generalize, communicate, document and reflect on visual information" (p. 75). Arcavi (2003) discusses mathematical visualization in a more figurative and deeper sense, as *seeing the unseen*, not only what comes *within sight* but also what we are unable to see. This author considers that mathematics deals with objects and entities quite different from the physical phenomena, raising the need to rely heavily on visualization in its different forms and at different levels. Following these ideas, Arcavi (2003) presents a definition of visualization blending the ones proposed by Hershkowitz et al. (1990) and by Zimmermann and Cunningham (1991):

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (p. 217).

This definition integrates many facets of visualization. Concurring with the perspective of Schmitz and Eichler (2015), we consider that the following aspects are especially relevant: (1) it integrates process and product; (2) it integrates different ways of how to handle visualization; (3) it mentions diverse kinds of representations; thus, everything not being completely symbolic could be considered as visual; and (4) it defines visualization as goal-oriented and enumerates purposes like discovery and understanding.

The use of visual representations as a crucial element in problem solving is not a new idea as it is stated by some authors (e.g. Polya, 1945; Schoenfeld, 1985; Stylianou & Silver, 2004). Polya proposed a set of strategies to achieve success in problem solving, being one of the most prominent in this list of suggestions to *draw a picture*; this translates the importance attributed by the author to visual aspects in mathematical thinking and problem solving. At this respect, Polya (1945) stated:

> Figures are not only the object of geometric problems but also an important help for all sorts of problems in which there is nothing geometric at the outset … Even if your problem is not a problem of geometry, you may try to draw a figure. To find a lucid geometric representation for your non-geometric problem could be an important step toward the solution. (p.108)

Considering the focus of this paper, it is useful to clarify the meaning of visual and non-visual methods of solution. Based on the ideas of Presmeg (2006), we consider that visual solutions include not only inscriptions in the form of pictures but also more abstract spatial depictions, involving different visual representations (e.g. graphics, charts, figures, drawings). A non-visual solution is one that involves mainly algebraic, numeric and verbal representations. There are many problems, usually of visual nature or context, that have great potential for visual solutions. It is to such problems that we propose the use of an additional and specific strategy we call *seeing*. In the same sense, Stylianydes and Silver (2009) also refer to the importance of visual representations as a possible problem-solving strategy. We consider that *seeing* involves an activity that may be associated with a more traditional range of strategies (e.g. draw a picture or diagram, solve a simpler problem, look for a pattern), but it is specifically considered as a strategy of thought that involves visual perception of mathematical objects and is blended with knowledge and past experiences. Moreover, s*eeing* includes imagining, which is related with having creative insights or Aha experiences and intuitions; it can also be expressed in terms of drawing, which means translating one's ideas in some visual form. In problem solving, this strategy that recurs to visual representations normally must be complemented with numerical and verbal explanations. According to Duval (2006) "…it is necessary to combine the use of at least two representation systems, one for verbal expression of properties or for numerical expressions of magnitude and the other for visualization" (p. 108). The use of a variety of representations contributes to flexibility of thought and creates the opportunity to think in different ways, which is so important to the development of creative visual solutions.

The use of visualization may be an important aid to all types of problems, including problems in which the visual component is not evident. In these cases, *seeing* may lead to the development of intuition and the ability to establish new relationships, thus breaking with mental fixations and enabling creative thinking. The *seeing* strategy does not replace any other traditional problem-solving strategy; it is rather a way to approach a problem. However it is not always encouraged by teachers although it can be very useful. Many researchers agree that there is a strong relationship between visualization and the ability to solve problems, since visualization often contributes, along with other strategies, to solve problems and to increase student repertoire (Ben-Chaim et al., 1989).

11.4.2 Potentialities and Constraints of Visualization in Problem Solving

A visual method is one that involves a visual image with or without a diagram as an essential part of the method of solution. A non-visual method, on the contrary, doesn't depend on a visual image. The discussion about the role played by visualization in students' reasoning can be complex.

The potential and limitations of visual reasoning are recognized as part of the mathematics classroom culture (e.g. Presmeg, 1999; Zodik & Zaslavsky, 2007) related to the considerations and dilemmas underlying the choice of visual representations. As we have been saying, among the community of mathematics educators and researchers, it is rather consensual that visualization is fundamental and has great potential, in the sense that enhances a global and intuitive perspective and understanding in different areas of mathematics. It is also clear that different individuals may have different thinking styles, and, when solving problems, some prefer to ground their reasoning in visual features (Presmeg, 1999), which reinforces the importance of not overlooking this ability.

In this section, we identify some roles/potentialities of visualization, that is, looking for a visual solution, and seek to explain and justify our choice with some examples of tasks that illustrate such roles. Most of them are associated with solutions that evidence divergent thinking and originality, which are components of creativity we also want to stress.

1. *To understand the problem*

   Non-visual methods are more likely to be used in such a way that prevents students to adequately understand the problem. Visualization is used to help understanding what the problem is about, allowing students to go deep into the situation to support their understanding before any generalization can happen.

   The Big Ben lasts 5 s to beat 6 h. How much time does it last to beat midnight?

   To solve this problem, students usually use proportional reasoning that leads to an incorrect answer: they suppose that it lasts 10 s. An easy way to prove that the numerical method is not appropriate is to make a scheme, as illustrated in Fig. 11.1.

   Starting from the data and through the visual representation of the problem, students are easily conducted to the correct answer of 11 s.

2. *To attribute a meaning to a numerical or algebraic expression*

   The tasks that require students to see the arrangement in different ways and translate it in a numerical or algebraic expression can be used to attribute a meaning through visualization to such numerical expression, connecting previous knowledge about numbers and basic geometric concepts and their relationships (Vale et al., 2012).

   The sea girl organized the shells she caught yesterday as the figure shows. Can you find a quick process to count them? Discover as much ways as you can.

   There are different ways to count the arrangement of the shells, and each counting can be, respectively, written through a numerical expression that translates the students' thinking and *seeing*. Figure 11.2 illustrates the summary of the most common solutions, with the expressions corresponding to each way of *seeing*.

Fig. 11.1 Solution of the Big Ben problem

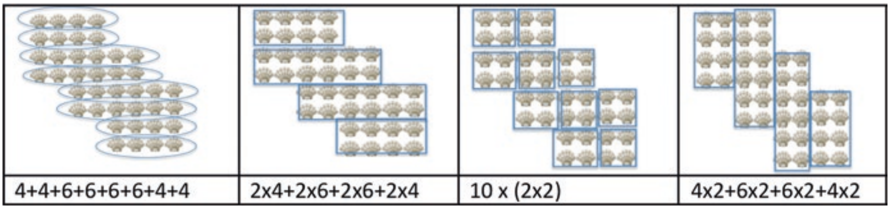


Fig. 11.2 Summary of the students' most common responses to the task

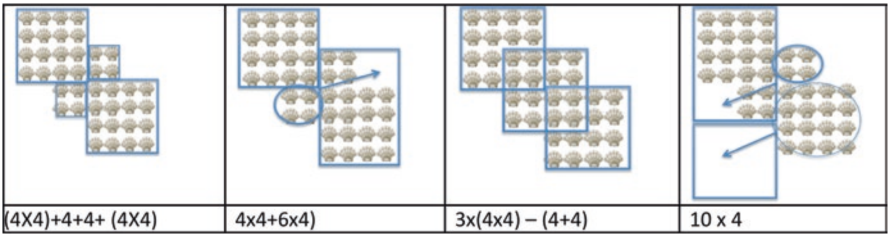


Fig. 11.3 Summary of the students' most original responses to task

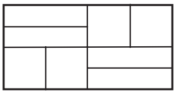
These expressions can be verbalized as the following: "I see the shells in horizontal rows, each one with 4, 4, 6, 6, 6, 6, 4 and 4 shells" or "I see ten squares of 2 by 2". It is important that teachers allow students to discover that each expression illustrates one way of *seeing* but they are all equivalent and correspond to the same number of shells, 40. These kind of tasks have other advantages if many solutions are requested in the statement, since, through visual attempts of solution, students may foster their creative skills, mainly the fluency and the originality. Figure 11.3 illustrates the most original responses.

We argue that a previous work with counting tasks in figurative settings can be a particularly good way to develop skills of *seeing* (identification, decomposition, rearrangement) to facilitate similar processes in growing pattern tasks (Vale & Pimentel, 2011).

3. *To avoid computation*

Seeing some features of the problem can possibly allow to justify numerical conclusions without making computations.

All the segments are traced by the midpoints of the respective rectangles. Find a relation between the areas of the several rectangles.



Students must discover if these eight figures are, or not, equivalent. Usually they begin to attribute numerical values to the dimensions of the initial rectangle and apply formulas. In order to avoid computation and break with fixation in the connection area formula (Haylock, 1997; Presmeg, 2014), the teacher may suggest looking at the figure, trying to see relationships that could be explained without computations. Students can discover that the initial rectangle is divided into four congruent rectangles, and then, each one, a fourth of the initial, is divided into two halves. So, in every case, he/she obtains a rectangle whose area is 1/2 × 1/4 of the initial rectangle. This way, the figures are all equivalent.

4. *To plan ahead*

   Students can visualize during the problem-solving process in order to anticipate, that is, wondering: "What will be the consequence, if I do this?". The problem *Towers of Hanoi* is an example of this.

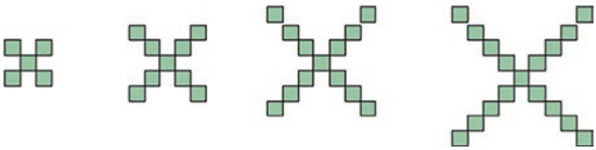
   Move all the discs from tower 1 *to tower* 3. *Tower* 2 *can be used as intermediate, but you cannot place a larger disc onto a smaller one.*

   This proceeding relates to the problem posing question "What if?". To answer a question like "What happens if I move the disc to here?", students would need to be aware that this is one of several possibilities and therefore have a visualization of the wider context of the problem, opposed to a blind route of choosing the first thing that comes into their head.

5. *To bridge with a formal mathematical representation*

   The following example shows that the work with figurative patterns is especially suitable to visualization. The sequence is a growth pattern since each term changes predictably over the previous. This type of patterns provides mathematically rich and varied explorations and is a privileged context for the development of algebraic thinking, particularly in the transition from arithmetic to algebra. Here we intend to enhance the relationship between visual/figurative patterns and the respective numerical patterns showing how the visual appropriation of the regularity, and the search for consistency with the numerical representation, allows a degree of generalization to the discovery of the general term that would not be possible for young students, by simple observation of the numeric sequence.

   Consider the first four figures of a given sequence:

   How many squares will have the 199*th figure? And the nth figure?*

   Previous to being in contact with visual approaches, students tend to convert figures into numbers, in order to find a rule based on the numeric relations.

$$5, 9, 13, 17, 20, \ldots$$

   But, observing the figurative pattern, the students can see that "we have a square in the middle and in each arm of the X we have the same number of squares as the order of the figure", as shown in Fig. 11.4.

Fig. 11.4 One way of *seeing*

Figure	Number of squares
1	1+4
2	1+4x2
3	1+4x3
4	1+4x4
5	1+4x5
...	...
199	1+4x199
n	1+4xn

Fig. 11.5 Construction of a table

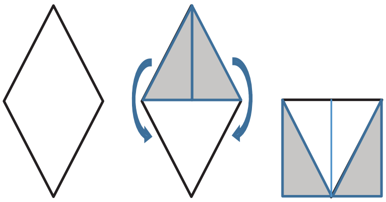


Fig. 11.6 Decomposition of the rhombus

The students frequently dispose in a table the numerical result. But, in order to facilitate far generalization, it is not the final number that must be recorded but the way the construction is seen (Fig. 11.5).

As we can see, this perspective is much more powerful to reach far generalization. Such a discovery includes a type of relational thinking involving the concept of function which leads to the general term of the sequence. In each case the teacher must use the language and the most appropriate concepts for the proposed goals and level of schooling.

6. *To transform the problem into a mathematical form*

   Mathematical formulas may be obtained from the visual representation. Let us see an example:

   Discover a general formula to the area of a rhombus.

   A good way to deduce a formula is to make a drawing (Fig. 11.6).

So, if we label D and d, the two diagonals of the rhombus, the area will be $A = \frac{D}{2} \times d$ which conducts to the classical form of the area of the rhombus, $A = \frac{D \times d}{2}$.

Through this dynamic decomposition, the rhombus is transformed into a rectangle whose area is known. That is, the visual transformations used consist of an operation of reconfiguring the original figure. This process is purely visual, occurring without resource to a mathematical property. The visual evidence obtained can be used in teaching, since it intuitively explains a mathematical result (Duval, 2006).

7. *To explain or prove an assertion*

Tasks with figurative patterns can be a good example of approach to complex processes, mostly conjecturing and generalizing, since they allow a friendly interaction either for teachers or for students. But can this process ensure, by itself, the truth of the conclusion? We argue that it does.

Stylianides and Silver (2009) defend that the structure provided by the figurative representation of the sequence, since the way it is constructed is univocally determined, can guarantee and justify the mathematical law discovered. That is, the mathematical structure of the pattern, provided by the figure and by the statement that makes the pattern defined, eliminates the need of checking the conjecture in more data. In these cases we can trust the pattern discovered without the need to examine individual cases.

With this issue we raise the question whether which is developed through visual argumentation support is a proof, that is, if it can be considered deductive reasoning. To explore a little more of this aspect, we present an example.

Triangular numbers

Observe the figure and try to discover a mathematical law.

A numerical approach can be done:

$$1,3,6,10,15,\ldots$$

$$t_n = 1 + 2 + 3 + \ldots\ldots + (n-1) + n$$

or, using commutative property, $t_n = n + (n\text{–}1) + \ldots + 3 + 2 + 1$.

Thus, adding both equalities, we have: $2t_n = (n + 1) + (n + 1) + \ldots + (n + 1)$ and, consequently, $t_n = \frac{n \times (n+1)}{2}$.

But, based on visual reasoning, we can also conclude that figure n is half a rectangle $n \times m$, as is shown in Fig. 11.7.

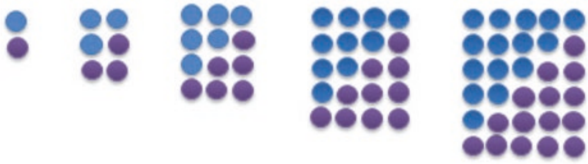


Fig. 11.7 Visual proof

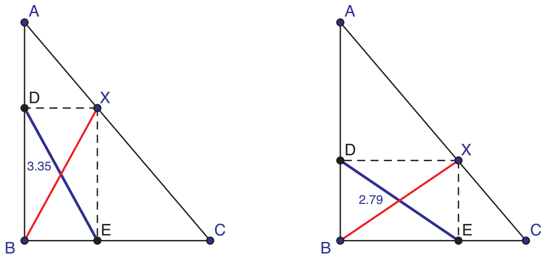


Fig. 11.8 Aha solution

So, its law is also $t_n = \frac{n \times (n+1)}{2}$.

What are the differences between the two approaches? Both are based on the concept of multiplication. In the first it seems to be no doubt that it is a mathematical proof. Thus, due to the parallel made between both, we argue that the second process, fully explained by viewing, can also be considered a mathematical proof (Nelsen, 1993).

8. *To enhance Aha experience*

To make a drawing can obviously facilitate the understanding of a problem, but it is not always sufficient to solve the problem. It is also necessary an Aha experience, as we can see in the example below:

[ABC] is a rectangular triangle and X is a mobile point in the hypotenuse [AC]. Each of the points D and E is the projection from X towards each cathetus. Discover the location of X in order that [DE] length is minimum.

In fact, a drawing with a dynamic geometry programme can be a good starting point to explore this question, but the solution depends on the Aha, which consists in choosing the other diagonal, not in the figure.

In the rectangle [BEXD], the diagonals are congruent, and, this way, we may seek the minimum length of [XB], which happens in such a position of point X that makes [XB] the distance from B to the hypotenuse, that is, when [XB] is perpendicular to the hypotenuse [AC], as shown in Fig. 11.8.

We highlighted eight situations where we aimed to exhibit some potentialities and benefits of visualization. However, we can ask whether the visual is always helpful. In the words of Hershkowitz (1989), the use of this ability can work in two directions: on one hand, visualization is needed to help form mental images of a concept; but the visual elements may, in some cases, narrow down the perspective of an individual about a certain concept.

Cognitive science studies on the problem-solving process show that, although a visual representation may contain the same amount of information than any other form of representation, it may become, by this fact, less explicit. It is precisely because visual representations may contain so much information that can be difficult for students to interpret, construct or use in mathematical problem solving. Thus, there is a need for students of an explicit teaching that puts these visual abilities in evidence.

It is very common to assume that students are introduced to a concept through the observation/exploration of a few examples only. This is called the prototype phenomenon (e.g. Hershkowitz, 1989)**.** We can say that the examples related to a certain concept satisfy its definition and contain its critical attributes; however they are different from one another visually and psychologically. The prototype is designated by Hershkowitz (1989) as being the "popular example".

The selection of certain examples may have implications, when some cases are preferred and others are neglected, which may result in different types of reasoning (Hershkowitz, 1989): (1) the prevalence of visual reasoning based on appearance, leading to a visual judgement that can result in correct or incorrect responses; (2) the fixation on noncritical attributes, which are associated to the prototype example only; and (3) reasoning based on critical attributes. The difference between these last two categories is leading to incorrect or correct responses, respectively.

Following the perspective of this author, also Yerushalmy and Chazan (1990) approached the issue of visualization obstacles, dividing them according to (1) the particularity of diagrams, (2) considering standard figures/diagrams as models (prototypical thinking) and (3) the inability to see a figure/diagram in different ways. These researchers suggest that figures/diagrams are not always easy to interpret and certainly one example or a particular case is not enough to form a complete and accurate mental image of a given concept. Thus it is recommended to explore many different examples, not only prototypes, which highlight critical and noncritical attributes, allowing individuals to broaden their visual perceptions (Zodik & Zaslavsky, 2007).

A problem can be presented to students with or without the support of a figure. When a visual representation is given, the figure presented to illustrate the problem may influence the way a student approaches that specific problem and to what extent he/she will be successful when solving it (Zodik & Zaslavsky, 2007). Some choices end up being helpful, reducing the cognitive demand of the problem, while others may increase the level of difficulty. The possible potential or limitations of the visual aid are related to the type of information that the student can convey from the figure, what he/she understands to be translated by the figure or not.

In a more broaden scenery, it is also pertinent to reflect about the potential and difficulties of using visualization in problem solving even when there is no figure to start with.

Presmeg (1999) starts by identifying the mnemonic advantages of prototype examples but also refers that concrete images are effective in parallel with non-visual methods, like the application of formulas or logical analysis.

Alongside Hershkowitz (1989), Presmeg (1999) also identifies situations related to prototype examples that can lead to some misconceptions. The author also mentions that a standard image of a figure can induce the lack of flexibility in reasoning, interfering with the recognition of a concept in a non-standard figure. These difficulties rely on the fixation for a persisting image preventing the individual to be open to other paths of solution. It is also stated by Presmeg (1999) that a concrete image, involving particular characteristics of a figure, can be associated by students to irrelevant details or may even introduce false assumptions. This type of situation occurs, for example, when students rely on an inaccurate drawing, taking for granted some relations that may not exist. Bell and Janvier (1981) discuss the implications of a similar phenomenon described as "pictorial distractions", stating that, despite the meanings underlying a certain figure, the judgement focuses on visually salient clues perceived by an individual.

Another aspect that should be taken into consideration, in terms of the difficulties arising from the use of visualization, whose attention is brought up by Presmeg (1999), is the possible lack of rigour of a certain representation in the sense that is vague, that is, an image that is not associated to a precise process of analytical reasoning can be of little help.

Visual intuitions can be very valuable, since they have meaning and they can constitute a referent to the individual, which is normally absent from the rigorous formal/abstract approach. However, intuitions aren't infallible. They need to be checked in the sense that they can mislead us to think that certain statements are obvious or self-evident when, in reality, they are dubious or false (Hersh, 1997).

Summarizing, we can assume that all these situations are related with the difficulty of generalizing an image, which is, naturally, a concrete case (Presmeg, 1999).

Both potentialities and constraints of visualization pointed out along this section are important contributions for classroom practices. In order to facilitate the use of visual thinking by students and contribute for them to use it in a more effective way, it's crucial to enhance the roles of visualization. The teacher must be aware of these different roles, to help students in the problem-solving process.

Following the words of Goethe, *what we see is what we know* (Arcavi, 2003), which can be determined by one's previous knowledge but also by the context within which the observation is made. For these reasons teachers should value the use of figures, recur to different types of visual images (e.g. drawings, gestures, movement) and motivate students to apply them but also present several visual examples of the same concept (e.g. varying aspects like orientation). To conclude, visualization must be valued and assume a central role in mathematics education; however, it is also necessary to understand the possible mishaps of its use, adjusting teaching practices, in order to indubitably consider it a key component of reasoning.

11.5 The Teaching of Seeing to Foster Creativity

Creativity is a dynamic characteristic that students can develop if teachers provide them appropriate learning opportunities. Our challenge as teacher educators is to find ways for developing creative and innovative education, as a means to foster creative competences and innovative skills among the next generation. So we must teach for creativity. Teachers are the main vehicle for doing this because they have the power to unlock the creative, innovative and critical potential of young students. If we believe that learning mathematics is strongly dependent on the teacher and creativity is connected with problem solving and problem posing, it is necessary, thus, to offer (future) teachers diverse experiences, in order to develop their problem-solving skills to apply in their teaching practice.

The teaching and learning process must give students the opportunity to "think outside the box", but this is only possible if teachers believe that creativity is teachable and know how to do that.

In this section we will discuss some aspects of the work we have been developing within the context of studies that we carried out with elementary pre-service teachers' education.

11.5.1 Teachers and Creativity

What students learn is largely influenced by the tasks used in the classroom that provide the starting point for the mathematical activity. Teachers can create some problematic situations for very specific purposes and allow others to arise in a less planned manner. Mathematical challenging tasks are not just difficult tasks or with a higher level of mathematization, but much of the challenge may be provided by the teacher (e.g. Stein & Smith, 1998; Vale & Pimentel, 2011). So, teachers should carefully plan what they want to work on and what cognitive challenges they wish to provide their students. Teachers also must decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise and how to support students without taking over the process of thinking for them and thus without eliminating challenge, in particular, when they explore problem-solving tasks (Vale et al., 2012). A task is good when it serves to introduce fundamental mathematical ideas, is an intellectual challenge for students and allows different approaches (NCTM, 2000). Tasks must develop new approaches and creative ideas, so they must provide multiple solutions in order to raise the student flow of mathematical ideas, flexibility of thought and originality in the responses. Teachers must encourage students to create, share and solve their own problems, as this is a very rich learning environment for the development of their ability to solve problems and their mathematical knowledge. The most successful problem solver is the individual who can apply diverse approaches. It is important that future teachers become themselves creative thinkers and they must be aware to act in the same way with their own students. Our concern was not to categorize students but to identify potentialities in the tasks to develop creativity in students, detecting their mathematical strengths or weaknesses.

To develop in future teachers the ability to look at a problem from different angles, we focus on the visual strategies, whose use is not always privileged in mathematics classes (Stein & Smith, 1998) but which have a great creative potential. To accomplish this goal, we advocate the analysis in the classroom of many of such strategies, presented and shared among students or, if it doesn't emerge naturally from the students' products, highlighted by the teacher. Some of those strategies don't occur without several attempts and reflection. This implies that teachers must encourage students to engage in suitably challenging problems over a prolonged period of time, thereby creating the opportunities for the discovery of an insight. Those good tasks are also concerned with knowledge because the more different are the situations we propose to students, the greater must be his/her repertoire in order to have the intuition towards the solution.

Psychologists and mathematics educators interested in mathematics problem-solving strategies have classified solvers according to their mode of thinking. Krutetski (1976) considers two modes: verbal-logical and visual-pictorial. According to this author, it is the balance between these two ways of thinking which determines how an individual operates on mathematical ideas, so students can be placed in a continuum with regard to their preference for thinking. In consequence we can consider three types of students depending on their thinking preference in mathematical problem solving: (a) *verbalizers* (*analytical)* – those students, who have a preference for the use of non-visual solution methods, preferring to use verbal-logical modes of thinking, which involve algebraic, numeric and verbal representations, even with problems that would yield to a relatively simple way to solve through a visual approach; (b) *visualizers (geometric)* – those students, who have a preference for the use of visual solution methods, preferring the use of visual-pictorial schemes, which involve graphic representation (namely, figures, diagrams, pictures), even when problems are easily solved by analytical means, i.e. they have preference for an extensive use of visual methods to solve a mathematical problem that can be solved either by visual or non-visual methods; and (c) *harmonic (mixed or integrated)* – those students who have no specific preference by either verbal-logical or visual-pictorial thinking. They have an integrated thinking style because they combine analytical and visual reasoning (e.g. Borromeo Ferri, 2012; Krutetski, 1976; Presmeg, 2014).

These issues have great implications in the classroom practice.

Students often tend to use analytical ways of processing information rather than visual ones, even when the former are more complex – a trend that sometimes leads to bad results, because students often have no mathematical knowledge to give a complete analysis of the problems (e.g. Eisenberg & Dreyfus, 1991). Some studies (e.g. Clements & Del Campo, 1989; Whiteley, 2004) show that students taught in a visual way tend to learn to use visual methods. Although depending on the problems and modes of thinking of solvers, the activity of *seeing* is something you can create, develop, teach and learn.

We arrived at the confluence of three domains – problem solving, visualization and creativity. The act of *seeing*, adopted as a problem-solving strategy, can be taught and developed, provoking the visual reasoning associated with intuition, the Aha experience, the ability to invent and divergent thinking, essential characteristics of creative thinking.

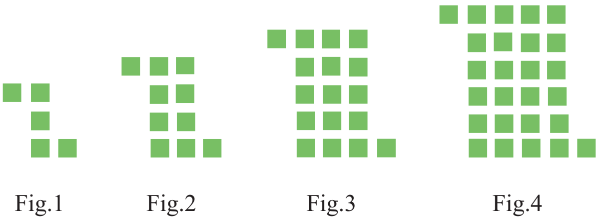
Our concern during classroom practice is to promote mathematical problem solving, encouraging thinking and creativity. We use a set of good tasks that challenge the students, providing opportunities for making connections, and for the teacher to build upon the knowledge that students bring to discuss through their products. The different strategies used to solve a problem are a suitable way for the teacher to highlight connections and to practice flexible thinking, an important feature of creative thinking. Along this activity we expect students to produce elegant solutions to a given problem situation. Certain classroom tasks seem to be able to motivate weak students, not only for learning mathematics but also to engage in mathematical thinking. They have some characteristics that make them less threatening and allow weaker students, in particular, to be less anxious and to focus on the tasks with a clearer mind, keeping nonetheless good students challenged. In our work we reached the same conclusions, i.e. when we use tasks that appeal for visual solutions (e.g. Barbosa & Vale, 2014; Vale et al., 2012, 2014), many solutions emerge, some of them very original. Without radical changes in teaching methodologies, we can have students think creatively.

The two tasks analysed below can be solved by both visual and non-visual methods. And visual solutions may be emphasized precisely when both options of solution exist. For some problems there is only a visual way to interpret and solve them. It is among them the *goat problem*, which statement can be, for example: *In a square house wall of* 6 *m situated in a big field, there is a goat tied to point A by a tether of length* 8 *m. What area of the field can be grazed?*

However, in many problems we have a choice for a visual or a non-visual strategy. We claim that, in most problems, visual solutions are simpler and more elegant, show a deeper understanding and are more likely to be creative. Yet, as most students are non-visualizers, they do not spontaneously reason this way. So we must teach them to see. With this goal of developing *seeing* and creative thinking in problem solving, in our teacher education courses, we seek to create a teaching environment that stands out visualization. Such a methodology lies on the following issues: (a) to choose carefully the tasks, privileging the ones that have multiple solutions and allow several approaches; (b) to suggest students to solve the problem by using more than one solution, hosting many ideas to tackle the problem, even if they seem unreasonable, and to follow each one for a while; and (c) to promote a discussion period, where the visual solutions must be presented and processed to get them highlighted, such as imagining; drawing pictures and schemes; painting them if useful; looking at the picture from another point of view, rotating the paper; decomposing and recomposing given pictures; and exploring symmetry.

We have verified that visual reasoning, or *seeing* to think, is learnt. Students can learn to develop and apply visual skills since such a work of visual enhancement is done in class. We present two problems and some solutions obtained before and after this work.

Example 11.1 *The Zs*



Assume that the pattern goes on. How many little squares can you find in the 5th figure of the sequence? Generalize to a figure of any order n. Present as many solutions as you can.

This example related to a pattern shows that the work with figurative patterns is especially suitable to visualization. Actually, in face of such a problem, traditionally students transform the figures into numbers obtaining the sequence

$$5, 10, 17, 26, 37, \ldots$$

for which it is not easy to discover a general mathematical law, for example, in a linear pattern as

$$3, 5, 7, 9, 11, \ldots$$

However, if the sequence is approached in a figurative way, the situation is quite different.

A visual approach to this pattern allows students to find a process of construction of the successive terms of the sequence that leads them to the discovery of a mathematical law. There are many ways of *seeing* to make this generalization; the example of Fig. 11.9 illustrates one of these modes.

The search for consistency between the visual pattern and the numerical representation makes the numerical sequence take the form shown in Fig. 11.10, which gives meaning to the numerical sequence above.

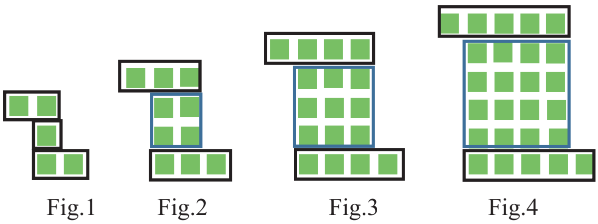


Fig. 11.9 First way of *seeing*

Fig.1	$1^2 + 2x2$
Fig.2	$2^2 + 2x3$
Fig.3	$3^2 + 2x4$
Fig.4	$4^2 + 2x5$

Fig. 11.10 Expressing the way of *seeing*

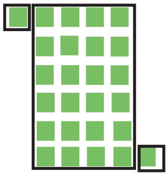


Fig. 11.11 Second way of *seeing*

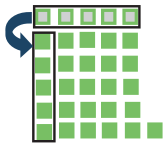


Fig. 11.12 Third way of *seeing*

Now a formula naturally emerges from generalization to any term:

$$a_n = n^2 + 2(n+1).$$

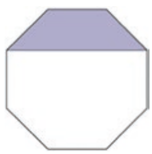
In alternative, many students see this arrangement as a rectangle plus two little squares. We show in Fig. 11.11 the decomposition made in the 4th figure of the sequence.

This generates the numerical sequence $1 \times 3 + 2$, $2 \times 4 + 2$, $3 \times 5 + 2$, $4 \times 6 + 2$, … and the generalization process gives rise to the algebraic expression

$$a_n = n(n+2) + 2.$$

There are, however, other ways of *seeing*. The rules above are equivalent to $a_n = (n+1)^2 + 1$, which translates another more dynamic and simpler way of *seeing*. This is shown in Fig. 11.12:

Example 11.2 *Regular octagon*



The painted region of the regular octagon has area 3. *What is the area of the octagon? Present as many solutions as you can.*

Since this problem is geometric, it is not possible to avoid the use of visualization. However, there are great differences among solutions. We must say that most students presented only one solution and in many cases even that one is not complete. The most typical attempt before stressing the importance of visual solutions relies obviously in the figure, but the word *area* immediately leads to algebraic formulas. Figure 11.13 shows the decomposition of the octagon:

$$A_T = 3 \Leftrightarrow \frac{2x+y+y}{2} \times x = 3 \Leftrightarrow x^2 + xy = 3$$

By the Pythagorean theorem, $x^2 + x^2 = y^2$.

So $A_R = (2x + y) \times y = 2xy + y^2 = 2xy + 2x^2 = 2(xy + x^2) = 2 \times 3 = 6$ and finally $A_{oct} = 3 + 6 + 3 = 12$

Another solution, although relying in an algebraic method like the first one, begins by considering that the given polygon can be obtained from a square from which we cut four triangles (Fig. 11.14).

After the teacher enhanced the search for visual solutions, some others appeared. The students, reasoning by symmetry, answered (Fig. 11.15):

> The octagon is composed by four overlapped trapezoids and a central square. The overlapped region is constituted by four triangles, whose area is precisely equal to the area of the central square, divided into four triangles. So the area of the octagon coincides with the area of the four trapezoids.

Afterwards, one student who finished the reasoning concluded that the area of the four triangles is equal to the area of the square. This conclusion, however, didn't rely on visualization as in the solution presented below, but in calculations using the Pythagorean theorem. So this solution is mixed, using both visual reasoning and calculations.

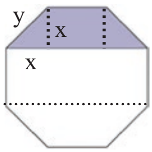


Fig. 11.13 Decomposition of the octagon in the most frequent solution

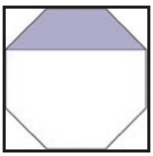


Fig. 11.14 Framing the initial polygon

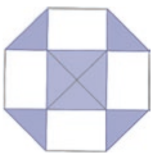


Fig. 11.15 The overlapped trapezoids

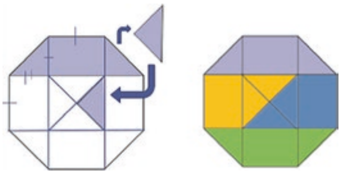


Fig. 11.16 Reconfiguration of the initial trapezoid

There was also a completely visual solution:

> Decomposing the figure, we obtain 8 congruent rectangular triangles and 4 congruent rectangles. The sum of the areas of 1 rectangle and 2 triangles is 3. So, the total area is 12 (Fig. 11.16).

In the last two examples, the students invented a dynamic solution that involved the displacement of parts of the figure, obtaining a different view of the whole.

The results obtained by students in these examples can be identified with the processes most used and are consistent with what is referred in several studies (e.g. Eisenberg & Dreyfus, 1991; Presmeg, 2014), confirming that most students, when having the opportunity to choose the process of solving a task, do not choose the visual solution, by preferring algebraic methods, and so fitting in the category of non-visual.

Those examples show that problems can be solved by more traditional means – which are those that the students prefer; however, some solutions involve Aha experience. Actually, the *seeing* strategy allows a flexible approach to the problem that leads to a much more concise and elegant solution (Liljedahl, 2004). In those examples, students were encouraged to search for visual solutions, and this happened either after the discovery of a numerical solution or after trying without achieving any solution. So we can say they had a time of incubation before the flash of insight that characterizes the Aha experience (Gardner, 1978; Liljedahl, 2004). Some of them were able to restructure the information to see beyond what was given (Wertheimer, 1959) so that the solution became simpler and more elegant. This route lies with creative thought, since the illumination stage is inventive by nature. Also, as students are required to present more than one solution, they can develop fluency and even flexibility when they turn to a different category of a solution (Vale et al., 2012) – for instance – turning from a numerical to a visual solution; originality is more likely when the solution is the result of a flash, in which there is a reorganization of the given data. And even in a set of visual solutions, there are some more original. In the Zs problem, the horizontal or vertical scanning in the initial figure is more common, and, on the contrary, those solutions that make a rearrangement of the initial figure are less frequent.

So, to foster creativity, we, as teachers, must encourage our students *to see beyond what they initially see*, either basing in the figure presented in the statement or in a visual representation made by the student.

11.6 Closing Comments

Several authors (e.g. English et al., 2008; Lesh & Zawojewski, 2007; Schoenfeld, 1985) consider that research on problem solving has not evolved since the late 1990s and early this century. Moreover, the research conducted does not seem to have allowed building up an efficient body of knowledge about how to effectively promote problem solving in and beyond the classroom. We argue that problem solving should be an integral part of the mathematics curriculum, helping in the comprehension of a given concept or mathematical process to enable students to think mathematically, which involves creating and interpreting a situation (describe, explain, communicate) at least as much as it involves calculating, performing procedures and reasoning deductively (Lesh & Zawojewski, 2007). Thus we call for a mathematics teaching *with* problem solving, where we value in particular the *seeing* strategy. Although some problems can be solved using both a visual and a non-visual approach (e.g. Eisenberg & Dreyfus, 1991; Presmeg, 1986), many of them can be more efficiently solved by visual methods, since they allow students to see the data, otherwise, rearranging them in a personal way, expressing flexible and original thinking, which are components of creativity. Thus, we can say that the approaches that use visual representations bring out the development of creative thinking (Presmeg, 2014).

Preliminary results of our work show that the tasks used are appropriate for our goal because they highlight the strength of the *seeing* strategy as being the one that provides more creative solutions. These tasks have multiple solutions, involving various contents and processes, which favour flexibility and fluency; but among the possible strategies, there are some clearly simpler ones involving *seeing* and *aha!* (e.g. Fischbein, 2002; Freiman & Sriraman, 2007; Presmeg, 2014).The thinking styles used by the students were mainly *analytic* and *integrated*. There is still a long way to run either towards creativity or to the use of visual strategies. Here the teacher's role is critical: when these strategies do not appear naturally, the teacher must necessarily demonstrate that way of *seeing*, in order to develop the visual skills of students and to increase the students' strategies repertoire (Arcavi, 2003). These results are in accordance with Presmeg (2006, 2014), which shows that students are not reluctant to visualize, by the contrary, for some this is the only way to solve a problem. But we realize that for sociocultural reasons we have yet a problem in the acceptance of this mode of thinking in the classroom.

Our work has been focused, in a first phase, especially in students with advanced schooling, who normally don't choose to use visual processes, since during their course of learning, this approach was not valued by teachers. It is our conviction that, with young students without this brand of education and with a teaching that uses tasks that put into evidence the potential of this approach, students can use these strategies in a more systematic way, which allows to develop their critical thinking and creativity.

At the end of this paper, we can answer the questions raised earlier. We defend a mathematics teaching and learning approach *with* problem solving, where teachers use a practice that enhances visualization and develops this ability through tasks that have many solutions. Those tasks can help students overcome some difficulties with mathematical concepts and procedures.

With this work we can get a repertoire of tasks, with at least a visual solution, to use during the mathematics classes, which help students to obtain a more dynamic use of their mathematical knowledge, being fluent and flexible in the transference to new situations and giving them opportunity to show their originality.

The teachers' challenge is to provide an environment of practice with problem solving that enables students to think creatively, so teachers should seek appropriate curricular materials to develop such ability. It would be interesting to extend this discussion to other resources like DGEs, analysing the potentialities of their visual impact in dynamical constructions.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215–241.

Barbeau, E. (2009). Introduction. In E. J. Barbeau & P. J. Taylor (Eds.), *Challenging mathematics in and beyond the classroom. The 16th ICMI Study* (pp. 1–9). New York: Springer.

Barbosa, A., & Vale, I. (2014). The impact of visualization on functional reasoning: The ability to generalize. *RIPEM, 4*(3), 29–44.

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2*(1), 34–42.

Ben-Chaim, D., Lappan, G., & Houang, R. (1989). Adolescent's ability to communicate spatial information: Analyzing and effecting students' performance. *Educational Studies in Mathematics, 20*, 121–146.

Borromeo Ferri, R. (2012). *Mathematical Thinking styles and their influence on teaching and learning mathematics*. Paper presented at the 12th International Congress on Mathematical Education, Seul, Korea. Retrieved March, 2015 from http://www.icme12.org/upload / submission/1905_F.pdf.

Burton, L. (1999). Why is intuition so important to mathematics but missing from mathematics education? *For the Learning of Mathematics, 19*(3), 27–32.

Cai, J., & Lester, F. (2010). Why is teaching with problem solving important to student learning? *Research Brief, 14*, 1–6.

Campbell, K., Watson, J., & Collis, K. (1995). Visual processing during mathematical problem-solving. *Educational Studies in Mathematics, 28*, 177–194.

Clements, M., & Del Campo, G. (1989). Linking verbal knowledge, visual images, and episodes for mathematical learning. *Focus on Learning Problems in Mathematics, 11*(1), 25–33.

Davis, P., & Hersh, R. (1981). *The mathematical experience*. Boston: Birhäuser.

Dreyfus, T. (1995). Imagery for diagrams. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 3–17). Berlin, Germany: Springer.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thoughts. *For the Learning of Mathematics, 6*(1), 2–10.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Eisenberg, T., & Dreyfus, T. (1991). On visual versus analytical thinking in mathematics. In L. Burton & C. Hoyles (Eds.), *Proceedings of the 10th International Conference of the International Group for the Psychology of Mathematics Education* (pp. 153–158). London.

English, L., Lesh, R., & Fennewald, T. (2008). Future directions and perspectives for problem solving research and curriculum development. In M. Santos-Trigo & Y. Shimizu (Eds.), *ICME,* *Topic Study Group 10, Research in Problem Solving in Mathematics Education* (pp. 46–58). Monterrey, Mexico.

Fischbein, H. (2002). *Intuition in science and mathematics: An educational approach*. New York: Kluwer Academic Publishers.

Freiman, V., & Sriraman, B. (2007). Does mathematics gifted education need a working philosophy of creativity? *Mediterranean Journal for Research in Mathematics Education, 6*, 23–46.

Gardner, M. (1978). *Aha! insight*. New York: W. H. Freeman and Company.

Guilford, J. P. (1956). The structure of intellect. *Psychological Bulletin, 53*, 267–293.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. *Proceedings of the 20th PME Conference, 1*, 3–19.

Guzmán, M. (1990). *Aventuras matemáticas*. Lisboa, Portugal: Gradiva.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Dover Publications.

Hatfield, L. (1978). Heuristical emphases in the instruction of mathematical problem solving: Rationales and research. In L. Hatfield & A. Bradbard (Eds.), *Mathematical problem solving: Papers from a research workshop* (pp. 21–42). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Haylock, D. (1997). Recognizing mathematical creativity in school children. *International Reviews on Mathematical Education, Essence of Mathematics, 29*(3), 68–74.

Hersh, R. (1997). *What is mathematics, really?* New York: Oxford University Press.

Hershkowitz, R. (1989). Visualization in geometry—Two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61–76.

Hershkowitz, R., Arcavi, A., & Bruckheimer, M. (2001). Reflections on the status and nature of visual reasoning—The case of the matches. *International Journal of Mathematical Education in Science and Technology, 32*(2), 255–265.

Hershkowitz, R., Ben-Chaim, D., Hoyles, C., Lappan, G., Mitchelmore, M., & Vinner, S. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 70–95). Cambridge, UK: Cambridge University Press.

Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–15). Hillsdale, MI: Erlbaum.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam, Netherlands: Sense Publishers.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.), *The second handbook of research on mathematics teaching and learning* (pp. 763–804). Charlotte, NC: Information Age Publishing.

Liljedahl, P. (2004). *The AHA! Experience: Mathematical contexts, pedagogical implications.* Unpublished doctoral dissertation, Simon Fraser University, Burnaby, British Columbia, Canada.

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For The Learning of Mathematics, 26*(1), 20–23.

Mann, E. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30*(2), 236–260.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2015). *Procedural fluency in mathematics*. Reston, VA: NCTM.

Nelsen, R. (1993). *Proofs without words*. Washington, D.C.: MAA.

Pehkonen, E. (1997). The state-of-the-art in mathematical creativity. *ZDM, 29*(3), 63–67.

Pitta-Pantazi, D., Sophocleous, P., & Christou, C. (2013). Spatial visualizers, object visualizers and verbalizers: Their mathematical creative abilities. *ZDM, 45*(2), 199–213.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.
Presmeg, N. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6*(3), 42–46.
Presmeg, N. (1999). Las posibilidades y peligros del pensamiento basado en imágenes en la resolución de problemas. *Suma, 32*, 17–22.
Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205–235). Rotterdam, The Netherlands: Sense Publishers.
Presmeg, N. (2014). Creative advantages of visual solutions to some non-routine mathematical problems. In S. Carreira, N. Amado, K. Jones, & H. Jacinto (Eds.), *Proceedings of the Problem@Web International Conference: Technology, Creativity and Affect in Mathematical Problem Solving* (pp. 156–167). Faro, Portugal: Universidade do Algarve.
Rivera, F. (2011). *Toward a visually-oriented school mathematics curriculum: Research, theory, practice, and issues*. Dordrecht, Netherlands: Springer.
Schmitz, A., & Eichler, A. (2015). Teachers' individual beliefs about the roles of visualization in classroom. In K. Krainer & N. Vondrova (Eds.), Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (pp. 1266–1272). Prague, Czech Republic.
Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.
Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM, 29*(3), 75–80.
Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268–275.
Stylianides, G., & Silver, E. (2009). Reasoning-and-proving in school mathematics: The case of pattern identification. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades - a K–16 perspective* (pp. 235–248). New York: Routledge and Reston.
Stylianou, D., & Silver, E. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning, 6*(4), 353–387.
Vale, I., Barbosa, A., & Pimentel, T. (2014). Teaching and learning mathematics for creativity through challenging tasks. In S. Carreira, N. Amado, K. Jones, & H. Jacinto (Eds.), *Proceedings of the Problem@Web International Conference: Technology, Creativity and Affect in mathematical problem solving* (pp. 335–336). Faro, Portugal: Universidade do Algarve.
Vale, I., & Pimentel, T. (2004). Resolução de Problemas. In P. Palhares (Coord.), Elementos da Matemática para professores do Ensino Básico (pp. 7–52). Lisboa, Portugal: Lidel.
Vale, I., & Pimentel, T. (2011). Mathematical challenging tasks in elementary grades. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh congress of the European Society for Research in Mathematics Education* (pp. 1154–1164). Rzeszow, Poland: ERME.
Vale, I., Pimentel, T., Cabrita, I., Barbosa, A., & Fonseca, L. (2012). Pattern problem solving tasks as a mean to foster creativity in mathematics. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 171–178). Taipei, Taiwan: PME.
Wertheimer, M. (1959). *Productive thinking*. New York: Harper & Row.
Whitley, W. (2004). Visualization in mathematics: Claims and questions towards a research program. Retrieved February, 2015 from http://www.math.yorku.ca/~whiteley/ Visualization.pdf.
Yerushalmy, M., & Chazan, D. (1990). Overcoming visual obstacles with the aid of the supposer. *Educational Studies in Mathematics, 21*, 199–219.
Zimmermann, W., & Cunningham, S. (1991). *Visualization in teaching and learning mathematics*. Washington, DC: Mathematical Association of America.
Zodik, I., & Zaslavsky, O. (2007). Is a visual example in geometry always helpful? In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 265–272). Seoul, South Korea: PME.

Chapter 12
Creativity and Problem Solving with Early Childhood Future Teachers

Yuly Vanegas and Joaquín Giménez

12.1 Introduction

Reports on children's mathematical achievement pointed out that many children at early childhood level successfully engaged in mathematical thought and these skills are predictors of later school mathematics achievement (Munn, 1997). The process of learning and instruction of early problem-solving skills are considered a way of improving basic capacities. In such a perspective, research and expert practice indicate that certain concepts and skills are both challenging and accessible to young children (Seo & Ginsburg, 2004).

For a long time, mathematics and creativity have appeared as incompatible terms even for some educators (Upitis, Phillips, & Higginson, 1997). In addition, many of the current studies refer mainly to exceptionally talented children. However, how do we start preparing creative pupils from preschool onwards? In this chapter we talk about the need to prepare future preschool teachers to consider creativity aspects when working with children since early ages, particularly when future teachers analyse what children do when they are involved in problem-solving activities and identify which mathematical ideas are behind the children's statements and their arguments.

It is generally assumed that in order to support high-quality mathematics education, the institutions, programme developers and policymakers should create a more

Y. Vanegas (✉)
Department of Linguistic and Literary Education, and Teaching and Learning of Experimental Sciences and Mathematics, Universitat de Barcelona, Barcelona, Spain
e-mail: ymvanegas@ub.edu

J. Giménez
Department of Linguistic and Literary Education, and Teaching and Learning of Experimental Sciences and Mathematics, Universitat de Barcelona, Barcelona, Spain
e-mail: quimgimenez@ub.edu

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_12

effective early childhood teacher's preparation and continuing professional development. Consequently, it is important to improve teachers' ability to identify children's behaviour, in order to support children's thinking and take the best professional decisions. There is a large lack of research about the impact of an early acquirement of didactical analysis competence of the future teachers, which are non-specialists in mathematics education, in particular, about how the future teachers interpret relevant events that are difficult to see, because of the early use of mathematical language, as happens with the early childhood education.

In such a framework, our main aim in this chapter is to characterize some trends of the initial process of how future teachers notice strategies and mathematical content that children developed when faced with problem-solving activities (Jacobs, Lamb, & Philipp, 2010). At the same time, we aim to understand the importance of promoting creativity through interactional means in the early ages.

In particular, we are interested to see if in the observations of future teachers, some creativity traits appear when they analyse children's strategies. The school experience presented to the future teachers is a part of an arithmetic problem-solving workshop done in preschool. The school experience was developed within an atmosphere of total freedom to choose manipulative and procedures of resolution, in which children invent their own strategies, discuss them within the group and decide which strategies will be the "official" ones for the group as a negotiating process (De Castro & Escorial, 2007). We structure the chapter into two parts: the theoretical presentation, explaining the main constructs used, and a case study experience with future teachers, offering results and perspectives.

12.2 Creativity and Problem Solving

Previous research found that the level and the complexity of preschool children's mathematical understanding are strongly underestimated by some teachers (Rudd, Lambert, Satterwhite, & Zaier, 2008). Some other authors, such as Craft (2002), highlight the role of the teacher in providing the optimum balance between structure and freedom of expression for young children.

Unfortunately, in many cases, future teachers assume that mathematical abilities develop by themselves without any support and problem-solving strategies are far from the possibilities of childhood (Clements, 2004). On the other hand, we know that for a mathematics teacher, a problem means an attractive question whose steps or ways of solution students do not know but the pupils do have the necessary preliminary information (Schoenfeld, 1989). In this sense, problem solving not only means finding the result of a mathematics question but also facing new conditions and finding flexible, effective and elegant solutions for these conditions. We assume a theoretical investigative approach (Ponte, 2007) to analyse the teaching and learning of mathematical processes in preschool (Baroody, 2003), in particular, for problem-solving activities.

When preschool children are situated in problem-solving situations, it is nearly always within a very familiar context, for example, a problem involving the application of a mathematical concept that students have just been taught. In such situations, learners are aware of boundaries and have been given some clear leads about what knowledge to apply. Such problems often seem closed, with very little room to explore and be creative. Therefore, in our perspective we are talking about challenging new problems, using different methods and strategies.

In such activities, children will need a teacher or another person (one committed to fairness) to guide them through the steps at first. This help will not be as necessary as children gain experience by practicing their newly learnt skills. The ability to communicate is a prerequisite skill for social problem solving, but children as young as 3 years of age need a mediator most of the time. Teachers should also work with parents and caregivers to encourage them to model problem solving for children. With this, they can promote and support children's efforts to practice problem solving at home.

We interpret creativity as a group of elements that helps to include mathematics in the educational process, as something that helps to develop flexible thinking. This kind of thinking is possible when future teachers promote the construction of problems and situations and promote the solution of real problems. All this appears in an environment where both teacher and student enjoy mathematics and where the students feel free to make mistakes and to learn from them. It gives the opportunity to improve the imagination of the children. Training teachers to teach mathematics creatively means fostering the ability to solve problems that allow the development of structures, recognizing their generation, in order to encourage future teachers in both mathematics production and difficulties' confrontation (Sequera, Giménez, & Servat, 2005).

12.2.1 Creativity, Problem Solving and Preschool Settings

It is regularly assumed that investigative approaches enable children to experience mathematics (in particular, problem solving) within a context that is meaningful to them. To do this, it is fundamental for the teachers to increase their personal subject knowledge, in order to fully extend the potential learning of the children in an investigative approach. Creativity in the mathematics classroom is not just about what pupils do but also about what we do as teachers. Very simple activities may be a problem for one child but not for others. Interesting problems involve alternative solutions using different mathematical ideas. If we are thinking creatively about the mathematical experiences we offer our pupils, we can open up opportunities for them to be creative (Piggott, 2011).

Emphasis must be put upon the creative process, rather than judging the quality of children's "products". However, it is necessary to control the reflective statements given during the process, as an adult game in which we should be finding validated solutions. In fact, creativity becomes more visible when adults try to be more attentive to the cognitive processes of children than to the results they achieve in various fields of doing and understanding (Malaguzzi, 2001).

In order for teachers to act as an informed other and scaffold the children's potential learning, they need to draw on their knowledge base about what the children know, understand and are able to learn (Clements, 2004). In this investigative framework, the creativity is considered an inherent characteristic of the mathematical knowledge building, and it can be widely promoted in the school population in general by means of problem-solving activities (Silver, 1997).

In such an approach, it is important to start with a worthwhile task, one that is interesting and creates a real need to learn and to practice. Experiencing mathematics in context is not only more interesting to children but more meaningful for children (Baroody, 2000), where students feel free to make mistakes and to learn from their mistakes. In our mathematical practices, we see the importance of using ways to represent what children are doing, giving opportunities for creativity. In fact, mathematical graphics support children's thinking when we encourage them to use their own ways of representing their personal ideas. Adults who acknowledge difficulties and which also demonstrate perseverance can help children to persist and become positively and creatively problem solvers (Gifford, 2005).

Imagery and systematization are also important for creative problem solving, particularly during the incubation-illumination stage of the process. Deloache and Brown (1987) found that when looking for a lost camera, some 3-year-old children used systematic strategies by searching only in places visited since the camera was seen there. We are convinced that children should be prepared for complexity since the early years. Mathematics educators should overcome traditional ways of teaching openness to unexpected answers (Jalongo & Hirsh, 2012) because in elementary school, creativity decreases.

Young children can also plan reflectively for problem-solving purposes. According to Gura (1992), children who were more experienced with block play tended to plan before building, by selecting the blocks they would need. At any given point in their development, children will use five or six different strategies. Higher and lower order strategies coexist and compete. Thus, what is important is not the process of moving from one strategy to the next (Baroody, 2000) but building several appropriate strategies in order to show a good interpretation of the problem needed to obtain a result.

In this chapter, we also assume that mathematical thinking is not primarily about developing isolated arithmetic skills but about posing and solving problems, which require ability in quantitative and relational understanding (Van Oers, 2004). We (as teachers) also acknowledge that children need to feel in control of the outcome. Instead, children's actions often are focused on pleasing the teacher (Carr, Peters, & Young-Loveridge, 1994).

Creativity is related to motivation, clarity, richness and openness. We know that a rich problem may have more than one solution and can be solved using a range of methods at different levels (Davis & Pepper, 1992).

12.2.2 Creativity, Equity and Problem Solving

Developing children's problem-solving strategies and confidence becomes an equal opportunity issue: teachers will need to find problems for engaging children. The issue is not so much who thinks of the solution first, but whether the children engage with the problem and come to see it as their own problem. Having a rich context is not enough for encouraging children's creativity. Some children may find it difficult to express their creativity. These ideas and many other similar comments defend the idea that equity is necessary for promoting creativity. We do not only interpret creativity associated with gifted children (Meissner, 2003).

Problem solving considered as play, in which questions are understandable, gives opportunities for all children to provide answers and control them as well. An arithmetic problem-solving workshop as a game leads to the expression of ideas and contrast in a democratic way, as a universal task which cannot be reduced to numbers (Carlsson-Paige, 2013). Some children may need to learn to stand up for their own ideas, especially when these do not conform to those of the rest of the group. In this moment when new curricula ask for inclusion, problem solving is a powerful task to improve equity.

We already know that many future teachers come from a curricular perspective in which such principles were not established. Besides, the difficulties of teachers in mathematical issues lead us to think that problem-solving tasks can be inappropriate for early childhood. Nevertheless, we should overcome such a belief. Emotional processes, such as emotional fantasy in play, pleasure in challenge, involvement in tasks and tolerance of anxiety, are characteristics of problem-solving activities for early childhood. Problem solving in an open democratic environment promotes creativity and enables children to make connections between different aspects in mathematics and thus extend their understanding. Both creativity and play require imagination, insight, problem solving, divergent thinking and the ability to experience emotion and to make choices (Russ, 2003).

12.2.3 Construction of Noticing Skills

The process of noticing is an important aspect in professional teacher development and in particular, as a first point for doing an in-depth didactical analysis. We assume that noticing relates two important professional skills (Sherin, Jacobs, & Philipp, 2011): (a) the extent to which future teachers attend to the mathematical details in children's strategies and (b) the extent to which the teachers' reasoning is consistent with both the details of the specific children's strategies and the research on children's mathematical development. This feature of teacher professional competency involves the cognitive ability to identify and interpret the salient features of the students' output in order to make informed decisions (Llinares, 2013).

The notion of noticing helps to characterize what the mathematics teacher knows and how this is put into practice in the classroom. It is an important topic in mathematics education (Ponte & Chapman, 2006). The skill "notice professionally" requires that the teacher be able to identify relevant aspects of the teaching situation, use knowledge to interpret the events and establish connections between specific aspects of teaching and learning situations and more general principles and ideas about teaching and learning (Jacobs et al., 2010; Mason, 2002; Sherin et al., 2011).

One particular aspect of teacher notice is the ability to be attuned to the students' mathematical thinking (Llinares, 2013). The main aim of noticing is that future teachers were able to understand and analyse the selecting and designing of suitable mathematical tasks. The future teachers should interpret and analyse the students' mathematical thinking, the practice of mathematics teaching initiating and guiding the mathematical discourse and the role of interactions. Although there are different theoretical ideas about noticing, a common approach involves highlighting the way in which teachers interpret mathematic teaching situations.

Most students' classroom experiences of mathematics involve studying materials and working through tasks set by their teachers, in which they are passive observers of mathematics (Boaler, 1997), leaving little room for the entrepreneur or creative thinker. Therefore, it is difficult for future teachers to identify traits of creativity in preschool. This includes aspects such as *being willing to notice patterns*, as well as making decisions about problem solving, including planning, checking, changing strategy and reviewing (OFSTED, 2013).

We assume that for detecting or noticing traits of creativity, it is fundamental to identify evidences from (a) children's problem-solving strategies developed in different school dialogues and (b) deep observations and interpretation about arithmetic knowledge built by individuals and identified processes. The discourse analysis generated by future teachers enables them to show what they consider relevant, and group discussion offers the possibility of proposing other formulations to others, raising objections and so on (Fernandez, Llinares, & Valls, 2013).

We consider that future early childhood teachers have previous personal knowledge about what children and the teacher are doing, but they need theoretical materials to improve their analysis (Fernandez et al., 2013). For instance, future teachers should know elements about semantic analysis coming from problem-solving research.

12.2.4 Children's Creativity and Pre-Service Teachers

In order to develop creativity in mathematics education, both teachers and students need much more than solid mathematics knowledge (Meissner, 2000). It is in this sense that Kubínová (1999) points out that in order to produce creative, self-confident pupils, it is necessary to produce creative and self-confident teachers, that is, teachers who are able to develop their pupils' skills to use their knowledge and react adequately to the changing conditions of their world.

Identifying creativity relates to the recognition of children using unusual strategies, and showing if children establish relations and structures, even it is not explicitly. In our framework we want to see how future preschool teachers recognize how children are creative thinkers going beyond algorithms (Leikin, 2009), by identifying what is behind the children's statements and their arguments. Educators have to be creative in planning and implementing the curriculum, and they must nurture creativity in children, which is all about identifying people's strengths and establishing an education community (ACECQA, 2011).

Reflective thinking is necessary for creative future teachers as the process of carrying one's own experience into meaning formation and the relation and connection of bringing a deep understanding into other experiences and ideas (Rodgers, 2002). Training aimed at developing creativity provides self-confidence to the participants (Scott, Leitz, & Mumford, 2004). Perception always creates a barrier in almost everything possible, and for one to improve creativity, one needs to have his/her thoughts clear. Perceptions can cause a limit to one's eventual level and means of reasoning. Problems in perception may not allow one to reach the necessary different levels of reasoning.

Creativity is an important aspect for early childhood education. In fact, children should have opportunities to imagine and create, propose theories and reasons, master skills, have meaningful experiences, express thoughts and ideas, solve problems, engage in reflective thinking and explore diverse ways of knowing, thinking and learning (ACECQA, 2011).

In many curriculum standards, the need for 'Children express ideas and make meaning using a range of media' (EYLF, 2009) is highlighted. It should include the following principles: (a) the need of imagining and creating role scripts and ideas; (b) sharing stories and symbols of their culture; (c) re-enacting stories; (d) using the creative arts such as drawing, painting, sculpture, drama, dance, movement, music and storytelling to express ideas and make meaning; (e) experimenting with ways of expressing ideas and meaning using a range of media; and (f) beginning to use images and approximations of letters and words to convey meanings. Assuming these principles, it was considered that number problems are an important issue and because future teachers tend to have a stronger competence in this field, more so than with geometry problems.

To categorize future teachers' comments, we assume the following indicators of creativity: *originality*, *flexibility*, *fluency* and *elaboration* (adapting which are used for tasks in Sequera et al. (2005) and Leikin (2009)). Such categories are explained as follows.

Originality is the ability to think in an unusual way, producing new and unique ideas (Silver, 1997). It means a reference about promoting novelty, unpredictability and surprise. It causes the student to evade a previous conceptual system and build a new one that affects both ideas and people. It brings about innovative ideas, statistically infrequent.

Flexibility is the ability to think in different ways to produce a variety of different views on the same issue. It means considering different kinds of knowledge, by recognizing that children use different representations, even methodologies, to interrelate interpretations and promote openness in problem solving (Callejo, 2003).

Fluency is the ability to produce a large number of different solutions for the same task. It is difficult to find more than one process being explicit. Effective *fluency* relates to the generation of ideas, communication and ability for judgement in challenging situations (Callejo, 2003). Therefore, we assume that for children it is possible to consider fluency when the children can accept or modify other's solutions. It means also to generate alternatives, approaches, combinations, association capacity, completion and production of relations when solving problems.

Elaboration is understood as the ability to face complex situations as an integrated configuration of meanings (Reigeluth & Stein, 1983). It appears when the generating process is focused, with attention to detail, carefully and meticulously. It allows the pre-service teacher to organize the content, to imagine the following steps once the images, thoughts and statements have been conceived.

In a previous research, we used such categories to analyse the potentiality of professional tasks used for future primary teachers (Sequera et al., 2005). What will happen with early childhood education? Therefore, we now explain the power of a professional task not only to analyse the task itself but also to see how it contributes to professional development on creativity aspects. For our purposes, in the training experience explained below, we considered just the first three components of creativity (originality, flexibility and fluency) as in the research of Amaral and Carreira (2012) which analyse problem-solving strategies with primary school students. In fact, it is difficult for future teachers to find elaboration and complexity in preschool classrooms.

12.3 An Experience with Future Teachers

Now we will describe a case study with future teachers, in which we exemplify our research development. In this study, we worked with 30 pre-service teachers of third-year in the Early Childhood Education Program at the University of Barcelona. We propose two professional tasks in order for future teachers to recognize and to interpret some classroom situations, by analysing problem-solving strategies and mathematical content behind the scholastic activity. In the analysis of pre-service teachers' answers of these two tasks, we focus on two questions related to noticing (a) how pre-service teachers integrated mathematical elements in the written text produced relating the characteristics of the problem and the strategies and (b) how to improve creativity findings by means of collaborative discussions during didactical analysis in two steps (naïf description, and including theoretical framework for analysis). The third aspect of noticing, which is taking decisions, could not be implemented because the topic appears before the school practicum.

A specific professional learning environment for future teachers was designed to establish our instructional goal as a basis for improving noticing skills. This learning environment requires that pre-service teachers first think about the issue, interacting among themselves in order to discuss aspects of the teaching of problem-solving strategies and numbers' operations at preschool stage, reflecting on theoretical issues and analysing new proposals. Our chapter refers to the first two steps as a first didactical analysis introducing noticing reflective process. The Moodle platform provided by the university gives the opportunity to collect all the future teachers' reflections. We also videotaped the sessions to record group discussions.

12.3.1 A First Professional Task

For designing the professional task, we selected a school practice that takes place in an environment that we believe helps the development of creativity. In this practice, the teacher proposes open problems that can be solved in different ways. The teacher also gives the opportunity to choose materials and procedures of resolution in a way in which it is possible that flexibility appears. It is important that future teachers are aware of these aspects. We assume that the teacher idea of promoting explanations by group discussion leads to children's *fluency*. When the teacher helps to decide on an official common strategy accepted by all children, *elaboration* seems to be promoted. The availability of materials leads to *originality* and metaphoric use of different representations for initiating the process of idealization. Thus, we imagine that in such a school practice, the teacher continuously provides challenging situations for children to promote creativity.

The professional task (see Fig. 12.1) starts by explaining the context of the problem-solving school practice selected (De Castro & Escorial 2007), called a problem-solving workshop. Previously, future teachers had been involved in readings about the importance of sharing ideas and communication in mathematics problem solving (Bushman, 2003; Parrish, 2010).

After describing the environment of school practice, the trainer gave the future teachers the transcription of the dialogues produced in the problem-solving school practice. Future teachers should respond to two questions in order to pursue the first analysis: Which strategies and mathematical content were developed during each dialogue? What aspects of the dialogue favour the process of problem solving?

As a second part of the professional task, the personal answers of future teachers were after discussed in small groups in order to arrive at an agreement. It was expected that natural strategies about adding, missing values, multiplication and sharing processes would be observed.

12.3.2 Expected Creativity Values of the School Experience

We will explain some parts of the school practice and also an expert analysis about creativity values through the description of some of the children's dialogues in order to establish expectations about future teachers' observations and findings.

PROFESSIONAL TASK

DESCRIPTION OF THE SCHOOL PRACTICE

- The experience consists of a workshop solving arithmetic problems developed with children in their last year of Early Childhood Education.
- The environment in which the experience develops is free to choose materials and resolution procedures, children invent their own strategies, discussed within the group, and decide what could be the "official" group strategy.

Description about the context

The dialogues discussed below occur in work sessions that last about an hour. In each of the sessions, one or two problems are solved. The teacher (Beatriz) takes part as little as possible, and always indirectly. She does not give the right solution, or propose a procedure as appropriate.

Description about the forms of action

When a child gives an answer that does not make sense within the context of the problem, the teacher tries to make him see through a question. On the contrary, when a procedure seems worthy of being highlighted, she asks the student to explain again and asks if everyone have understood. The teacher also, takes part to assess the validity of the explanations given by children about the procedures used, highlighting the criteria of this assessment.

READING OF THE FIRST PROBLEM

Beatriz: We had three pot-bellied pigs.
Nacho: What are pot-bellied pigs?
Beatriz: Oh, sure! You were not at the farm last year
Nacho: No. I got sick
Beatriz: The pot-bellied pigs are smaller than normal pigs, and they are of black colour
Diego: I went!, I went! There were many pigs!
Beatriz: (reading the letter) In the farmer school, we had three little pigs before and were born four more. How many pigs we have altogether?

AN EXAMPLE OF DIALOGUE

Cristina: I have done it like this: 1, 2, 3. Then I went up to four (1, 2, 3, 4) with the fingers and then I saw that is 7.
Beatriz: Have you put 3 in one hand and four in another?
Cristina: No. I put 3 in one hand and then I have counted 4, 5, 6 and 7.

QUESTIONS FOR FUTURE TEACHERS

Look at the dialogues, and tell us which processes you can see in each of the dialogues.
1. Which strategies and mathematical content have been developed during each dialogue?
2. What aspects of the dialogue favour the process of problem solving? Especially observe the role of the teacher and students.

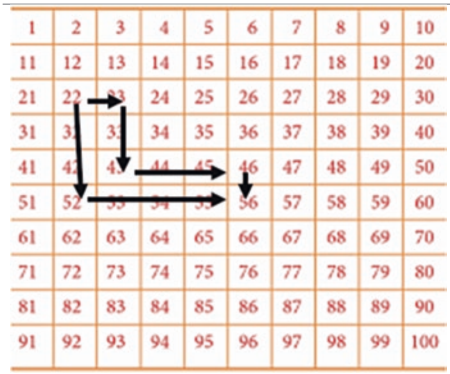
Fig. 12.1 Components of the professional task

As we mentioned above, the teacher (Beatriz) starts the first workshop by motivating the children. A letter appeared from a character that the children met when they went to a farm. The letter was written by an elf named Pitutin, who takes care of the animals on the farm and proposes problems to children. The preschool teacher encouraged children to help the elf solve the proposed problems and to send letters with the elaborated answers, as agreed on by the group. It is also assumed that the use of manipulative resources can create more mathematical opportunities, prompting children to choose the answers according to their previous knowledge and to explore different combinations and arrangements using them. Here we see the value of communicating mathematical ideas by using written texts. It contributes to a flexible and elaborated mathematical thinking.

A first problem proposed tells us about someone who started with three pot-bellied pigs and then four more were born. *How many pigs do we have now?* The teacher explains that children solve this problem without any difficulty.

A second problem proposes a new addition problem. "*In the school farm, there are four male ducks and five female. How many ducks in total?*" Three different answers appear: ten, nine and seven. The one who said ten explains to the others "I had thought wrong! Because I heard you say five and five". Let us here say something about potential creativity in this part. The naturalistic way of acting gives opportunities for self-regulation and control. It promotes fluency in the sense of building meaning for addition and also building the coherency of the strategy used. Even making a mistake, the answer ten corresponds to five plus five answer.

One child argues that he used a table of numbers from 1 to 100 to solve the problem. Why does it contribute to creativity? Children did not discuss addition and its symbolic use before the classroom experience, but they can solve problems by using materials such as the number Table 1–100. Future teachers never used the Table 1–100 before. Therefore, we decided to introduce such material in training class in order to see its mathematical contribution. Thus, future teachers will learn about the similarities and differences between the 1–100 table, the number line or a rekenrek abacus as a semiotic mediator (Vigotsky, 1978).



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	3	3	34	35	36	37	38	39	40
41	4	4	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Using the table, the order of numbers is maintained. Going down a row in the table means that we add 10. Going one step to the right is one more. The children see the order of numbers and the structure of each group of ten. After some trials they start to discover equivalence representations like:
+ 30 + 4 = +4 + 30 = +1 + 20 + 3 + 10.
That is the starting of algebraic thinking. Thus, we consider that it provides opportunities for flexible thinking.

Therefore, reflecting on this kind of observations, we hope that future teachers not only recognize the need for computing addition but also realize that children should transform the data into a different representative form such as the use of manipulative material. It is important to notice that the use of manipulation is a consequence of a "way of doing" as a problem-solving strategy, not directly promoted by the problem statement.

When the trainer discusses such a dialogue, the relevance of introducing questions to reflect about the conceptual scaffolding (Coltman, Petyaeva, & Anghileri, 2002) is considered by future teachers. In this case, we use questions like the following: "What was the child's thinking when using the 1–100 table" and "Why do you think some children answered seven?"

With this type of above questions, we want to identify if future teachers recognize mathematical knowledge behind students' answers, as an important part of noticing. According to an expert position, it opens the possibility for identifying flexibility and fluency as traits of creativity. In fact, the children use original strategies, as an initial point for building structures. Accepting other ways of doing things, children should highlight conjectural thinking and promote flexibility of strategies. Refutations appear through sustainable learning, by hearing the students' voices. It is also the starting of building meanings for validation processes and the elaboration part of creativity.

Children's discussions with materials promote the use of original strategies during the school practices of multiplication and division problems. In such an experience, of course, some children may rush towards a solution without going through preliminary or reflective stages. They can use materials as coloured woods, or cubes, for instance, to solve the problem. However, we also expect to find a sense for strategies of grouping in a natural way, according to the situation of presented division. In fact, sharing's problems are very useful because they are familiar and purposeful, and the level of difficulty can be easily adapted from everyday life (Anthony & Walshaw, 2007).

A final dialogue about a subtraction problem is the following: *eight ducks on the farm and one day a few more came. Since then there are 14 ducks on the farm. How many ducks came over?* Christine says, "Was it fourteen [I point the finger]? No, sorry. It was eight [points to eight and counts from there to 14] and then: one, two, three, four, five and six. Therefore, it is fourteen. Mary explains that ducks were... (I use two cubes together) and... Those who came flying are these (showing 1cube). Moreover, I counted six. Peter uses blocks and he made a mistake, because he counted to nine".

In such sentences, we find original strategies, because children need to use material for metaphorical help. Some children use a counting related strategy, but others imagine the situation in a more abstract way. Surpassing the border of ten presses the children to find a "going back strategy" or starting at the end (according to Polya's sense). It also promotes future teacher's flexibility by building structural images.

Let us say something about multiplication and division problems. Recent research about early multiplication and division (Mammede & Soutinho, 2012) identify a decomposition strategy related to semantics about grouping and finding out about fairness as a way to reduce the problems to a set of counting strategies in some division problems.

After analysing the children's dialogues, we – as researchers – could assume that children use at least three different types of images when we analyse the inscriptions in a semiotic way (according to Presmeg, 2006):

(a) *Kinaesthetic imagery* (involving physical movement) when they use the blocks to express a subtraction that identify what is typical in the situation. It is the case of Ainvar, which uses 14 blocks and leaves 5 on 1 part of the table. Then he counts 9.
(b) *Pattern images* when Diego tells us that he made *a tower of 14, altogether, and I took out five (1, 2, 3, 4, 5)* to represent the ducks that Pitutin had lost.
(c) *Changing representations*. We can also state, according to the educator, that some children change from one representation to another. For instance, Sandra moves from the use of a number line to verbal writings and words (Gagatsis & Elia, 2003); thus she is approaching dynamic imagery. Nacho also moved from the number line idea to writing a series of numbers and suggests that deleting five last numbers gives the solution.

Cristina represented both numbers separately. She says the result is 19. Then, the following dialogue appears:

Teacher: Let us see. 14 ducks, 5 are lost…thus…19?
Child 2: Mm. (Observes that Cristina did a mistake).
Teacher: Does he have more ducks? [Cristina says yes by gesturing] Some ducks are lost, and he has more? When you lose something, do you have more or less?
Cristina: Less…I did a mistake.

The case of Cristina is an example to show that the kinaesthetic visualization strategy sometimes inhibits the child from getting a good solution, because she cannot interpret the lost ducks as a distance model, because she sees 14 and 5 and perhaps forgets the question. What is interesting in the global dialogue is that the teacher does not focus on the action but on the validity of the proposed result. It is not possible that if you lose something, then you have more.

Could we say that offering visual imagery possibilities will improve creativity values and attitudes in early childhood education? Not always, but an attitude of control seems to help even Cristina to improve this "proving" attitude in problem solving in order to better use images and visual reasoning. From now, what is most important for our discourse analysis is to recognize that future teachers see it and look at such details as an important way to promote originality and fluency aspects.

12.3.3 Introducing a Theoretical Framework

After a global dialogue, about the results of the first part, we start by explaining to the future teachers a set of theoretical categories about problem solving:

(a) Classical notions of heuristics, according to Polya.
(b) List of counting strategies (Baroody, 2000).
(c) Semantic classical classification of verbal problems of addition and subtraction (Carpenter, Fennema, Peterson, & Carey, 1988).
(d) Categories about the use of mediators: gestures, concrete materials, diagrams or digits as numerical representations.
(e) Categories of creativity as fluency, originality, flexibility and elaboration.

The first part of this second task was to discuss some personal answers about the type of strategies or ideas used by children exposed during the first task, according to the recent theoretical perspectives above cited. After initial personal corrections, it is proposed to discuss in groups of four in order to give a common answer about one of the dialogues. Globally speaking, the group reflects on the use of individual processes, representations and personal meanings in each dialogue. For that purpose, a grid was used to collect the information (see Fig. 12.2).

The main aim for both professional tasks is to know about what future teachers identify and interpret the children's strategies observed when reading the mathematical dialogues, in order to improve an individual perspective of noticing analysis. During a first professional task, each future teacher sees all the dialogues and explains which of the strategies could be relevant. During the second task, each group should analyse one of the dialogues, by using the categories introduced. The future teachers discussed their ideas as a communicative group debate. A third part of that professional task confronts future teachers with a creativity framework above explained in a group discussion.

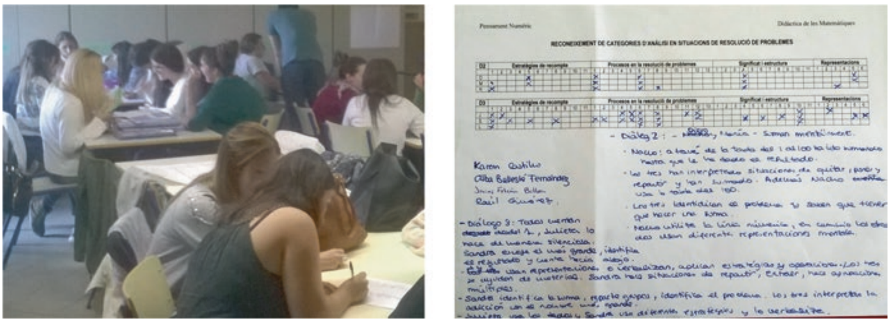


Fig. 12.2 Future teachers involved in reflection about strategies in problem solving

12.3.4 Research Analysis Method

A team of three researchers tried to reach a consensus on knowing if the future teachers have observed traits of creativity above explained in children's behaviour. In order to know future teachers' knowledge, we collected all the future teachers' writings (posted in the Moodle platform for the first task and written comments for the second task), some group discussion and class discussions, which were audio-taped and constituted data for our research process. Finally, a team of researchers found commonalities in the conjectured approaches, because of noticing processes always based upon future teacher texts (Fernandez et al., 2013).

The answers proposed by the future teachers were codified according to three categories associated to creativity:

(a) Future teachers *identify originality* when they reflect about some unexpected and intelligent children's responses from remote premises and when they assumed that children use mathematical connections to interpret images or diagrams and/or use them when solving problems. An unexpected answer is theoretically interpreted as the ability to promote no immediate relations and far connectivity among problem-solving strategies (Callejo, 2003).
(b) Future teachers *identify flexibility* when they talk about children using appropriate mathematics when they see the children looking into a problem from different angles and when children use different effective representation systems and/or use connections between representations and interrelate given data and goals in order to give a solution.
(c) Future teachers *identify fluency* when they interpret the fluency in children's reasoning and communicate the problem-solving process done by the children. We also consider that future teachers identify fluency when using a specific mathematical content knowledge to describe the children's behaviour having a direct influence over the solution. We assume that it is difficult to say, in this category, that preschool children develop and explore mathematical concepts and procedures in different ways.

Finally, the research team discusses about the assigned codes and interprets the results observed qualitatively and quantitatively by seeing the mathematical elements introduced in future teachers' explanations. The identification of these mathematical elements is the first step in the teacher's ability to interpret the students' level of problem solving and the creativity issues observed. In some occasions, we understand the category is present even if the explanation is not fully explicit.

12.4 Future Teachers' Interpretations

First, we will explain how the future teachers (called from now on St-#) identify heuristic strategies used by the children and how they understand arithmetic content knowledge recognized in the dialogues. We will argue which traits of creativity we can see in their explanations. After that, we will explain some observations about the role of the teacher promoting creativity.

12.4.1 About Traits of Creativity Found in Task 1

In some comments, we see the future teachers explaining the strategies used by the children, but without explicit comments referred to creativity uses. Now we will explain some examples.

The main heuristics observed by the future teachers are the use of diagrams and guessing/proving strategy without talking about novelty. For instance, they say, "*children go testing and finding out the solution to obtain the definitive answer*" (St-9). In many cases, future teachers confuse strategies with content knowledge. This situation is evident in sentences like "*In the dialogue... we see the children subtracting*" (St-1).

Although the going back strategy was unstated, we can interpret it as being observed in some arguments of future teachers like "*Children have difficulties observing the missing number solution, as is supposed to be, perhaps because it is easy to count back if you know the list of numbers going down*" (St-21). Almost all the future teachers talk about some children (Sandra) as being cognitively different from the other children, because she formulates *hypothesis and conjectures* (St-14, St-16 and others). In addition, they give arguments relating different representations: "*She used cubes to graphically do the addition, because the digits of two hands are not enough in order to achieve the result*" (St-12). Such comments are related to a restricted interpretation of what the process of conjecturing means. Therefore, we assume that, in this case, it is not a trait of creativity found.

As it was expected, the future teachers do not say anything about guessing as being different from retrieval by heart. This is the case of a child explaining that she guessed, but she really looks at the distance from 8 to 14. The child does not attempt to retrieve an answer from memory, but sometimes simply provides any number that comes to mind, which implies that the number is randomly generated. To sum up, future teachers intuitively assume the following as heuristics: counting strategies, representational issues and some heuristic ideas, rather than semantic categories that are not so easy to identify.

We assume surprising comments as being approaches to flexibility as "*I was surprised by the strategy of using cubes*" (St-4). Two future teachers talk about analysis/synthesis strategy before group discussion. However, they do not explain where they see this in the dialogue. In some addition problems, future teachers talk about children interpreting the meaning of operations, for example, *adding as putting together by using designs or material, because it is stated in the classroom dialogue* (St-16). It is also a reference about the strategies, but not to the creativity. Let us say that the future teachers talk about counting as a common sense strategy, because they never received any specific training about such counting strategies.

Future teachers found that *Carmen demonstrates a trial-error strategy starting using groups of two with the design, by using cubes* (St-14). None of the future teachers explain that it is important for her to visualize the representation of pigs in order to have the result but also for controlling the result. Some other future teachers talk about *Diego using trial-error to verify his hypothesis. He modifies the strategy until he finds the good solution. He never disconnects from the context. He used a random behaviour to find the solution (St-12, St-13, St-14 and St-15)*. In these cases, we are not fully convinced that such comments are expressing a creativity remark.

In some cases, the future teachers identify some interesting strategy, without any explicit assumption of surprise. Diego tries to share by giving three to each, plus one. After that, he decomposes by giving 4, 3, 3, 2 and 0. After that, he uses a complete sharing process by giving one to each pig and then another one and perceiving that the problem is solved by giving three acorns to each pig. In our interpretation, the future teachers do not identify the use of pattern images (Presmeg, 2006) as an original use of the strategy in the same observation, interpreting the semantics of sharing besides the trial and error strategy.

Some future teachers explain the need for stability in acquiring strategies as a basis for creative thinking in other moments or in the future learning process. Future teachers considered this idea as a common sense strategy adapting real-life experience. In this case, it is not interpreted as a part of a creativity process.

12.4.1.1 About Flexibility

Many future teachers recognize that in some dialogues, addition strategy involves some different strategies as *counting each of the addends separately, then counting up to the first addend and continuing to count by the number indicated by the second addend* (St-24). Others explain that "*we can find the composition and decomposition strategy*" (St-18). There is evidence that children spontaneously activate the sum of two numbers as counting both subsets, so it seems likely that guessing somehow involves consciously not attempting to retrieve an answer. Some future teachers also identify counting from the first strategy involving "*counting on from the first addend by the number indicated by the second addend*" (St-19 and St-22). After analysing all such future teachers' explanations, it seems that they refer to flexible use of solving methods, but it is not explicit.

Many other comments relate to flexibility. It is the case of the use of fingers. In fact, future teachers identify that finger recognition involves "*putting up fingers to represent each of the addends*" (St-20, St-26) and assuming that "*the child recognises the total, which is different from using fingers to assist counting*" (St-17). None of the future teachers interpreted children's behaviour as counting on from the larger of the two addends by the number indicated by the smaller of the addends. The use of structural images as fingers gives opportunities for bringing a personal computation. It seems completely different from the use of fingers for evocation of the traditional number sequence. It also relates to the flexibility of strategies.

When future teachers talk about subtraction, they considered the observation of *counting down using the cardinal principle by observing the use of material* (St-23). Another future teacher said: "*In the case of Diego, he uses multilink pieces to find the answer, putting 14 as a pile, and taking off five, before counting*" (St-21). We interpret such future teacher comments as some way of noticing the use of external representations, as not only helping to solve the problem but also revealing some originality issue. In fact, not every child needs to associate the cubes with the situation.

To understand the case of the problem having 15 magic acorns and wanting to share them between 5 pigs, the question was as follows: How many acorns correspond to each pig in order for everybody to have the same amount? Not all children answered correctly. Some future teachers consider that *Cristina's argument is good because she uses the number line in an original way. She searched for 15. She made groups of three, using hands. It seems that she tried first groups of two, or identified that using groups of two, you do not arrive at 15, but 10* (St-21, St-22, St-23, St-24). Some other future teachers talk about the use of cubes, without explaining if this relates to flexibility or not.

12.4.1.2 About Originality

We found that it is not easy to identify if several children's actions and statements reveal some specific new or unusual mathematical strategy. Moreover, we also can conclude that future teachers do no assume imagery perspectives. Five future teachers identify that children use counting on from one and counting on from the last number in one dialogue as something usual.

In many explanations, the expression "surprising" more than "unusual" appears. When the future teachers talk about subtraction, they assume *the strategy of deleting from the highest number as unusual* (St-2, St-13, St-21). Such explanation provides an example of a child successfully bridging his/her own knowledge of informal and formal symbolic representation for the calculation. In this case, we assume the future teacher recognizes some originality. Instead of that, some future teachers' comments are too general: "*all children identify that it is necessary to do a sum*" (St-17, St-18, St-19, St-20).

Some future teachers talk about the 1–100 table as new and surprising material for them. Is it a suspicion that there is some originality in such a comment? Or do they have difficulties to assume that children know more in this case than future teachers?

In many comments analysed, we observe that it is difficult for future teachers to identify some of the strategies as unexpected or interesting by themselves. In many cases, the future teacher describes the solving of the problem in terms of the operations carried out, but he does not relate these operations to the structure of the problem. The future teacher, therefore, does not recognize that the solution was incorrect for this type of problem.

The future teachers consider unexpected what is different from their own strategy to solve. They used sentences such as "*I cannot understand... because it's easier to solve...in such a way*" (St-15 and St-16).

12.4.1.3 About Fluency

They also find that "The *more you use it, the more you are confident with such a mediator*" (St-18). Only a few of them talk about its relation with arithmetic counting strategies: "S*uch a strategy is intermediate between counting everything, and count from first*" (St-5 and St-6). "*...A child, instead of counting as the others, does use colours*" (St-4, St-5, St-7, St-8). Future teachers notice that it is a fluency comment because it helps future different strategies to solve problems.

We identify nine future teachers making several comments about flexible use of strategies relating to the use of multiple representations to solve the division problems. Let us mention some examples: in some cases, there are general comments about "*the use of materials for abstracting processes*" (St-18). In other cases, there are connections with building structures: "*I was surprised…the child organized groups of two eggs without representing the amount of hen*" (St-9) *or uses a way for representing equal amounts* (St-19, St-20). In some other cases, connecting representations "*…materials give opportunities to find solutions by themselves*" (St-16).

It is difficult that future teachers identify fluency in some children's episodes, because it is not easy for preschool children to explain verbally mathematical relations used to solve the problem. We found some evidences of flexibility, when future teachers interpret a separation process to solve the problem by relating eggs to hen, such as "*the interesting aspect is that the child separates [the pieces] and evidences the use of representations in sharing situations, without making the animal associated to eggs explicit*" (St-6, St-7). In two of the discussion groups, they focus on the importance of the material as helping the strategies to solve the problems without entering in details.

Using fingers as representation to solve the problem is not a common strategy for children. Therefore, it is difficult for future teachers to identify it as unexpected strategy. However, some future teachers tell us "*I was surprised with Cristina's talk because of the use of fingers on one hand, but as we need four digits, we remove one digit to have four plus*" (St-24). In this case, we consider that future teachers assume this comment as an example of flexibility, even when it is not fully explicit.

When they discuss in a group to make a common opinion, they mainly observed self-control of results in many of the analysed children. These cases clearly relate to flexibility and fluency. One of the groups of future teachers explains that *Cristina interpret sharing situations,* but future teachers think that *she has internalized the process, because she verbalises all the steps of her reasoning without teacher's help* (St-10, St-11, St-12, St-13). In this case, the future teachers group interprets such autonomy in terms of solving strategies, not as a way to gain effective fluency.

12.4.2 Traits of Creativity during the Second Activity

During the future teachers' group discussion after knowing theoretical aspects, they continue to assume some misunderstandings as misinterpretations of the problem statements. Nevertheless, they consider some deep understandings about creativity aspects.

Let us see some examples observed during task 2. Future teachers are excited when children can solve a problem of six hens having two eggs each one, by "using a representation of eggs, without representing the hen" (St-13 and St-14). Carmen's behaviour surprises almost everybody "*joining the two blocks to represent the two eggs of each hen; even Diego accepts Carmen's proposal. We think that it signifies an original process because they abstract some mathematical idea with such a gesture*" (St-15 and St-16).

Although future teachers do not see the strategy "counting by pairs", they are close to this immediate strategy, identifying the group of two assigned metaphorically to one hen. It means that a trend of mathematical originality is that the children can enter into the dialectics of idealization/materialization (Godino & Font, 2010) to approach the context solution. Some future teachers explain that *the problem of eggs is similar to the problem of counting legs, when children have plastic animals* (St 8&9). The context is used as a referent in the sense that they know how to approach a solution in these problems.

The table used as a research tool for concentrating on the assignments of any of the indicators is not enough to see changes about traits of creativity, but the written justifications are. They find flexibility related to the fact of having an open problem. Even when recording the conversations, sometimes it is not easy to understand the adopted meanings to classify some children's behaviours to marks and explanations given. For instance, a pair of future teachers told us about Sandra's original strategy, but others did not.

Some disagreements appear, because of the different interpretations of problem-solving strategies or categories. After being introduced to theoretical categories and explanations, we see that not every indicator used to analyse the problem-solving activity was fully understood. For instance, the future teachers associated an unexpected multiple grouping with a subtraction problem. They found in dialogues 5 and 6 the situations in which some interesting or "unexpected strategies or flexible behaviours" are more explicit than others, probably because in such problems there are interactions suggesting different ways of solving.

The 66.6% of the future teachers think that the use of manipulative aids or fingers is crucial for solving the problems but just 36.6% tell us about them as alternative or unexpected strategies. It sounds similar to what is expressed by some authors telling us that "whereas children are adept at forming varied and unusual images, adults tend to have advantages when it comes to storing and retrieving information, drawing upon experience, and making judgments about what is benefit or appropriate" (Jalongo & Hirsh, 2012).

When using theoretical indicators about representations, 90% of future teachers found the use of the number line as the main tool for solving the most difficult problems. The future teachers did not relate this comment to flexibility, but to improving problem solving itself.

When analysing children's behaviour with the lens of theoretical tools provided during the second task, every future teacher knows that "*adding to*" and "*taking away*" means knowing that adding to a collection makes it larger and subtracting makes it smaller. Moreover, they see addition and subtraction as the same structure. Therefore, they find *new originality aspects*, valuing some children's actions. Some future teachers accept that children use such ideas as common sense strategy combined with counting and (de)composing. It was assumed that children could solve simple problems with increasing efficiency because of classroom dialogues.

In fact, children's creativity is different from adults, in which expertise and habits dominate. For children, it is related to playing and interacting with the real world in a fresh way having lasting effects on the brain, stretching pathways, as neuroscientists suggest (Jensen, 2006), and enlarging the connections in the brain when children experiment and observe (Gopnik, Meltzoff, & Kuhl, 2001). It is also important to know that formulating good problems that invite creative thought enables children to understand and express abstract concepts (Jensen & Kiley, 2003).

12.4.3 The Teaching Role in Promoting Challenge and Creativity

Future teachers found that the educator uses an environment creating a community of collaborative problem solvers and promoting the ability of communicating ideas in project-based learning in the mathematics classroom. The future teachers also identify that emergent mathematical ideas are present in children's arguments, which reflect the processes worked through to understand and to communicate the strategies used when solving the problems.

Future teachers assume (not always explicitly stated) a set of teaching strategies (Epstein, 2007) contributing to improving the dialogue as a way of building collective knowledge as a challenging situation, giving opportunities for creativity. The dialogue has the following characteristics:

- The teacher *did not pressure children to choose a particular solution*, *even if children refused a number of possible solutions offered by the other child* (St-21).
- The teacher quietly observed from a short distance to be able to intervene if necessary, but using words that *did not judge* either side of the disagreement.
- "Beatriz (the school teacher), *accepted and heard the children's solutions* if both parties were satisfied, even if the solution did not make sense to the adults" (St-26).
- The teacher also *accurately defines the situation*, which, in itself, sometimes moved children towards solving the problem.

- The teacher promotes inquiry, by asking questions and *asking for clarification and consolidation*s that are good strategies to involve children into problem-solving participation.
- The teacher uses challenges and promotes the idea of a non-unified truth on mathematical knowledge (St-18). In such comments, we assume they are talking about flexibility issues.

Any one of the future teachers interprets the importance of helping children intuitions of properties by themselves, giving them the vocabulary to describe the properties. It only appears during the whole group discussion after small group discussion.

Future teachers do not really have a founded knowledge about the possibility for children to solve structured problems since early years. In fact, they see some of the dialogues as a surprise. The most impressive answer for future teachers is to *find the result of a subtraction by counting backwards after writing the numbers as a number line series* (Nacho's proposal). Nevertheless, in such an original aspect, future teachers identified drawing graphical representations as fluency comment.

The future teachers found semiotic gestures and number lines as giving a chance to stimulate number and calculating discussions. Future teachers considered the use of democratic dialogues as opportunities for building visualization tools when developing metacognitive reflection and as opportunities to relate previous knowledge with new ideas and standard written symbols. Just during discussion after having theoretical tools, we interpret that they notice the role of semiotic tools as representations or gestures promoting effective interactions.

The use of dialogical methodology in the teacher training process and the fact of having purposes behind the methodology used lead us to conjecture a set of principles induced by the teacher:

(a) Scaffolding process using structural dependences in order to analyse and have arguments.
(b) Helping for sharing patterns.
(c) Planning strategies hearing children's voices, needed to build efficient good images.
(d) Adapting sensitive perceptions to build metaphoric constructions, as a play situation.
(e) Truth as something not previously decided and argumentation being the only way for controlling.
(f) Self-control as a way to touch personal beliefs.

12.4.4 About Evolution in Traits of Creativity

According to the future teachers' writings, we notice how many of them assume positions related to originality, by assuming that he or she considers at least one children's case. For flexibility and fluency, it is assigned the codification by the team of researchers who decide about the evidences, because it is not so clear as originality.

Table 12.1 Percentage of future teachers identifying the categories of creativity

N = 30	Identifying originality	Identifying flexibility	Identifying fluency
First approach (task 1)	36.6%	30%	16.6%
After group discussion(task 2)	53.3%	66.6%	40%

To see how the evolution of future teachers occurs, we use the same categories to identify creativity as explained above to see their comments on the second task after discussing about theoretical tools before introducing any explicit reflection about creativity in task 2. In Table 12.1, we see the different percentage of future teachers that we find in each category.

Let us see an example of five students, which during group discussion, considered originality in Cristina's behaviour, but not before. To notice new fluency aspects is really a consequence of group discussion, because future teachers did not recognize some of the children's actions in terms of learning many possibilities for new strategies in the future. The main changes in terms of flexibility interpreted by future teachers are related to representational observations.

A few comments suggest the advances on enrichment of mathematical processes. Comments like "*I see that the complementary use of representations as number line and table 1-100 gives opportunities that I never expected*" or "*We never had this opportunity to have objects for solving problems…I never had discussed about others' strategies during my primary school. It was not usual to use inquiry based approach when I was a kid*" (St-16 to St-20) appeared.

As we can see in the table, after group discussions, more students talk about some traits of creativity. The main aspect relates to the role of counting strategies, and the stability of the use of processes needed to solve the problems as evidences for originality and flexibility. One of the interesting changes when group discussions occur is that, in both moments, the idea of cause-effect as a necessary principle to interpret an equality problem that seems to be related to pattern images, fruitful for problem solving, appears. We interpret such comments in terms of fluency. There is also a general observation that in preschool children sometimes solves simple multiplication (grouping) and division (partitioning) problems by direct modelling with objects.

Future teachers do not identify the need for negotiation according to previous pedagogical reflections as easily. St-4 (24-year-old future teacher) explains: "*It was becoming clear that children were learning the process of finding solutions on their own. In some cases, I see children negotiating the operational meaning*". Some future teachers interpret that the teacher uses the dialogue to help children feel empowered to express their needs.

During the third part of task 2, where creativity features were described, we observed some changes in five groups of future teachers. They notice specific relations between semiotic perspective of imagery (Presmeg, 2006) and originality. They continue observing fluency more related to the classroom attitude organized by the educator.

Table 12.2 Findings about the use of metaphors, creativity strategic aspects and mathematical thinking promoted during the school experience

Type of metaphors used in the classroom	Strategic aspects observed	Mathematical thinking promoted
Relativity of representations and having referents	Ordering and relational properties	Mathematical properties and number sense
Physical and kinaesthetical observations and gestures	Trial and error strategies	Applications and tools for controlling
Free use of materials showing equivalences	Pattern recognition and classification	Algebraic thinking and design
Motivating by everyday situations	Semantic and structural observations	Embodied cognition
Heuristics and visual intuitions	Conjecturing and heuristic strategies	Hypothetical reasoning
Observing changes by stories and music	Operational understanding	Relational and functional thinking

As we see in Table 12.2, the dual process of materialization and idealization found in the mathematical practice appears to be consistent and seems to give opportunities for increasing not only creativity but growing mathematical thinking. We interpret the use of metaphors as children's actions, leading to certain aspects observed as strategic knowledge, and some aspects of mathematical thinking promoted.

Future teachers theoretically accept that throughout the early years of life, children notice and explore mathematical dimensions of their world. However, it seems initially difficult for them to evoke rich mathematical ideas appearing in the dialogues without any specific preparation about analysing children's texts. The future teachers also emphasized that creative education should start at an early age to become permanent. They understand that powerful and strong use of imagery allows the person to engage with all of the possible senses. In addition, future teachers use metaphors in expressing views and opinions in graphic imagery at least in a naïf way.

12.5 Final Reflections

The study contributes to the research based on how pre-service teachers make sense of preschool mathematics in a specific learning environment. The work done reinforces the hypothesis that interaction focused on a specific goal can help pre-service teachers to develop the noticing skill. The observed problems engaged children in a range of mathematical skills and ideas, such as counting, subitizing, comparing and recognizing numerical relationships. Future teachers considered that the school experience was nice to analyse.

Future teachers spontaneously identify some traditional problem-solving strategies and increase their observational perspective about how to interpret children's behaviour, when they have some theoretical and methodological tools of analysis and group discussion. Nevertheless, it is not clear if their ability to notice creativity issues is really growing. In the context researched, deep interpretation occurs, when being aware of categories of creativity. It is built upon deep understanding about the importance of giving children the opportunities to share and compare the strategies used.

The future teachers consider each of the creativity skills as a method of instruction to be expressed that would reach efficient, permanent and meaningful learning. They emphasized the individualization of learning and enriching the educational climate with the help of these techniques. They see more aspects, but they assume interpretations not directly associated with the evidences found in the dialogues.

We suspect that future teachers are more worried with schoolteacher intervention than recognizing mathematics behind children's comments. They look at the problem-solving process as a global set of actions leading to a positive end. Let us say that as an initial study, we explain neither the growing of future teachers when improving their didactical analysis nor the role given to dialogue in developing problem-solving capacities here.

The study reveals the difficulties future teachers have noticing authentic traits of creativity in problem-solving activities because of methodological difficulties in overcoming their own perspectives about evidences. This demonstrates how important it is for future teachers to develop an explicit "awareness" of the mathematical elements involved in solving problems and their role in determining the students' reasoning.

The future teachers assume that the development of creativity is time-consuming and it is quite normal that the applied techniques take time to improve and understand. Such a result is in agreement with the research from Sternberg (1996).

Future teachers interpreted problem solving as an important element in classroom life, enriching the children's experiences, language and learning. Problem-solving activity is not easy. It is one of the hardest skills to build. It takes a lot of self-control for the teachers *and* the children (Gross, 2005). The considered satisfactory strategies are the following: getting a feel for the problem; looking at it holistically; checking if they have understood (e.g. talking it through or asking questions); planning, preparing and predicting outcomes; monitoring progress towards the goal (e.g. checking that the bears will fit the houses); being systematic; methodically trying possibilities without repetition, rather than at random; trying alternative approaches and evaluating strategies; and refining and improving solutions (e.g. solving a puzzle again in fewer moves) (Gifford, 2005:153).

The problems and workshop style allow a wide variety of representations and the development of inductive processes, starting with particular cases. Oral conversation occurred. Finally, we discovered that future teachers believe that it is possible to learn and improve creativity (Houtz, 2003) via activities of ongoing training and creative problem solving. The trainers play a vital part in creating an environment, in which creativity and reasoning individuality are valued and utilized, but also offering situations in which students can reveal their strengths and highlight areas where they need greater support. Additional research is needed on the factors that constrain and/or promote the development of creativity understanding in a noticing process, while better theoretical models must be developed that enable us to understand it.

Acknowledgement This work has been funded by the Spanish and Catalan governments. Yuly Vanegas' participation (EDU2015-65378-P; SGR2017-101). For Joaquin Giménez (EDU2015-64646-P; SGR2017-1181).

References

ACECQA–Australian Children's Education and Care Quality Authority. (2011). *Guide to the National Quality Standard*. Sydney: ACECQA.

Amaral, N. & Carreira, S. (2012). An essay on students' creativity in problem solving beyond school: Proposing a framework of analysis. In *Pre-Proceedings of the 12th International Congress on* Mathematical Education *(ICME 12) – Topic Study 3*. Seoul, 8–15 July 2012, (pp. 1419–1428). Seoul, South Korea: ICMI.

Anthony, G. & Walshaw, M. (2007). *Effective pedagogy in mathematics/Pangarau: Best evidence synthesis (BES)*. Wellington, NZ: Ministry of Education. Retrieved from http://educationcounts.edcentre.govt.nz/publications/curriculum/bes-eff-pedagogy.html

Baroody, A. J. (2000). Does mathematics instruction for 3 to 5-year olds really make sense? *Young Children, 55*(4), 61–67.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills* (pp. 1–33). Mahwah, NJ: Lawrence Erlbaum Associates.

Boaler, J. (1997). *Experiencing school mathematics*. Buckingham: Open University Press.

Bushman, L. (2003). *Share and compare: A Teacher's story about helping children become problem solvers in mathematics*. Reston: NCTM.

Callejo, M. L. (2003). Creatividad matemática y resolución de problemas. *SIGMA, 22*, 25–34.

Carlsson-Paige, N. (2013). TEDx Talk: "When Education Goes Wrong: Taking Creativity and Play out of Learning". Retrieved 12 May 2016 from https://www.youtube.com/watch?v=BZzFM1MHz_M.

Carpenter, T., Fennema, E., Peterson, P., & Carey, D. (1988). Teacher's pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19*(5), 385–401.

Carr, M., Peters, S., & Young-Loveridge, J. (1994). Early childhood mathematics: Finding the right level of challenge. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers* (Vol. 1, pp. 271–282). Wellington: Wellington College of Education.

Clements, D. (2004). Major themes and recommendations. In D. H. Clements, J. Sarama, & A. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 7–72). New Jersey: Lawrence Erlbaum.

Coltman, P., Petyaeva, D., & Anghileri, J. (2002). Scaffolding learning through meaningful tasks and adult interaction. *Early Years, 22*(1), 39–49.

Craft, A. (2002). *Creativity and early years education*. London: Continuum.

Davis, G., & Pepper, K. (1992). Mathematical problem solving by pre-school children. *Educational Studies in Mathematics, 23*, 397–415.

De Castro, C., & Escorial, B. (2007). Resolución de problemas aritméticos verbales en la Educación Infantil: Una experiencia de enfoque investigativo. *Indivisa, Boletín de Estudios e Investigación*, Monografía*, IX*, 23–47.

Deloache, J. S., & Brown, A. L. (1987). The early emergence of planning skills in children. In J. Bruner & H. Haste (Eds.), *Making sense. The child's construction of the world* (pp. 71–86). London: Methuen & Co..

Epstein, A. S. (2007). *The intentional teacher: Choosing the best strategies for young Children's learning*. Washington, DC: NAEYC.

EYLF - Department of Education, Employment and Workplace Relations. (2009). *Belonging, being and becoming: The early years learning framework for Australia*. Canberra: Australian Government.

Fernandez, C., Llinares, S., & Valls, J. (2013). Primary teachers professional noticing of students' mathematical thinking. *The Mathematics Enthusiast (Special Issue: International perspectives on problem solving, Research in Mathematics Education), 10*(1&2), 441–468.

Gagatsis, A., & Elia, I. (2003). The use of number line in conducting arithmetic operations: A comparative study between Cypriot, Italian and Greek primary pupils. In J. Navotna (Ed.), *Proceedings of the international symposium in elementary mathematics teaching* (pp. 70–73). Prague: Charles University.

Gifford, S. (2005). *Teaching mathematics 3–5: Developing learning in the foundation stage*. Maidenhead: Open University Press/McGraw-Hill Education.

Godino, J., & Font, V. (2010). The theory of representations as viewed from the ontosemiotic approach to mathematics education. *Mediterranean Journal for Research in Mathematics Education, 9*(1), 89–210.

Gopnik, A., Meltzoff, A., & Kuhl, P. (2001). *How babies think: The science of childhood*. London: Phoenix.

Gross, C. (2005). *Promoting Problem Solving with Young Children: An Approach to Community* Building and Classroom Management in One Classroom. (Unpublished PhD thesis). Teachers College, Columbia University, New York, United States of America.

Gura, P. (1992). *Exploring learning: Young children and blockplay*. London: Paul Chapman Publishing Ltd.

Jacobs, V., Lamb, L., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.

Houtz, J. C. (2003). *The educational psychology of creativity*. Cresskill, NJ: Hampton Press.

Jalongo, M. R., & Hirsh, R. A. (2012). Reconceptualising creativity thought process in young children. In O. Saracho (Ed.), *Perspectives on research in creativity in early childhood education* (pp. 89–108). Charlotte: IAP.

Jensen, R., & Kiley, T. (2003). *Teaching, leading and learning in pre K-8 settings* (2nd ed.). Boston: Houghton Mifflin.

Jensen, D. (2006). Metaphors as a bridge to understanding educational and social contexts. *International Journal of Qualitative Methods, 5*(1). Retrieved June 13th 2015 from http://www.ualberta.ca/~iiqm/backissues/5_1/pdf/jensen.pdf,), 36.

Kubínová, M. (1999). The position of didactics of mathematics in the training of mathematics teachers. In Ernest, P. (Ed.) *Philosophy of mathematics education journal 12*. Retrieved from http://webdoc.sub.gwdg.de/edoc/e/pome/pome12/default.htm

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 130–144). Rotterdam: Sense Publishers.

Llinares, S. (2013). Professional noticing: A component of the mathematics Teacher's professional practice. *Sisyfus. Journal of Education, 1*(3), 76–93.

Malaguzzi, L. (2001). *La educación infantil en Reggio Emilia*. Barcelona: Octaedro.

Mammede, E., & Soutinho, F. (2012). Focusing on young children's additive and multiplicative reasoning. *Journal of the European Teacher Education Network, 7*, 21–34.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge.

Meissner, H. (2000, July–August). *Creativity in mathematics education*. Paper presented at the meeting of the International Congress on Mathematical Education, Tokyo, Japan.

Meissner, H. (2003). Stimulating Creativity. *Proceedings of the Third International Conference "Creativity in Mathematics Education and the Education of Gifted Students"* (pp. 30–36). Rousse, Bulgaria.

Munn, P. (1997). Children's beliefs about counting. In I. Thompson (Ed.), *Teaching and learning early number* (pp. 9–20). Philadelphia: Open University Press.

OFSTED–Office for Standards in Education. (2013). Mathematics in school inspection January 2013: Information pack for training. Retrieved 3 Mar 2015 from https://www.whatdotheyknow.com/request/additional_guidance_for_inspectors.

Parrish, S. (2010). *Number talks: Helping children build mental math and computation strategies, grades K 5, updated with common Core connections*. Sausalito: Math Solutions.

Piggott, J. (2011). Cultivating creativity. Retrieved 4 June 2013 from https://nrich.maths.org/5784.

Ponte, J. P. (2007). Investigations and explorations in the mathematics classroom. *ZDM, 39*(5–6), 419–430.

Ponte, J. P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 461–494). Rotterdam/Taipei: Sense Publishers.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205–235). Rotterdam: Sense Publishers.

Reigeluth, C., & Stein, F. (1983). The collaboration theory of instruction). In C. Reigeluth (Ed.), *Instructional design; theories and models. An overview of their current status* (pp. 335–381). Hillsdale: Erlbaum.

Rodgers, C. (2002). *Defining reflection: Facilitating reflection in novice teachers*. (Doctoral dissertation). Stanford University, Stanford, CA.

Rudd, L. C., Lambert, M. C., Satterwhite, M., & Zaier, A. (2008). Mathematical language in early childhood settings: What really counts? *Early Childhood Education Journal, 36*, 75–80.

Russ, S. W. (2003). Play and creativity: Developmental issues. *Scandinavian Journal of Educational Research, 47*(3), 291–303.

Sequera, E., Giménez, J., & Servat, J. (2005). Detecting traits of creativity potential in mathematics tasks with prospective Primary Mathematics Teachers. In H. Meissner (Ed.), *EARCOME 3*. Shangai: East China Normal University [CD].

Schoenfeld, A. (1989). Reflections on problem solving theory and practice. *TME, 10*(1&2), 9–33.

Scott, G., Leitz, L. E., & Mumford, M. D. (2004). The effectiveness of creativity training: A quantitative review. *Creativity Research Journal, 16*(4), 361–378.

Sherin, M., Jacobs, V., & Philipp, R. (Eds.). (2011). *Mathematics teacher noticing. Seeing through teachers' eyes*. New York: Routledge.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM – International Journal on Mathematics Education, 29*(3), 75–80.

Seo, K., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements & J. Sarama (Eds.), *Engaging young children in mathematics: Standards for early mathematics education* (pp. 91–104). Mahwah, NJ: Lawrence Erlbaum.

Sternberg, R. J. (1996). *How to develop student creativity*. Alexandria, VA: Association for Supervision & Curriculum Development.

Upitis, R., Phillips, E., & Higginson, W. (1997). *Creative mathematics exploring Children's understanding*. New York: Routledge.

Van Oers, B. (2004). Mathematisches Denken bei Vorschulkindern. In: W. Fthenakis (Hg.), *Frühpädagogik international: Bildungsqualität im Blickpunkt* (pp. 313–329). Wiesbaden: Verlag für Sozialwissenschaften.

Vigotsky, L. (1978). *Mind in society. The development of higher psychological processes*. In M. Cole, V. John-Steiner, S. Scribner, E, Souberman. London: Harvard University Press.

Chapter 13
Stimulating Mathematical Creativity through Constraints in Problem-Solving

Christian Bokhove and Keith Jones

13.1 Introduction

In the project entitled *Mathematical Creativity Squared* (MC-squared; www.mc2-project.eu), we began exploring the premise that enabling teachers to be more creative in their design of tasks to be used in their lessons can help their learners in these teachers' classrooms be more creative (see, e.g. Bokhove, Jones, Mavrikis, Geraniou, & Charlton, 2014; Bokhove, Mavrikis, & Jones, 2015). In working on the design of mathematical tasks during the project, we pondered how the use of so-called open problems often appears to be emphasised in deliberations about mathematical problem-solving (Boaler, 1998; Pehkonen, 1997; Silver, 1995; Sullivan, Clarke, & Clarke, 2012; Sweller, Mawer, & Ward, 1983). The reason seems to be that problem-solving is deemed best supported by the provision of open-ended tasks. On top of that, it is further argued that the openness of problems is more conducive to students' mathematical creativity than closed tasks. Silver (1997, p. 77), for instance, argued that the development of learners' 'creative fluency' is 'likely to be encouraged through the classroom use of ill-structured, open-ended problems that are stated in a manner that permits the generation of multiple specific goals and possibly multiple correct solutions'. Similarly, Kwon et al. (2006, p. 51), in their study, concluded that the use of open-ended tasks 'may provide a possible arena for exploring the prospects and possibilities of improving mathematical creativity'.

In this chapter we problematise this idea that problem-solving and creativity are seen as best supported by providing open-ended tasks. In doing so we take a somewhat different, but related, approach to that proposed by Haught-Tromp and Stokes (2016) who describe what they call 'constraint pairs', where one constraint precludes something, while its 'pair' directs the search for a substitute (such as the next

C. Bokhove (✉) · K. Jones
University of Southampton, Southampton, UK
e-mail: c.bokhove@soton.ac.uk; d.k.jones@soton.ac.uk

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_13

most efficient strategy) as a way to inform lessons that help students develop creativity. In contrast, we make the case for what might be called ‘constraints-based’ task design. In this latter approach, which we relate to research in economics on scarcity (and the American television series character *MacGyver*), we show how tasks that are ‘moderately closed’ (being neither fully ‘open’ nor fully ‘closed’) can provide for creative mathematical thinking and problem-solving. In some cases, the use of feedback can provide cues to students. Using examples from a range of topics, we explore examples of ‘constraints-based’ creativity such as producing geometry constructions solely with ruler and compass, ways of tackling number puzzles and solutions to sets of equations. We conclude that such examples demonstrate that classroom tasks for mathematical problem-solving and creativity need not be restricted solely to open-ended problems; rather, we argue that tasks with suitable constraints can serve as creativity-inducing problem-solving tasks as well.

13.2 Problem-Solving, Creativity and Task Constraints

13.2.1 Problem-Solving Tasks

While problem-solving is a key component of the school mathematics curriculum internationally, the nature of what constitutes the ‘problem’ (or task) to be solved during mathematical problem-solving is not necessarily straightforward. Borasi (1986), for example, attempted to clarify the notion of a mathematics ‘problem’ in terms of the following structural elements:

(a) The formulation of the problem; the explicit definition of the task to be performed.
(b) The context of the problem; the situation in which the problem itself is embedded.
(c) The set of solution(s) that could be considered acceptable for the problem given.
(d) The methods of approach that could be used to reach the solution.

Although this clarifies some aspects of the notion of a mathematics ‘problem’, it leaves much under-explored. For example, Christiansen and Walther (1986, p. 244) argue that even though classroom work on mathematical problems in the form of ‘exercises’ [which they describe as ‘drill and practice in relation to previously described concepts and procedures’] is ‘centrally placed at all levels of mathematics teaching’, such use ‘rests on an inadequate and insufficient differentiation with respect to the relationships between the concepts *task* and *activity*’ (ibid., pp. 245–46), where the task is set by the teacher and the activity is what the learners engage in. In this, they argue (ibid., p. 275), the completion by learners of *tasks* that are ‘exercises’ *does not* ‘contribute to a genuine development of knowledge’ in terms of learner *activity*, while the use of nonroutine *tasks* provides ‘optimal conditions’ for cognitive development of new knowledge. Thus, as Boaler (1998, p. 42) explains, supporters of the use of nonroutine tasks contend that 'if students are given open-ended, practical and investigative work that requires them to make their own decisions, plan their own routes through tasks, choose methods and apply their mathematical knowledge, the students will benefit in a number of ways'.

Sullivan et al. (2012, p. 57) differentiate task goals into 'closed' or 'open', where a closed goal 'implies there is only one acceptable response', whereas an open goal 'has more than one (preferably many more than one) possible response'. Table 13.1 provides an illustrative example of what might be considered a 'closed' task (one with a goal that is likely to be closed) and an 'open' task (one with a goal that is rather open). Such different types of tasks reflect different qualities and priorities in task design. This can include the degree to which the task is constrained for the student (for more on this, see Stoyanova, 1998) or the role the task may play in relationship with problem-solving (for more see Silver, 1995). More recently, Yeo (2017) proposed a framework to characterise the 'openness' of tasks with five elements: goal, method, task complexity, answer and extension. All this suggests that 'closed-open' is one of the main distinctions that is made about the form of mathematical tasks that can be used in the mathematics classroom.

In what follows we examine the case for what might be called 'constraints-based' task design by exploring tasks that are 'moderately closed' (neither fully 'open' nor fully 'closed') and how these can provide for creative mathematical thinking and problem-solving. We begin by examining the notion of creativity (both broadly and in mathematics education) and from that go on to explore the role and nature of constraints in mathematical creativity and problem-solving.

13.2.2 *Defining Creativity*

Creativity is a frequently used term in education in general and in mathematics education in particular. Despite this, and the increasing interest in promoting creativity in all sectors and across all individuals and groups, the term remains a vague and difficult construct both to define and to operationalise (Mann, 2006; Sriraman, Haavold, & Lee, 2014). This is mainly because of the wide range of theoretical approaches and disciplines that have been used to address creativity (Cropley, 1999). There is currently an array of theories and perspectives on creativity that provides critical insights into a better understanding of the phenomenon. For example, in the field of psychology, many co-existing paradigms been developed, ranging from linear to integrative approaches to creativity. Sternberg (2003), for instance,

Table 13.1 Illustrative examples of 'closed' and 'open' tasks

Illustrative example of a 'closed' task	Illustrative example of an 'open' task
You have two dogs; one eats 1 can of dog food per day, while the other eats 3 cans. If a can of dog food costs 80p, what is the total of feed the two dogs each day?	How much does it cost to keep a pet?

identified at least eight frameworks for viewing creativity, from ‘mystical’ approaches to those entailing psychodynamic, cognitive, psychometric, pragmatic, social personality, evolutionary and confluence models of creativity.

A major distinction that is often made is whether creativity is ‘Big-C’ creativity (Creativity with a capital C) or ‘everyday’ or ‘little-c’ (creativity with a little c). The former, ‘Big-C’, has a tradition in creativity research that is typically concerned with the exceptionally creative activity of some really very talented but rare individuals (who might get referred to as ‘geniuses’) (Simonton, 2010). Yet creativity can also be seen as a potential that all people are capable of displaying and which can find expression in various situations of everyday life; this is ‘everyday’ or ‘little-c’ creativity (Simonton, 2013, 2017). Boden (1994) provides another term for ‘little-c’ creativity: ‘psychological creativity’. This is when something is identified as creative at least by the creator themselves, without being necessarily an outstanding contribution to some specific domain.

Notwithstanding these distinctions, most definitions of creativity found in research literature (Runco & Albert, 1990; Runco & Pritzker, 1999; Kaufman & Sternberg, 2010) include two structural elements: (1) novelty (originality, unexpectedness) of the creative work and (2) its value (relevance, appropriateness, significance, usefulness, effectiveness). These elements are apparent in the *Handbook of Creativity* (Kaufman, Glăveanu, & Baer, 2017; Sternberg, 1999; Kaufman & Sternberg, 2010), which summarises contemporary creativity research. Some definitions that show this common pattern are:

1. ‘Creativity is the ability to produce work that is both novel (i.e. original, unexpected) and appropriate (i.e. useful, adaptive concerning task constraints)’ (Sternberg & Lubart, 1999, p. 3).
2. ‘[our definition] involves novelty and value: The creative product must be new and must be given value according to some external criteria’ (Gruber & Wallace, 1999, p. 94).
3. ‘A creative idea is one that is both original and appropriate for the situation in which it occurs’ (Martindale, 1999, p. 137).
4. ‘Creativity from the Western perspective can be defined as the ability to produce work that is novel and appropriate’ (Lubart, 1999, p. 339).

The 1999 edition of the handbook contained a comparison table compiled by Mayer (1999, p. 450) which shows how these two elements are prevalent in definitions of creativity.

While these existing definitions of creativity can be used to help specify creativity in mathematics and mathematics education (Mann, 2006), no single and widely accepted definition of mathematical creativity exists. One distinction often made in the definition question is whether it is the *process* or the *product* that is in focus. Hadamard (1945), for example, refers to the mathematicians’ creative process using the four-stage Gestalt model: preparation – incubation – illumination – verification. Liljedahl (2013) extended that model by also emphasising the inventive process of mathematical creativity by adding the phenomenon of the AHA! experience. In particular, Liljedahl and Sriraman (2006) describe mathematical creativity at the school level, as (1) the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problem and/or (2) the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle. The product approach to creativity focuses on the outcomes that result from creative processes. It starts from the premise that to identify a process or activity as creative, one has to discern the existence of some creative outcome. Sternberg and Lubart (1999) seem to support this view when saying creativity is the capability to produce unexpected, original, appropriate and useful pieces of work. Leikin and Pitta-Pantazi (2013) give a broad overview of the 'state of the art' regarding creativity and mathematics education, arguing that the focus can be on the creative person, the creative process, the creative product or the creative environment. Liljedahl and Sriraman (2006) make the case that the creative process and creative product are inseparable and indistinguishable.

Silver (1997), among others, sees creativity as an orientation or disposition towards mathematical activity that can be broadly fostered in the general school population. This approach is more associated with long periods of work and reflection rather than rapid and exceptional insight. This has consequences for how one might approach creativity as it suggests that creativity-enriched teaching and other support might be conducive for a broad range of students. Here, creativity is not merely something for a few exceptional individuals. This view is often accompanied by thoughts about problem-posing and problem-solving (Bonotto & Dal Santo, 2015; Singer & Voica, 2016).

More specifically, problem-posing and problem-solving are closely related (Singer, Ellerton, & Cai, 2015). It is in the interplay of formulating, attempting to solve, reformulating and eventually solving, a problem that is seen as creative activity. Features of creativity that could be detected within this frame are novelty in the problem formulation or solution, shift in direction during the problem-solving process and number of different solution paths or different solutions.

In his assessments of 'divergent thinking', Guilford's (1967) *Alternative Uses Task* distinguished four components of creativity that could be applied to the domain of creative mathematical thinking:

- Fluency as it relates to the continuity of ideas, flow of associations and use of basic and universal knowledge. This entails the student's ability to pose or come up with as many responses as possible (ideas or configurations).
- Flexibility as associated with changing ideas, approaching a problem in various ways and producing a variety of solutions. It describes the student's ability to vary the approach or suggest a variety of different methods towards a problem or situation.
- Originality/novelty as seeking for unique ways of thinking and unique products of a mental or artistic activity. The criterion pertains to the student's ability to try novel or unusual approaches towards a problem.
- Elaboration as relating to the capability to describe, illuminate and generalise ideas. Such a criterion should assess the person's capacity to redefine a problem or situation to create others.

Sternberg and Kaufman (2010) emphasise another important criterion, namely, ‘quality’ or ‘usefulness’. They describe how, under the criteria of novelty, drawing random lines on a piece of paper could be deemed creative as ‘in all likelihood, no one has ever before created quite the same pattern of lines’ (p. 467). At the same time, it is not a given that random lines on a piece of paper would necessarily pass the test of quality or usefulness. Sternberg and Kaufman assert that this is a definitional *constraint*, if only because novelty is ‘in the eye of the beholder’. In general, there are many constraints on creativity, something which some see as an impediment for creativity. The issue for us in this chapter is the role and impact of constraints in creativity and problem-solving. It is to this issue that we turn next.

13.2.3 Constraints in Creativity and Problem-Solving

Sternberg and Kaufman (2010) contend that the concept of creativity is subject to many constraints. These constraints are, in a sense, already embedded in the definition of creativity itself – work can be creative only if it is both novel and useful in some way. Apart from the constraints of being novel and useful in some way, there also are constraints internal to the problem-solver such as risk-taking or motivation. Society also imposes external constraints. Such internal and external constraints interact. Sternberg and Kaufman (2010, p. 481) give the following example: ‘someone who is constantly beaten down as a result of being creative may give up and simply decide not to be creative again’.

This chapter runs with the idea that constraints do not necessarily harm creative potential. As constraints are part of the construct of creativity itself, we rather see them as a mechanism by which creativity can be stimulated. Here, the words of Sternberg and Kaufman (2010) are pertinent: ‘Many consider the haiku [a traditional form of Japanese poetry that consists of three lines with the first and last lines having five syllables each and the middle line seven syllables] to be an ultimate creative expression precisely because only a handful of words is allowed. What makes a person or product creative is the flair of originality constrained by usefulness, and the benefit of usefulness constrained by originality’ (p. 481).

There are numerous examples where constraints have played a positive role regarding creativity. In an informal way, one can think of how in economics the ‘scarcity’ of certain goods pushes individuals and organisations to be creative (e.g. Legros, Newman, & Proto, 2014). The ‘MacGyver’ meme (humorous illustrations or pieces of text spread rapidly by Internet users) shows how television personality *MacGyver* (from the American television series of the same name) always manages to escape from a serious predicament by ‘just using a paperclip’; the message being that MacGyver’s inventive character can always find a creative solution for any challenge. Yet Amabile and Kramer (2011a & b) would likely not call this ‘scarcity’; rather, they would call it ‘necessity’ because, they say, constraints can ‘stoke the innovation fire’ (Amabile & Kramer, 2011b). A similar view was expressed by Yahoo! CEO Marissa Mayer (2006) writing for *Businessweek*:

'Constraints shape and focus problems, and provide clear challenges to overcome as well as inspiration. Creativity loves constraints, but they must be balanced with a healthy disregard for the impossible'.

Such views are not only apparent in the entertainment and business realm. A study by Marquc, Förster, and Van Kleef, (2011) showed that tough obstacles can prompt people to open their minds, look at the 'big picture' and make connections between things that are not obviously connected. Participants in the study were asked to play a computer maze game. For one group the maze contained an obstacle, severely constraining the number of possible routes to escape. The other group had no obstacle. Using the *Remote Associates Test* as the creativity measure, the group *with* the obstacle performed 40% better. It was hypothesised that the constraint had forced members of the obstacle group into a more creative mindset. Stokes (2001) reports that related experimental research suggests two things. One is that along with learning how to do something, people learn how to do it in different and variable ways, so they can continue doing it. A second thing Stokes shows is that constraints play a role because high variability is caused by constraints. Stokes gives the example of Claude Monet's painting: his high level of variability in painting was acquired during the first part of his life and was maintained throughout his adult career by a continuous series of task constraints imposed by the artist on his own work.

In her book, Stokes (2005) lists different types of constraints. The first set of constraints is domain constraints. Individuals in any field can only be creative if they first acquire expertise in the field. Acquiring expertise requires some agreed-upon performance criteria of a field, criteria that are seen as goal, subject and task constraints. Goal constraints specify a particular style, subject constraints involve content and task constraints refer to the particular materials that are used in a domain. It is seen as the foundation upon which variations can be produced. A second set of constraints concerns cognitive constraints. These pertain to the limitations of the human mind. It is here where developing expertise is relevant as this involves overcoming cognitive limitations. A third set of constraints is about variability constraints. These specify how differently something must or should be done. This suggests that maintaining a high-variability level is likely to be conducive for divergent thinking. Finally, the fourth set involves talent constraints. A domain might require special talents and capabilities. If someone does not have such talents and capabilities, this is likely to constrain their capacity to achieve, whereas if someone does have the talents and capabilities, this is likely to help.

Rosso (2014) notes there is a paradox in the tension between freedom and constraint in the creative process. Although, in the abstract sense, there is the view that the ideal creative process is unstructured, open-ended and free of external limitations, there is evidence that, in some cases, creative individuals and teams can benefit from constraints. Here we have space to elaborate on only some of the many different types, with one of the most researched ones being *time constraints*. Typically, the presence of deadlines has been seen as a negative influence on creativity as it is not considered conducive to exploration and tends to reinforce fixed ways of working (Amabile, 1996). Nevertheless, there also are indications to the contrary that suggest that time constraints can have a positive impact on creativity, subject to conditions (e.g. Amabile, Hadley, & Kramer, 2002; Baer & Oldham, 2006). One prerequisite though, according to Hennessey and Amabile (2010), is that creators are protected from distractions and the feeling that they are on a mission. The same ambiguous findings are reported for *resource constraints*. While sufficient material resources need to be available to be maximally creative (e.g. Amabile, 1988, 1996), researchers such as Csikszentmihalyi (1997, p. 321) warn that excess resources can also make people too comfortable and that this has a 'deadening' effect on creativity. Another resource aspect that has been studied is the impact of standardised routines and processes. Here again some research indicates that standardised routines and processes might hurt creative efforts, while other studies – especially in team work – indicate that standardised processes can be conducive to creativity. Groups might structure or bound their work in ways that enhance their creativity (e.g. Hargadon & Sutton, 1996; Stokes, 2005).

In sum, empirical research on the impact of constraints on creativity suggests that constraints do not necessarily impede creativity. In fact, subject to conditions, constraints may even be conducive to creativity. In the next sections, we give concrete examples for mathematics education of ways in which this might work. For these examples, we adopt the view, adapting Sternberg, that creative mathematical thinking is a combination of fluency, flexibility, originality, elaboration and usefulness.

13.3 Cases Illustrating Task Constraints

13.3.1 Case Example 1: Restricting Geometry Tools and Operations

Our first case example is about restricting geometry tools and operations. There are three ancient geometry problems that are considered to be extremely influential not only in the development of geometry but also on mathematics as a field (Coxeter & Greitzer, 1967; Sibley, 2015). These problems, each of which could be tackled using only a compass and straightedge, are the following: squaring the circle, doubling the cube and trisecting an angle.

The problem of 'squaring the circle' was to construct a square with the same area as a given circle; see Fig. 13.1 for an illustration. While the ancient Greeks did find exact constructions for this problem (and the other problems listed above) using methods beyond a straightedge and compass (it is less clear that in ancient times such constraints were imposed), nowadays, these classic problems such as 'squaring the circle' are formulated as problems which have to be solved using solely a compass and straightedge (for more on these classic problems from a computer-orientated position, see Meskens & Tytgat, 2017).

In tackling geometrical construction problems solely with compass and straightedge, the argument is that this provides a rich environment for problem-solving

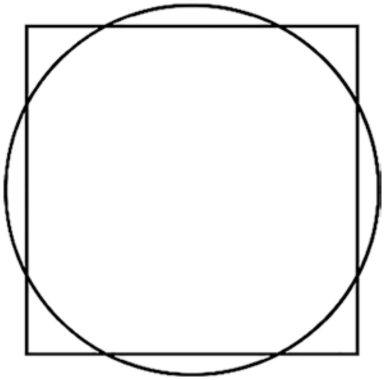


Fig 13.1 'Squaring the circle'; construct a square with the same area as a given circle using only a compass and straightedge

because 'the restriction to using compass and straightedge only, 'forces' the solver to exercise higher-order thinking skills such as analysis, evaluation, hypothesising, [and] organising' (Lim, 1997, p. 144). This remains the case even though in the nineteenth century it was proved that the three constructions listed above are actually impossible to solve with only a compass and unmarked straightedge.

In contemporary mathematics curricula, it is ruler (a marked straightedge) and compass constructions that continue to be specified for classroom teaching. So-called 'standard' constructions include constructing the perpendicular bisector of a line segment, constructing a perpendicular to a given line from/at a given point and bisecting a given angle. This continues to be the case even though forms of dynamic geometry software (DGS) (such as Cabri, Cinderella, GeoGebra, Geometer's Sketchpad and others) have been available since the mid-1990s. In such software, there is the decision of the designer about which menu items to provide in the software (such as a perpendicular line or a circle through three points), and then there is the decision of the task designer (or the teacher) about whether or not to constrain the choice of menu items that is available to learners (Jones, Mackrell, & Stevenson, 2010; Mackrell, 2011).

For example, Mariotti (2001) reports on a research project where the decision was made to start the teaching with the software having a menu of tools that was empty. Menu items were subsequently introduced only after having been the subject of classroom discussion. Mariotti gives the example of the menu item giving a perpendicular line. This, says Mariotti, 'incorporates the piece of theory concerning the definition of perpendicularity and the theorem validating the existence of a perpendicular line', which 'may too complex [for learners] to grasp all at once' (p. 715). The approach of beginning with an empty menu, and only adding menu tools when these had been explored and understood, is one way of designing geometrical construction tasks that have constraints on what learners do. The reason for having constraints, as Kortenkamp and Dohrmann (2010, p. 59) describe, is the tensions within problem-solving: 'if students learn to combine several steps into a more complex operation (say, instead of using the compass twice and connecting the intersections to find the midpoint between two points, one can use the midpoint operation provided by the software) this reduces their workload when doing constructions, but it increases the complexity of their software use'.

The issue of constraints in geometry problems is not restricted to DGS. Figure 13.2 provides an example of a project created using the software *Desmos*, while Fig. 13.3 shows a task from our enGasia project (http://engasia.soton.ac.uk). In the latter, the number of 'stars' indicates the number of different proofs that are valid. Returning to the issue of the provision of menu items in DGS, there are parallels that can be drawn between this issue and the notions of procedures and sub-procedures in the 'turtle geometry' microworld available in *Logo* (for more on procedures, and sub-procedures, in *Logo*, see Papert, 1972).

In each of the examples in this section, tasks are set in a constrained domain, with constrained operations. Nevertheless, in each case there are multiple ways to approach and solve the tasks. The latter is the space for creativity.

13.3.2 Case Example 2: Operations and 'Countdown'

Our second case example focuses on how elements of a popular TV game show illustrate elements of constraints in creativity and also how similar elements are apparent in two online digital applications (the WisWeb application 'number factory' and a digital book from the MC-squared project). In addition, for the latter application, we show how the constraints can be operationalised for creativity.

Countdown is a British game show involving word and number puzzles that is based on the French game show *Des chiffres et des lettres* (Numbers and Letters), created by Armand Jammot. The game show has a so-called numbers round in which a contestant is asked to select 6 of 24 shuffled tiles. There are two groups of numbers: one group with four 'large numbers' (25, 50, 75 and 100) and one group with 'small numbers' with two of each of the numbers 1–10. The contestant first

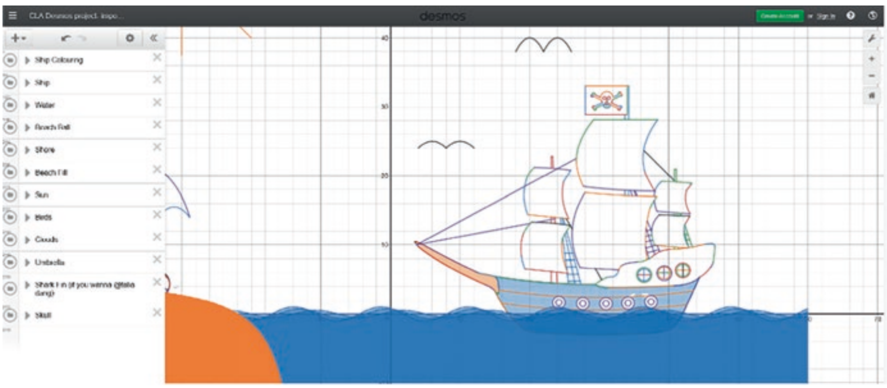


Fig. 13.2 An example of a project created with *Desmos* (and used with permission); see https://www.desmos.com/calculator/umgyltokag

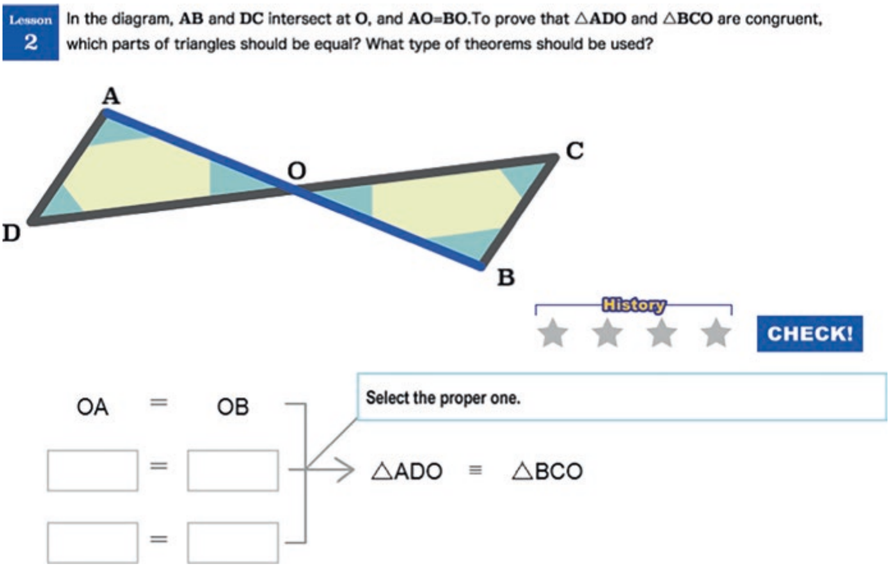


Fig. 13.3 An example of a task from the enGasia project (http://engasia.soton.ac.uk/); used with permission

chooses how many large numbers he or she wants, ranging from zero to four; the rest will be a small number. Then a random three-digit target is generated, and the contestant is asked to find a calculation with the six numbers that gets as close as possible to the target number using addition, subtraction, multiplication and division. Not all numbers need to be used; fractions are not allowed, only positive integers. In some games, there are multiple solutions; other games are unsolvable. This aspect demonstrates how constraints can lead to creativity, after all there are severe constraints regarding numbers, operations and time (in the game show a contestant has 30 s to come up with an answer). Yet this does not hold back the possibility of creative answers, as this actual example of a solution demonstrates (for the TV episode, see https://www.youtube.com/watch?v=pfa3MHLLSWI):

Numbers given: 25 50 75,100 3 6
Target: 952

The contestant came up with:
100 + 6 = 106
3*106 = 318
318*75 = 23,850
23,850–50 = 23,800
23,800/25 = 952

In the 'numbers round' game, Alliot (2015) shows the intricacies by providing a complexity analysis of the game, an analysis of solution algorithms and the presentation of a new algorithm that increases resolution speed by a factor of 20.

The core idea behind the numbers round is also the basis for a WisWeb application of 'number factory', which has been used in Dutch mathematics education. As with *Countdown*, there are numbers and operations, and a student has to make the target number.

Figures 13.4 and 13.5 show screenshots of a digital book, a so-called c-book, from the MC-squared project (www.mc2-project.eu). Figure 13.5 shows a task that

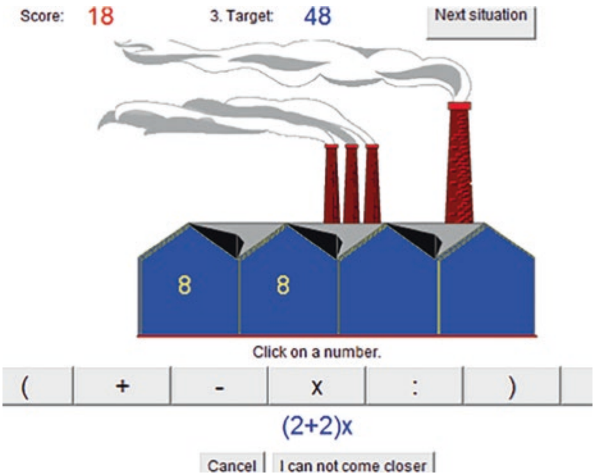


Fig. 13.4 Screenshot of a digital book ('c-book') with the 'number factory' application. (From WisWeb, www.wisweb.nl and the MC-squared project; used with permission)

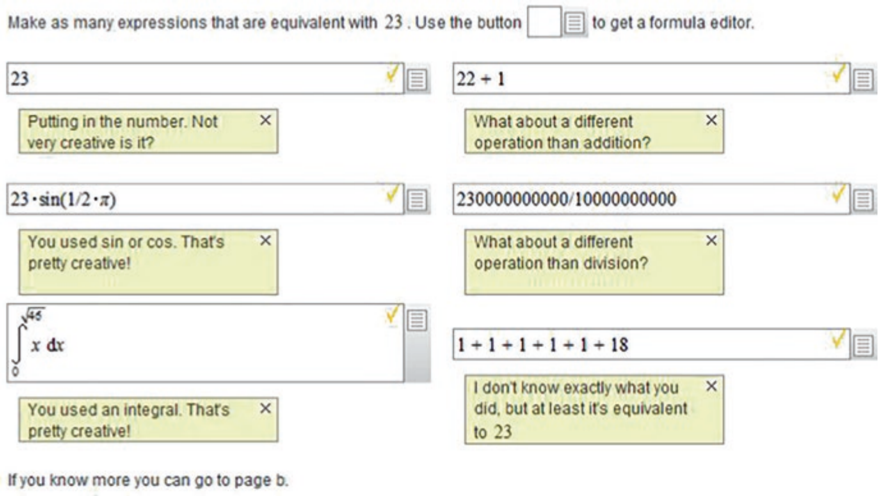


Fig. 13.5 Screenshot of a digital book ('c-book'). (From the MC-squared project; used with permission)

is proposed, in this case ‘make 23’, along with prompts and responses from the c-book system. The chosen number can be randomised, so it shows a different number, or even expression, every time a student visits the page. The student is presented with several boxes and asked to give as many expressions that are equivalent to the target number. The example answers, not from a real student, by the way, but proof of concept, show that the solution strategies can be more varied. This is the case because there are fewer constraints on the operations. The figure shows answers with numbers, basic operations, trigonometric functions and even integrals. If desired, the generation of these answers could also be subjected to time constraints. In this particular case, feedback can be provided as well, to further stimulate students to think more widely than just the current answers. In our view, this again gives an example of how constraints might be used to trigger creativity. A decent amount of ‘closedness’ (and a balancing amount of openness) seems helpful.

The fact that creativity is there can also be seen by applying the creativity criteria described previously: fluency, flexibility, originality/novelty and elaboration. A lot of fluency (i.e. the student’s capacity to pose or come up with as many responses as possible) is required. The whole point of the key activity in the c-book is to provide as many equivalent expressions to a given expression. With regard to flexibility (i.e. the student’s capacity to vary the approach or suggest a variety of different methods towards a problem or situation), the task provides the opportunity for students to provide many alternative answers. This is demonstrated in the figure as well: integral signs, trigonometric functions and many solutions are acceptable. Of course, this does not mean that students necessarily do this, and this is where feedback could help to trigger more creative thinking. This aspect also touches on originality/novelty, as the student is allowed to try novel or unusual solutions (in this case, equivalent expressions) towards a problem or situation. Elaboration, in this task, has not been catered for yet but could be addressed by making variations of this page with a progression from simple numbers towards more generalised algebraic expressions.

13.3.3 Case Example 3: Diophantine Equations

Our third case example of creativity through constraints is the mathematics game *Wuzzit Trouble* from the company *InnerTube Games*, a venture initiated by Keith Devlin (from Stanford University and known as *The Math Guy* on US National Public Radio). With this game, Devlin builds on work from his book *Mathematics Education for a New Era* (Devlin, 2011). The free game follows the structure that is well-known from many modern ‘apps’ such as *Angry Birds*: there are levels (75 in total in the case of *Wuzzit Trouble*) that require the user to solve a puzzle. While the storyline is about Wuzzits, ‘cute characters’ that have to be saved, the better the user solves each puzzle (in the case of *Wuzzit Trouble*, this relates to the minimum number of moves), the more ‘stars’ are earned. The differing numbers of stars to be collected can be seen as a way to encourage multiple solution strategies.

Fig. 13.6 The mathematics game Wuzzit Trouble. (From the company InnerTube Games; used with permission)

The levels themselves consist of target numbers that need to be constructed by turning a cog. In the case shown in Fig. 13.6, the first target number is 5. The cog can be turned to the left and to the right. The keys have to be collected by making the numbers by turning the cog.

The topic of the game, through using cogs, entails integer partitions (though not, of course, in the formal sense). Actually the game relates to Diophantine equations, polynomial equations of the form that seek whole number solutions – the most famous example being the problem in Fermat's Last Theorem (does $a^n + b^n = c^n$ have any integer solutions for $n>2$?). Players are unlikely to feel that they are solving these equations; rather it is a puzzle. The tension is immediately apparent: there is a constrained domain, namely, integer partitions, with constrained operations, namely, turning cogs. Nevertheless, there are multiple ways to solve the tasks.

13.4 Discussion

In this chapter we examined a number of mathematical problem-solving scenarios, ranging from producing geometrical constructions to ways of tackling number puzzles and solving sets of equations. Through our analysis of these forms of tasks, we argue that classroom tasks for mathematical problem-solving and creativity need not be restricted solely to open-ended problems but that tasks with suitable constraints can serve as creativity-inducing problem-solving tasks as well.

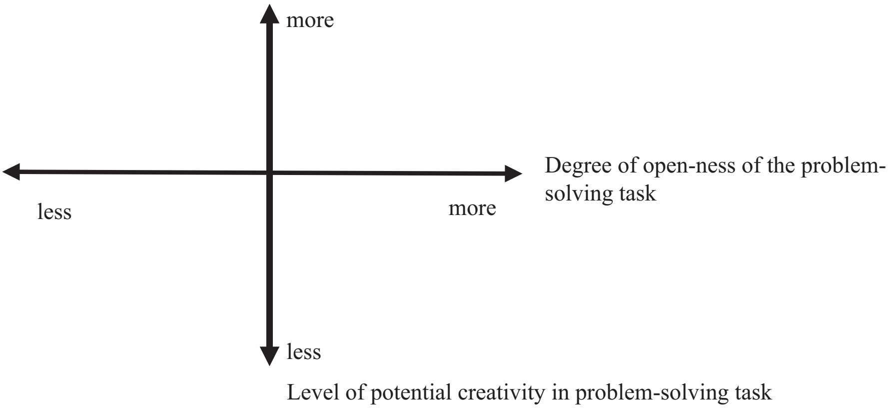


Fig. 13.7 Task openness and level of potential creativity

Notwithstanding the categorisation of Sullivan et al. (2012, p. 57) that a task 'goal' (taken as 'the result that students seek as a product of their activity in response to the task statement') can be open or closed, perhaps tasks can be positioned on a two-way grid, as shown in Fig. 13.7, in terms of the degree of 'openness' (on one axis) and the level of potential creativity in problem-solving task (on the other axis). This entails a more nuanced analysis of task design than an 'open' versus 'closed' opposition, or an opposition of creativity versus problem-solving, or creative versus noncreative. This would likely imply something more sophisticated than 'more open is more creative', or 'less open (so more closed) is less creative'. Of course it is not necessarily the case that tasks are more or less open; it might be preferable to talk about being less or more creative with less or more open tasks, in other words, moderately closed (or moderately open) tasks. As such, it may not be a case of placing the tasks discussed above at specific locations on the graph in Fig. 13.7; it may be more that the axes in Fig. 13.7 could be helpful in considering learner activities with any one of the tasks (and with other moderately closed, or moderately open, tasks).

This approach of relating task openness and level of potential creativity aligns with Lubart and Mouchiroud (2003, p. 142) who show how 'all problem-solving is not creative problem-solving' and that 'there exists a continuum' between the two extremes of open-closed in that 'some problem-solving relies heavily on existing [known] procedures but [also] requires some novelty, some enhancements to existing structure'. Mason (1991, p. 14) suggested that task qualities such as interesting or relevant, open or closed, belong 'not to questions, but arise only in the presence of people', while Guilford (1967) proposed that 'real' problem-solving involved actively seeking and constructing new ideas that fit with constraints imposed by a task or more generally by the environment. Nevertheless, there are also dangers; Wu (1994, p. 122-123), for example, argued that while the use of open-ended problems started off as 'a well-intentioned pedagogical device', the issue of mathematical substance 'got lost somewhere' such that there is a 'very real possibility of [open-ended problems] being an educational liability instead'. Another parallel is with the notion of 'moderately challenging' tasks developed by Carreira, Jones, Amado, Jacinto, and Nobre, (2016) in their project on mathematical problem-solving with technology. Also related is the proposal of Yeo (2017) to characterise the 'openness' of tasks.

All this raises a question regarding the way in which 'creativity' is often measured using the pioneering work of Torrance (1962) and the notions of fluency, flexibility, originality and elaboration. Our reconceptualization of creativity through constraints might indicate that such a measure of creativity may need some revision. Here the work of Ohlsson (2011) might be useful as he identifies four issues that he suggests need to be addressed by a successful theory of creativity: how are novel ideas possible; what are the key features that distinguish creative processes and justify calling them creative; and what gives direction to the creative process, what are the limiting factors, and why is it difficult to create.

13.5 Conclusion

In this chapter we have subjected to critical scrutiny the idea that problem-solving and creativity are seen as best supported by providing open-ended tasks. In doing so we have made the case for what might be called 'constraints-based' task design. In this latter approach, which we have related to research in economics on scarcity (and the American television series character *MacGyver)*, we have examined how tasks that are moderately closed (being neither fully open nor fully closed) can provide for creative mathematical thinking and problem-solving. In some cases, the use of feedback can provide cues to students.

Based on a range of examples from across a number of mathematical topics, our argument is that such examples demonstrate that classroom tasks for mathematical problem-solving and creativity need not be restricted solely to open-ended problems; rather, we argue that tasks with suitable constraints can serve as creativity-inducing problem-solving tasks as well. As Sullivan et al. (2012, p.14) argue, this entails classroom tasks that 'provide appropriate contexts and complexity; that stimulate construction of cognitive networks, thinking, creativity, and reflection; and that address significant mathematical topics explicitly'. As we have shown in this chapter, based on our experience of designing task on the MC-squared project (www.mc2-project.eu), and on other projects such as enGasia (engasia.soton.ac.uk), moderately closed (or moderately open) tasks meet these requirements and can serve as creativity-inducing problem-solving tasks.

Authors' Note and Acknowledgements The authors contributed equally; our names are in alphabetic order by family name.

We acknowledge and thank our many colleagues in the MC-squared project (www.mc2-project.eu) and the enGasia project (engasia.soton.ac.uk). The research leading to these results received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement n° 610,467 for the project MC-squared, and the enGasia project gained support from British Academy's International Partnership and Mobility Scheme. The views expressed are entirely the authors. Figs. 13.1 and 13.7 are the work of the authors; permissions were granted to use the other five figures.

References

Alliot, J-M. (2015). *The (Final) countdown*. Available on arXiv, http://arxiv.org/abs/1502.05450

Amabile, T. M. (1988). From individual creativity to organizational innovation. In K. Gronhaug & G. Kaufmann (Eds.), *Innovation: A cross-disciplinary perspective*. Oslo: Norwegian University Press.

Amabile, T. M. (1996). *Creativity in context*. Boulder, CO: Westview Press.

Amabile, T. M., & Kramer, S. J. (2011a). *The progress principle: Using small wins to ignite joy, engagement, and creativity at work*. Harvard: Harvard Business Review Press.

Amabile, T. M., & Kramer, S. J. (2011b). Necessity, not scarcity, is the mother of invention. *Harvard Business Review Blog, March, 25*, 2011. https://hbr.org/2011/03/necessity-not-scarcity-is-the

Amabile, T. M., Hadley, C. N., & Kramer, S. J. (2002). Creativity under the gun. *Harvard Business Review, 80*, 52–63.

Baer, M., & Oldham, G. R. (2006). The curvilinear relation between experienced creative time pressure and creativity: Moderating effects of openness to experience and support for creativity. *Journal of Applied Psychology, 91*(4), 963–970. http://dx.doi.org/10.1037/0021-9010.91.4.963

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education, 29*(1), 41–62.

Boden, M. A. (1994). What is creativity? In M. A. Boden (Ed.), *Dimensions in creativity* (pp. 75–118). Cambridge, MA: MIT Press.

Bokhove, C., Jones, K., Mavrikis, M., Geraniou, E., & Charlton, P. (2014). Co-designing electronic books: boundary objects for social creativity. In K. Jones, C. Bokhove, G. Howson, & L. Fan (Eds.), *Proceedings of the International Conference on Mathematics Textbook Research and Development (ICMT-2014)* (pp. 167–172). Southampton, GB: University of Southampton.

Bokhove, C., Mavrikis, M., & Jones, K. (2015). The potential of authoring creative electronic mathematics books in the MC-squared project. In N. Amado & S. Carreira (Eds.), *Proceedings of the 12th International Conference on Technology in Mathematics Teaching* (pp. 513–515). Faro: University of Algarve.

Bonotto, C., & Dal Santo, L. (2015). On the relationship between problem posing, problem solving and creativity in the primary school. In F. Singer, N. F. Ellerton, & J. Cai (Eds.), *Mathematical problem posing* (pp. 103–124). New York: Springer.

Borasi, R. (1986). On the nature of problems. *Educational Studies in Mathematics, 17*(2), 125–141.

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematical problems with technology: The results and implications of the problem@web project*. New York: Springer.

Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 243–307). Dordrecht: Reidel.

Coxeter, H. S. M., & Greitzer, S. L. (1967). *Geometry revisited*. Washington: The Mathematical Association of America.

Cropley, A. (1999). Definitions of creativity. In M. Runco & S. Pritzker (Eds.), *Encyclopedia of creativity* (pp. 511–524). San Diego: Academic Press.

Csikszentmihalyi, M. (1997). *Creativity: Flow and the psychology of discovery and invention*. New York: Harper Collins.

Devlin, K. (2011). *Mathematics education for a new era: Video games as a medium for learning*. Natick, MA: A. K. Peters.

Gruber, H. E., & Wallace, D. B. (1999). The case study method and evolving systems approach for understanding unique creative people at work. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 93–115). Cambridge: Cambridge University Press.

Guilford, J. P. (1967). *The nature of human intelligence*. New York: McGraw-Hill.

Hadamard, J. (1945). *Essay on the psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hargadon, A., & Sutton, R.I. (1996). Brainstorming groups in context: Effectiveness in a product design firm. *Administrative Science Quarterly, 41*(4), 685–718.

Haught-Tromp, C., & Stokes, P. D. (2016). Constraints, competency and creativity in the classroom. In R. Leikin & B. Sriraman (Eds.), *Creativity and giftedness: Interdisciplinary perspectives from mathematics and beyond* (pp. 105–116). Rotterdam: Springer.

Hennessey, B. A., & Amabile, T. M. (2010). Creativity. *Annual Review of Psychology, 61*, 569–598.

Jones, K., Mackrell, K., & Stevenson, I. (2010). Designing digital technologies and learning activities for different geometries. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology: Rethinking the terrain: The 17th ICMI study* (pp. 47–60). New York: Springer.

Kaufman, J. C., & Sternberg, R. J. (Eds.). (2010). *Cambridge handbook of creativity*. Cambridge: Cambridge University Press.

Kaufman, J. C., Glăveanu, V. P., & Baer, J. (Eds.). (2017). *Cambridge handbook of creativity across different domains*. Cambridge: Cambridge University Press.

Kortenkamp, U., & Dohrmann, C. (2010). User interface design for dynamic geometry software. *Acta Didactica Napocensia, 3*(2), 59–66.

Kwon, O. N., Park, J. H., & Park, J. S. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. *Asia Pacific Education Review, 7*(1), 51–61.

Legros, P., Newman, A. F., & Proto, E. (2014). Smithian growth through creative organization. *Review of Economics and Statistics, 96*(5), 796–811.

Leikin, R., & Pitta-Pantazi, D. (2013). Creativity and mathematics education: The state of the art. *ZDM: International Journal on Mathematics Education, 45*(2), 159–166.

Liljedahl, P. (2013). Illumination: An affective experience? *ZDM: International Journal on Mathematics Education, 45*(2), 253–265.

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For the Learning of Mathematics, 26*(1), 17–19.

Lim, S. K. (1997). Compass constructions: A vehicle for promoting relational understanding and higher order thinking skills. *The Mathematics Educator, 2*(2), 138–147.

Lubart, T. I. (1999). Creativity across cultures. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 339–350). Cambridge: Cambridge University Press.

Lubart, T. I., & Mouchiroud, C. (2003). Creativity: A source of difficulty in problem solving. In J. Davidson & R. Sternberg (Eds.), *The psychology of problem solving* (pp. 127–148). Cambridge: Cambridge University Press.

Mackrell, K. (2011). Design decisions in interactive geometry software. *ZDM: International Journal on Mathematics Education, 43*(3), 373–387.

Mann, E. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30*(2), 236–260.

Marquc, J., Förster, J., & Van Kleef, G. A. (2011). Stepping back to see the big picture: When obstacles elicit global processing. *Journal of Personality and Social Psychology, 101*(5), 883–901.

Mariotti, M. A. (2001). Influence of technologies advances on students' mathematics learning. In L. English, M. G. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 695–723). Mahwah, NJ, USA: Lawrence Erlbaum.

Martindale, C. (1999). Biological bases of creativity. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 137–152). Cambridge: Cambridge University Press.

Mason, J. (1991). Mathematical problem solving: Open, closed and exploratory in the UK. *ZDM: International Reviews on Mathematical Education, 23*(1), 14–19.

Mayer, M. A. (2006). Creativity loves constraints. *Businessweek*.

Mayer, R. E. (1999). Fifty years of creativity research. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 449–460). Cambridge: Cambridge University Press.

Meskens, A., & Tytgat, P. (2017). *Exploring classical Greek construction problems with interactive geometry software*. Cham, CH: Birkhauser.

Ohlsson, S. (2011). *Deep learning: How the mind overrides experience*. Cambridge: Cambridge University Press.

Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. *International Journal of Mathematical Education in Science and Technology, 3*(3), 249–262.

Pehkonen, E. (1997). Use of open-ended problems in mathematics classroom. *Research Report* 176. Helsinki: University of Helsinki.

Rosso, B. D. (2014). Creativity and constraints: Exploring the role of constraints in the creative processes of research and development teams. *Organization Studies, 35*(4), 551–585.

Runco, M. A., & Albert, R. S. (Eds.). (1990). *Theories of creativity*. Newbury Park, CA: Sage.

Runco, M. A., & Pritzker, S. (Eds.). (1999). *Encyclopedia of creativity*. San Diego: Academic Press.

Sibley, T. Q. (2015). *Thinking geometrically: A survey of geometries*. Washington: The Mathematical Association of America.

Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *ZDM: International Reviews on Mathematical Education, 27*(2), 67–72.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM: International Reviews on Mathematical Education, 29*(3), 75–80.

Simonton, D. K. (2010). Creativity in highly eminent individuals. In J. C. Kaufman & R. J. Sternberg (Eds.), *The Cambridge handbook of creativity* (pp. 174–188). Cambridge: Cambridge University Press.

Simonton, D. K. (2013). What is a creative idea? Little-c versus big-C creativity. In K. Thomas & J. Chan (Eds.), *Handbook of research on creativity* (pp. 69–83). Cheltenham: Edward Elgar Publishing.

Simonton, D. K. (2017). Big-C versus little-c creativity: Definitions, implications, and inherent educational contradictions. In R. Beghetto & B. Sriraman (Eds.), *Creative contradictions in education* (pp. 3–19). New York: Springer.

Singer, F. M., Ellerton, N., & Cai, J. (2015). *Mathematical problem posing: From research to effective practice*. New York, NY: Springer.

Singer, F. M., & Voica, C. (2016). When mathematics meets real objects: How does creativity interact with expertise in problem solving and posing? In R. Leikin & B. Sriraman (Eds.), *Creativity and giftedness: Interdisciplinary perspectives* (pp. 75–103). New York: Springer.

Sriraman, B., Haavold, P., & Lee, K. (2014). Creativity in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 109–115). Berlin: Springer.

Sternberg, R. J. (1999). *Handbook of creativity*. Cambridge: Cambridge University Press.

Sternberg, R. J. (2003). The development of creativity as a decision-making process. In R. K. Sawyer, V. John-Steiner, S. Moran, R. J. Sternberg, D. H. Feldman, J. Nakamura, & M. Csikszentmihalyi (Eds.), *Creativity and development* (pp. 91–138). New York: Oxford University Press.

Sternberg, R. J., & Kaufman, J. C. (2010). Constraints on creativity: Obvious and not so obvious. In J. C. Kaufman & R. J. Sternberg (Eds.), *Cambridge handbook of creativity*. Cambridge: Cambridge University Press.

Sternberg, R. J., & Lubart, T. I. (1999). The concept of creativity: Prospects and paradigms. In R. J. Sternberg (Ed.), *Handbook of creativity*. Cambridge: University Press.

Stokes, P. D. (2001). Variability, constraints, and creativity: Shedding light on Claude Monet. *American Psychologist, 56*(4), 355–359.

Stokes, P. D. (2005). *Creativity from constraints: The psychology of breakthrough*. Berlin: Springer.

Stoyanova, E. (1998). Problem posing in mathematics classrooms. In A. McIntosh & N. Ellerton (Eds.), *Research in mathematics education: A contemporary perspective* (pp. 164–185). Edith Cowan University: MASTEC.

Sweller, J., Mawer, R. F., & Ward, M. R. (1983). Development of expertise in mathematical problem solving. *Journal of Experimental Psychology: General, 112*(4), 639–661.

Sullivan, P., Clarke, D., & Clarke, B. (2012). *Teaching with tasks for effective mathematics learning*. New York: Springer.

Torrance, E. P. (1962). *Guiding creative talent*. Englewood Cliffs, NJ: Prentice Hall.

Wu, H. (1994). The role of open-ended problems in mathematics education. *Journal of Mathematical Behavior, 13*(1), 115–128.

Yeo, J. B. (2017). Development of a framework to characterise the openness of mathematical tasks. *International Journal of Science and Mathematics Education, 15*(1), 175–191.

Chapter 14
Linking Mathematical Creativity to Problem Solving: Views from the Field

Deborah Moore-Russo and Erica L. Demler

14.1 Introduction

According to Pólya (1981), problem solving involves "some degree of independence, judgment, originality, [and] creativity" (p. xi). Bolden, Harries, and Newton (2010) propose that there are many creative opportunities for students in the mathematics classroom, citing several examples including how mathematics solutions for mathematical problems are generated and how mathematical situations, plans, and outcomes are hypothesized. This would suggest that the development of mathematical creativity in students should be a goal of mathematics education (Lev-Zamir & Leikin, 2011).

In fact, Silver (1997) suggests that "inquiry-oriented mathematics instruction which includes problem-solving and problem-posing tasks and activities can assist students to develop more creative approaches to mathematics" (p. 75). Problem posing is both the transformation or modification of given problems and the generation of new problems (Brown & Walter, 1993, 2005, 2014; Silver, 1994). It has been considered as a means to measure mathematical creativity (Cai, Hwang, Jiang, & Silber, 2015) and is also seen as a didactic tool that might be used to foster creativity. In addition, creativity is more apt to be cultivated in classrooms that use open problems (Pehkonen, 1997), whose solution paths are not immediately known but require more than mere algorithms and routines (Haylock, 1987), and are not limited but rather allow for creative exploration (Mann, 2006).

D. Moore-Russo
University of Oklahoma, Norman, OK, USA
e-mail: dmr@ou.edu

E. L. Demler (✉)
University at Buffalo, Buffalo, NY, USA
e-mail: ericadem@buffalo.edu

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_14

Aiken (1973) sees teachers as the keys to unlocking creativity in the classroom, as he believes creative teachers will produce creative students. Recent studies have confirmed that mathematics teachers also believe this to be true and consider themselves primarily responsible for enhancing creativity in their students (Bolden et al., 2010; Kattou, Kontoyianni, & Christou, 2009; Leikin, Subotnik, Pitta-Pantazi, Singer, & Pelczer, 2013). It is clear from the literature, however, that more than creative teaching is necessary to advance student creativity (Bolden et al., 2010; Starko, 2013). Starko (2013) delineates between "creative teaching" and "teaching to develop creativity" – what others have deemed respectively as teaching creatively and enhancing students' creativity (Leikin et al., 2013), creative teaching and creative learning (Bolden et al., 2010), and teacher-directed creativity and student-directed creativity (Lev-Zamir & Leikin, 2011, 2013).

Whereas creative teaching refers specifically to acts made by teachers, whether they be mathematical or pedagogical in nature (Lev-Zamir & Leikin, 2011), teaching to develop creativity requires a shift in focus to "essential creativity on the part of the students" (Starko, 2013, p. 20). One of the key responsibilities of mathematics teachers is to provide meaningful mathematical activities in the classroom (Fennema & Romberg, 1999). If creativity is central to genuine mathematical endeavor (Silver, 1997) and if it plays a role in problem solving, then teachers should be cultivating creativity in their classrooms. They should be helping students "pay attention to their wonderings… [and to] capture their ideas and build on them" (Starko, 2013, p. 144).

The practices teachers use in the classroom are primarily influenced by their conceptions or implicit theories (Sternberg, 1985) of creativity. It has been found that these implicit theories are used by teachers to identify and describe creativity both in themselves and in others (Runco & Bahleda, 1986; Sternberg, 1985). Kowalski (1997) argues that implicit theories are extremely important and should be considered concurrently with any initiative to enhance students' creativity. Hence, it is important to study teachers' views on creativity and its role in the classroom. Recent studies have examined teachers' conceptions of creativity (Bolden et al., 2010; Kampylis, Berki, & Saariluoma, 2009; Leikin et al., 2013; Lev-Zamir & Leikin, 2011, 2013) and have reported that preservice mathematics teachers in the United Kingdom have a limited understanding of creativity and how to encourage and assess it in the classroom (Bolden et al., 2010); less than a quarter of Greek teachers feel well-trained to facilitate students' creativity (Kampylis et al., 2009); certain factors of mathematical creativity are culturally dependent (i.e., the amount of attention devoted to creativity), while others seem to reach across cultures (i.e., the relationship between creativity and problem solving) (Leikin et al., 2013); teachers hold both teacher-directed and student-directed conceptions of creativity (Lev-Zamir & Leikin, 2011); and teachers who are more student-directed display a greater congruence between their declared and enacted conceptions of creativity (Lev-Zamir & Leikin, 2013).

Creativity is recognized in these studies (and many others) as being a very complex construct that is "widely used but vague" so researchers "face difficulties when required to put its meaning into words" (Kampylis et al., 2009, p. 16). Despite this recognition, each of the aforementioned studies settles on a single definition of creativity to rely on throughout the research. They examine creativity in the classroom through a single lens (e.g., considering certain components of creativity), which may or may not be a particularly good filter as it only captures a limited notion of creativity.

Our approach is unique in that we are viewing creativity through a variety of lenses in order to get a more complete representation of the complex nature of the construct. As such the purpose of this study was to examine mathematics instructors' conceptions of creativity through four different lenses, which are discussed below. The following research questions were explored:

1. How do mathematics educators define creativity, and which components of creativity do they identify?
2. How do they feel creativity should be cultivated in the mathematics classroom?
3. Who benefits from creativity-enriched instruction?

In the following sections, we consider the existing literature on these topics, both generally and more specifically, in relation to mathematical problem posing and solving. We then present our findings and discuss significant themes that emerged from the data while also considering the implications these results have for mathematics instruction.

14.2 Literature Review

14.2.1 Categories of Creativity Definitions

Definitions of creativity abound. Treffinger, Young, Selby, and Shepardson (2002) write, for instance, that there are more than 100 contemporary definitions of mathematical creativity. Nevertheless, Plucker and Beghetto (2004) note that there are two key features of creativity that are seen throughout the literature in general: originality and usefulness. Sriraman (2005), in an attempt to conceptualize mathematical creativity, writes that creativity is "the process that results in unusual and insightful solutions to a given problem, irrespective of the level of complexity." In other words, mathematical creativity can be defined as novel and useful work in mathematics. The specific meaning of "novel" and "useful" would depend on the context and on the user.

According to Bolden et al. (2010), "creativity is a personal activity intended to produce something new" (p. 143). Creativity is the ability to combine ideas, things, techniques, or approaches in a novel way (Romney, 1970) and to distinguish between acceptable and unacceptable patterns (Birkhoff, 1969). Plucker and Beghetto (2004) call it "the interplay between ability and process by which an individual or group produces an outcome or product that is both novel and useful as defined within some social context" (p. 156). Specifically in relation to mathematical thinking, Poincaré (2000) and Hadamard (1954) maintain that creativity is an intuitive "flash of insight," a subjective experience (Aiken, 1973). It is clear from these examples that there is a lack of consistency and precision in how creativity is defined in the literature, which Sriraman (2005) attributes to the inherent complexity of the construct itself.

In short, creativity may mean different things to different people. For example, there is evidence in the literature of varying degrees of creativity. Boden (2004) distinguishes between historical and psychological creativity, which are respectively linked to the absolute and relative creativity discussed by Leikin (2009). Absolute or historical creativity is more universal as it relates to profound discoveries and great historical works at the global level. Relative or psychological creativity, on the other hand, is more personal and refers to individual discoveries that are novel to a particular person. One of the most interesting publications on creativity is the recorded back-and-forth dialog between Liljedahl and Sriraman (2006) where the two discuss creativity and its many definitions. Like good art, creativity is something you recognize when you see but is difficult to define.

In looking at definitions of creativity, Getzels (1969) found that they fit in three categories. Creativity can be seen as a product, a process, or a subjective experience.

Creativity can be thought of as a *product*. The root word of creativity is crea, which comes from the Latin word creare meaning "to create." Creativity is therefore naturally linked to creation; something is produced that is unique or unexpected but also useful (MacKinnon, 1962; Plucker & Beghetto, 2004; Sternberg & Lubart, 1999).

Creativity is also a *process*. Many definitions characterize creativity as the ability to go against and/or beyond traditional thinking and accepted practices. Creativity is the process by which individuals engage in nonalgorithmic decision-making (Ervynck, 1991); it is the ability to discern patterns and find similarities and differences by analyzing a problem in a variety of ways (Birkhoff, 1969; Laycock, 1970). While this process often requires divergent thinking (Runco, 1999) to be fruitful, it is not completely unfounded. Rather, people move from the familiar to the unfamiliar by utilizing and combining knowledge they already have in unique and novel ways (Laycock, 1970; Romney, 1970).

Distinct from the notions of creativity as a product or a process stands the view that creativity is a *subjective experience*. Definitions in this category consider creativity to be a flash of insight, the "Aha!" or "Eureka!" moment where clarity is gained (Hadamard, 1954; Poincaré, 2000). This conception is associated with the "genius view" of creativity, where creative acts are considered to be rare mental feats achieved by extraordinary individuals, rapidly and effortlessly (Weisberg, 1988). Under this notion, creativity is seen as an innate ability that certain people possess while others simply do not. It follows that creativity-enriched instruction would only benefit select students and only insofar as it would help them to unlock and/or access this ability. Otherwise, instruction designed around the principles of the "genius view" is likely to have little or no effect on creativity. Although this view has dominated western culture's beliefs about creativity, Silver (1997) acknowledges that it has been challenged in recent research, giving rise to a new, less restrictive conception of creativity.

Recent research has argued that creative acts are actually the result of extended periods of mathematical activity and reflection based on the use of deep and flexible content knowledge (Gruber & Wallace, 1999; Holyoak & Thagard, 1996; Sternberg, 1988). This conception is more consistent with definitions that characterize creativity as a product or process, rather than a subjective experience, as it emphasizes prolonged engagement with a problem situation and reflection on the processes used and the products created. It also suggests that instruction can influence creativity. Haylock (1997) and Sriraman (2009) maintain that, in order for people to be creative, they need the ability to think outside the box and also an understanding of how to apply their deep and flexible knowledge of mathematics in seeing opportunities in problem situations. Teachers can use problem-posing and problem-solving activities in the classroom not only to demonstrate how to do this but also to provide students with opportunities to practice themselves. In stark contrast to the "genius view," this contemporary view of creativity supports the argument that creativity can be cultivated and nurtured and that all students – not merely a few exceptional individuals – can benefit from creativity-enriched instruction (Cropley, 2001; Silver, 1997; Smith, Ward, & Finke, 1995).

14.2.2 Components of Creativity

Even though definitions of creativity can differ considerably, there are some underlying themes that can be identified. One such theme apparent in each of the examples above is originality; creativity is about coming up with something new, about not thinking like everyone else. Originality, or novelty, is just one piece of the puzzle, though.

Silver (1997) discusses three components of creativity – fluency, flexibility, and novelty – and the implications they have for mathematics instruction. *Fluency* relates to the amount or quantity of ideas that are generated in response to a prompt and the continuity or flow of those ideas. *Flexibility* is being able to shift approaches, consider situations from different perspectives, and utilize multiple forms of representation when solving, or presenting a solution to, a problem. *Novelty* is concerned with the generation of new, original ideas.

Mathematical problem posing and solving are naturally connected to creativity. As Silver (1997) notes, "it is in this interplay of formulating, attempting to solve, reformulating, and eventually solving a problem that one sees creative activity" (p. 76). In fact, well-established means of assessing creativity, such as the Torrance Tests of Creative Thinking (TTCT) (Torrance, 1966, 1974), often rely on problem-posing and problem-solving activities to measure the three identified key aspects of creativity (fluency, flexibility, and novelty). While some may question the applicability and appropriateness of using problem-posing and problem-solving activities in the classroom (i.e., Sriraman, 2005), others argue that their use can actually help develop these components of creativity. Silver (1997) and Lev-Zamir and Leikin (2011) recognize that the use of ill-structured and open-ended problems by both teachers and students can help to build fluency and flexibility because these types of problems generally allow for multiple interpretations, methods, and solutions. For a given problem situation, students can first try to define as many different problems as possible (fluency). Then as students solve a particular problem, they can look for additional methods that could also be used to solve it (flexibility). Voica and Singer (2013) also found a connection between problem-posing activities and cognitive flexibility in above-average mathematical students. Working with ill-structured and open-ended problems should help students engage in exploration that may help them learn to appreciate (and even develop) novel, original ideas. Therefore, it can be argued that students should be exposed to an environment where teachers and students pose problems, problems are solved in multiple ways, solution ideas and paths are shared with peers, and originality is championed.

14.2.3 Principles for Maximizing Creativity

Based on his work with professional mathematicians, Sriraman (2005) identifies five overarching principles to maximize creativity and suggests ways to incorporate them in the K–12 mathematics classrooms. These principles are gestalt, aesthetic, free market, scholarly, and uncertainty.

The *gestalt principle* mainly refers to the work of Hadamard (1945) and Poincaré (2000), who believed there were four stages to mathematical creativity: (a) preparation or the introduction to the problem situation and formulation of problem; (b) incubation or the time to unconsciously think about the problem situation by letting the mind be occupied by other things; (c) illumination or a flash of insight, the Aha! moment (what Liljedahl (2013) related to as an affective experience); and (d) verification or the refining of a final solution. The gestalt principle relates to recognizing how the unconscious mind works on latent ideas (Haavold, 2016). When applied to the mathematics classroom, this conception of creativity calls for allowing students freedom of time and movement when working with problems (Sriraman, 2005). Students need to know that time, energy, and effort are needed to solve mathematical problems (consistent with the contemporary view of creativity as discussed above), and they need to be exposed to an environment where they are allowed to experience this.

The *aesthetic principle* is concerned with the beauty of mathematics. Although many people, including mathematics teachers, tend to associate creativity and beauty more with the arts than mathematics (Bolden et al., 2010), others contend that mathematical work has more in common with art than the sciences (Littlewood, 1986). Hardy (1940) exclaims:

> The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (p. 85)

Students should be able to recognize and appreciate that some solutions are more elegant than others and that there can exist a simple solution to a seemingly complex problem. One example from mathematics history that highlights this elegance and simplicity is Gauss' technique for adding all the natural numbers from 1 to 100. He paired the numbers up as sums equaling 101 (i.e., 1 + 100, 2 + 99, 3 + 98, and so on), recognized that 50 pairs would result, and so multiplied 101 by 50 to arrive at the answer of 5050. Sriraman (2005) argues that students should be exposed to notable examples from mathematics history such as this to help develop their appreciation of the beauty of mathematics and also to encourage them to look for patterns and simpler solutions in problems they are working on.

The *free market principle* derives from the environment in which professional mathematicians work, an environment where they are expected to develop and defend solutions to problems under scrutiny from their peers. This often requires mathematicians to engage in atypical thinking as they work on new, unsolved problems. They must also be willing to take the risk of presenting their atypical thinking and solutions to others in the field, who will surely take the time to verify such solutions. As it translates to the mathematics classroom, the free market principle states that students should be encouraged to take risks in presenting and defending their solutions – both in the classroom and in other venues, such as mathematics contests and competitions – and that they should be provided a safe environment in which to take those risks (Sriraman, 2005). Starko (2013) echoes these sentiments as she contends that developing a creativity-friendly classroom "entails creating a classroom atmosphere in which flexible thinking and seeking and solving problems are welcomed…and a safe space for risk taking" is provided (p. 139). Students are not likely to put themselves out there in front of their peers unless they are sure their unique contributions will be valued. Atypical thinking about problem situations should be encouraged and even modelled by the teacher, and students should not fear reprimands for trying a creative method and failing.

The *scholarly principle* is based on the notion that creative problems and solutions help push the field of mathematics forward by contributing to, challenging, and extending the body of mathematical knowledge, often through paradigm shifts (Juter & Sriraman, 2011). Students need to see that mathematical knowledge is continuously being constructed (as opposed to being stale) and that they can actually make valuable contributions to the classroom (instead of relying solely on the teacher and/or textbook as sources of knowledge) (Haavold, 2016). Alternative approaches, or "creative deviance," are needed to ensure that a field continues to progress. Creative deviance, as Sriraman (2005) refers to it, involves being flexible and open to using multiple approaches to solve a problem. Teachers can help cultivate creative deviance by allowing students to work on open-ended problems without any explicit instruction. They should also "nurture a classroom environment in which students are encouraged to debate and question the validity of both the teachers', as well as other students', approaches to problems" (Sriraman, 2005, p. 28).

The final principle identified by Sriraman (2005) to maximize creativity in the classroom is the *uncertainty principle*. This principle again derives from the work of professional mathematicians, who must confront uncertainty and ambiguity on a regular basis both when posing problems and also when solving them (Haavold, 2016). Creating knowledge is not an easy task and requires perseverance, and students need to understand that uncertainty and discomfort are part of being creative (Sternberg, 2003). In order to expose students to the difficulty and uncertainty inherent in mathematics, Sriraman (2005) suggests using examples from the history of mathematics where ideas evolved over extended periods of time and required contributions from multiple mathematicians. He also recommends utilizing ill-structured and open-ended problems that are messy and need definition or assumptions to be made before solutions can be considered. Recent research has shown that these practices do indeed extend creative thinking in both grade school students (Amit & Gilat, 2012) and prospective mathematics teachers (Shriki, 2010).

As previously mentioned, these five principles were derived from what professional mathematicians saw as invaluable characteristics of mathematical creativity (Sriraman, 2005). While Sriraman successfully translates each of them from the world of professional mathematicians to the K-12 mathematics classroom, little attention is given to the pedagogical concerns associated with promoting creativity in the classroom. Lev-Zamir and Leikin (2011) found that teachers' conceptions of creativity for mathematics instruction are comprised of two distinct facets – teacher-directed creativity and student-directed creativity – and that student-directed creativity is dependent upon the teacher's ability to be creative. Sriraman's (2005) five principles are all related to student-directed creativity in that they prescribe creative activities for the students to partake in; none of the principles refer to the creativity needed by the teacher in order to effectively implement those activities. Starko (2013) addresses this gap by more directly considering the role of the teacher in developing a creativity-friendly classroom.

14.2.4 Keys to a Creativity-Friendly Classroom

Teachers are aware that they play a key role in enhancing students' mathematical creativity (Bolden et al., 2010; Kattou et al., 2009; Leikin et al., 2013) but are often unsure of how to actually accomplish this (Bolden et al., 2010). Starko (2013) provides insight into three keys she believes teachers should use to create a classroom in which creativity can flourish: teaching the *general skills and attitudes* of creativity, teaching *discipline-specific methods* of creativity, and creating an *appropriate atmosphere*.

Teaching the *skills and attitudes* of creativity involves explicit instruction on creativity. This instruction can take several forms. Teachers should introduce students to creative individuals and discuss the nature of the creative processes they use (Starko, 2013). These individuals can be notable people from history who are deemed to have been creative, or they may be contemporary professionals who work in fields who are reliant on creative thinking. Teachers should also teach students about the strategies that can be used to spawn creative ideas. This requires teachers to be familiar with those strategies so they can effectively model them for the students.

The second key, teaching *discipline-specific methods* of creativity, is similar to but has a narrower focus than the first key (teaching the skills and attitudes of creativity). Here, the teacher is encouraged to teach students how individuals are creative in relation to a specific discipline, such as mathematics. Professional mathematicians are a great resource as they can talk about the uncertain nature of their work, the stages of problem posing and solving, the amount of time and energy they devote to a single problem, and the joy associated with creating new knowledge and pushing the field forward, all reminiscent of the five principles (Sriraman, 2005) discussed above. It is important to realize that mathematics teachers are also part of the mathematics field and are capable of displaying both mathematical flexibility and fluency to their students (Lev-Zamir & Leikin, 2011).

In order to fashion an *appropriate atmosphere* for creativity in the classroom, a teacher "uses varied and flexible teaching methods, provides experiences with choice, offers informational feedback in assessment, encourages self-assessment, uses rewards thoughtfully, teaches both cooperation and independence, and promotes questioning and experimentation" (Starko, 2013, p. 139). While some of these features were discussed above in relation to the *free market* and *scholarly principles* espoused by Sriraman (e.g., questioning and experimentation), Starko's description is more akin to teacher-directed creativity (Lev-Zamir & Leikin, 2011). Teachers must be flexible, both mathematically and pedagogically, in order to do mathematics in different ways and to transform instructional settings and content to meet the needs of the students. They must also be original in order to generate tasks beyond the textbook and to make the lessons interesting and enjoyable for students (Lev-Zamir & Leikin, 2011). In short, teachers must be creative themselves so as to enhance the creativity of their students.

14.3 Methodology

14.3.1 Participants

Data were collected from a convenience sample of 13 faculty and staff members associated with a gifted mathematics program in the United States. While the faculty and staff members work with mathematically gifted and talented middle and high school students (i.e., students typically aged 12–18) in this unique program, they all also work with, or have worked with, students in regular educational settings. All of the participants had successfully completed varying amounts of graduate coursework in either mathematics or mathematics education. Of the 13 participants, 6 held PhD and 5 held Masters degrees. All had taught mathematics at either the secondary or tertiary level prior to being employed by the program.

14.3.2 Setting

The gifted mathematics program (GMP) is a selective, after-school program that is housed on a university campus in the northeastern part of the United States. This 6-year program is designed for students in grades 7–12 (ages 12–18). Its curriculum is both accelerated (by corresponding to the state mathematics requirements but moving through them more rapidly than is usually done) and enriched (by introducing topics, such as set theory, not included in state requirements). In the final 2 years of the program, students take university mathematics (i.e., Calculus I, II, and III as well as Linear Algebra).

Each of the six levels of GMP includes students from the same age group. They are kept together and move through the program as a cohort. During the school year, GMP meets twice a week, for 2 h and 20 min each meeting. GMP replaces the required math courses at the students' home schools. Approximately 60 students enter in the first year of the program from over 30 schools in and around the metropolitan area where the university is located. In any year, there are about 250 students in GMP, and inevitably, there is some attrition as the student cohort progresses in the program, primarily due to competing interests as the students grow older and have more extracurricular options in their lives.

14.3.3 Data Collection and Coding

The data were collected via surveys. The surveys contained a series of intentionally open-ended items, which are displayed in Table 14.1. Participation in the survey was optional, and participants were given 3 weeks to submit either handwritten or word-processed responses. All responses were de-identified and input in a spreadsheet. In total, there were 13 participants and 65 responses (5 per person). Responses to the five items were used as the units of analysis.

Multiphase coding was based on four different schemes. The first scheme came from the three categories of definitions of creativity outlined by Getzels (1969) as a *product*, *process*, or *subjective experience*. Table 14.2 outlines the codes that were used. The second coding scheme examined the components of creativity offered by Silver (1997) for creativity as including *fluency*, *flexibility*, and *novelty*. The coding

Table 14.1 Data collection items

	Item
1.	What is creativity? What does creativity involve?
2.	What role does/should creativity play in mathematics and mathematics education?
3.	What role does/should creativity play in mathematical problem solving?
4.	How should creativity be promoted/facilitated in mathematical instruction?
5.	Is the promotion/facilitation of creativity something that is only applicable to certain students, or is it something that you see as valid for all mathematical learners?

Table 14.2 Coding phase one – Getzels's (1969) categories of definitions of creativity

Codes	Descriptions	Representative comments[a]
Creativity as		
Product	An outcome or result Something creative that is accomplished or created be it an artefact, commodity, idea, substance, or some other noun	*It's not enough to just try things that are weird, novel, unique – you eventually need to accomplish whatever you are trying to accomplish. There has to be some end game*
Process	A series of actions or steps Something that occurs during the creative "doing" whether that be creating, choosing, communicating, discerning, distinguishing, representing, thinking, or some other verb	*Creativity is a process that involves the ability of going beyond what is typically done or known pushing past commonly accepted ways of doing and thinking*
Subjective experience	A flash or burst of insight Something inspired that occurs in an "Aha!" moment	None

[a]Note that representative comments were also often assigned other codes (in the different phases) than the code for which they are provided as an illustration. Underlining was part of the coding, not part of the original comment; it relates to key elements in the comment to the code

Table 14.3 Coding phase two – Silver's (1997) components of creativity

Codes	Descriptions	Representative comments[a]
Creativity can include		
Fluency	Ability to generate a number of solutions or ideas. Purposed or obvious multiplicity in solving and posing problems	*This multiplicity in problem solving needs to help learners see that there are many ways to solve problems and there are many ways to communicate those solutions*
Flexibility	Ability to shift in approach or method used to solve a problem Purposed or obvious utilization of alternative mathematical representations when solving or posing problems	*In problem solving, creativity should play a role in trying … different ways to find a solution to a problem*
Novelty	Ability to create something that is original, unique, or unexpected Purposed or obvious uniqueness of methods and solutions when solving or posing problems	*Creativity involves thinking outside of the box and producing distinct ideas and unconventional solutions never before seen*

[a]Note that representative comments were also often assigned other codes (in the different phases) than the code for which they are provided as an illustration. Underlining was part of the coding, not part of the original comment; it relates to key elements in the comment to the code

criteria are described in Table 14.3. The third coding scheme was based on the principles that Sriraman (2005) suggested could be used to maximize creativity: *gestalt*, *aesthetic*, *free market*, *scholarly*, and *uncertainty*. In addition, a sixth principle for guiding creativity was eventually introduced: *instructional design*. Table 14.4 outlines the coding criteria that were used. Finally, the fourth coding scheme was based

Table 14.4 Coding phase three – Sriraman's (2005) principles to maximize creativity

Codes	Descriptions	Representative comments[a]
Creativity is maximized when teachers		
Gestalt principle	Allow students' freedom of time and movement; let students know that often considerable time, energy, and effort are needed to solve mathematical problems (and provide ample time and space for students to experience this)	*Creativity is often the result of persistence and collaboration. Students should be given a challenging problem then encouraged to "chew on" this mathematical problem for a while*
Aesthetic principle	Help students see the beauty of certain solutions; expect students to recognize, appreciate, and assess aesthetic appeal in mathematics; facilitate students' recognition and evaluation of elegance in mathematical ideas, theorems, and solutions	[*Creativity*] *allows students to see that there are different approaches to mathematics, some of which are more elegant (or even fun) than others – and sometimes this is a matter of personal preference*
Free market principle	Encourage students to take risks and provide a safe environment for them to do so in the classroom; promote other venues were students can take risks (e.g., mathematical contests); encourage students to engage in atypical thinking	*Create an environment where students feel free to take risks*
Scholarly principle	Help students understand creativity as contributing to, challenging, and extending the existing body of knowledge; expect students to be flexible and open to alternative approaches to problems since this can push mathematics, as a field, forward	*Creativity … is needed to extend the boundaries of what is known*
Uncertainty principle	Help students accept and grow comfortably with open-ended, messy, ill-defined problems; teach students to tolerate ambiguity and uncertainty in mathematical problems	*Creativity really flourishes when questions are open ended and left for interpretation by the reader*

(continued)

Table 14.4 (continued)

Codes	Descriptions	Representative comments[a]
Instructional design principle[b]	Design instruction so that creativity is purposely both: 1. Demonstrated and valued by the teacher and 2. Fostered during the teacher's interactions with students	*Teachers can … allow for problem posing in the classroom. By asking students to think about the problem in a slightly different way* (e.g., *using the what-if-not strategy), students can often start to view the original problem in a new light* *Creativity should be promoted by the types of questions that teachers/instructors ask of their students* *[M]ultiplicity …means more than a teacher showing her students three or four solutions to a problem. Teachers need to select examples that allow for different representations, with multiple solutions… [s]tudents might come to see that this is more than one way to tackle a problem (and share its solution). This is most easily done if problems are not "lock-step" but more open in nature.*

[a]Note that representative comments were also often assigned other codes (in the different phases) than the code for which they are provided as an illustration. Underlining was part of the coding, not part of the original comment; it relates to key elements in the comment to the code
[b]This code was added to the original five suggested by Sriraman (2005); additional representative comments are provided

on the keys to a creativity-friendly classroom outlined by Starko (2013): teaching *general skills and attitudes*, teaching *discipline-specific methods*, and creating an *appropriate atmosphere*. Table 14.5 outlines the codes that were assigned.

Two coders, the authors of this chapter, read and independently coded the first response using the three definitions of creativity. They then used the three components of creativity and the five original principles for maximizing creativity. Then on the last phase, they coded the keys to developing a creativity-friendly classroom. When there was 100% agreement on the first coding phases, the two coders decided to code the remaining phases together after reading each response at least three times and after discussing their ideas for the response. The coding team met six times and spent over 14 h assigning codes to the responses. During this process, it was determined that an additional code was needed for the principles for maximizing creativity. That is when the sixth code, *instructional design*, was added. The coding team then revisited the responses and assigned this code appropriately. Tables 14.2, 14.3, 14.4, and 14.5 provide direct quotations taken from the participants' responses that are representative of the codes assigned in each of the coding phases.

Table 14.5 Coding phase four – Starko's (2013) keys to a creativity-friendly classroom

Codes	Descriptions	Representative comments[a]
Creativity-friendly classrooms are developed by teachers'		
General skills and attitudes	Teaching: 1. Creativity in an explicit manner 2. The nature of the creative process 3. Strategies to generate creative ideas	*By asking students to think about the problem in a slightly different way* (e.g., *using the what-if-not strategy)*
Discipline-specific methods	Teaching about: 1. How others have been creative in mathematics (e.g., using historical examples in mathematics) 2. The kinds of questions that mathematicians ask and the methods they use to do mathematics 3. Obstacles to mathematical progress	*I feel that this perspective can be brought about by teaching the history of solving problems*
Appropriate atmosphere	Providing: 1. A safe space for risk taking 2. Organizational structures that support motivation 3. Experiences that involve student choice 4. Feedback and assessment while also encouraging self-assessment 5. An environment that promotes questioning and experimentation	*…the classroom needs to be a safe place where students are open to thinking outside the box*

[a]Note that representative comments were also often assigned other codes (in the different phases) than the code for which they are provided as an illustration. Underlining was part of the coding, not part of the original comment; it relates to key elements in the comment to the code

14.3.4 Data Analysis

All of the assigned codes were placed in a spreadsheet by person and by item. The data were then studied to see how many of the comments were assigned each code. Next, the data were studied to see how many of the participants were assigned a code in any of their responses to the five items.

14.4 Results

Our aim with this study is to offer a more complete picture of teachers' conceptions of mathematical creativity by using four coding schemes as lenses: Getzels' (1969) three categories of creativity definitions, Silver's (1997) three components of creativity, Sriraman's (2005) five principles for maximizing creativity, and Starko's (2013) three keys to a creativity-friendly classroom. The results of our multiphase

Table 14.6 Coding counts by participant and by response

Aspects of creativity	Participants ($n = 13$)	Responses ($n = 65$)
Getzels' (1969) definitions of creativity		
Process	13	39
Product	8	14
Experience	0	0
Silver's (1997) components of creativity		
Novelty	12	25
Flexibility	10	23
Fluency	7	9
Sriraman's (2005) expanded principles to maximize creativity		
Scholarly	10	28
Instructional design	9	19
Uncertainty	8	14
Free market	6	7
Gestalt	5	6
Aesthetic	3	5
Starko's (2013) keys to creativity-friendly classroom		
General skills and attitudes	7	16
Discipline-specific methods	6	10
Appropriate atmosphere	6	8

analysis, including the counts for each coding scheme, are provided in Table 14.6. We now will consider each of the three research questions that guided this study.

14.4.1 Definitions and Components of Creativity

Using Getzels' (1969) work as a lens, we found that all of the participants thought of creativity as a *process*, while only 8 of the 13 also thought of it as an end *product* or outcome. None of the participants provided evidence of seeing it as a subjective *experience* where rare mental feats occur in a burst of insight. In terms of the 65 responses, 39 were identified as participants adopting a *process* view, 14 a *product* view, and 0 an *experience* view of creativity.

When Silver's (1997) work was used as a lens, we noted that at least 7 of the 13 participants identified *fluency*, *flexibility*, and *novelty* as components of creativity. Participants were most likely to comment on the value of a unique or original solution method (i.e., *novelty* coded for 12 participants, 25 responses). They also commented on creativity as involving a variety or shift in the methods used to solve problems (i.e., *flexibility* coded for 10 participants, 23 responses) more than on the generative processes that can be part of problem solving and posing (i.e., *fluency* coded for 7 participants, 9 responses).

Within the responses that were coded as showing evidence of Silver's (1997) *novelty* component and Getzels' (1969) *process* view of creativity, it was noted that there was a general consensus among the participants that creativity involves moving from the known to the unknown and requires foundational knowledge or a certain level of scaffolding to occur. Responses from 8 of the 13 participants allude to this idea, which seemed heavily rooted in constructivism and related to a relative view of creativity; below are some representative excerpts with underlining that was not part of the original response but was added to highlight key elements in the comment:

> *For a student to be able to solve a problem in a* <u>unique or personal way</u> [novelty]*, it often means that they have fully synthesized the problem and are* <u>constructing a solution from previous knowledge</u> [process].
>
> *Creativity consists in …* <u>the application of previously learned systems of experiences</u> [process] <u>onto new unfamiliar</u> [novelty] <u>ones by way of analogy</u>.
>
> *Exposing students to creativity-enriched instruction in which they are challenged to come up with multiple problems and solutions and to persevere in looking for those solutions will help them think more adaptively and learn how to* <u>apply skills and knowledge acquired</u> [process] <u>in one setting to problems in another setting</u> [novelty].
>
> *Creativity is often perceived as the ability to see things differently, or perceive the world in* <u>new ways</u> [novelty]*, which in math is…* <u>applying existing theories to solve new, difficult problems</u> [process].

The participants who held a more absolute view of creativity, meaning creativity was seen as pushing the boundaries of the field, also showed evidence of Silver's (1997) *novelty* component of creativity and Getzels' (1969) *product* view of creativity in their responses. While the overall message in the response below suggests the absolute view of creativity, the individual components were related to *novelty* and *product* coding:

> *At this point, not every student is at the level where they can come up with a* <u>truly unique or creative</u> [novelty] <u>solution</u> [product]. *That doesn't mean that every student can't appreciate the ability for others to do so and strive to achieve that level of understanding.*

14.4.2 Cultivation of Creativity

Using Sriraman's (2005) work as a lens, we found that each of the five original principles to maximize creativity, as well as the added *instructional design* principle, was noted in the participants' responses. That said, three of the principles were not commonly coded. Participants were least likely to mention the elegance or beauty of solutions (i.e., *aesthetic* principle coded for only three participants and five responses), how students need space and time for creativity to blossom (i.e., *gestalt* principle coded for only five participants and six responses), or the need to provide an environment where students can take risks and defend their ideas (i.e., *free market* principle coded for only six participants and seven responses).

The other principles were all mentioned by over half of the participants and were noted in a greater number of responses. The most common (10 participants, 28 responses) was the *scholarly* principle where participants thought of creativity as pushing the boundaries of knowledge. Note, however, that a number of participants did mention that the extension of knowledge could be at the individual level and does not require engaging in pursuits that push the entire field of mathematics forward in order to be considered creative. The second most common (9 participants, 19 responses) was the *instructional design* principle where participants denoted the need for components of creativity (i.e., fluency, flexibility, and novelty) to be modelled by instructors and fostered in instructor-student interactions. The third most common (8 participants, 14 responses) was the *uncertainty* principle where students are taught to tolerate messy problems that are marked with ambiguity and uncertainty.

When Starko's (2013) work was used as a lens, we noted that about half of the participants were recorded as touching on all three keys to creativity-friendly classrooms. Topping the list was the teaching of *general skills and attitudes* of creativity where there is an overt emphasis on the nature of the creative process and general strategies are explicitly taught; this was evidenced in 7 participants and 16 responses. Teaching *discipline-specific methods* by considering how others have been creative in mathematics and what this looks like in terms of methods used and questions asked was found for six participants and ten responses. Providing an *appropriate atmosphere* where students have an environment that promotes questioning, where students feel safe to take risks, and in which student are allowed to make choices was noted for six participants and eight responses. It was noted that all of the responses except one coded as *general skills and attitudes* of creativity were also assigned the *instructional design* principle (Sriraman, 2005) code. Also, all of the responses coded as addressing the *free market* principle (Sriraman, 2005) were also coded as *appropriate atmosphere* code.

Some of the participants used vibrant examples and language to describe creativity and how it is cultivated through problem solving. Here are examples from three of the participants:

> *Play is that murky area wherein we feel ideas out in our heads, turning them around and trying to draw analogies to bring it into a world we understand, and also expanding the world of our understanding…teachers should be constantly challenging students to solve problems they haven't been given solutions to… being given the tools to solve a problem [rather] than being given directions…*
>
> *Creativity should be interwoven in the very fabric of problem solving. To push this analogy further, how can we expect mathematical artistry if we do not allow learners to view many types of fabrics with varying patterns and threads of different textures and colors? This multiplicity in problem solving needs to help learners see that there are many ways to solve problems and there are many ways to communicate those solutions. Often algebraic methods are often over emphasized in the classroom. In privileging the algebraic, other types of solutions methods (be they more numeric/combinatoric or more visual) are not being fostered.*
>
> *Students should be allowed to think outside of the box on a problem they are given before just being given an algorithm for solving it. They should be consistently told the mathematical playground has a strict set of rules, but should be allowed to think creatively about a problem within that playground as opposed to being told immediately how the playground works.*

Participants often couched their responses in terms of what not to do. They commented on what has traditionally been done in school settings that should be altered to promote creativity. There seems to be strong feelings about this held by many of the participants, as illustrated in the following quotes by five participants:

> *If we "show and tell" students, as often happens, how to do all problems then we take away [students'] ability to think critically and their ability to problem solve. In addition, this process of show and tell gives the students a false sense that there are no other ways to solve the same problem…*
>
> *Unfortunately, creativity does not play as much of a role in K-12 (or even university) mathematics, as it is typically taught, as it should. There is nothing that kills creativity quite like the formula-first, example-second loop that is all too common in mathematics instruction.*
>
> *… the kinds of problems we actually need to solve are not the kind we know directions for (if we did, they'd be solved already), but we do have a repertoire of tools and techniques that can be used to tackle many problems. Problem design needs to ride this line between too few tools and too much guidance. By designing and assigning problems which are solvable but novel to the student and require them to play, we give them the opportunity to draw their own analogies and gain a deep understanding and competence with material that is totally missed if we tell them how to solve a problem before giving them a chance to try.*
>
> *Creativity is usually associated with problem solving with either challenging problems or their solutions to them [that are] original and non-standard. Almost all classroom instruction aims at reducing the creativity necessary to solve problems, in effect necessarily making the problems pedestrian so that students can achieve on standard content exams.*

Other participants took a more positive approach, by talking about their own instruction and how they foster creativity, but in essence, they alluded to the same ideas as shown in the following excerpt:

> *It is a regular occurrence that students solve a problem in a different way than I solved it. On these occasions, I am sure to have students present their methods and discuss the merits of their method with the class.*
>
> *… students benefit because they get to see a side of mathematics that is often hidden in school – the "living" side where problems are posed and contemplated and the solutions are not obvious or easy to determine. Students have a tendency to give up on a problem or think they are incapable of "doing" math when they don't readily know the answer to a problem, or how to approach it, or if they don't understand the process that has been presented to them. Exposing students to creativity-enriched instruction in which they are challenged to come up with multiple problems and solutions and to persevere in looking for those solutions will help them think more adaptively…*

14.4.3 Audience for Creativity-Enriched Instruction

The final survey question posed to participants asked them to identify who in particular they felt would benefit from creativity-enriched instruction. Each of the 13 participants believed that all students should be exposed to instruction that fosters creativity. This is seen below in the responses from five of the participants:

> *All human beings are capable of being creative and promoting creativity….*

Creativity is something that should be cultivated and fostered in all learners. All levels of learners benefit from creativity.

Everyone can benefit from their [activities that foster creativity] use.

It [creativity-enriched instruction] should be available to all students.

Not every student is at the level where [he or she] can come up with a truly unique or creative solution that doesn't mean that every student can't appreciate the ability for others to do so and strive to achieve that level of understanding.

Recall that every participant had worked, or was still working, with regular education students, but at the time of the study, all were associated with a program that exclusively serves mathematically gifted students. Despite this connection, there were no explicit references in the responses to suggest that participants viewed creative acts solely as the work of gifted and talented students or that even hinted to a relationship between creativity and giftedness. This was somewhat surprising in light of the connections that are often made between creativity and giftedness (Hong & Aqui, 2004; Kattou, Kontoyianni, Pitta-Pantazi, Christou, 2013; Kontoyianni, Kattou, Pitta-Pantazi, & Christou, 2013; Leu, & Chiu, 2015; Sriraman, Haavold, & Lee, 2013) and the fact that creativity has been connected with high mathematical proficiency (Leikin, 2009; Sheffield, 2000).

However, as can be noted in the preceding excerpt, there was a good deal of variance evident in responses about whether participants thought that all students are capable of being creative. These ideas are denoted in the following excerpts from participants' responses:

I want to believe that it [creativity] is valid for all. Certainly, the ability to be creative will vary greatly among students.

In the most optimal classroom, all students should get to the point of creative problem solving.… Students should know the power of thinking creatively about a problem even if they never get to the point where they are able to do it themselves.

Some students really do have a higher aptitude for creativity… than others.

Students' aptitudes for creativity were often tied to whether a participant adopted a more relative or absolute (Boden, 2004; Leikin, 2009) view of creativity. Below are exemplars of responses where participants seemed to hold a relative view of creativity:

However, this is not just something that is applicable to the field of mathematics. The same case could be made at the individual level. Creativity does not mean that a learner is necessarily doing something in a way that no person has ever done it before. This learner could just be doing something in a way that is creative for him or her, in a way that extends past what he or she has ever seen or done before.

For a student to be able to solve a problem in a unique or personal way, it often means that they have fully synthesized the problem and are constructing a solution from previous knowledge. … Every student likes to be recognized when they come up a method or solution that the teacher hasn't yet taught or discussed.

If in the course of history of solving a particular problem, as a student, I realize that I would have proposed the same paths that yielded to breakthroughs at the time, I might feel incline to do more math.

Several participants took a more absolute stance, viewing creative acts as those that would push the field forward. These participants were often less optimistic that every student is capable of such creativity. However, their responses seemed to suggest that relative creativity is something that all students would be able to achieve. Some representative examples are shown below:

> *…creativity is the ability to generate completely new theories … My personal view is that … the existence of a solution already limits the degree of creativity that can be applied to a problem.*
>
> *All human beings are capable of being creative and promoting creativity if there exists a real desire to achieve something no one has yet!*

14.4.4 Limitations

Based on our findings and on the research of others (Boden, 2004; Leikin, 2009; Lev-Zamir & Leikin, 2011), it seems that instead of a single "thing" to be studied, there are actually two related constructs: absolute creativity and relative creativity. By using only a single word for creativity, it is hard to tease apart which of the two conceptions participants are actually espousing in their responses. In hindsight, we acknowledge that this study could have yielded more illuminating findings had the distinction between absolute creativity and relative creativity been made in the data collection instrument.

This study was very much contextualized in the gifted mathematics program from which its participants were drawn. To a certain extent, it serves as an initial point of departure for design-based research on creativity and the role and promotion of creativity in today's mathematics classrooms. As such, we are hopeful that this study might serve as a springboard for more iterative designs of future research, which will take into account a variety of overlapping lenses that can and should be used when considering creativity. We do not propose that this particular study puts the mathematics education community anywhere near to the stage of being able to develop domain theories, design frameworks, or even design methodologies (Edelson, 2002); rather, it serves as one small, initial step in that process.

The small, focused sample used for this study was not meant to be representative of the larger population of mathematics educators. While this might be deemed a limitation, the intention of the study was to glean more detailed information and to see if the exploration of creativity warranted the use of multiple phases of qualitative analysis through a variety of lenses. In this way, the study does contribute to the corpus of literature on creativity.

14.5 Summary

Kern (2010) reports that, on a survey conducted by IBM, 1500 chief executives identified creativity as the most important leadership competency. "It is evident that creative thinking skills, openness to change, flexibility and the ability to cope with challenging tasks are essential for integration in today's society and workplace" (Barak, 2009, p. 345). Since teachers are preparing the workforce and leaders of tomorrow, from a school-to-work perspective, the promotion of creativity in students should be desirable. Moreover, creativity and creative ways of problem solving are also of value in the study of higher-level mathematics (Sriraman, 2005). For these reasons, teachers and teacher educators should recognize that creativity-enriched instruction is important and deserves to be researched more aggressively. It is easy to note that teachers can set classroom norms where students come to expect messy problems that are marked with ambiguity and uncertainty or that teachers can ask students to engage in problem-posing activities in addition to problem-solving activities. Yet, such prescribed teacher actions are not quick remedies. Kaufman and Sternberg (2007) report that creating a learning environment that allows and encourages creativity "is sometimes seen as irrelevant to educational practice… [where] creative teachers and those who encourage creativity in the classroom often are accused of being idealists or missing the big picture" (p. 55). Creativity and problem-solving activities are often incorporated in the classroom only when time allows and often only for certain students. This is a culture that must change.

One strategy to encourage the regular and universal promotion of creativity in the mathematics classroom is to change teachers' conceptions of who exactly is capable of being creative. According to Haavold (2016), "a widespread belief among both teachers and students is that mathematics is created only by very prodigious and creative people; others just have to try to learn what is handed down" (p. 7). This belief ties in with both the absolute and the "genius" views of creativity. The genius view is often associated with definitions that classify creativity as an *experience* (Silver, 1997); those who see creativity as being merely a flash of insight or an Aha! moment tend to think that only select individuals are capable of being truly creative and pushing the boundaries of knowledge. While all of the participants in this study seem to believe that, to varying degrees, students should be capable of exhibiting signs of creativity, those who were less optimistic about it assumed a more absolute view of creativity. The genius view of creativity was challenged, as all 13 participants believe that every student is capable of engaging in and benefitting from creativity-enriched instruction. This finding is supported by the fact that none of the participants classified creativity as a subjective *experience*; instead, all perceived it to be a *process*. It is interesting to note, however, that only five participants cited the *gestalt* principle (i.e., freedom of time and movement) as being necessary for the promotion of creativity. Processes take time to complete, and teachers need to be willing to allow students time to work on solving and posing problems. "It is well known in the literature that creativity tasks as well as tasks linking mathematics to other disciplines typically require more time to be completed" (Leikin et al. 2013, p. 318).

There was considerable overlap between some of Sriraman's (2005) principles for maximizing creativity and Starko's (2013) keys to creativity-friendly classrooms. We noted that Starko's categories were broader than Sriraman's, and, while the two did have some commonalities, the lenses were different enough in focus that each brought value to the study. As discussed above, Sriraman's principles were tied to student-directed creativity (Lev-Zamir & Leikin, 2011), whereas Starko's (2013) work was geared more toward teacher-directed creativity (Lev-Zamir & Leikin, 2011). Moreover, the results of the current study reported in this chapter suggest that certain views of creativity might be linked to specific components of creativity (e.g., the absolute view and the *novelty* component) and the creative problem-solving and problem-posing activities teachers use. More investigation is needed to consider the definitions and components of creativity that teachers value in light of their absolute or relative views of the construct. In turn, all of this should be studied in relation to the assumptions teachers make and the methods they use to promote creativity in the classroom.

Creativity is a complex construct with a plethora of definitions that are often vague, marked by a variety of viewpoints that can be conflicting. Recent research on teachers' conceptions (Bolden et al., 2010; Kampylis et al., 2009; Leikin et al., 2013; Lev-Zamir & Leikin, 2011, 2013) tend to offer partial insights into mathematical creativity as each study relies on a single lens which filters all analysis. This might lead to an underdeveloped understanding of creativity and how those in mathematics education view it. Mann (2006) argues that the lack of a clear and consistent definition has impeded research efforts to comprehend mathematical creativity and its role in the classroom. We contend that trying to reduce such a multifaceted construct down to a single definition, and a single coding scheme, actually undermines attempts to grasp the true nature of creativity and how it can be cultivated.

In fact, this particular study could have benefitted from the inclusion of a fifth lens for analyzing creativity: achievement emotions. Pekrun (2006) defines achievement emotions as "emotions tied directly to achievement activities or achievement outcomes" (p. 317) and argues that positive control and value appraisals are essential for creative problem solving. In order for students to find enjoyment while engaging in activities that promote creativity, they must highly value that type of activity while also feeling like they have some control over it. Although consideration of the role emotion plays in creativity was not part of the initial study design, we recognize its importance and see it as being related to Starko's (2013) keys to creativity-friendly classrooms. In fact, upon further review of the data, there were instances when a few of our participants did reference emotional aspects of creativity and problem solving (i.e., value, *creativity is rewarding. People love to express their ideas. We could make use of this motivational tool better*; control, *we should want kids to be problem solvers and we shouldn't restrict the manner in which those problems are solved*). Perhaps the inclusion of additional items in the data collection instrument could have been worded to prompt more participant input related to achievement emotion. This would have yielded more data (from participant responses) tied to achievement emotions; future research could hone-in on this aspect of creativity.

The strength of this study is that, rather than reducing creativity down to a single dimension, it embraced the multifaceted construct that has been so elusive and undervalued in educational research. In fact, the multiple phases of coding and the variety of coding schemes allowed the research team to look through many different lenses and develop a more comprehensive understanding of how different views of creativity impact the way educators consider methods for fostering creativity in mathematics classrooms.

References

Aiken, L. R. (1973). Ability and creativity in mathematics. *Review of Educational Research, 43*(4), 405–432.

Amit, M., & Gilat, T. (2012). Reflecting upon ambiguous situations as a way of developing students' mathematical creativity. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 2–19). Taipei, Taiwan: PME.

Barak, M. (2009). Idea focusing versus idea generating: a course for teachers on inventive problem solving. *Innovations in Education and Teaching International, 46*(4), 345–356.

Birkhoff, G. (1969). Mathematics and psychology. *SIAM Review, 11*(4), 429–469.

Boden, M. A. (2004). *The creative mind: Myths and mechanisms*. London: Routledge.

Bolden, D. S., Harries, T. V., & Newton, D. P. (2010). Pre-service primary teachers' conceptions of creativity in mathematics. *Educational Studies in Mathematics, 73*(2), 143–157.

Brown, S. I., & Walter, M. I. (1993). *Problem Posing: Reflections and Applications*. Hillsdale, NJ: Lawrence Erlbaum.

Brown, S. I., & Walter, M. I. (2005). *The art of problem posing* (3rd ed.). New York: Routlege.

Brown, S. I., & Walter, M. I. (2014). *Problem posing: Reflections and applications*. New York: Psychology Press.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In F. Singer, N. F. Ellerton, & J. Cai (Eds.), *Mathematical Problem Posing* (pp. 3–34). New York: Springer.

Cropley, A. J. (2001). *Creativity in education & learning: A guide for teachers and educators*. London: Psychology Press.

Edelson, D. C. (2002). Design research: What we learn when we engage in design. *The Journal of the Learning Sciences, 11*(1), 105–121.

Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 42–53). Dodrecht, The Netherlands: Kluwer Academic Publishers.

Fennema, E., & Romberg, T. A. (1999). *Mathematics classrooms that promote understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Getzels, J. W. (1969). Creativity. In R. L. Ebel (Ed.), *Encyclopedia of Educational Research* (4th ed., pp. 267–275). New York: Macmillan.

Gruber, H. E., & Wallace, D. B. (1999). The case study method and evolving systems approach for understanding unique creative people at work. In R. Sternberg (Ed.), *Handbook of Creativity* (pp. 93–115). Cambridge, UK: Cambridge University Press.

Haavold, P. Ø. (2016). An empirical investigation of a theoretical model for mathematical creativity. *The Journal of Creative Behavior.* https://doi.org/10.1002/jocb.145

Hadamard, J. (1945). *The mathematician's mind: The psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hadamard, J. (1954). *An essay on the psychology of invention in the mathematical field: Mineola*. New York: Courier Corporation.

Hardy, G. H. (1940). *A mathematician's apology*. Cambridge, UK: Cambridge University Press.

Haylock, D. W. (1987). A framework for assessing mathematical creativity in school chilren. *Educational Studies in Mathematics, 18*(1), 59–74.

Haylock, D. W. (1997). Recognising mathematical creativity in schoolchildren. *ZDM, 29*(3), 68–74.

Holyoak, K. J., & Thagard, P. (1996). *Mental leaps: Analogy in creative thought*. Cambridge, MA: MIT press.

Hong, E., & Aqui, Y. (2004). Cognitive and motivational characteristics of adolescents gifted in mathematics: Comparisons among students with different types of giftedness. *Gifted Child Quarterly, 48*(3), 191–201.

Juter, K., & Sriraman, B. (2011). Does high achieving in mathematics = gifted and/or creative in mathematics? In B. Sriraman & K. H. Lee (Eds.), *The elements of creativity and giftedness in mathematics* (Vol. 1, pp. 45–65). New York: Springer.

Kampylis, P., Berki, E., & Saariluoma, P. (2009). In-service and prospective teachers' conceptions of creativity. *Thinking Skills and Creativity, 4*(1), 15–29.

Kattou, M., Kontoyianni, K., & Christou, C. (2009). Mathematical creativity through teachers' perceptions. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 297–304). Thessaloniki, Greece: PME.

Kattou, M., Kontoyianni, K., Pitta-Pantazi, D., & Christou, C. (2013). Connecting mathematical creativity to mathematical ability. *ZDM, 45*(2), 167–181.

Kaufman, J. C., & Sternberg, R. J. (2007). Resource review: Creativity. *Change, 39*(4), 55–58.

Kern, F. (2010). What chief executives really want. *Bloomberg businessweek 18.* Retrieved from http://www.businessweek.com/innovate/content/may2010/id20100517190221.htm

Kontoyianni, K., Kattou, M., Pitta-Pantazi, D., & Christou, C. (2013). Integrating mathematical abilities and creativity in the assessment of mathematical giftedness. *Psychological Test and Assessment Modeling, 55*(3), 289.

Kowalski, S. A. (1997). *Toward a vision of creative schools: teachers' belief about creativity and public creative identity* (Ph.D thesis). University of California, Los Angeles.

Laycock, M. (1970). Creative mathematics at Nueva. *The Arithmetic Teacher, 14*, 325–328.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam, The Netherlands: Sense Publishers.

Leikin, R., Subotnik, R., Pitta-Pantazi, D., Singer, F. M., & Pelczer, I. (2013). Teachers' views on creativity in mathematics education: an international survey. *ZDM, 45*(2), 309–324.

Lev-Zamir, H., & Leikin, R. (2011). Creative mathematics teaching in the eye of the beholder: Focusing on teachers' conceptions. *Research in Mathematics Education, 13*(1), 17–32.

Lev-Zamir, H., & Leikin, R. (2013). Saying versus doing: Teachers' conceptions of creativity in elementary mathematics teaching. *ZDM, 45*(2), 295–308.

Leu, Y.-C., & Chiu, M.-S. (2015). Creative behaviours in mathematics: Relationships with abilities, demographics, affects and gifted behaviours. *Thinking Skills and Creativity, 16*, 40–50.

Liljedahl, P. (2013). Illumination: An affective experience? *ZDM, 45*(2), 253–265.

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For the Learning of Mathematics, 26*(1), 17–19.

Littlewood, J. E. (1986). The mathematician's art of work. In B. Bolloba (Ed.), *Littlewood's miscellany*. Cambridge, UK: Cambridge University Press.

MacKinnon, D. W. (1962). The nature and nurture of creative talent. *American Psychologist, 17*(7), 484.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30*(2), 236–260.

Pehkonen, E. (1997). The state-of-art in mathematical creativity. *ZDM, 29*(3), 63–67.

Pekrun, R. (2006). The control-value theory of achievement emotions: Assumptions, corollaries, and implications for educational research and practice. *Educational Psychology Review, 18*(4), 315–341.

Plucker, J. A., & Beghetto, R. A. (2004). Why creativity is domain general, why it looks domain specific, and why the distinction does not matter. In R. J. Sternberg, E. L. Grigorenko, & J. L. Singer (Eds.), *Creativity: From Potential to Realization* (pp. 213–225). Washington, DC: American Psychological Association.

Poincaré, H. (2000). Mathematical creation. *Resonance, 5*(2), 85–94.
Pólya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving*. New York: John Wiley & Sons.
Romney, W. D. (1970). What is your creativity quotient. *School Science and Mathematics, 70*(1), 3–8.
Runco, M. A. (1999). Divergent Thinking. In M. A. Runco & S. R. Pritzker (Eds.), *Encyclopedia of creativity* (Vol. 2). San Diego: Elsevier.
Runco, M. A., & Bahleda, M. D. (1986). Implicit theories of artistic, scientific, and everyday creativity. *The Journal of Creative Behavior, 20*(2), 93–98.
Sheffield, L. J. (2000). Creating and developing promising young mathematicians. *Teaching Children Mathematics, 6*(6), 416.
Shriki, A. (2010). Working like real mathematicians: Developing prospective teachers' awareness of mathematical creativity through generating new concepts. *Educational Studies in Mathematics, 73*(2), 159–179.
Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.
Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM, 29*(3), 75–80.
Smith, S. M., Ward, T. B., & Finke, R. A. (1995). *The creative cognition approach*. Cambridge, MA: MIT Press.
Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? *Prufrock Journal, 17*(1), 20–36.
Sriraman, B. (2009). The characteristics of mathematical creativity. *ZDM, 41*(1–2), 13–27.
Sriraman, B., Haavold, P., & Lee, K. (2013). Mathematical creativity and giftedness: a commentary on and review of theory, new operational views, and ways forward. *ZDM, 45*(2), 215–225.
Starko, A. J. (2013). *Creativity in the classroom: Schools of curious delight* (5th ed.). New York: Routledge.
Sternberg, R. J. (1985). Implicit theories of intelligence, creativity, and wisdom. *Journal of Personality and Social Psychology, 49*(3), 607–627.
Sternberg, R. J. (1988). *The nature of creativity: Contemporary psychological perspectives*. Cambridge, UK: Cambridge University Press.
Sternberg, R. J. (2003). *Wisdom, intelligence, and creativity synthesized*. Cambridge, UK: Cambridge University Press.
Sternberg, R. J., & Lubart, T. I. (1999). The concept of creativity: Prospects and paradigms. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 3–16). New York: Cambridge University Press.
Torrance, E. P. (1966). *Torrance tests of creative thinking: Normstechnical manual: Research Ed.: Verbal Tests, Forms A and B: Figural Tests, Forms A and B. Flere Materialer*. Princeton: Personnel Press.
Torrance, E. P. (1974). *The Torrance tests of creative thinking-TTCT Manual and Scoring Guide: Verbal test A, figural test*. Lexington, KY: Ginn.
Treffinger, D. J., Young, G. C., Selby, E. C., & Shepardson, C. (2002). *Assessing creativity: A guide for educators*. Storrs, CT: National Research Center on the Gifted and Talented.
Voica, C., & Singer, F. M. (2013). Problem modification as a tool for detecting cognitive flexibility in school children. *ZDM, 45*(2), 267–279.
Weisberg, R. W. (1988). Problem solving and creativity. In R. J. Sternberg (Ed.), *The nature of creativity: Contemporary psychological perspectives* (pp. 148–176). New York: Cambridge University Press.

Chapter 15
Problem-Solving and Mathematical Research Projects: Creative Processes, Actions, and Mediations

Inés M. Gómez-Chacón and Constantino de la Fuente

15.1 Introduction

Inquiry-based learning is a complex inquiry-based student-centred pedagogical approach. There exists a general consensus among researchers (Artigue & Baptist, 2012; Artigue & Blomboj, 2013; Palatnik & Koichu, 2015; Schoenfeld & Kilpatrick, 2013) about the desirability of mathematics research (in its various meanings) which are present in the mathematics classroom, because it provides opportunities for students to implement thinking processes which are useful in solving problems or unfamiliar situations while illustrating the working modes and methods used by professional mathematicians in their daily work.

The design and analysis of Mathematical Research Project (MRP) are complicated tasks. Among the main challenges teachers face are the implementation and organization of students' projects, which may last up to several months and may be influenced by various factors, which are difficulties even for an experienced teacher.

In this chapter, suggestions are provided for resolving these challenges. It focuses on the process of transforming a problem-solving task (PST) in a Mathematical Research Project (MRP) carried out with high school students (17 years old, senior year). The process of generating a Mathematical Research Project (MRP) for a particular problem will not only require creativity on the part of the students but also the mediation of the teacher for the establishment of a suitable creative mathematical working space.

I. M. Gómez-Chacón (✉)
Instituto de Matemática Interdisciplinar, Facultad de Ciencias Matemáticas,
Universidad Complutense de Madrid, Madrid, Spain
e-mail: igomezchacon@mat.ucm.es

C. de la Fuente
Universidad Complutense de Madrid, Madrid, Spain

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_15

Creativity is, according to Sriraman, Haavol, and Lee (2013), "a buzzword of the 21st century and seen as a major component of education". These authors consider that research on creativity, specifically in mathematics, is scarce, because there is not an agreement on a consistent definition of creativity and aspects of mathematical creativity need be distilled (aspects of problem-solving, problem posing, and problem sequencing).

We would like to point out that the focus on creativity research has shifted over time, moving from an early dominant focus on genius, towards giftedness in the middle of the twentieth century, to a more contemporary emphasis on originality of thought and work (Getzels, 1987). Our work lies firmly in the latter category, in that we do not see these individuals as having "extraordinary" creative powers but rather having what may be accessible to most, if not all of us. Creativity is fundamentally defined as a state relating to or involving imagination or original ideas. Ideas are generated by thinking, and skills for thinking can be learned. Our assumption is that mathematics creative thinking and idea generation are skills that can be learned. As Mishra and Henriksen (2014) noted, "Creativity is the end result of the manner in which human cognition works—and is available to all people. Finally, as with other human cognitive skills, there is a significant level of variability between individuals, and it can be learned and developed with practice" (p. 15).

Based on this idea, the creative process is a process that involves skills that can be learned and developed with practice; a key issue are the relations and mediations between the instrument (in our case the Mathematical Research Project (MRP) is the instrument) and the individual or the individuals involved in the mediated activity (student and teacher). So, one of the fundamental points in this chapter focuses on the activity of the student on the design and development of the Mathematical Research Project (MRP) identified by the teacher.

In recent decades, research in cognitive ergonomics, in the French tradition, has devoted considerable effort to the issue of inventiveness as it is manifested in the activity of users confronted with a technique. The theory of activity mediated by an instrument (Rabardel, 1995, 2005; Rabardel, & Béguin, 2005; Winograd & Flores, 1986) and the instrumental approach in mathematics education are part of this perspective (Artigue, 2002; Gueudet & Trouche, 2008; Guin & Trouche, 2007). In the present research, this approach allows us to highlight the articulations and connections between action and activity, namely, activity and identified theories (see Sect. 15.3). The cognitive ergonomics approach indicates four dimensions that characterize activity in a situation: inventiveness, mediation, and the developmental and social nature of instrument-linked activities. Among the premises of this approach lies the inventiveness of a user using an instrument as a key characteristic in the design and use of instruments and the apprehension or the subjective grasp of reality by individuals in a situation.

Based on this perspective, we define the mediated activity. We look at the mediator and suggest conceptualizing it as a mixed functional entity: the instrument –the Mathematical Research Projects (MRPs) – elaborated by the student. The identification of student activity will be done by the teacher through the activity mediated by the instrument (different versions of the MRP developed by the student until the final version). We consider the constructive and productive dimensions of the teacher activity in the relationships that take place between the teacher and the development of the MRP by the student. We examine the output and development of the modalities of mediations during processes of instrumental genesis working from problem-solving tasks (PST) to Mathematical Research Projects (MRPs). So, mediations produced by the subject (teacher) and the goal of the activity (MRP obtained by the student) are analysed.

Although different authors have proposed different models for inquiry-based learning with projects (Artigue & Blomboj, 2013), we believe it can be helpful to study learning through Mathematical Research Projects (MRPs) for high school students, considering the cognitive ergonomics approach of using an instrument. We believe that conceptualization in terms of mediated activity is part of a broader evolution of models that aim to account for human behaviour.

In this chapter, the modelling that takes place in the development of a Mathematical Research Project (MRP) is presented, that is, the process (instrumental and documentary genesis) of the transformation of the resource (the student productions – drafts and scientific final report) into a final document that will be the Mathematical Research Project (MRP). This intends to answer the question: What teacher mediations articulate student inventiveness in Mathematical Research Projects? And it will specify (1) identification of the student activity by the teacher – the operational mathematical invariants of the students' schemes, cognitive processes that the student puts into action, and domains of mathematical knowledge – and (2) mediated activity between teacher and students regarding instrument, object, and subject.

In Sect. 15.2, we present our understanding of Mathematical Research Projects (MRPs). In Sect. 15.3, we define the mediated activity, and we will distinguish different types of mediations conceptualized as mixed functional entities: the instrument and the instrumental and documentary genesis. These concepts are based on analytic methods of the development of Mathematical Research Projects (MRPs). Section 15.4 illustrates a project, and the chapter will end with some didactic considerations for the design of Mathematical Research Projects (MRPs).

15.2 From Tasks to Mathematical Research Projects

The basis of the following tasks and mathematical research projects is problem-solving, developed in the second half of the twentieth century (Guzmán, 1991; Polya, 1957; Schoenfeld, 1985, 1992a, 1992b), and studies that conceptualize the action of mathematical thinking (Mason, Burton, & Stacey, 1982) where the idea of mathematical structure (as identification of general properties, relationships between elements) and the disposition to use, explain, and connect these characteristics in mathematical thinking require attention structures of the individual (Mason, 2010).

We understand the idea of mathematical research in high school with two meanings: (1) through a research project characterized by a series of processes (exploration; consideration of special cases; inductive thinking; generalization; speculation, along with its proof or refutation; and approaching new problems arising from the initial problem) and (2) the transformation of a small math problem (or set of problems) in serious research work (Braverman, 2010; Braverman & Samovol, 2008). We present an illustrative example that, after several years of experimentation with students, has generated a theory that generalizes the idea of traditional arithmetic progressions (a.p.).

> *Initial problem-solving task*: Is it possible to fill the empty spaces in the table with positive integers so that the numbers in each row and in each column form arithmetic progressions?

In Sect. 15.4, we will show the MRP "arithmetic progressions in space" generated by students taking as a starting point this initial problem-solving task. Here we summarize some of the questions that led to several research projects with different students, described in Sect. 15.4, generated by several students as a result of working on the initial problem shown in Fig. 15.1.

In an *arithmetic progression*, we can deduce the whole progression if we know two terms. In a table of numbers like this, if we know four terms, can we identify everything else? Under what conditions? In three-dimensional space, into a rectangular parallelepiped shape, consisting of cubes in each of the three dimensions, if we know eight elements, could we know all the elements of the figure? Under what conditions? If we consider a similar N-dimensional figure, what could we say about it?

As in the case of an *arithmetic progression* for which we know formulas that identify a general term a_n, the sum of the first n terms, and the interpolation of arithmetic means, could we obtain similar formulas for two-dimensional, three-dimensional, and N-dimensional tables?

15.2.1 Types of Mathematical Research Projects

Following the conception of the aforementioned mathematics research, our proposal covers two types of tasks or research projects:

	74			
				186
		103		
0				

Fig. 15.1 Example of an initial problem-solving task

(a) *Mini-Mathematical Research Projects (mMRPs)*. They constitute a first approach of high school students to mathematics research and are based, among others, on the idea of scaffolding (Wood, Bruner, & Ross, 1976). The idea of mMRP is related to scaffolding, as they provide the student with a guiding script to enlighten a possible way forward (De la Fuente, 2013, 2016; De la Fuente, Gómez-Chacón, & Arcavi, 2015). After 3 or 4 weeks, the student must submit a scientific report that is reviewed by the teacher and returned with written comments. Subsequently, the student submits the final report.

These mMRPs are usually of two types: projects for the discovery, demonstration, and possible generalization of some abstract and general property derived from a particular mathematical context and proposals for the discovery of effective mathematical models for solving real problems. We present an example of the beginning of a guiding script proposed by the teacher for the problem of a table, the rows and columns of which are arithmetic progressions (it is not fully shown because of space limitations in this document).

> If the number above the 0 were x, could you fill the table in terms of x? Considering the problem data, calculate the value of x and solve the problem.
>
> Solve the problem in other ways. We suggest that one of them will come from the difference in the *arithmetic progression* of the first row, h_1, and from the difference in *arithmetic progression* of the first column, v_1. Study successions (h_i) and (v_i)) i = 1, 2, 3, ... Do they have any property in common?
>
> Find out if any table of this kind can be completed if we know four of its elements. What about if we know fewer than 4 elements?

(b) *Mathematical Research Projects* (MRPs). These are the real research (De la Fuente, 2016). From the initial problem statement, a process analogous to scientific research follows, with the teacher acting as project manager. It lasts about 4 months, and at the end of this process, the student must submit a scientific report on the results achieved, reasons for the work, and personal reflections on it. The mathematical content of MRP is usually of two types: the deepening of some mathematical knowledge on which the student built the research problem or the application of mathematical knowledge to solve a real problem. In Sect. 15.4, this example of MRP "arithmetic progressions in space" is presented.

15.3 Theoretical Framework

The theoretical conceptualization of the research presented is based on the latest developments of the person-tool interaction. One of the key points of the approach is that the main relation is that of the subject with the object of his activity. The tool, whether it is the traditional pencil and paper or digital technology, is in an intermediary mediator between the subject and the object. In this context, the main ideas of the theoretical framework are *the theory of activity* and within them the *activities* *mediated by an instrument.* In the latter, in the person-instrument interaction, actions of *epistemic*, *pragmatic*, and *heuristic mediation* take place, which give rise to the documentary genesis, through the functions of instrumentation and instrumentalization to take place (Gueudet & Trouche, 2011; Rabardel & Bourmaud, 2003).

15.3.1 Mediated Activity

When the relationship between the subject and object of the activity is not direct but is carried out through an instrument, the idea of *mediated activity* arises (Wertsch, 1998). We share Wertsch's point of view of mediated activity as a unit of analysis that retains the characteristic properties of individuals, tools, and contexts.

In this context, working with the instrument (in the light of the object of the activity) produces the relations of mediation. In our study on the implementation of Mathematical Research Projects, mediated activity will be a good candidate for a unit of analysis for the exploration of issues of mediation and the development of students' projects in the classroom. We consider this mediation as the central fact that transforms psychological functions in the creative activity. Following Rabardel and Bourmaud (2003), we propose to distinguish three main orientations of mediation in instrument-mediated activity: towards the object of the activity, towards other subjects, and finally towards oneself.

In Fig. 15.2, the dotted arrows represent the three orientations of mediation in the instrument-mediated activity. The linear arrows represent non-mediated relations. Depending on the content and to what or to whom mediation relations are addressed, these can be classified into three types: *epistemic*, *pragmatic*, and

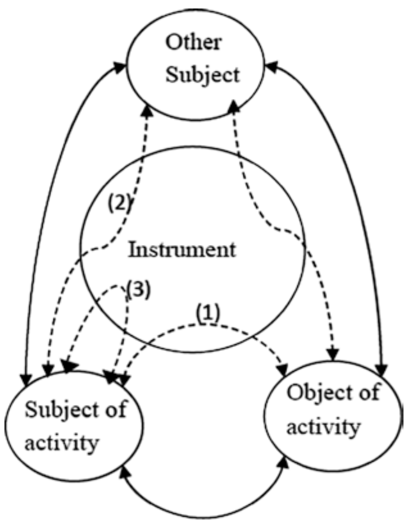


Fig. 15.2 Orientations of mediation in the instrument-mediated activity

reflexive or *heuristic* (Rabardel & Bourmaud, 2003, p. 670). The mediations are described below for each one of the agents of mediated activity:

1. *Towards the object of the activity*. In this case, the *epistemic mediation* is directed to knowing the object of the activity. The subject here is aware of the nature of the goal of the activity. It occurs in the analysis phase, in the process of the functioning of the artefact component of the instrument. The *pragmatic mediation* is directed to knowing the action and performance of the object. It can lead to the transformation (review, modification, adaptation) of the object or goal of the activity.
2. *Towards other subjects*. "Interpersonal mediations" may also be epistemic or pragmatic in nature, depending on whether it is a question of getting to know others or acting upon them. *Epistemic mediation* is directed to seeking the knowledge of other subjects, ideas, arguments, mindsets, learning, etc., while the *pragmatic mediation* tries to act on them producing transformations and changes, new learning, changing their mindsets, etc.
3. *Towards oneself*. In this case, the most important mediation that occurs is *reflexive* or *heuristic mediation*, aimed at management, activity monitoring, and decision-making, depending on the evolution of what is being done and/or obtained. This mediation is specific in writing notes and making annotations for the subject himself. In the case of a mediated activity where the subject (or aspect thereof) is also the object of activity, under these conditions, the *epistemic mediation* is directed to the knowledge of this aspect of the subject, which forms the object of the activity. Also, *pragmatic mediation*, which will seek transformation and change in the object (subject) of the activity, will be part of it.

15.3.2 Instrumental and Documentary Genesis

In the "instrument-mediated activity" approach (Beguin & Rabardel, 2000; Rabardel & Bourmaud, 2003), the design of an instrument cannot be confused with an artefact. An instrument is seen as a composite entity made up of "the artefact", in its structural and formal aspects, and the subject's social and private schemes. According to Rabardel (1995), the notion of artefact includes everything that has undergone a transformation, however small, of human origin. Its significance is not restricted to material objects (physical world) but includes symbolic systems that can also be instruments. The two components of the instrument, artefact and scheme, are associated with one another, but they are also in a relation of relative independence.

The instrumental genesis could be considered a way of conceptualizing the processes of apprehension. In this process of apprehension, the subject internalizes something that is on the outside (the artefact), associating it with their own means (the scheme), creating the transformation from artefact to instrument.

Likewise, in the instrumental genesis (Guin & Trouche, 2007; Rabardel & Bourmaud, 2003), two dual processes occur:

1. *The instrumentation*. The instrumentation dimension includes the artefact and the subject and is reflected in the changes that occur in the schemes of action of the individual to use the artefact-instrument. Changes in the schemes are produced by the *pragmatic* (action and use), *epistemic* (knowledge), and *heuristic* (management and decision-making) functions of the schemes.
2. *The instrumentalization*. This dimension goes from the subject to the instrument and is manifested by the changes that occur in the artefact as it is customized by the subject, in the gradual process of transformation of the artefact into an instrument. Figure 15.3 represents the instrumental genesis and the two processes that structure.

For the implementation of this approach by a teacher who wants to perform MRP, there are several actions: (a) think of the educational use of artefacts in order to understand, accompany the work, and enhance the work of the students and (b) establish an instrumental orchestration that involves designing various configurations of devices and multiple operating modes, adjusted to the successive stages of an educational activity (e.g. different stages of solving a mathematical problem) and the objectives of the teacher in each of these phases (stimulate individual exploration efforts, or the exchange of ideas, or giving proofs, or a reflective feedback, etc.).

Moreover, the so-called resources and documents play an important role in the teacher's role. Via this documentary work, the teacher constructs the subject to teach; therefore, this work is the bearer of professional development.

It is within this interrelated process that documentary focus emerges, which is an extension of the instrumental perspective. The concept has been outlined by Gueudet and Trouche (2010, p. 2), who consider that the heart of the teacher's activity is:

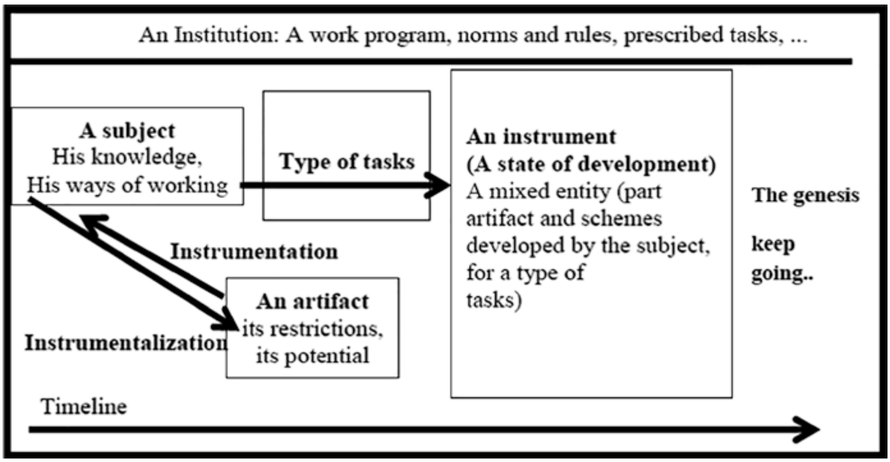


Fig. 15.3 Instrumental genesis taken from Guin and Trouche (2007, p. 203)

> documentary work that consists of seeking, combining, designing, sharing and revising his teaching resources. In this task, the teacher interacts with groups of resources. These resources may be devices, which span a wider range; in this way, a student's answer is an essential resource for the teacher. During these interactions, the teacher develops what we call a document, which incorporates both the combined resources and their patterns of use, saturated with professional experience and knowledge.

This process of transformation, from the resources to the documents, is defined as documentary genesis, which also contains the two dimensions of instrumentation and instrumentalization, which play an important role in the context, form, and type of the proposed activity.

15.3.3 Organization Levels in the Domain of Activity

Instruments are not isolated. All of us know this intuitively from experience. As soon as the activity has a certain complexity, its authors use a range of instruments. They are mobilized in line with operational goals and needs as given. The functional complementarity relations between instruments and the logic of the activity generate temporary sequences in the use of instruments, in a successive or simultaneous manner.

But the instruments are mobilized not only in specific situations, but they are also structurally linked to the invariable dimensions of some types of activities (Rabardel, 2001). The instrumental genesis is grounded in these invariants, and the instruments developed allow the management of their peculiarities.

Types of situations that organize themselves into domains of activity (or intervention) for which there are corresponding groups of instruments. Rabardel and Bourmaud (2003, p. 678) exemplify with an intuitive experience of this in everyday life: "the tool box we keep in the trunk of our car or the sewing kit kept handy, etc. are groups of instruments that allow us to deal with the main situations requiring intervention in these limited everyday domains".

In the professional analysis of the relationships between instruments and situations, three levels of organization and analysis (Rabardel & Bourmaud, 2003) are distinguished:

(a) The *situations* with sufficiently similar characteristics, either by the tasks or by situations that should be considered, can be grouped, resulting in *types of situations*. These *types* have associated instruments adapted to the peculiarities of each of them. They give rise to forms of activity that are relatively stable for the same type and are differentiated.

(b) The *types of situations* can be organized in a higher-level cluster, constituting *families of activities*. These *families* gather and organize all types of situations that correspond to the same type of general purpose of the activity. At this level, there may be *types of situations* belonging to more than one family of activities.

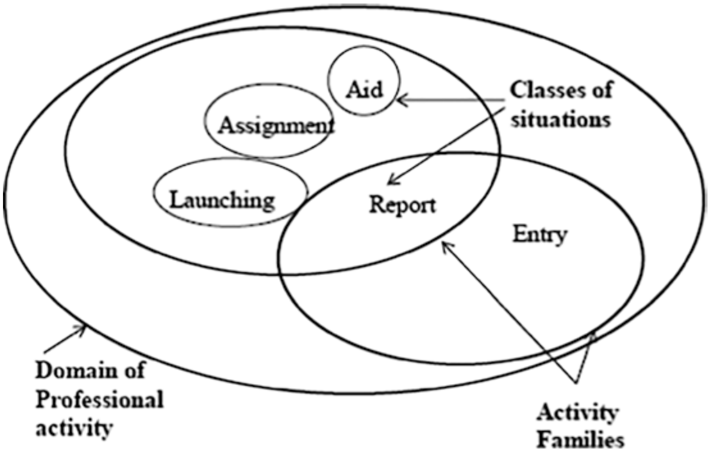


Fig. 15.4 Organizational levels in the domain of activity

(c) The group of all *activity families* constitutes a third level of organization and analysis: the professional domain of activity. This domain includes all activity families along with the kinds of situations that compose them, and for each there are corresponding groups of instruments.

Figure 15.4 (Rabardel & Bourmaud, 2003, p. 678) outlines the ideas graphically: the activity families, types of situations, and domain of professional activity

15.3.4 Productive Dimension (Instrumentalization) and Constructive Dimension (Instrumentation)

The documentary approach focuses its attention on the analysis of teachers' work starting with their production activity: for a type of situations (e.g. "teach the notion of function to a high school class"), the teacher gathers resources, works on them, and develops teaching materials, which he/she implements in the classroom. This implementation is not a single action; it will happen again with other students and other academic courses. At various points in the development and implementation, changes take place: in a movement of instrumentalization, the resources involved are constantly reviewed by the teacher; in a process of instrumentation, the teacher's knowledge is questioned by the resources, by the implementation, and by the resulting effects (Gueudet & Trouche, 2011).

Thus, the productive activity of the teacher is also a constructive activity (Samurçay & Rabardel, 2004) because it is not only limited to producing transformations of objects in the external world (specifically, the resources documents) but also he himself becomes transformed, thus enriching his repertoire of resources and knowledge, both the content and the *didactical content*. That is, the teacher learns what he/she teaches; this is the constructive role of the activity (Pastré & Rabardel, 2005).

Moreover, during the process in which the constructive functions arise, the teacher conducts an internalization (Wertsch, 1998) which is not always observable from the outside nor verbalizable by him.

The same applies to the instrumentalization, which also does not necessarily have immediate consequences. This is why it is necessary to look at the invisible work (Blombreg, Suchman, & Trigg, 1996). But besides being difficult to see, not all components of the instrumentation and instrumentalization functions are directly named.

This is particularly true in the case of schemes that, in the instrumental or documentary approach, must be taken into account. The *action* is the knowledge that activates the “already incorporated knowledge”, and this is much more than being able to explain it. This dimension also applies to schemes when they are supported by operational invariants (Vergnaud, 1996) that are implicit in the subject and have not been externalized.

As for the relations between the productive and constructive functions and the professional activity of the teacher, Gueudet and Trouche (2008, p. 16) explain it the following way:

> This dialectic of the productive and constructive allows taking into account the complexity of these processes: a completed activity has a production goal that is nothing more than the realization of the given task. In this activity, the subject also constructs oneself and modifies the conditions of later production.

This dialectic does not mean that we simply consider the documentary work as capable of generating professional development, but it also represents a real change of perspective. This is not to consider documentary work as a necessity directed towards the work in the classroom; the documentary work is at the heart of the professional activity of the teacher, which takes place inside and outside classroom. Thus, classwork is analysed as a moment of documentary enrichment. From this perspective, the documentary genesis is the essence of professional development.

At this point, one should highlight the connections between the constructive role of *documentary genesis* and the *heuristic activities mediations*, mediated by an instrument. Heuristic functions, which are directed towards the subject himself, are those that identify operational invariants (knowledge in action) of the teacher’s schemes, by interacting with the instrument (document). These mediations are those in which the subject is the teacher himself and allow the exposure of the invisible work. At this point the two major approaches of the theoretical framework, mediations and documentary genesis, are connected to give more rigour and consistency to the research. This potential is exemplified and illustrated in the following section.

15.4 Exemplification of a Mathematical Research Project: "Arithmetic Progressions in Space"

In this section, the process of transforming a problem-solving task into a MRP is presented. The aim of this section is the characterization of this transformation process in two key aspects: (a) the operational mathematical invariants of the students' schemes – mathematical and cognitive processes that the student puts into action – and (b) mediated activity between teacher and students regarding the instrument, object, and subject that allows the structure and the phases that form the backbone of the structure to be identified.

Student activity is a complex process that it is part of the resolution of a problem-solving task (PST) in order to obtain a Mathematical Research Project (MRP) by proposing the corresponding research question. This process is finished when the student makes the research problem explicit, obtained as an extension of the work on the problem-solving.

In order to facilitate better understanding, we explain the terminology used. We consider the MRP report as the "instrument" (artefact + scheme), i.e. it is the documentary resource that collects student productions (two draft reports and the final report). The object is the MRP, and the subjects are the teacher and the students (see Fig. 15.2).

We base our presentation on experimentation conducted during two academic years, 3 months in each year, of the MRP: "arithmetic progressions in space". The empirical data presented are supported by the response of a student, PI, the first year (TN-PI) who developed the MRP individually – and the response of a group of five students (document RPA-TG) who developed the MRP together as a group. In both cases we will use the final document of the MRP.

All the participants are students in their senior year of high school (17 years old), who selected the line of studies towards science and technology. They have an average high performance in mathematics – above average with respect to their peer group. One of them is a participant in a mathematical talent programme, and the other four are students who regularly take part in Mathematics Olympics.

The teacher proposes to each student an initial problem-solving task (PST) to be resolved and expanded gradually to formulate and develop an MRP. The students must submit successive drafts every 3 weeks. Drafts collect the accumulated work carried out by the student so far, with help and guidance given by the teacher, and the results achieved so far. The teacher reads and analyses each of the drafts submitted and adds comments in the margin of the documents, returning them to the students through interviews at the end of the following week. These ideas and suggestions given by the teacher should be taken into account by the students, who must incorporate them into the working document and develop them. In total, each student completes three versions of the document: two drafts and the final report of the MRP at the end of the third month.

The cases presented have been selected for the following reasons, among others:

1. The initial PST develops an elementary mathematical content, adequate for the students' academic level, which is rich and complex once one delves further into the topic.
2. The MRPs obtained are all based on the same initial PST; therefore, they offer different problems to be addressed relating to a common mathematical content.
3. The final reports of these students are of a high quality, and the level of achievement of the students is rather satisfactory. So, they can be considered key reporters.

Furthermore, these projects have been developed as part of the teaching process, their completion being compulsory for all the students in the class, although their topics may not all be the same nor all the projects completed in the classroom.

With respect to the role of the researchers, we note that one of them is a teacher-researcher involved in the teaching process, and the other researcher manages the investigation process. Selected tasks for the implementation and data analysis were worked on by both researchers.

In the following section, we describe the example of the MRP: arithmetic progressions in space. The section is divided into several subsections. In Sects. 15.4.1, 15.4.2, and 15.4.3, the mathematical resolution of the initial problem for the MRP formulation, both in the case of PI and in the case of RPA group (Sects. 15.4.1 and 15.4.2), and the levels of organization of situations and domains of mathematical activity (Sect. 15.4.3) are presented. The data allow the teacher to establish the mathematical invariants and the student' schemes put into practice in the development of the MRP. In Sect. 15.4.3, the mediated activity between the teacher and the students regarding the instrument, object, and subject is presented.

15.4.1 Mathematical Resolution from the Problem-Solving Task to the Formulation of the MRP: The Case of the Student PI

The productive and constructive functions of the teacher will be included in Sects. 15.4.1 and 15.4.2, through the analysis done by the teacher regarding the mathematical resolution of the initial problem for the MRP formulation, elaborated by the students in the final version of the document.

The student PI is a high school senior (17 years old), who has selected science and technology as the main line of study. We will take her final report of MRP (TN-PI). The initial problem posed by the teacher to the students was presented in Fig. 15.1.

The table of contents of the MRP final report submitted by the student is shown in Fig. 15.5. Prior to this final document, the student made two drafts. Both were discussed in the tutorial sessions with the teacher. The processing time until the final version was 3 months.

1. *Introduction*
2. *A problem*
    2.1. *Entry - Initial exploration*
    2.2. *Attack*
    2.3. *Review-Extension*
        2.3.1. *Alternative 1*
        2.3.2. *Alternative 2*
        2.3.3. *Alternative 3*
        2.3.4. *Alternative 4*
3. *Questions, conjectures, new ideas*
    3.1. *Some form of generalization*
        3.1.1. *A more general problem*
        3.1.2. *Some more general concepts*
    3.2. *Variations on the problem*
        3.2.1. *A case not determined*
        3.2.2. *Some cases with equal differences*
    3.3. *Problem Structure*
4. *Concepts, properties, and theorems*
    4.1. *Arithmetic networks*
    4.2. *Unidimensional Arithmetic Networks and arithmetic progressions*
    4.3. *Two-dimensional Arithmetic networks and matrices*
    4.4. *Priscila number associated with an arithmetic network*
        4.4.1. *For two-dimensional arithmetical networks*
        4.4.2. *The case of three-dimensional arithmetic networks*
        4.4.3. *The case of n-dimensional arithmetic networks*
    4.5. *Back to the initial problem*
5. *Open problems*
6. *Final Reflection*
7. *Bibliography*

Fig. 15.5 Table of contents of the MRP report submitted by student PI

Firstly, one can see that the student, during the process of solving the initial problem, uses the terminology of Mason, Burton, and Stacey (1985) in the titles of Sect. 15.2. This use of this terminology is due to the guidance given by the teacher in her first draft. In point 3, the student deepens into the initial problem, and in point 4, she presents the concepts of a new theory, which generalizes the traditional idea of arithmetic progression (a.p.[1]), defining similar concepts in space and presenting some properties or theorems.

We note that points 2 and 3 make the process of transforming a problem-solving task into a MRP. To illustrate this fact, we will look at points 3.1.2. and 3.3. The student starts solving the problem (item 2 in the table of contents, Fig. 15.5) in

[1] The arithmetic progression will be denoted by a.p.

These "arithmetic squares" are figures in the plane (two-dimensional space) that have their analogues in one-dimensional space and in three dimensional space. We will expose this in a graphical way:
Given the following arithmetic square (fig. 16), the analog line is the traditional arithmetic progression (Figure 17) and space is the arithmetic parallelepiped (figure 17a).

y	t	
x	z	

Fig. 16: arithmetic square

...•...•...........
x y

Fig. 17: arithmetic line

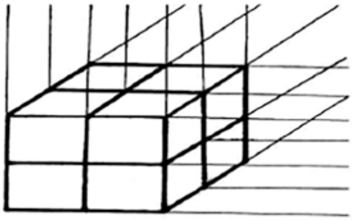


Fig 17 bis: arithmetic parallelepiped

Analogous to the arithmetic table in three-dimensional space the figure would be the "cube, or arithmetic parallelepiped", which would be a network of cubes, and each cell would be a number. The minimum number of cubes that would give us to complete the simple cube looks like it should be eight, by analogy with the plane and straight, but this is conjecture.
At this point, we also wonder if all this analysis for arithmetic progressions can be used for geometric progressions, for what we should change the addition by multiplication, subtraction by divisions and products for powers.

Fig. 15.6 Extract of the MRP of student PI from the final report

several ways (TN-PI, p. 4–7[2]). Specifically, she proposes four different ways to solve the problem: first, using as an auxiliary an unknown x, the element located above 0, and applying the properties of the a.p., solving an equation with one unknown. In other forms of solution, other auxiliary unknowns – differences of a.p.-row or a.p.-column, etc. – are used, solving the systems of linear equations with two or three unknowns. Later some generalizations are proposed. In order to illustrate this, we will look at Sect. 3.1.2 (TN-PI, p. 8) entitled *Some More General Concepts* (Fig. 15.5).

In this excerpt from the final report of the PI (Fig. 15.6), it can be seen that the student, using analogy, contextualization, connections between contexts, and abstraction, outlines questions. She poses some conjectures about possible general results and connects the context of arithmetic progressions with geometric progressions. This is a first approach to obtaining general questions or first formulation of the mathematical research problem.

[2] The final report of MRP will be denoted by TN-PI. The pages of the presented extracts are included.

Returning to the table of contents of the MRP, the student continues to item 3.2. (TN-PI, p. 9), *Variations on the problem*, which presents the study of different cases obtained by modifications of the initial conditions of the problem, and then finishing with item 3.3 (TN-PI, p. 12) entitled *Problem Structure*. The student wrote what is shown in Fig. 15.7.

The process ends with the statement of a Mathematical Research Problem (MRP) or, equivalently, with the approach of a possible MRP: “Under what conditions can we build: a cube, a parallelepiped, a square, a line, or an arithmetic rectangle?” The work of the student continues trying to pose a possible MRP.

It is convenient to note that the subject-object mediations of the activity (the teacher and the obtaining of the MRP) via the instrument have allowed the specific identification of the operational constants of the students’ schemes and the levels of organization for the conditions and domains of the developed activities.

With respect to this last aspect, we note that the teacher-student interviews and the teacher’s analysis of the student’s drafts prior to the definitive report of the MRP also provide results about productive and constructive functions but will not be included here due to the space limitation of this chapter, which is exclusively concerned with the process of obtaining an MRP via a PST. In any case, let us note that the teacher’s types of orientations and commentaries, in the analysis of the student’s successive drafts, focus on suggestions of improvements for the attainment of the project objectives: assess the results, propose essay improvements and suggestions for the elimination of errors and modification of results, suggest new paths, pose new questions, etc.

We note that the mediations between the subject and the object of activity (teacher and obtaining of the MRP) through the instrument facilitate the following functions: (a) productive (instrumentalization), i.e. when the teacher checks the instrument, assesses their adequacy with the goal, and identifies and proposes the

---

After all the above, we are able to analyze the structure of the problem so that we can observe all its richness, complexity, etc.

The internal structure of the problem includes:

- *Square.*
- *Numerical values given: Four integers*
- *Arithmetic progressions with their respective differences.*

The structure of the problem varies when we change any of the variables given:

- *Square: Rectangle, Straight, cube, parallelepiped, etc.*
- *Numbers given: four. It will be the minimum number? If we get 3 ... And if we are given values of the differences ...*
- *Arithmetic progressions: Geometrics*
- *Positive integers: whole numbers, fractional numbers, etc.*

All issues listed refer to a more open problem that could have the statement: "Under what conditions can we build: a cube, a parallelepiped, a square, a straight or arithmetic rectangle?"

Fig. 15.7 Extract of the MRP report of student PI, from Sect. 3.1.2. of the final report characterization of the process, and (b) constructive (instrumentation), when the teacher clarifies the structure and phase in the passage from the PST to a MRP done by the student according to his mathematical knowledge schemes.

15.4.2 Mathematical Resolution from the Problem-Solving Task to the Formulation of the MRP: The Case of the Group

In this section, we will explain the process of the mathematical resolution of the initial task to the formulation of an MRP, taking the solution given by the group of five students in their senior year (17 years old) who selected science and technology as their line of study.

Regarding the final report of the group of five students (document RPA-TG), we note that they know the work developed by the student PI in the previous academic year and start working with this knowledge. They solve the initial problem without contributing anything new to what was done by the student PI, but they add a second initial problem, which has some analogy or resemblance to the first problem. It is a context of a numerical table in which lines are also a.p. They know the second problem as it was proposed earlier in a high school Mathematics Olympiad, in which some of them participated. This problem is as follows (RPA-TG, p. 6):

> We have a set of 221 real numbers the sum of which is 110721. They are set in a rectangular table, such that all rows and the first and last column are arithmetic progressions of more than one item. Prove that the sum of the elements of the four corners is 2004.

Starting from the resolution of this problem, along with the results obtained by the student PI, in the previous year, students propose the attainment of formulas similar to those used in the traditional a.p., to calculate the sum of all the elements of a table with these features, the expression of the general term of a table, the interpolation of arithmetic terms in a table, and the resolution of some similar problems for geometric progressions. This is summarized in the table of contents of their final report (RPA-TG, p. 3) (Fig. 15.8).

The most commonly used mathematical processes, to transfer the context of traditional a.p. to arithmetic tables, which they call arithmetic networks, are contextualization, analogy, abstraction in context, establishing connections between contexts, and generalization.

The result of all this is the originality and creativity achieved in some results, such as

$$a_{i,j} = d_p \cdot (i-1) \cdot (j-1) + h_1 \cdot (i-1) + v_1 \cdot (j-1) + a_{1,1}$$

The general term of a bi-dimensional arithmetic table, aij, may be expressed as a function of h_1, the difference of the arithmetic progression (a.p.) first row, the difference between the arithmetic progression (a.p.) first column v_1, and the Priscila

Introduction and background 5
    1.1 - Starting point 6
2 - Objectives and methodology 9
    2.1 - Objetives.. 9
    2.2 - Methodology.. 9
3 - Results.. 11
3.1 - A general term for a Arithmetic Network 11
    3.1.1 - A general term NA2D (Two-dimensional Network)
    3.1.2 - A general term NA3D (Space Network)
    3.1.3 - A general term NAND (N-dimensional Network)
3.2 General-term for geometric networks 18
    3.2.1 - A general term NG2D (Two-dimensional Network)
    3.2.2 - A general term NG3D (Space Network)
    3.2.3 - A general term NGND (N-dimensional Network)
    3.3-Sum of the elements of a Arithmetic Network.................................... 21
    3.3.1 - Sum of the elements of a NA2D
        - From the elements at the corners
        - Depending on the central term
        - Through integrals
    3.3.2 - Sum of the elements of a NA3D
        - From the elements at the corners
        - Depending on the central term
        - Through integrals
    3.3.3 - Sum of the elements of a NAND
        - From the elements at the corners
        - Depending on the central term
        - Through integrals
3.4 - Interpolation in Arithmetic Networks 27
    3.4.1-Interpolation in a NA2D
    3.4.2-Interpolation in a NA3D
    3.4.3-Interpolation in a NAND
3.5 - Interpolation in Geometric Networks 30
    3.5.1-Interpolation in a NG2D
    3.5.2-Interpolation in a NG3D
    3.5.3-Interpolation in a NGND
4 - Conclusions.. 32
5 - References.. 33
6 - Anexs... 34

Fig. 15.8 Table of contents of the final report of the MRP submitted by the students group RPA

difference, d_p, which is the difference of the a.p. formed by the differences of the a.p. rows or columns (RPA-TG, p. 6):

$$a_{i,j} = d_p \cdot (i-1) \cdot (j-1) + h_1 \cdot (i-1) + v_1 \cdot (j-1) + a_{1,1}$$

This formula generalizes the $a_n = a_1 + d\,(n-1)$. It is obtained from the previous for the case $i = 1$ or $j = 1$.

The sum of the terms of a two-dimensional arithmetic table depends on the terms in the corners (RPA-TG, p. 16):

$$S_{n,m} = \frac{n \cdot m}{4} \cdot \left(a_{1,1} + a_{n,1} + a_{1,m} + a_{n,m}\right)$$

This formula generalizes the traditional a.p. $S_n = \frac{n\left(a_1 + a_n\right)}{2}$ resulting in the latter of the above for the case $m = 1$.

This idea of generalizing the formula is not only for two-dimensional tables but also for tables of any size. This is recurring throughout the report.

Finally, we note that in this case (working group case), the mediations between the subject and the other subjects (i.e. mediations of teacher and students) foster in the teacher two functions: (a) productive (instrumentalization), to allow him to review the instrument, assessing its suitability to identify operational invariants (knowledge in action) of students (use of contextualization, analogy, generalization, and establishing connections between contexts), and (b) constructive (instrumentation), making explicit the value of knowledge in action by students and making clear the role of analogy, generalization, and the connection between contexts, within the appropriate mathematical working space to solve the problem of the MRP.

15.4.3 Organizational Levels of Situations and Domains of Mathematical Activity

According to the theoretical framework (Sect. 15.3), the levels of organization and domains of activity are described. These domains and levels are those that support the transformation of the problem-solving task into the MRP "arithmetic progressions in space" by the student PI. These organizational levels refer to the work done by the student in TN-PI.

Domains of Mathematical Activity

To analyse the activities of the process carried out by the student, we have used the concepts of domain of activity (in this case there is a domain D1) defined in Sect. 15.3, following Rabardel and Bourmaud (2003).

This idea includes the activity families (in the example four families were studied, $F_{1,i}$ with i varying between 1 and 4) and the kinds of situations (in the example there are various kinds of situations $C_{1,i}$, j with i varying from 1 to 4 and j varying from 1 and 4 depending on i). In Fig. 15.9, this is summarized.

Mathematical Operational Invariant and Mathematical Schemes

The final reports of the student PI and of the group of five students (RPA) have allowed the collection of the main operational invariants implemented by them and part of their schemes (psychological part of the instrument) (Table 15.1).

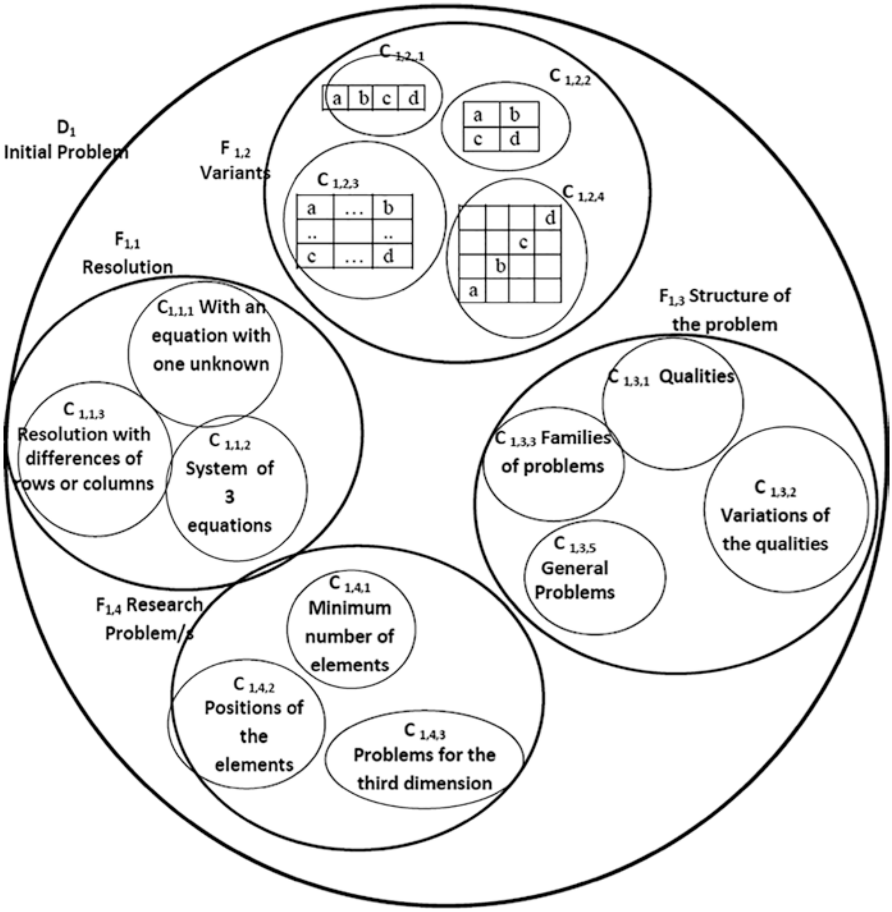


Fig. 15.9 Domains of mathematical activity for the initial problem of MRP "arithmetic progressions in space"

If these operational invariants are compared to the cognitive processes of the model by Yeo and Yeap (2009) which proposes a model for research tasks, characterizing, on one hand, the stages of student work and, on the other, the cognitive processes that occur and interactions that occur among them, we can see that in the passage from a RPT to a MRP, there are also these cognitive processes, along with others that allow extending the RPT to make explicit the research problem.

Main Mathematical Processes Implemented

As a result of reflection on the cases studied, Fig. 15.10, a first explanation is presented. It is a graphic approach, with the phases and key mathematical processes that the student can implement to reach an MRP.

Table 15.1 Mathematical operational invariants

Mathematical operational invariants
Particularization. Generic examples. Special cases
Patterns and invariants. Similarities, differences
Generalization. Abstraction in context
Conjectures: construction and justification
Mathematical proof
Mathematization processes-modelling
Variations on the problem. Strategies: what else? what if? etc.
Contextualization
Connections between contexts
Other heuristic strategies for problem-solving: number of cases, appropriate language (graphical, algebraic, etc.)

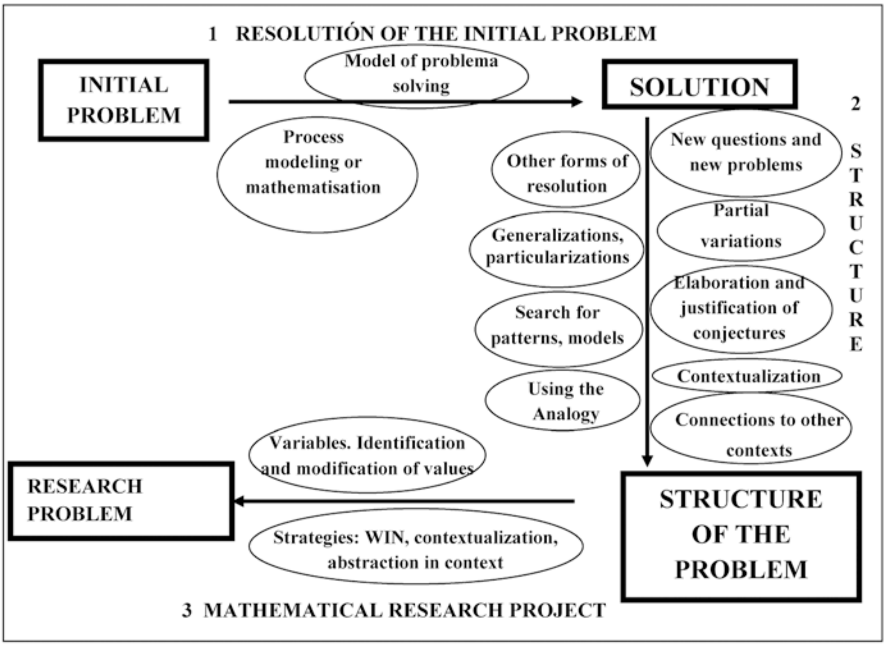


Fig. 15.10 Main mathematical processes implemented

Phases of the process The first phase is the initial problem resolution through processes of the mathematization-modelling (with its phases) or through the use of a problem-solving model (with its phases). The second phase is the search for the structure of the initial problem by deepening into some aspect of it (other forms of resolution, raising new questions about the problem, solving connected problems obtained by variations in its data, studying specific cases, looking for patterns, conjecturing, etc.). The third phase is the preparation of the research project, which is considered completed once it has been achieved to pinpoint the problem or research problems. This phase is very effective to analyse the free variation of the qualities of the initial problem, generalization of the context of the initial problem, use of appropriate strategies, etc.

Mathematical processes Mathematical cognitive processes that the student implements are, on the one hand, explicit academic knowledge that has been acquired in the process of teaching and learning. These are explicit in the student mind (reasoning, action rules, concepts, etc.). On the other hand, however, it also implements thinking processes that are not explicit knowledge, not formal, and not part of the scientific subject formalized knowledge or operational invariants, which are those that have been identified in the study of the cases. These mathematical processes are part of the activities carried out by the student in the process of extension from a PST to a MRP.

15.4.4 Mediations of the Teacher in Transforming PST into MRP

As we indicated in the introduction to the process of generating a Mathematical Research Project from a specific problem, it will not only require creativity on the part of the students but the mediation of the teacher for the establishment of a creative suitable working space. In this section, based on the theoretical foundations (presented in Sect. 15.3.1), teacher-experienced mediations in transforming a problem-solving task into a Mathematics Research Project (MRP) are described. In each of the experiments, carried out by the student PI and the group of five students, different types of mediations occur.

In Fig. 15.11, mediation lines that have a prominent role in each of the experiments can be seen: both for the student PI in the first experiment and in the case of group work in the second experiment:

(a) *Epistemic mediations*. In the first experiment, between the teacher (subject of activity) and the student (another subject), line (1) will allow identification, through the mediation of the epistemic type, mathematical processes that the student implements. Along this line, in the second experimentation, analogous to the above situation, the same occurs but in this case with five students, so that line (1) through the mediation of epistemic type will allow contrasting the mathematical processes implemented by the five students. These identified mathematical processes are part of operational invariant schemes of the students in both experiments. In this case, epistemic mediations can recognize the types of knowledge in action implemented by the student or by the group: to search for contexts related to the problem-solving variants of the initial problem, to establish analogies with other problems or other contexts, to generalize, to draw analogies with known formulas, etc.

Fig. 15.11 Mediated activity by the instrument

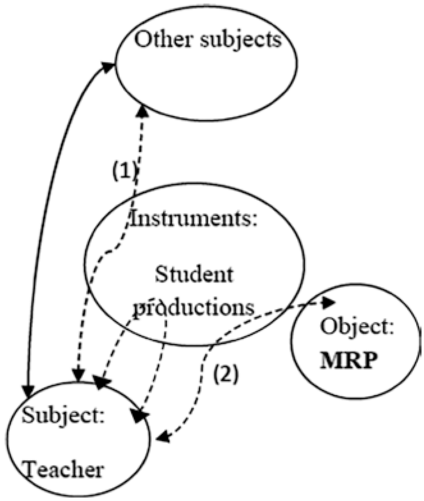


Table 15.2 Results of teachers' mediation work

Mediations	Results for the teacher
Towards the student (epistemic)	To make explicit the operational invariants (implicit) of their schemes To know the mathematical processes implemented
Towards the goal (pragmatic)	To deepen into the nature of the goal To know the structure (phases) of the process To describe the actions associated with each phase To provide dynamism to the process by using effective strategies

(b) *Pragmatic mediations*. In the first experiment, this type of mediation between the teacher (subject) and the MRP (object of activity), line (2) allows the teacher, through the pragmatic mediation, to identify the way to move from the initial task to achieve the object of the activity: MRP. The teacher values, above all, the appropriateness and effectiveness of the processes carried out by the student (in the case of the first experiment) and by the five students (in the case of the second experiment). In this case, the pragmatic mediations allow the teacher, on the one hand, to deepen into the nature of the object of the activity, i.e. to know better the process and requirements involved and, on the other hand, to know the adequacy and effectiveness of knowledge in action, implemented by students to achieve the object of the activity, i.e. for formulation of the research problem(s) corresponding to the MRP. This knowledge in action has been identified through epistemic mediations.

A summary of these mediations from the point of view of the teachers' work is presented in Table 15.2.

As shown in Table 15.2, the epistemic mediations (addressed to the student) allow the teacher to detect what kind of knowledge in action the student has, which is a key issue to later, through pragmatic mediation (directed towards the goal), characterize the process: phases and cognitive processes implemented by the student, which is the goal of experimentation.

Epistemic and pragmatic mediations by the teacher with both the instrument and the subject lead students to put into practice the procedural knowledge: strategies for problem-solving, posing questions, making connections between contexts, determination of the specific cases, detection of pattern, its extension and generalization, etc.

15.5 Didactic Considerations for Implementing a MRP

Creativity plays a vital role in the full cycle of advanced mathematical thinking. It contributes in the first stages of development of a Mathematical Research Project when possible conjectures are framed and in the formulation of the MRP.

Among the didactic considerations arising from this research, two are highlighted:

1. Detection of domains of activity and mathematical invariants
2. The mediated activity: teacher-instrument-subject

Regarding the first, we agree with authors who have indicated that generalization is a form of mathematical creativity (Ervynck, 1991; Kruteskii, 1976; Sriraman, 2009). Hence, we look at mathematical creativity essentially as the ability to create mathematical objects, together with the discovery of their mutual relationships. In the transformation of a problem-solving task (PST) into a Mathematical Research Project (MRP), creativity demands an extension of this context in a way that has not been conceived before. It therefore requires the individual to create new ideas and put old ideas together in a new way. It is something that can be carried out if it is involved not only in instrumental understanding but also in relational understanding, which involves a meaningful grasp of the relationships between the concepts.

In the passage from the PST to MRP, the student extends their schemes and gives two fundamentally different kinds of generalization: the expansive generalization, which broadens the applicability of the theory without changing the nature of the cognitive structure, and the constructive generalization which requires the knowledge structure to be reorganized. While the first one may be relatively easy, even when it occurs creatively for the first time, the latter involves a cognitive transition of great difficulty which requires not only special personal qualities to overcome the struggle but also a special mediation by the teacher.

Concerning the second, mediated activity tends to promote professional mathematical activity. The examples illustrate epistemic and pragmatic mediations. These mediations help in student's activity of meta-process; a meta-process, which acts upon and generates new mathematics that several authors, such as Poincaré or Hadamard, explicitly attempted to describe in their ideas relating to mathematical creativity. In addition, we have seen that there are certain requirements of mathematical creativity which seem to link with the full range of advanced mathematical thinking, for which a work of incubation and metacognitive processes are required, in our case accompanied by the teacher.

We summarize some specific considerations by way of practical suggestions for any teacher interested in implementing MRPs in the classroom. These are as follows:

1. Teachers interested in MRPs, or in other tasks which develop research processes, must consider a gradual approach, starting with mMRPs; with less complex tasks, open problems, and the extension of problem-solving tasks; or with research with an illustrative outline for the students. In all of these, the students will need to produce a report which brings together the process, the results, and their reflections.
2. The process of generating an MRP is one of the most fitting and simple ways for teachers to introduce secondary school students to the process of discovery in mathematics, putting into practice specific methods typical of professional mathematicians' everyday tasks.
3. The MRPs consist of tasks which are very valuable for the development of students' mathematical competencies: solving problems; thinking mathematically; reasoning with logical correctness; representing mathematical ideas using different language; modelling real-life situations; using codes, symbols, and mathematical notations; and communicating mathematical ideas.
4. The MRPs give the students a leading role in their education: they solve the problems which they pose to themselves; make decisions about the paths, strategies, and focuses of the investigation; reflect on the process and the results obtained; and communicate their ideas in a scientific document.
5. The teacher's analysis of the students' productions, their drafts and final report, is one of the most effective resources in the process of subject-object mediation of the activity. This strategy allows the teacher to identify the levels of organization which the student adopts in different situations and the operational constants of their schemes of thought.

Finally, the case described in this chapter shows a framework where the individual, the domain, and the mediated activity all work together to define the creative act (Csikszentmihalyi, 1996). The individual (each of these students) imbibes the contours of the domain, even while they seek to transform it. The inventiveness, as it is manifested in the activity of users confronted with a technique, i.e. the mediated activity (teacher-instrument-their peers), negotiates and demands high-quality work as defined by the domain – which in itself is transformed by the work of these individuals. Creativity is where the individual, the domain, and the mediated activity operate in harmony.

Acknowledgements This study was funded by the Spanish Ministry of the Economy and Competitive Affairs under project EDU2013-44047-P entitled "Characterization of specialized knowledge in Mathematics Teachers" and by special action grant from Cátedra UCM Miguel de Guzmán (Spain) under project "Mathematical Working Space" (UCM-CmdeGuzman-2015-01).

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Artigue, M. & Baptist, P. (2012). *Inquiry in Mathematics Educations*. The Fibonacci Project. France. www.fibonacci-proyect.eu

Artigue, M., & Blomboj, M. (2013). Conceptualizing inquiry-based in mathematics educations. *ZDM-The International Journal on Mathematics Education, 45*, 797–810.

Beguin, P., & Rabardel, P. (2000). Designing for instrument-mediated activity. *Scandinavian Journal of Information Systems, 12*, 173–190.

Blombreg, J., Suchman, L., & Trigg, R. H. (1996). Reflection on a work oriented design project. *Human–Computer Interaction, 11*(3), 237–266.

Braverman, A. (2010). *Systematic implementation of inquiry projects in secondary school mathematics for the development of students' creativity*. Doctoral Dissertation. University of Ben-Gurion in Negev.

Braverman A., & Samovol, P. (2008). Creativity: Transforming problem solving to the research. In M. E. Saul & M. Applebaum (Coords.), Proceedings of the 5th international conference on creativity in mathematics and the education of gifted students. (pp. 398–399). Haifa.

Csikszentmihalyi, M. (1996). *Creativity: Flow and the psychology of discovery and invention*. New York: Harper Collins.

De la Fuente, C. (2013). De la resolución de problemas a las investigaciones matemáticas. Algunas estrategias didácticas. *Uno - Revista de Didáctica de las Matemáticas, 64*, 91–100.

De la Fuente, C. (2016). Invariantes operacionales matemáticos en los proyectos de investigación matemática con estudiantes de secundaria. Doctoral dissertation. Complutense University of Madrid, Madrid.

De la Fuente, C., Gómez-Chacón, I. M., & Arcavi, A. (2015). Tareas de investigación matemática con estudiantes de Secundaria. In I. Gómez-Chacón et al. (Eds.), *Mathematical Working Space, Proceedings Fourth ETM Symposium* (pp. 555–574). Madrid: Publicaciones del Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid.

Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 42–53). Dordrecht, The Netherlands: Kluwer.

Getzels, J. W. (1987). Creativity, Intelligence, and Problem Finding: Retrospect and Prospect, In Scott G. Isaksen (Ed.) *Frontiers of Creativity Research: Beyond the Basics*, (pp. 88–102). Buffalo, NY: Bearly.

Gueudet, G., & Trouche, L. (2008). Du travail documentaire des enseignants : genèses, collectifs, communautés. *Éducation et didactique, 2*, 3 http://educationdidactique.revues.org/342. Accessed 15 Jan 2015

Gueudet, G., & Trouche, L. (2010). Ressources en ligne et travail collectif enseignant : accompagner les évolutions de pratique. In L. Mottier Lopez, C. Martinet, & V. Lussi (Eds.), *Congrès Actualité de la Recherche en Education* (pp. 1–10). Genève, Suisse: Université de Genève.

Gueudet, G., & Trouche, L. (2011). Renouvellement des ressources et de l'activité des professeurs, renouvellement du regard sur une profession. In *Colloque le travail enseignant au XXie siècle*. Lyon, France: INRP http://www.inrp.fr/archives/colloques/travail-enseignant/contrib/3.htm,

Guin, D., & Trouche, L. (2007). Une approche multidimentionnelle pour la conception collaborative de ressources pédagogiques? In M. Baron, D. Guin, & L. Trouche (Eds.), *Ressources numériques et environnements d'apprentissages informatisés: conception et usages, regards croisés* (pp. 197–228). Paris: Ed. Hermés.

Guzmán, M. (1991). Para pensar mejor. Barcelona: Edit. Labor.

Kruteskii, V. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Mason, J. (2010). Attention and Intention in Learning About Teaching Through Teaching. In R. Leikin & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 23–47). New York: Springer.

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking mathematically*. Wokingham, UK: Addison-Wesley.
Mishra, P., & Henriksen, D. (2014). Revisited and remixed: Creative variations and twisting knobs. *TechTrends, 58*(1), 20–23.
Palatnik, A., & Koichu, B. (2015). Exploring insight: Focus on shifts of attention. *For The Learning of Mathematics, 35*(2), 9–14.
Pastré, P. & Rabardel, P. (Eds). (2005). *Modèles du sujet pour la conception – Dialectiques activités développement*, Toulouse:Octarès.
Polya, G. (1957). *How to solve it?* New York: Doubleday Anchor Books Edition.
Rabardel, P. (1995). *Les hommes et les technologies, une approche cognitive des instruments contemporains*. Paris: Armand Colin.
Rabardel, P. (2001). Instrument mediated activity in situations). In A. Blandford, J. Vanderdonckt, & P. Gray (Eds.), *People and computers XV—Interaction without Frontiers* (pp. 17–30). New York: Springer-Verlag.
Rabardel, P. (2005). Instrument subjectif et développement du pouvoir d'agir. In P. Rabardel & P. Pastré (dir.), *Modèles du sujet pour la conception. Dialectiques activités développement* (pp. 11–29). Toulouse: Octarès.
Rabardel, P., & Béguin, P. (2005). Instrument mediated activity: From subject development to anthropocentric design. *Theoretical Issues in Ergonomics Sciences., 6*(5), 429–461.
Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*, 665–691.
Samurçay, R., & Rabardel, P. (2004). Modèles pour l'analyse de l'activité et des compétences: propositions. In R. Samurçay & P. Pastré (Orgs.), *Recherches en didactique professionnelle*, (pp. 163–180). Toulouse, France : Octarès.
Schoenfeld, A., & Kilpatrick, J. (2013). A U.S. perspective on the implementation of inquiry-based learning in mathematics. *ZDM – The International Journal on Mathematics Education, 45*(6), 901–909.
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
Schoenfeld, A. H. (1992a). Learning to think mathematically: Problem solving, meta-cognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.
Schoenfeld, A. H. (1992b). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53–70). Hillsdale, NJ: Erlbaum.
Sriraman, B. (2009). The characteristics of mathematical creativity. *ZDM-The International Journal on Mathematics Education, 41*(1&2), 13–27.
Sriraman, B., Haavold, P., & Lee, K. (2013). Mathematical creativity and giftedness: A commentary on and review of theory, new operational views, and ways forward. *ZDM-The International Journal on Mathematics Education, 45*(2), 215–225.
Vergnaud, G. (1996). Au fond de l'apprentissage, la conceptualisation. In R. Noirfalise & M.-J. Perrin (Eds.), *Ecole d'été de didactique des mathématiques* (pp. 174–185). Université Clermont-Ferrand II, France.
Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.
Winograd, T., & Flores, C. F. (1986). *Understanding computers and cognition: A new foundation for design*. Norwood, NJ: Ablex.
Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*, 89–81.
Yeo, J. B. W., & Yeap, B. H. (2009). *Investigating the processes of mathematical investigation*. Paper presented at the 3rd Redesigning Pedagogy International Conference, Singapoor.

Chapter 16
Mathematics Education and Creativity: A Point of View from the Systems Perspective on Creativity

Cleyton Hércules Gontijo

16.1 Introduction

The field of research in creativity is continuing to develop, and, because of this, several conceptions of creativity are evident in the literature. Lubart (2007) points out that multifaceted ways of analysing creativity have been proposed as it is "the result of a convergence of cognitive, conative and environmental factors" (p.16). This highlights the perspective that, rather than resulting from specific factors of each individual, creativity arises from a multiplicity of factors. According to Alencar and Fleith (2003a, p. 13), "it can be noticed that one of the main dimensions present on the several definitions of creativity implies the emerging of a new product, be it a new idea or an original invention, be it re-elaboration and perfection of existing products and ideas".

Studies investigating creativity aim to comprehend the variables that allow it to be expressed or inhibited, measuring it in order to establish strategies for its development. Studies usually focus on one of the elements involved in creative production. In this way it is possible to focus on the following perspectives (Feldhusen & Goh, 1995): the person (cognitive characteristics, emotional and personality qualities, life experiences), the product (verifying if it is new, has social value and utility, causes impact), the process (the stages of development of a creative product) and the environment (environmental elements involved in the promotion or inhibition of creative capabilities: physical, emotional, social and cultural factors, etc.).

Sternberg and Lubart (1999) emphasize that, in order to comprehend creativity, a multidisciplinary approach is needed. This is because isolated studies can only provide a partial and incomplete view of the phenomenon. According to Alencar and Fleith (2003b, p. 2), "in order to comprehend why, when and how new ideas are

C. H. Gontijo (✉)
Cleyton Hércules Gontijo, Universidade de Brasília, Brasilia, Brazil
e-mail: cleyton@unb.br

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_16

produced it is necessary to consider variables both internal and external to the individual".

In this sense, the aim of this chapter is to approach creativity in mathematics from the theoretical model of Csikszentmihalyi's (1988, 1996, 1999a, 1999b) systems perspective on creativity. This "systems perspective" relates the creative effort by individuals to the state of the domain in which they are working and to the characteristics of those who assess the worth of the creative endeavour in the field concerned. As such, it offers a way to generate a penetrating analysis of how a creative endeavour emerges within a social field.

16.2 Mathematical Creativity Analysed from the Systems Perspective

According to Csikszentmihalyi's systems perspective, creativity is not the result solely of an individual's action; rather, it emerges from the interaction between the individual and socio-historical-cultural environment. According to Csikszentmihalyi, creativity depends more on the social and cultural context than on the individual. Genetic differences and personal experiences may be involved, but these do not determine the expression of creative production. In this way, the systems perspective accepts the importance of individual characteristics for determinating creative production, but, nevertheless, it associates two other elements to the creativity domain and field, which, together, make its achievement possible. By Csikszentmihalyi's proposition, creativity is considered as resulting from the interaction of three systems: the person (genetic background and personal experiences), domain (cultural and scientific production) and field (social system). In the next section, these systems are described, pointing to a way of thinking creativity in mathematics based on this model.

16.2.1 Person

In the systems perspective, the person is approached by means of several aspects of their development and the relation between these and creativity. Nakamura and Csikszentmihalyi (2003) analysed three aspects of a creative person: their cognitive process, their personality and their values and motivation. Their view is that "every person is potentially creative" (p. 189).

Cognitive processes are related to psychological processes involved in coming to know, to understand, to perceive, to learn, etc. Nakamura and Csikszentmihalyi refer to the way the person deals with stimuli from external world: the way the subject sees and perceives, the way they register information and the way they add new information to previously registered data. Personality characteristics refer to curiosity, independence, positive self-concept, attraction for complex problems and lack of fear of taking risks. Motivation can be described by the interest, pleasure and satisfaction of having accomplished a task. It can also be seen when the person searches for information about their field of interest; they thus develop their domain capabilities. Another characteristic resulting from motivation is a person's capacity to take risks and break free from commonly employed styles of idea production (Amabile, 2001).

These characteristics may lead the person to a creative production, since environmental conditions are favourable to this production. That is why it is important to be in an environment which stimulates creative production, values the learning process, offers opportunities of accessing and updating knowledge and opens the way to mentors and resources such as books, computers, etc. Regarding the model being considered, which involves the person, the field and the domain, the person has the function of promoting domain variations.

Carlton (1959, quoted in Gontijo, 2007, p. 45), when specifically discussing individuals with mathematical creative potential, lists a set of characteristics to describe such individuals. The list includes aesthetic sensitivity to observe mathematical patterns and relations, capacity to solve and pose problems which go unnoticed by other people, willingness to work independent from teachers and other students, pleasure in communicating mathematical ideas, capacity to speculate or elaborate more than one hypothesis to a given problem, pleasure in adding something new to knowledge produced in the classroom or a different solution to a problem already solved, pleasure in working with mathematical language, the tendency to make generalizations, the capability to visualize an entire solution at once, the capacity to insert imagination into the process of producing mathematical ideas, conviction that every problem must have a solution, persistence in look for solution to the problems, boredom with repetitive activities and the capability to perform several mathematical operations in a short period of time, among others. According to Biermann (1985), one of the most important characteristics of creative mathematicians of the seventeenth to nineteenth centuries was fascination with the subject matter and consequent extreme motivation.

16.2.2 Domain

The domain is a body of formally organized knowledge that is related to a certain field of knowledge. Alencar and Fleith (2003b, p. 6), analysing this system in Csikszentmihalyi's work, affirm that "domain consists of a set of symbolic rules and procedures culturally established; that is, knowledge accumulated, structured, conveyed and shared in a society or by many societies". The function of the domain is preserving knowledge selected by a group of specialists (in the field) in order to convey it to new generations.

Mathematics is as an important field of knowledge that can contribute significantly to personal and scientific growth, granting the person the development of competences and capabilities that instrument and structure thought, enabling the person to comprehend and interpret situations; to master specific languages; to argue, analyse, evaluate and come to personal conclusions; and to make decisions and generalizations. At the same time, it provides the person with techniques and strategies to be applied to different sciences, including mathematics itself, and contributes to the advancement of knowledge and to the comprehension and resolution of everyday problems. As D'Ambrósio (2001) puts it, mathematics appeared as "a strategy developed by human species throughout its history to explain, to understand, to handle and deal with the sensitive, perceptive reality and its imaginary, naturally in a natural and cultural context" (p. 82).

According to Pais (2001), scientific knowledge is related to academic life and to knowledge produced by universities and research centres, in accordance with a formal, encoded language. School knowledge, in contrast, is constituted by knowledge and practices shared by a given community, where the mathematics content is determined by the curriculum and presented in teaching programmes and didactic books. By means of didactic transposition (i.e., "the work that makes an object of knowing how to teach into an object of teaching" (Chevallard, 1991, p. 39)), knowledge that constitutes the mathematics domain is made accessible to students.

Having access to this domain may enable the instrumentalization of the individual to use mathematics techniques and strategies that can be applied to the different sciences, including mathematics. This contributes to the advancement of knowledge and to the comprehension and solving of problems found in everyday life. It is paramount that students are able to exploit this domain, having the experience of constructing mathematics knowledge and not only reproducing what has historically been accumulated by humanity. In this sense, culture and scientific production may be accessible to the students so that they are able to interact with certain study objects, both to appropriate them and to change them in order to incorporate them to the domain of school mathematics.

When mathematics content is presented to the student isolated from real life, it becomes devoid of true educational meaning. When this connection with reality is lacking, the student's approach to mathematics becomes more complex or even difficult to happen (Freitas, 1999). Unfortunately, the way pedagogical work is sometimes conducted in schools has been causing in the students a lack of interest and indifference regarding this domain and creating, throughout the student's school life, a feeling of failure and inability to understand and solve mathematical problems (Gontijo, 2007).

16.2.3 Field

A field is composed of all those who can affect the structure of the related domain. Its primary function is preserving the domain as it is, and its second function is to select carefully new approaches to be incorporated to the domain. In each field of knowledge or of production (artistic, cultural, industrial, etc.), there is a group of specialists who, because of their experience and knowledge, are considered qualified to analyse and judge the elements that may be incorporated to the domain.

In mathematics, as in other fields of knowledge, the field is made up of different levels of specialists, from university researchers to teachers of basic schools; that is, it includes those who are producing and/or conveying knowledge by means of teaching at all the levels. Considering school mathematics, teachers represent the specialists who organize activities that make it possible for the students to enjoy the mathematical experience, and they are responsible for evaluating what the student produces. This way, representations and beliefs that these teachers may have regarding mathematics may result in an action that fosters mathematical creativity (this is besides, of course, the theoretical mastery they have that allows them to teach/judge properly).

Hence, teachers must develop competences to create an environment appropriate to mathematics learning. In developing these competences, there is a vital role for initial and continued teacher education to inform teacher behaviour in the classroom. For example, it is important that mathematics teachers have a clear view of the nature of mathematics, of mathematical activity, of mathematics learning and of what constitutes an environment suitable for learning mathematics (D'Ambrósio, 1993). These things are important because, in considering if students' mathematical work is creative, it is evaluated and validated by the teacher acting according to their conceptions, beliefs, values and attitudes. In order for the student's mathematical production to become learning and an expression of their creativity, it is necessary that the pedagogical work developed at schools stimulates the students. For Brousseau (1996), it is the teacher's duty to create a suitable situation that becomes a learning situation.

Cropley (1999, p. 634) identified some teacher behaviours that promote creativity:

> 1) possession of a fund of general knowledge; 2) knowledge of one or more special fields; 3) an active imagination; 4) ability to recognize, discover, or invent problems; 5) skill at seeing connections, overlaps, similarities, and logical implications (convergent thinking); 6) skill at making remote associations, branching out, seeing the unexpected, and so on (divergent thinking); 7) ability to think up many ways to solve problems; 8) a preference for accommodating rather than assimilating; 9) ability and willingness to evaluate their own work; ability to communicate their results to other people.

When dealing with mathematics field, Higginson (2000) considers that there are four prevalent conceptions in the teacher's practice as to how creativity is manifested in the classroom. The first sees creativity as a methodological resource to make the work dynamic. Thus, the class is considered creative when the teacher, for example, presents content to the students in an unusual, different and innovative way. The second considers as creative the construction of manipulative didactic materials, and the classroom becomes a laboratory to produce artefacts used to illustrate/demonstrate the mathematics being studied. The third relates to the atmosphere in the classroom, where the creativity is manifest when there are opportunities for uncovering ideas; that is, when students are allowed to express their ideas and interpretations of the mathematics with which they are working. The fourth refers to the

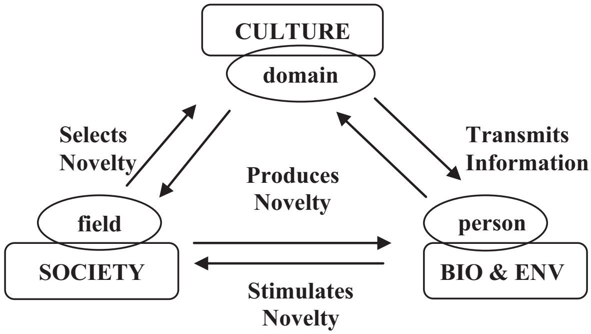


Fig. 16.1 The systems perspective. (Csikszentmihalyi, 1999a, p. 315)

activity of constructing symbolic models from problem situations. For that, problem-solving activities are proposed, and the students must suggest models of solution, indicating logical implications existing between the initial situation and variations performed until a solution is reached. Higginson (2000) affirms that, in educational practice, teachers mix these conceptions when they organize mathematics activities.

The systems perspective model considers that the person, the social system and the domain are marked by a dialectical interaction. Figure 16.1 illustrates the relation between the systems described by Csikszentmihalyi.

This suggests that the emerging of creativity in the process of teaching and learning mathematics depends on creating an environment favourable to mathematical activity – one that stimulates curiosity and enables learners' actions with mathematical objects. In the school environment, teachers and students are in permanent interaction, with learning as the primary goal. This interaction is facilitated by a dialectic contact (Brousseau, 2008) in which learners' social representations of mathematics determine their actions and guide their engagement in the work being done.

16.3 Some Strategies to Develop Creativity in Mathematics Education

The three systems proposed by Csikszentmihalyi to study creativity (the person's background and personal experiences, the domain of culture and scientific production and the field or social system) can be used to analyse creativity in mathematics. To stimulate creativity, it is helpful for teachers to be attentive to the experiences their students may have had, trying to identify elements that may have created a positive or negative stimulus as regards to mathematics and how these act in the construction of a positive representation of mathematics. It is also helpful for teachers to investigate the curriculum structure they teach in order to identify where it appeals to mathematical creativity and if its organization fosters creative processes. When teachers examine their own conceptions of mathematics and its teaching, this can help them to comprehend these in their dynamic characteristics whose essence is problem-solving.

Besides that, it is necessary to create a classroom environment favourable to the student's creativity, strengthening personality traits such as self-confidence, curiosity, persistence, independence of thought and courage to exploit new situations and deal with the unknown. Organizing the work in the classroom can be done in ways that aim at collaborating with students, helping them to let go of emotional blocks such as fear of making mistakes or being criticized and to overcome feelings of inferiority and insecurity (Alencar & Fleith, 2003a). The way in which the student (person), the teacher (field) and the knowledge (domain) interact in the classroom can favour the construction of a culture of learning and academic success, helping the creative production to happen.

In this sense, many strategies are being employed to assist the development of mathematical creativity. Studies favour problem-solving, problem posing and redefinition as didactic-methodological strategies that enable the development of mathematical creativity and at the same time allow evaluating this creativity (Gontijo & Fleith, 2014; Haylock, 1987). While using these strategies may not guarantee the development of creativity, it is worth considering each in turn.

16.3.1 Problem-Solving

Problems, according to Sarduy (1987), are situations that cannot be solved with automatic and direct use of previous experience. In other words, a genuine problem means that the person does not have immediate access to the answer but is forced to think and reason to find necessary knowledge that leads to the answer or, in a broad sense, to the solution of problem.

As Brito (2006) puts it,

> Solving problems is a complex form of combining cognitive mechanisms available from the moment the subject comes across a situation to which s/he must search alternatives of solution. It can be defined as a cognitive process aiming to transform a given situation in a situation directed to an objective, when an obvious method of solution is not available to the solver presenting four basic characteristics: it is cognitive, it is a process, it is directed to an objective and is personal because it depends on the person's previous knowledge. (p. 18)

Adopting problem-solving as a strategy for organizing the pedagogical work in mathematics enables the development of capabilities such as observation, establishing relations, communication, argumentation and validating processes, besides stimulating ways of reasoning such as intuition, induction, deduction and estimation. These capabilities are required in the students' everyday situations, which is when problem-solving demands a set of competences. This option has in it the conviction that mathematical knowledge acquires significance when students have challenging situations to solve and work to develop solving strategies. In this way even a simple problem may stimulate interest for mathematical activity as long as it raises in the student the interest for looking for solutions and stimulates curiosity, creativity and reasoning improvement, expanding mathematical knowledge.

A problem to motivate the student and arouse their creativity cannot be characterized as the direct application of an algorithm or formula but must involve invention and/or creation of any specific resolution strategy. The model suggested by Polya (1994) for solving problems inspires many that use this as a resource to conduct the mathematical learning process. The model presents four steps for solving a problem: (a) understanding the problem, (b) elaborating a plan of resolution, (c) executing the chosen strategies and (d) reviewing the solution.

Problem situations that aim to favour the development of mathematical creativity are formulated in contexts that make it possible for the student to generate numerous ways to resolve them. Thus, the preferred type of problem to develop creativity is the one that is often called an open-ended problem, that is, one that admits multiple solutions.

For solving open-ended problems, students must be responsible for decision-making, not leaving this responsibility to the teacher or to the rules and models presented in books (Gontijo, 2015). Deciding what kind of methods and/or procedures could be used can be based on previous knowledge and experiences students may have, especially resulting from prior works they had contact with or developed to solve similar problems. It is important to give the students the opportunity to construct their own models, test them and then come to the solution. It is also necessary to construct a strategy to communicate to schoolmates and teachers their experience in solving the problem, explaining the mental process and the way they revised the selected strategies to come to the solution. A successful communication depends on how deeply the student understood the problem. Nevertheless, successful communication enables the student to reflect about the methods chosen and, at the same time, how to use them in other problems and areas of mathematics.

An example of open-ended problem that could be posed to students in the field of geometry is formulated as follows:

> A rectangle has an area of 120 cm^2. Knowing that the length and width are whole numbers, answer: (a) What are the possible length and width of the rectangle? (b) What is the minimum perimeter possible?

Another example is the problem posed by Kattou, Kontoyianni, Pitta-Pantazi, Christou, & Cleanthous (2013) for a mathematical creativity test as it allows for multiple solutions:

> Look at this number pyramid. All the cells must contain one number. Each number in the pyramid can be computed by performing always the same operation with the two numbers that appear underneath it. Fill in the pyramid, by keeping on the top the number 35. Try to find as many solutions as possible (see Fig. 16.2).

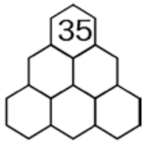


Fig. 16.2 Number pyramid. (Kattou et al., 2013, p. 172)

16.3.2 Problem Posing

Problem posing is described by Silver (1994) as the creation of a new problem or reformulation of certain problem presented to the students. Posing problems can happen before, during or after the solution of a problem. Problem posing must be based on concrete situations that express significant mathematical situations.

English (1997) considers that problem posing involves generating new problems and questions to explore a given situation and also reformulation of a problem during the process of solution. For the author, this strategy allows the teachers to have important insights about how the students understand mathematical concepts and processes, as well as their perception of the activities done and their attitudes regarding mathematics and the creative ability in mathematics.

To develop the capacity of posing problems, English (1997) points to three basic elements:

(a) Understanding what a problem is: this refers to the ability of recognizing a structure underneath a problem and finding these structures in corresponding problems, that is, realize which different problems have similar structures.
(b) Perceiving different problems: this element refers to aspects that may draw students' attention in everyday situations or not. Activities when students are able to express their perceptions regarding different problems and compare them to the different opinions their schoolmates may have can represent a powerful instrument to understand mathematics.
(c) Perceiving mathematical situation in different perspectives: interpreting a mathematical situation in more than one way is particularly important for the student to develop his/her ability to posing or reformulating problems.

Activities involving problem posing can be proposed in all teaching levels. For example, Silver and Cai (1996) presented, in a mathematical creativity test, the following situation for the students to create problems: "Write different questions that can be answered from the information: Jerome, Elliot, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliot. Elliot drove twice as many miles as Jerome. Jerome drove 50 miles" (Silver & Cai, 1996).

Teachers can also encourage students to formulate a new, different question about an existing mathematical problem related to the contents they are working with. Another activity can be asking the students to formulate mathematical problems based on a sport they may be interested in or in any everyday situation that they may be interested in investigating (Sternberg & Grigorenko, 2004).

16.3.3 Redefinition

This strategy consists in redefining a mathematical situation in terms of its features, in a varied and original way, coming up with many possibilities of representing it. Thus, students must be encouraged to, for example, present different ways of organizing numbers, objects and other significant elements based on their mathematical properties and features.

Redefinition activities are developed from preschool. Logic blocks are usually employed at this level of education, for they enable the children to come to different ways of classifying pieces of this material, take as a reference one of its features (shape, colour, thickness and size) or the combination of more than one feature.

Haylock (1985) suggested, in an activity to evaluate mathematical creativity in basic education students, that students took the whole numbers from 2 to 16 (including 2 and 16) and wrote the several subsets they were able to create with these numbers and indicated a rule for the formation of each of them, that is, indicating the characteristics numbers have and that allows that they form part of the same subset. This problem also represents an example of redefinition.

16.4 Final Considerations

In conclusion, mathematical creativity can be understood and stimulated taking into account the model proposed by Csikszentmihalyi, who considers creativity as a result of the interaction of three systems: person (genetic background and personal experiences), domain (culture and scientific production) and field (social system). Thus, in order to develop this creativity, attention needs to be paid to the experiences students have had, trying to identify elements that generate positive or negative stimuli in relation to mathematics and how they contribute to the construction of positive representation about this field of knowledge. The curriculum can be investigated to find out if its structure fosters mathematical creativity or if its organization encourages memorizing processes and reproduction of formulas and algorithms. It is also necessary to verify if the field members, who would be the teachers in school, understand that mathematics has a dynamic nature whose essence is problem-solving and that this strategy can help in educating students to be creative and competent to solve not only school problems but especially those that appear in their different everyday life contexts.

References

Alencar, E. M. L. S., & Fleith, D. S. (2003a). *Criatividade: múltiplas perspectivas* (2ª ed.). Brasília: Editora da Universidade de Brasília.

Alencar, E. M. L. S., & Fleith, D. S. (2003b). Contribuições teóricas recentes ao estudo da criatividade. *Psicologia: Teoria e Pesquisa, 19*(1), 1–8.

Amabile, T. M. (2001). Beyond talent: John Irving and the passionate craft of creativity. *American Psychologist, 56*(4), 333–336.

Biermann, K. R. (1985). Űber Stigmata der Kreativität bei Mathematikern des 17. bis 19. Jahrhunderts. *Rostocker Mathematik Kolloquium, 27*, 5–22.

Brito, M. R. F. (2006). Alguns aspectos teóricos e conceituais da solução de problemas matemáticos. In M. R. F. Brito (Ed.), *Solução de Problemas e a Matemática Escolar* (pp. 13–53). Campinas-SP: Editora Alínea.

Brousseau, G. (1996). Os diferentes papéis do professor. In C. Parra & I. Saiz (Orgs.). *Didática da Matemática: reflexões psicopedagógicas* (pp. 48–72). Porto Alegre: Artes Médicas.

Brousseau, G. (2008). *Introdução ao estudo da teoria das situações didáticas: conteúdos e métodos de ensino*. São Paulo: Ática.

Carlton, L. V. (1959). An analysis of the educational concepts of fourteen outstanding mathematicians, 1790–1940, in the areas of mental growth and development, creative thinking and symbolism and meaning. Doctoral thesis, Northwestern University, Illinois.

Chevallard, Y. (1991). *La transposition didactique: Du savoir savant au savoir ensigné*. Grenoble: La Pensée Sauvage.

Cropley, A. J. (1999). Education. In M. Runco & S. R. Pritzker (Eds.), *Encyclopedia of creativity* (Vol. 1, pp. 629–642). London: Academic Press.

Csikszentmihalyi, M. (1988). Society, culture, and person: A systems view of creativity. In R. J. Sternberg (Org.), *The nature of creativity* (pp. 325–339). New York: Cambridge University Press.

Csikszentmihalyi, M. (1996). *Creativity: Flow and the psychology of discovery and invention*. New York: Harper Collins.

Csikszentmihalyi, M. (1999a). Implications of a systems perspective for the study of creativity. In R. J. Sternberg (Org.), *Handbook of creativity* (pp. 313–335). New York: Cambridge University Press.

Csikszentmihalyi, M. (1999b). Creativity across the life-span: A systems view. In N. Colangelo & S. Assouline (Orgs.), *Talent development III* (pp. 9–18). Scottsdale, AZ: Gifted Psychology Press.

D'Ambrósio, B. S. (1993). Formação de professores de matemática para o século XXI: O grande desafio. *Pro-Posições, 4*(1), 35–41.

D'Ambrósio, U. (2001). *Etnomatemática: elo entre as tradições e a modernidade*. Belo Horizonte: Autêntica.

English, L. D. (1997). The development of fifth-grade children's problem-posing abilities. *Education Studies in Mathematics, 34*, 183–217.

Feldhusen, J. F., & Goh, B. E. (1995). Assessing and accessing creativity: An integrative review of theory, research and development. *Creativity Research Journal, 8*, 231–247.

Freitas, J. L. M. (1999). Situações Didáticas. In S. D. A. Machado (Org.) Didática da matemática: uma introdução. São Paulo: EDUC.

Gontijo, C. H. (2007). *Relações entre criatividade, criatividade em matemática e motivação em matemática de alunos do ensino médio*. Unpublished doctoral dissertation. Universidade de Brasília, Brasília, Brazil.

Gontijo, C. H. (2015). Técnicas de criatividade para estimular o pensamento matemático. *Educação e Matemática, 135*, 16–20.

Gontijo, C. H. & Fleith, D. S. (2014). Assessment of creativity in mathematics. In E. M. L. S. Alencar, M. F. Bruno-Faria, & D. S. Fleith. (Orgs.). *Theory and practice of creativity measurement* (pp. 65–84).Waco, TX: Prufrock Press Inc.

Haylock, D. W. (1985). Conflicts in the assessment and encouragement of mathematical creativity in schoolchildren. *International Journal of Mathematical Education in Science and Technology, 16*, 547–553.

Haylock, D. W. (1987). A framework for assessing mathematical creativity in schoolchildren. *Educational Studies in Mathematics, 18*, 59–74.

Higginson, W. (2000). *Creativity in mathematics education: The role of the teacher*. Paper presented at the 9th International Congress on Mathematical Education, Tokyo. Retrieved from: https://wwwmath.uni-muenster.de/u/meissne/WWW/creativity.crbh.doc

Kattou, M., Kontoyianni, K., Pitta-Pantazi, D., Christou, C., & Cleanthous, E. (2013). Connecting mathematical creativity to mathematical ability. *International Journal on Mathematics Education, 45*(2), 167–181.

Lubart, T. (2007). *Psicologia da criatividade*. Porto Alegre: Artmed.

Nakamura, J. & Csikszentmihalyi, M. (2003). Creativity in later life. In R. K. Sawyer (Org.), *Creativity and development* (pp. 186–216). New York: Oxford University Press.

Pais, L. C. (2001). *Didática da Matemática*. Belo Horizonte: Autêntica.

Polya, G. (1994). *A arte de resolver problemas: um novo enfoque do método matemático*. (Trans. Heitor Lisboa de Araújo). Rio de Janeiro: Interciência.

Sarduy, A. F. (1987). *Bases psicopedagógicas de la enseñanza de la solución de problemas matemáticos en la escuela primaria*. La Habana: Editorial Pueblo e Educación.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*, 19–28.

Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*, 521–539.

Sternberg, R. J., & Grigorenko, E. L. (2004). *Inteligência plena: Ensinando e incentivando a aprendizagem e a realização dos alunos*. Porto Alegre: Artmed.

Sternberg, R. J. & Lubart, T. I. (1999). The concept of creativity: Prospects and paradigms. In R. J. Sternberg (Org.), *Handbook of creativity* (pp. 3–15). New York: Cambridge University Press.

Chapter 17
Openness and Constraints Associated with Creativity-Directed Activities in Mathematics for All Students

Roza Leikin

17.1 Creativity as a Major Twenty-First-Century Skill

Modern society has been undergoing exponential innovative changes in all areas of life including sciences, technology, engineering and medicine. Computers, cell phones and Internet-based technologies influence our life endlessly. One of the central roles of the educational system is to allow people to adapt fluently to these changes and innovations. Obviously, mathematics is at the basis of technological progress, but school education should also ensure – along with the development of deep and robust mathematical knowledge – students' creative abilities. Moreover, teaching and learning of mathematics can and should make use of the advantages provided by technological progress to achieve this goal.

The NRC (National Research Council) committee defined a system of "transferable knowledge and skills in the 21st century" (Pellegrino & Hilton, 2012). The report defines three major dimensions of the twenty-first-century skills – cognitive, intrapersonal and interpersonal competences – which are interwoven and support each other in their development. Cognitive competencies include clusters of processes and strategies, knowledge and creativity associated with complex problem-solving. Strategical reasoning is described in terms of critical thinking, argumentation, decision-making and adaptive learning. The knowledge cluster includes information and ICT literacy. Creativity is one of the basic twenty-first-century cognitive skills that determine career readiness.

Intrapersonal competences encompass positive self-evaluation, self-regulation, responsibility and appreciation for diversity. Furthermore, intellectual openness, flexibility, adaptability, intellectual interest and curiosity, which are usually

R. Leikin (✉)
Department of Mathematics Education, Faculty of Education RANGE (Research and Advancement of Giftedness and Excellence) Center, University of Haifa, Haifa, Israel
e-mail: rozal@edu.haifa.ac.il

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_17

considered acute characteristics of a creative personality, are among the twenty-first-century intrapersonal skills. Interpersonal skills, e.g. teamwork and collaboration, have to be developed along with cognitive and intrapersonal competencies. These skills are critical for everyday life as well as for social, scientific and technological progress.

I call *creativity-directed activity* any activity which is aimed at developing, supporting or evaluating creativity skills and other accompanying (above-mentioned) skills. I argue that any creativity-directed activity *is a powerful tool for the advancement of twenty-first-century skills* in all the dimensions. I address the rich collection of chapters published in this section of the book in light of this argument. I discuss critical features of the Creativity-Directed Activities addressed by the authors. Finally, I suggest a framework for the analysis of types of openness, constraints and insights embedded in the creativity-directed mathematical activities and explain that creativity-directed activities are challenging – that is, cognitively demanding and interesting for the participants.

17.2 Components of Creativity-Directed Activities According to the Chapters Observed Here

Each of the eight chapters presents a unique view on creativity, while at the same time, they have many features in common. First, all chapters meaningfully intersect when it comes to the literature review: they address Guilford (1956) for divergent and convergent thinking and, with or without explicit reference Torrance, 1974, for the tetrad consisting of fluency, flexibility, originality and elaboration as integral components of the creative process. Second, within the creativity-knowledge paradox (Leikin, 2016), the authors accept the necessity of the knowledge base for creative production and consider creativity-directed activity to be a springboard for the development of mathematical knowledge.

The word cloud (created at https://www.jasondavies.com/wordcloud/) applied to the text of all the chapters in this section (Fig. 17.1) demonstrates that when the

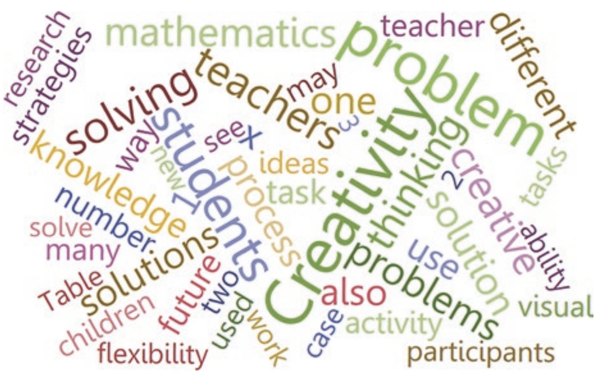


Fig. 17.1 Word cloud applied to the text of all the chapters in this section

authors analyse creativity-directed mathematical activities, both the teachers and the students who are involved in solving problems of different types play a central role. Furthermore, they address students' creative ability as related to their knowledge, visual skills and ability to comprehend different strategies and ways of solving the problems as well as flexibility and the raising of new ideas. Finally, in line with the vision of the goals of education in the twenty-first century, the authors agree on the importance of creativity for students' futures.

Like any activity, creativity-directed activities have particular motives and goals and are conducted under specific conditions (Leontiev, 1983). The activities are composed of actions and operations as determined by goals and conditions. The actions and operations (tasks and problems) are framed by a theoretical perspective on creativity chosen by facilitators of the activities. In spite of the differences in specific goals, conditions and groups of participants, the unifying motive of all the authors in this section is their commitment to the integration of creativity-directed activities in mathematics education for all and their understanding that creativity is a skill that can be learned and advanced.

Carreira and Amaral (Chap. 9) aimed to *develop a model for the evaluation of creativity* in the solving of nonroutine problems by young students. The study was conducted in the conditions of mathematical outside-the-classroom, web-based problem-solving competitions. The goal of the activity was both development and evaluation of students' creativity. Correspondingly, students' ability to produce solutions that were unique relative to their group served as an indicator of the originality of the solutions. Carreira and Amaral introduce a model for the evaluation of creativity which integrates dimensions of originality, activation of mathematical knowledge and activation of forms of representation. The authors devise their original visual representation of this three-dimensional model for the assessment of creativity.

Tabach and Levenson (Chap. 10) *demonstrate the power of mathematical tasks* with an *infinite number of solutions* for eliciting mathematical creativity. They examined problem solutions produced by adults with a range of mathematical backgrounds and resources within a time constraint of 5 min. Analysis of solutions was based on the diversity of problem-solving strategies and outcomes. Vale, Pimentel and Barbosa (Chap. 11) aimed to demonstrate the *power of visual solutions* for advancing creativity, deepening mathematical understanding and reasoning and developing problem-solving abilities. A problem-solving strategy that leads to a visual solution, named *seeing*, serves as a complementary strategy and thus requires flexible reasoning. Visual solutions are also shown to be connected to pattern recognition and generalisations, and the originality of solution is based on the ability to "think outside the box". Bokhove and Jones (Chap. 13) discuss the stimulation of mathematical creativity through *constraints in problem-solving*. They examine and exemplify the power of implementation of "constraint-based" task design in creativity-directed activity. They argue that constraints stimulate creative thought and describe several types of constraints accompanied by examples of constraint-based mathematical tasks. In this context, the requirement to produce a visual solution can be seen as a creativity-eliciting constraint of a task. The authors introduce the concept of potential creativity of a task and claim there is a continuum of openness which is relative with respect to personal experiences.

Vanegas and Giménez (Chap. 12) work with prospective mathematics teachers in the purpose of developing their *proficiency in creativity-directed teaching*. The authors develop what they call "professional asks" for teachers in which they are asked to evaluate student dialogues and responses. The goal of this study was to explore prospective teachers' conceptions of creativity. The authors find that the teachers develop understanding of the importance of the use of multiple representations as well as imagery for the development of students' mathematical competences and creativity. Three types of imagery were identified: kinaesthetic imagery, pattern images and changing representations.

Moore-Russo and Demler (Chap. 14) explore *educators' conceptions of creativity* at the gifted mathematics programme housed on a university campus. They suggest a multiphase coding for the analysis of teachers' conceptions of creativity based on four theoretical dimensions. The first dimension distinguished between product, process and subjective experience; the second dimension focused on fluency, flexibility and novelty; the third dimension included such perspectives as aesthetic, free market, scholarly and uncertainty; and the fourth dimension integrated instructional design principles for guiding creativity.

Gómez-Chacón and de la Fuente (Chap. 15) aimed to *develop teachers' creativity* and skills in the management of creativity-directed activities. They asked teachers to transform problem-solving tasks into a mathematical research project. Two creativity-directed activities can be observed in this work: one is task transformation itself, and the other is performing mathematical research projects. Three theoretical lenses for considering creativity are implemented: (1) creativity is defined, as a product, a process or an experience; (2) fluency, flexibility and novelty; and (3) principles that allow to maximise creativity. Gómez-Chacón and de la Fuente analyse types of mediation that support creativity-directed activity in students. Gontijo (Chap. 16) takes a *systems perspective on creativity* with a focus on the interaction between the three systems – the person, the domain and the field – with problem-solving, problem posing and redefinition being major actions in creativity-directed activities.

17.3 Openness and Constraints in Creativity-Directed Activities

While there is a consensus that the openness of the task is a critical feature of creativity-directed activity, Bokhove and Jones (Chap. 13), uniquely for this collection of works, explicitly adapt a position according to which constraints affect creativity-directed activities, actions and operations due to constraints being integrated in the definition of creativity: a requirement for novel and useful products. Constraints can be associated with domain, cognitive processing, variability, time limitations, availability of resources, tasks and personal talent. Despite the seemingly paradoxical contrast between the request for openness inherent to creativity-directed activities and the power of constraints in provoking creative reasoning, they seem to be two sides of a coin; the openness of a task can also be considered a particular type of task-design constraint of the creativity-directed activities. For example, the requirement to produce visual solutions in Vale et al. (Chap. 11) can be seen either as a constraint or as an opening to alternative ways of solution.

An interesting comparison of openness combined with constraint can be seen in the two following problems and I assume can be used to explain the power of constraints in creativity-directed activities.

Problem 1
In 5 min construct multiple polygons with an area of 15 sq. units (Tabach & Levinson, Chap. 10).

Problem 2
Construct all possible rectangles for which the area is 120 sq. units, and the sides' lengths are whole numbers of units (Gontijo, Chap. 16).

Both problems are combined open-start and open-end tasks. The openness of Problem 1 is related to the possibility of constructing geometrical figures of different types using available resources (e.g. paper and pencil and DGE). It includes time constraints only. Then the discussion of the collective solution space can be focused on the variety of answers and their rareness. I would suggest that a discussion of classes of figures and of the systematic strategies for solving this problem could raise mathematical challenge of the activity and, probably, allow for a process of generalisation. I assume that a requirement of describing possible systematic strategies for solving this problem could be of interest for participants. However, this definitely cannot be performed within a 5-min time limit.

Problem 2 seems to be "less open", though it is also a combined open-start and open-end task. It includes constraints related to the type of figures (rectangles) that should be constructed and the length of sides (whole number of units). This task allows flexible and systematic reasoning associated with different problem-solving strategies. Factorization of 120 into 2 factors can be performed in different ways, and the whole solution space can serve as an exciting base for the discussion of the fundamental theorem of arithmetic as a connector for all the produced solutions.

The two problems are definitely creativity-evoking and challenging. However, the different constraints included in these problems lead to different goals that can be attained by means of solving them.

The use of open problems in mathematics education has a long history. In 1993, Pehkonen led a PME discussion group (PME-17 in Japan) and then edited a special issue of ZDM in which four articles (Nohda, 1995; Pehkonen, 1995; Silver, 1995;

Stacey, 1995) analysed the use of open problems in mathematics teaching and learning. In this special issue, Nohda and Silver argued that the main goal of using open problems is promoting creativity and mathematical thinking. Later Silver (1997) explicitly connected problem-solving and problem posing with Torrance's (1974) component of creativity – fluency flexibility and originality.

Open problems include a broad range of mathematical problem types: "investigation problems, problem-posing tasks (e.g., problem variations, using the what-if-not method), real-life situations, projects, problem fields (problem sequences), and problems without questions" (Pehkonen, 1995, p. 55). Pehkonen suggested considering open problems as an "umbrella class" comprising problems of three main types, depending on the openness of the starting situation (open-start problems) and openness of the goal situation (open-end problems) defined by the task. Table 17.1 systematizes types of openness and constraints embedded in creativity-directed activities. It includes references to corresponding types of activities described in this book section.

Table 17.1 Types and examples of open creativity-directed tasks

Types of tasks		Task requirements	Examples of tasks can be found in[a, b]
Open-end	Multiple answer tasks	Finding all possible outcomes (for a finite number of solutions)	Gontijo (Chap. 16): Problem 2 above
		Finding as many answers as possible with a time constraint (for tasks with infinite number of answers)	Tabach & Levinson (Chap. 10): Problem 1 above
		Task the solution which is an inequation	Verschaffel and De Corte (1997): birthday party problem, school distance problem
Open-start	Nonroutine problems	Solving "new" problem (applying knowledge in a new situation)	Carreira and Amaral (Chap.9): *solving Olympiad problems*
	Insight-based problems	**Insight-based due to the newness of the problem**	Monck problem http://bredemeyer.com/RequisiteVariety/?p=154
	Multiple solution tasks	Solving/proving problems using multiple strategies base on: Different representations Different theorems/ properties Different tools Concepts from different fields **Insight-based solutions are possible**	Vale, Pimentel and Barbosa (Chap. 11): *visual solutions* Gómez-Chacón and de la Fuente (Chap. 15): *multiple representations* Bokhove and Jones (Chap. 13): *alternative solutions under constraints* Krutetskii 1968/1976 (e.g. Gilya's solutions, p. 208) Leikin (2007, 2009a, 2009b, 2016)

(continued)

Table 17.1 (continued)

Types of tasks		Task requirements	Examples of tasks can be found in[a, b]
Open-end and open-start combined	Multiple solution/answer tasks	Different strategies and different answers are possible	Tabach & Levinson (Chap. 10): Problem 1 Gontijo (Chap. 16): Problem 2
	Sorting tasks	Sorting criteria chosen by participants, different sorting criteria lead to different groups of objects	Zaslavsky and Leikin (2004)
	Problem posing	Posing problems (that include real-life situations) to match a mathematical model	Singer, Ellerton, and Cai (2015)
		Problem transformation	Gómez-Chacón and de la Fuente (Chap. 15): *transforming problem-solving tasks into mathematical research project*s
	Mathematical investigations research projects	Discovering different properties using different investigation strategies	Gómez-Chacón and de la Fuente (Chap. 15): *transforming problem-solving tasks into mathematical research*
		Insight-based solutions are often	Leikin (2015)

[a]The table does not cover all the possibilities
[b]There was no possibility to cite all the work in the field

Solving open problems makes a strong contribution to both the development of mental flexibility and openness and the development of problem-solving expertise. Expertise is a result of deliberate practice that leads to the acquisition of huge amounts of relevant knowledge and its systematic storage, which allows for better use of rapid processing as expressed in the ability to make fluent and flexible use of strategy-based processes as and when required (Ericsson & Lehmann, 1996; Schoenfeld, 1992). Creativity (it is commonly agreed) is expressed in the production of original work that is useful and adaptive (Sternberg & Lubart, 2000). It is associated with invention and insight, which is an experience of suddenly realizing "how to solve a problem" (Hadamard, 1945; Polya, 1954/1973).

The importance of creativity for child development was stressed by Vygotsky (1930/1982). Creativity (or "imagination" in Vygotsky's words) is the central mechanism in the development of children's knowledge and skills and thus in the development of expertise. Vice versa, the creative process requires deep and broad knowledge and rich experience since it presumes to expand existing knowledge into new territory, on the basis of one's existing knowledge. Moreover, according to Vygotsky, everything the imagination creates is always based on elements taken from a person's previous experience. These two connections between creativity and knowledge base create the *creativity-expertise paradox* (Leikin, 2016): one cannot be creative without a sufficient level of knowledge and expertise, while creativity is the main mechanism of knowledge development.

The creativity-expertise paradox emphasizes the importance of the integration of creativity-directed activities in the deliberate mathematical practices used in school mathematics. This integration requires that a distinction be made between absolute and relative creativity (Leikin, 2009a). *Absolute creativity* is associated with "historical works" as termed by Vygotsky (1930/1982, 1930/1984); *relative creativity* is associated with creativity *in* mathematics students and teachers.

The relativity of mathematical creativity in the education of students and teachers can be of different types that can be found in the chapters here – originality of solution (either problem-solving strategy or problem-solving outcome) can be evaluated *relative to*:

- A person's *previous experience* in his/her educational history
- A person's *reference group*, when a solution is unique within the collective solution space (Leikin, 2007)

Relativity of the first type is associated with mathematical insight linked to the implementation of known strategies in an unfamiliar situation or in an unexpected way and usually leads to the production of ideas that are new to the individual. Relativity of the second type is related to the uniqueness of the solutions, questions asked or ideas on the group level. In this sense, collective solution spaces are the major source for the development of individual solution spaces and thus of individual creativity and expertise. The two types of relativity are intertwined and reflect mutual relations between the development of creativity and expertise in mathematics and beyond.

17.4 Creativity-Directed Activity in Mathematics as a Springboard for the Development of Twenty-First-Century Skills

Mathematical challenge – defined as mathematical difficulty that a person is able and motivated (interested) to overcome at a particular moment – is an integral component of high-quality mathematical instruction. Analysis of creativity-directed activities above demonstrates that they are challenging due to the openness, constraints and mathematical insight embedded in the creativity-directed mathematical tasks. Levav-Waynberg and Leikin (2012) demonstrated that systematic implementation of multiple solution tasks in geometry significantly advanced learning achievements, fluency and flexibility. This finding emphasizes the importance of the integration of creativity-directed activities in mathematical instruction. The openness of the tasks also serves as a basis for differentiated instruction that is fitted to students' needs. Note that development of originality and of the ability to produce insight-based solutions is more complex. Our experience demonstrates that while flexibility and fluency are dynamic characteristics, which can be developed, originality seems to be related more to general intelligence (Lev & Leikin, 2017), and thus insight-based tasks require special attention and mediation.

Interestingly, creativity-directed activities not only constitute an effective tool for the development of cognitive skills but also seem to promote the development of intrapersonal and interpersonal competences. Pellegrino and Hilton (2012) recommend using multiple and varied representations of concepts and tasks, encouraging elaboration, questioning and explanation, engaging learners in challenging tasks and priming student motivation. All these activities are obviously described in the chapters observed here. When each student is provided with the opportunity to present their solutions to others and when a diverse array of solutions is discussed in the classroom, collective solution spaces become a major source for the development of individual solution spaces. Thus, when accompanied by teamwork and collaborative discussion, the suggested approaches and activities become a powerful tool for the development of responsibility, appreciation for diversity, intellectual openness, intellectual interest and curiosity – the twenty-first-century intrapersonal skills. More systematic, big-scale studies that explore creativity-directed instructional practices and devise recommendations on the effective integration of creativity-directed tasks in teaching and learning mathematics are needed.

A decade ago, a review of research on creativity in mathematics (Leikin 2009b) demonstrated that publications on creativity in mathematics were rare and the research on mathematical creativity was overlooked. The same review revealed a gap between the research on creativity as a general psychological phenomenon and research on mathematical creativity. Luckily, the situation has changed during the past decade. Several volumes were issued since then (REF). Mathematics educators have paid more and more attention to the nature and nurture of creativity in school mathematics and beyond, and more connections are drawn between theories in mathematics education and theories of creativity. The current collection is an exciting example of an additional meaningful step in the development of creativity-directed research and practice in mathematics education. I am certain that readers will enjoy the diverse range of creative ideas about creativity-directed activities and will use them in their practice.

References

Ericsson, K. A., & Lehmann, A. C. (1996). Expert and exceptional performance: Evidence of maximal adaptation to task constraints. *Annual Review of Psychology, 47*(1), 273–305.

Guilford, J.P. (1956) The Structure of Intellect. *Psychological Bulletin, 53*, 267–293.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Dover Publications.

Krutetskii, V. A. (1968/1976). *The psychology of mathematical abilities in schoolchildren*. (Translated from Russian by Teller, J.; Edited by J. Kilpatrick & I. Wirszup). Chicago, IL: The University of Chicago Press.

Leontiev, L. (1983). Analysis of activity. Vol. 14: *Psychology*. Moscow State University: Vestnik MGU.

Leikin, R. (2007). *Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks*. The Fifth Conference of the European Society for Research in Mathematics Education – CERME-5. (pp. 2330–2339) (CD-ROM and On-line). Available: http://ermeweb.free.fr/Cerme5.pdf

Leikin, R. (2009a). Bridging research and theory in mathematics education with research and theory in creativity and giftedness. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 383–409). Rotterdam, the Netherlands: Sense Publishers.

Leikin, R. (2009b). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students. (Ch. 9)* (pp. 129–145). Rotterdam, The Netherlands: Sense Publisher.

Leikin, R. (2015). Problem posing for and through investigations in a dynamic geometry environment. In F. M. Singer, N. Ellerton, & J. Cai (Eds.), *Problem posing: From research to effective practice* (pp. 373–391). Dordrecht, The Netherlands: Springer.

Leikin, R. (2016). Interplay between creativity and expertise in teaching and learning of mathematics. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the psychology of mathematics education Vol. 1* (pp. 19–34). Szeged, Hungary: PME.

Lev, M., & Leikin, R. (2017). The interplay between excellence in school mathematics and general giftedness: Focusing on mathematical creativity. In R. Leikin & B. Sriraman (Eds.), *Creativity and giftedness: Interdisciplinary perspectives from mathematics and beyond* (pp. 225–238). Cham, Switzerland: Springer.

Levav-Waynberg, A., & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *Journal of Mathematical Behavior, 31*, 73–90.

Nohda, N. (1995). Teaching and evaluating using "Open-Ended Problem" in classroom. *ZDM-International Journal of Mathematics Education, 27*(2), 57–61.

Pehkonen, E. (1995). Introduction: use of open-ended problems. *ZDM-International Journal of Mathematics Education, 27*(2), 55–57.

Pellegrino, J. W., & Hilton, M. L. (2012). *Educating for life and work: Developing transferable knowledge and skills in the 21st century. National Research Council*. Washington, DC: The National Academies Press. https://doi.org/10.17226/13398

Polya, G. (1954/1973). *How to solve it; A new aspect of mathematical method*. Princeton University Press, Princeton, N.J.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In *Handbook of research on mathematics teaching and learning* (pp. 334–370). Reston, V.A.

Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *ZDM-International Journal of Mathematics Education, 27*(2), 67–72.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM-International Journal of Mathematics Education, 29*(3), 75–80.

Singer, F. M., Ellerton, N., & Cai, J. (Eds.). (2015). *Mathematical problem posing: From research to effective practice*. New York: Springer.

Stacey, K. (1995). The challenges of keeping open problem solving open in school mathematics. *ZDM-International Journal of Mathematics Education, 27*(2), 62–67.

Sternberg, R. J., & Lubart, T. I. (2000). The concept of creativity: prospects and paradigms. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 93–115). Cambridge, UK: Cambridge University Press.

Torrance, E. P. (1974). *Torrance tests of creative thinking*. Bensenville, IL: Scholastic Testing Service.

Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modelling in the elementary school. A teaching experiment with fifth graders. *Journal for Research in Mathematics Education, 28*, 577–601.
Vygotsky, L. S. (1930/1982). Imagination and its development in childhood. In V. V. Davydov (Ed.), *General problems of psychology. The collected works of L. S. Vygotsky Vol. 2* (pp. 438–454). Moscow: Pedagogika (in Russian).
Vygotsky, L. S. (1930/1984). Imagination and creativity in adolescent. In D. B. Elkonin (Ed.), *Child psychology. The collected works of L. S. Vygotsky Vol. 4* (pp. 199–219). Moscow: Pedagogika (in Russian).
Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher-educators: Growth through practice. *Journal of Mathematics Teacher Education, 7*, 5–32.

Part III
Affect and Aesthetics in Mathematical Problem Solving

Chapter 18
Students' Attitudes in a Mathematical Problem-Solving Competition

Nélia Amado and Susana Carreira

18.1 Introduction

In this chapter, we present part of the research work developed in one of the strands of the Problem@Web Project that aimed to study attitudes and emotions related to mathematics and mathematical problem solving, both in school and beyond-school activities, involving students, parents, and teachers (Carreira, Jones, Amado, Jacinto, & Nobre, 2016). Here, we will be reporting part of the research results related to the attitudes revealed by the young participants in the mathematical competitions SUB12 and SUB14 towards mathematics, problem solving, in the context of their participation in those web-based inclusive competitions.

The students' participation in inclusive mathematics competitions such as SUB12 and SUB14 stands out as a fruitful field of affect involving mathematics (Amado, Carreira, & Ferreira, 2016), similar to what has been observed in other contexts with comparable characteristics (Applebaum & Leikin, 2014; Freiman & Lirette-Pitre, 2009; Freiman & Manuel, 2015). Our approach to the study of student attitudes moves away from the more traditional trend of research, which is very focused on searching for causal relationships between attitude and achievement and looking at mathematics from the point of view of a school activity (Di Martino & Zan, 2015). Instead, as highlighted in Fig. 18.1, our approach intends to study a *combination* of attitudes towards (i) mathematics, (ii) mathematical problem solving, and (iii) engagement with challenging mathematics within an activity that takes place beyond school.

In this way, we aim to get a better understanding of the attitudes of the young participants based on their participation in the competitions, by addressing different

N. Amado (✉) · S. Carreira
Universidade do Algarve and UIDEF, Instituto de Educação, Universidade de Lisboa, Lisbon, Portugal
e-mail: namado@ualg.pt; scarrei@ualg.pt

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_18

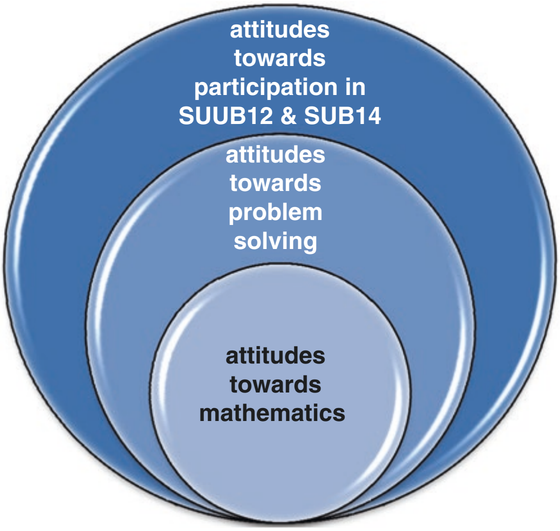


Fig. 18.1 A threefold perspective for the study of attitudes regarding mathematics and problem solving in a beyond-school context

dimensions involved, in an interconnected way, and by resorting to several kinds of data, which include interviews and questionnaires, together with spontaneous statements and reactions expressed in their e-mail messages generated in the course of the web-based interaction that consistently takes place in each edition. We expect this broadens our knowledge on the way attitudes relate to students' engagement with positive experiences with mathematics and mathematical problem solving beyond school.

Typically, mathematics and problem solving are associated with negative attitudes such as rejection, disinterest, or disaffection on the part of a large number of students (Freiman, Vezina, & Gandaho, 2005; Mata, Monteiro, & Peixoto, 2012). Along with such dominant picture, much attention has been given to the effects of attitudes on school performance and school attainment (OECD, 2013). However, little is yet known about other contexts of mathematical activities that are dissociated from the issue of school achievement but not indifferent to the issue of creating affective bonds with mathematics. As such, this project becomes particularly relevant insofar as it is more related to engagement with mathematics than to school performance, as it focuses on inclusive competitions that welcome young participants with different levels of achievement in school mathematics. Besides, it brings to the foreground the problem-solving activity as a way to attract students to the world of mathematics, both highest and lowest achievers. Recent reports (OECD, 2013) have already drawn attention to the weak involvement of students with mathematical activities outside the classroom, in pointing out that the vast majority of students only participate in such activities when these are a school requirement.

18.2 Underlying Aspects of the Research Field

The research field of this study – the mathematical problem-solving competitions SUB12 and SUB14 – undertakes particular characteristics that bear relevance to the specific domain of students' attitudes. First and foremost, it is inspired by the concept of mathematical challenge, and it embodies the idea of inclusion within a mathematical competition.

In this section, we start by clarifying our understanding of mathematical challenges and inclusive mathematical competitions.

18.2.1 Mathematical Challenges

The idea of mathematical challenges has been discussed by several authors regarding the different contexts in which they may arise. Although not a new discussion, it has been accentuated in recent years, mostly in light of the relevance, that this type of mathematical activity has aroused, both among students and teachers and more generally in society (Kenderov et al., 2009).

According to Protasov et al. (2009), mathematical challenges have always been present throughout history, in a more or less formal way, sometimes as a means to encourage the ingenuity and insight of those who were incited to solve difficult problems and could be granted some reward. This practice of putting mathematical challenges to others has taken on different conventional forms, being one of them the largely disseminated promotion of mathematical competitions directed at students. In fact, nowadays the presence of mathematical challenges has to be considered both in the classroom and beyond, including in public events. The recognition of its importance may be seen as reflected, for example, in the publication of the *Challenging Mathematics in and Beyond the Classroom: The 16th ICMI Study* (Barbeau & Taylor, 2009), which is devoted to the discussion of mathematical challenges in and out of school, as well as in the formation of a topic study group dedicated to mathematical competitions in the International Congress on Mathematical Education (ICME).

Although the literature offers several definitions of mathematical challenges, also referred to as rich mathematical problems, they share many aspects that allow us to recognize some common denominator (Freiman & Manuel, 2015; Leikin, 2007, 2014; Protasov et al., 2009; Taylor, 2006).

Leikin (2004) has identified a set of parameters, generated from the contributions of theorists and researchers who were determinant in the problem-solving domain (Charles & Lester, 1982; Polya, 1973; Schoenfeld, 1985), rendering a challenging mathematical task as one which should (a) be motivating, (b) not include readily available procedures, (c) require an attempt, and (d) have several approaches to a solution.

According to Taylor (2006), the most frequent meaning of mathematical challenge is grounded on the idea that a challenge is the experience of finding a new and unforeseen situation and attempting to solve it. The author also claims that by learning to overcome a challenge offered by a mathematical situation, the individual develops the ability to abstract, which will eventually be helpful in everyday situations without any mathematical context but still requiring a systematic approach.

> Challenge is not only an important component of the learning process but also a vital skill for life. People are confronted with challenging situations each day and need to deal with them. Fortunately, the processes in solving mathematics challenges (abstract or otherwise) involve certain types of reasoning which generalize to solving challenges encountered in everyday life. (p. 2)

Another form of interpreting this notion, as suggested by Barbeau (2009), includes that a challenge may be an opportunity for the individual to develop knowledge and understanding:

> …a challenge is a question posed deliberately to entice its recipient to attempt a resolution, while at the same time stretching their understanding and knowledge of some topic. (p. 5)

Thus, we can say that a mathematical challenge is an interesting and motivating mathematical difficulty that a person can overcome. A mathematical challenge is a core element of mathematical learning directed at the fulfilment of the learner's mathematical potential through integration of mathematical difficulty and positive affect in the learning process (Applebaum & Leikin, 2014).

There is a clear consensus among the various authors regarding the need to pay particular attention to the selection of the challenges posed to young people in and beyond the classroom. For example, Applebaum and Leikin (2014) consider that it is necessary to take into account all the variables having to do with the students' learning potential. Thus, a mathematical challenge must be adjusted to the students' abilities and therefore not too easy or too difficult. It should motivate students to persevere on the task and develop their mathematical curiosity and interest in the subject. It should also support and nurture students' beliefs about the creative nature of mathematics, the constructive nature of the learning process, and the dynamic nature of mathematical problems as having different solution paths and enabling individual learning styles and knowledge construction (Leikin, 2007).

Likewise, Barbeau (2009) brings forth other facets, such as the several processes involved in solving a good challenge:

> A good challenge will often involve explanation, questioning and conjecturing, multiple approaches, evaluation of solutions for effectiveness and elegance, and construction and evaluation of examples. (p. 5)

In short, it is possible to isolate some characteristics that make a challenge appropriate or not, such as the context in which it is proposed and the background of the recipient. A good mathematical challenge in the classroom may not be a good challenge beyond the classroom. In fact, there have been proposals to characterize good mathematical challenges, often from the point of view of the students and other times from the teachers' perspective. One of such models (Freiman & Manuel, 2015; Manuel, 2010) suggests a set of dimensions that constitute a good challenge or rich problem, each of which is deployed in several indicators. This model establishes the principles underlying a good mathematical challenge, according to which such a problem should be (i) open-ended, (ii) contextualized, (iii) ill-defined, (iv) complex, and (v) allowing multiple interpretations.

Our perspective of a mathematical challenge or rich mathematical problem is in line with the broad concept presented by many researchers in the field, and it contains, in particular, some of the characteristics that are present in the model offered by Manuel (2010). Another distinctive feature of good challenges, as suggested by Kenderov et al. (2009), is that they are suitable for participants with different levels of performance in mathematics:

> …Challenges can be adapted to all levels of achievement. Even those with limited abilities can benefit from a challenging environment. They will be involved in the investigation and strategizing from the outset, and so gain an intimacy with the mathematics involved. (p. 87)

Having in mind the main key features of a mathematical challenge, we may say that the mathematical problems selected, within the scope of the mathematical competitions SUB12 and SUB14, are:

1. Contextualized, because they present concrete situations (real or fictitious).
2. Reasonably complex, because they imply more than a single step, and often more than a mathematical concept, as well as require finding patterns and generalization.
3. Open-ended, in the sense that they allow several paths leading to the solution.
4. Likely to offer several points of entry and can be approached in different ways.
5. Suited to different skill levels, as they can be also solved through non-formal methods.

Finally, we know that mathematical challenges can be worked out in school and beyond. Our choice to look at solving rich mathematical problems in the context of a mathematical competition that takes place beyond the classroom has to do with the realization that there are opportunities and advantages that are specific to this context and that are relevant to identify positive attitudes regarding mathematics, which are desirable for mathematics learning. Table 17.1 illustrates some of the most obvious differences regarding the activity of solving rich mathematical problems when comparing the contexts of in and beyond the classroom, by taking as a reference the mathematical competitions SUB12 and SUB14.

18.2.2 Inclusive Mathematical Competitions

Mathematical challenges are closely linked to mathematical competitions as discussed above. But mathematical competitions have also been conceived as an activity to motivate students to mathematics and to promote mathematical talent.

Table 17.1 Rich mathematical problem solving in and beyond the classroom

Rich mathematical problem solving	
In the classroom	In the mathematical competitions SUB12 and SUB14
Limited time	Extended time
Possibility of addressing the teacher and using the classroom resources	Possibility of resorting to a large diversity of human and material resources
Strong linkage of problems with the official school curriculum	Independence of problems from the official curriculum
Solving the problems using formal methods or with the aim of obtaining new formal methods	Great independence and freedom in the creation of a resolution strategy and in the way of approaching the problem
Mathematical communication strongly linked to the mathematical language of the classroom	Freedom and creativity to express the thinking/ reasoning that leads to the solution
Predominantly use of formal and conventional representations	Very diverse use of representations, including diagrams, images, graphics, or just words, as well as others made possible by the use of digital technologies

In Portugal, as in the whole world, the number of mathematical competitions has increased, taking on the most diverse forms, contents, and durations and targeting ever-widening groups of students. Examples of well-known competitions are the national and international Mathematical Olympiads aimed for especially talented students, along with others like the Mathematical Kangaroo or the SUB12 and SUB14 problem-solving competitions, which clearly are of an inclusive character, that is, they are open to a large number of students with varying degrees of problem-solving ability.

One of the intentions of these so-called inclusive competitions is to create opportunities for the development of positive attitudes towards mathematics. This means, among other aspects, to motivate and to engage students with mathematics, promoting enjoyment and interest for this subject and involving them in activities that are pleasant for them (Kenderov et al., 2009; Stockton, 2012). Competitions of an inclusive character, such as the SU12 and SUB14, propose several problems throughout their course that are simple though often intriguing and set in familiar situations (Carreira et al., 2016).

The participation of the students in an inclusive competition must be based on their decision and willingness, as stressed by Protasov et al. (2009). No parent or teacher should force or make children or students participate. One of the distinguishing features of inclusive competitions, which envisage the participation of a large number of students, is that the proposed activities are intended for all, or in other words, they try to ensure that the mathematical abilities involved are within the scope of all students.

This may be a reason why we find opinions on the importance and the advantages of (inclusive) competitions being detached from the curriculum:

> One of the problems with the classroom is that a school curriculum is rather restricted and cannot suit all. Competitions enable students to be exposed to other aspects of mathematics

> and for them to apply the skills they have to new situations. Competitions enrich the learning experience of hundreds of thousands, in fact millions of students who participate in the inclusive competitions. (Kenderov et al., 2009, p. 64)

Some researchers have already shown that participation in mathematical competitions of an inclusive nature, particularly for the younger ones, has a positive influence on their motivation to learn mathematics (Freiman & Véniza, 2006; Kenderov et al., 2009; Koichu & Andzans, 2009; Wedege & Skott, 2007). Moreover, the benefits of participating in mathematical competitions beyond the classroom – which are not only challenging but also intended to be appealing and mathematically enriching projects – include developing a positive affective relationship with mathematics and problem-solving activity. And this happens for both students who perform well in school mathematics and those who experience a lesser degree of success in the school context (Freiman & Véniza, 2006; Freiman et al., 2005; Kenderov et al., 2009). Satisfaction, the sense of self-efficacy, and the enjoyment and interest in mathematics are some of the advantages stemming from the participation of young people in inclusive mathematical competitions that have been identified in previous research studies (Freiman & Applebaum, 2011).

18.3 A Theoretical Model of Attitude

In mathematics education, research on attitude has a very long tradition (Hannula, 2012; Hannula et al., 2016; McLeod, 1992), and in that tradition, there is a great concern of researchers with the relation between attitudes and performance, namely, school performance. Therefore, the efforts tend to be more evident in the persistent attempts at identifying, measuring, and correlating variables in the field of attitudes than at producing a theoretical conceptualization around the construct of attitude (Di Martino & Zan, 2015). The recognized difficulty in the theorizing of attitude is fairly understandable due to the enormous complexity, invisibility, and fluidity of this concept, which includes its interweaving with other affective dimensions, such as beliefs, emotions, self-confidence, motivation, anxiety, and so on. More recently, the efforts to deepen and construct the concept of attitude have become more accentuated, and new theoretical and methodological approaches to the study of attitudes have been affirmed in the field of mathematics education (Eligio, 2017).

In a sense, those efforts seek to break with some previous models that, although pioneering and allowing a remarkable development of the study of attitudes and other affective variables, paralyzed the theoretical discussion in favour of the development of instruments and their empirical validation. Among such models, it is impossible to ignore McLeod's theoretical approach in which "attitude refers to affective responses that involve positive or negative feelings of *moderate* intensity and *reasonable* stability" (p. 581, our emphasis). This model is based on a division of the affective domain into three categories, beliefs, attitudes, and emotions, and the major criteria to distinguish them are aspects such as the duration/intensity and the stability of the affective responses they provoke in individuals regarding mathematics. In this model, beliefs and attitudes are usually more stable, while emotions usually change more quickly. Emotions lie at a higher end of a continuum of intensity, with beliefs at the weaker intensity opposite end and attitudes at the intermediate points.

> Therefore, we can think of beliefs, attitudes, and emotions as representing increasing levels of affective involvement, decreasing levels of cognitive involvement, increasing levels of intensity of response, and decreasing levels of response stability. (McLeod, 1992, p. 579)

One of the consequences of this perspective, as we have emphasized before, is a load of fuzziness and ambiguity in the conceptualization of attitude, insofar as it is regarded as something that lies in an intermediate position between two extremes. It means that one cannot speak of attitude without putting it into some relative position to beliefs and emotions. Another critical issue that has been raised is the characterization of attitudes as positive/negative. Such dichotomy has been simplistically identified with an emotional disposition towards mathematics that could be summed up in the opposition between like and dislike. The work of McLeod (1992) and many other researchers tends to identify the positive and the negative with what is emotionally pleasant and unpleasant, thus making the definition of attitude unidimensional, which leads to attitudes being examined in terms of dislike of geometric proof, enjoyment of problem solving, or preference for discovery learning, among other examples. At the basis of this conceptualization is the assumption that attitudes can be analysed in terms of emotional responses that correspond to them. An example would be the following causal relationship: if students feel emotionally satisfied working with their classmates in solving problems, then they develop a positive attitude towards group work in learning mathematics. In other words, a positive emotion triggers a positive attitude towards mathematics or the learning of mathematics. One of the weaknesses of this hypothesis seems to be on the fact that several large-scale studies do not find a particular causal relationship that could be expected between a positive attitude towards mathematics and achievement in mathematics. This fact continues to generate puzzlement among researchers, particularly with regard to the results of the PISA studies (Eric, 2011).

The research work of Di Martino and Zan (2010, 2011, 2015) has been consolidated as a proposal to overcome the conceptual fragilities mentioned above. On the one hand, the authors embrace the need to consider attitude as an autonomous theoretical concept, which therefore ceases to be seen as a hybrid form between belief and emotion and is no longer defined on the basis of the instruments designed to capture it. On the other hand, they seek to overcome the unidimensional nature of attitude, connected to the duality of like/dislike, and therefore to the emotional disposition, by postulating that an attitude is negative when at least one of its various dimensions is negative.

Their theoretical formulation does not eradicate complexity or density from the concept of attitude; however, it makes the concept as the ground on which cognitive, emotional, and behavioural elements are played. Of course, the fact that relations can be found between attitudes and achievement is not to be ruled out; however, the

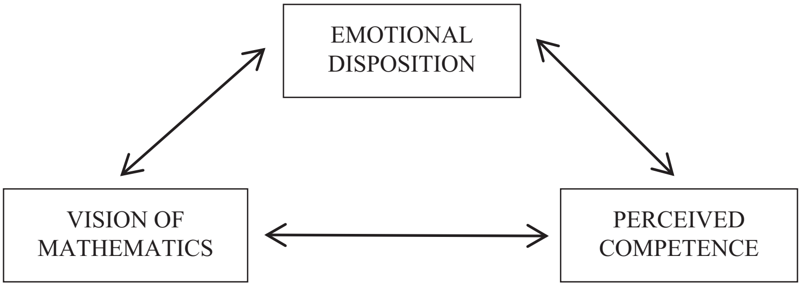


Fig. 18.2 The three-dimensional model for attitudes towards mathematics. (Di Martino & Zan, 2011, p. 476)

idea of causality and measurement is set aside in order to give primacy to ways of interpreting and understanding what a positive attitude or a negative attitude means (Di Martino & Zan, 2011). One of their intentions is to capture the subjectivity of the interaction between emotional disposition and other dimensions of attitudes that may allow forms of diagnosis and intervention in the face of negative attitudes or the implementation of projects to cultivate and develop positive attitudes towards mathematics (Di Martino & Zan, 2010).

In recent years, Di Martino and Zan (2010, 2011, 2015) have established what they have named as a three-dimensional model for attitudes (TMA) towards mathematics, which consists of three dimensions (Fig. 18.2): (1) emotional disposition towards mathematics, (2) vision of mathematics, and (3) perceived competence in mathematics.

When considering the model against the purpose of our study, it may be anticipated that the three proposed dimensions and also their interconnections are useful to examine the attitudes of the students who participate in a beyond-school mathematical activity, where mathematical challenge and communication are the fulcrum and where formal and informal mathematical knowledge are both relevant.

As such, we endorse the three-dimensional model in our approach to students' attitudes, by projecting our analysis of each dimension on a threefold empirical phenomenon: students' participation in the mathematical competitions, students' problem-solving activity, and students' performance and mathematics knowledge. This approach is schematically depicted in Fig. 18.3.

The links between the three dimensions that are seen as one of the advances of this model are potentially convenient to our study, especially in light of the available data gathered from multiple sources as we will describe in the following section.

18.4 The Research Context and Method

Specific details about the context of the mathematical competitions SUB12 and SUB14 will be presented; we then briefly describe the methodological approach to our study of students' attitudes.

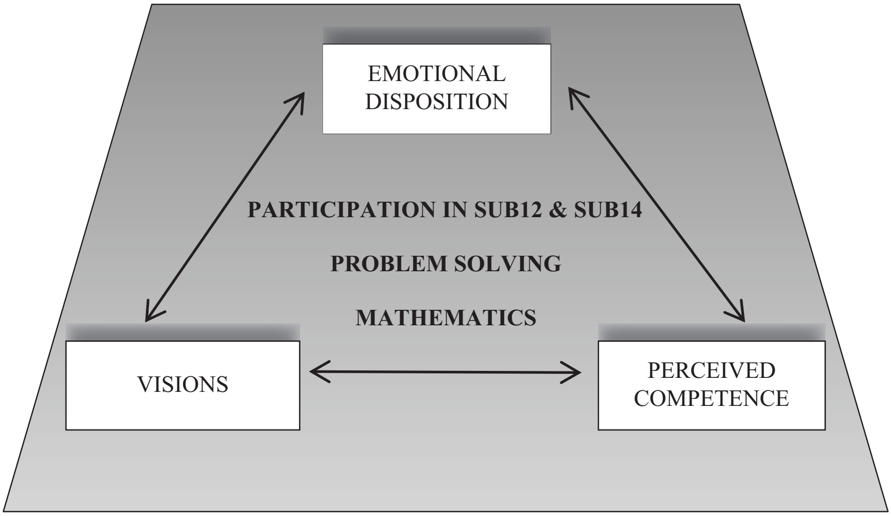


Fig. 18.3 The three-dimensional model adapted to the context of the present study

18.4.1 The Mathematical Competitions SUB12 and SUB14

The SUB12 and SUB14 are regional online problem-solving competitions promoted by the University of Algarve, having similar formats and running in parallel (Carreira et al., 2016). SUB12 targets fifth and sixth graders (10–12 years old), while SUB14 addresses students in seventh and 8th grades (12–14 years old). The competition is composed of two phases: the qualifying that takes place at a distance, between January and June, and the final that is on-site and takes place at the end of June. During the qualifying phase, a new challenge for each of the SUB's is posted online every 2 weeks. Participants have 15 days to e-mail their resolution and answer to the problem. Feedback is given to all the resolutions, and the participants have the chance to revise and resubmit their solutions within the established submission period. To get qualified to the final, any participant must attain the correct solution in eight of the ten proposed problems.

The competitions can be briefly characterized through ten distinctive features:

- The mathematical challenges proposed can be described as contextualized non-routine word problems and usually have various forms of resolution.
- Mathematical communication is valued by explicitly requiring the presentation of the strategy and process used to obtain the solution.
- It is intended that students have the necessary knowledge to solve the proposed challenge.
- The challenges are chosen independently of the school curriculum, that is, the problems are not designed to fit specific curricular topics.
- The support of parents and teachers to the young participants is encouraged.

- Formative and friendly feedback is offered, by suggesting clues or hints to encourage the student's revision or improvement of the solution, if needed.
- All types of media that students can use to produce their resolutions are accepted (from the use of digital tools to the scanning of resolutions made with paper and pencil).
- Persistence in the competition and parental involvement are encouraged, although communication is always directed at the participating pupils.
- Public recognition is given to expeditious, creative, elegant, and interesting resolutions through its publication on the competition webpage.
- The competitive part is confined to the final where students have to solve a set of problems individually and in limited time.

This raises the hypothesis which is that the combination of the various features of the SUB12 and SUB14 competitions has an impact on students' attitudes towards mathematics, in direct articulation with their attitudes towards problem solving and their participation in the competitions.

18.4.2 Methodological Approach

As referred above, the Problem@Web Project entailed a long and extensive process of data collection within the context of the mathematical competitions SUB12 and SUB14. A large part of the data concerning the participating students' attitudes is qualitative in its nature. Since it concerns online competitions that take place beyond school and without face-to-face contact with the participants, we soon found the need to listen to the voices of the participants and of some older students who had been former participants. We did this through conducting several semi-structured and extensive interviews to a number of participants at the time and to ex-participants who took part in former editions (in a total of 20 interviewees). A systematic collection of e-mail messages was also carried out. In these e-mails, we have signalled not only relevant information regarding the students' solutions to the problems proposed but also some emotional reactions and statements they spontaneously included in their messages, namely, about their participation in the competition, the challenges presented, and their enjoyment in solving the problems. At the same time, we obtained more extensive data from a questionnaire that was administered after the final of one of the editions of the SUB12 and SUB14. The survey was carried out electronically to all the participants in that edition of the competition. The number of validated questionnaires is $n = 350$ which corresponds to a 20% rate of return.

The questionnaire focused on questions related to the use of technologies in solving problems, students' relationship with mathematics and problem solving, and affective aspects involved in their participation in the mathematical competitions. It was mainly composed of 4-point Likert-type questions, after some closed-ended questions related to demographic data, namely, gender, age, year of schooling, and school grade in mathematics.

Aiming to outline the attitudes of the students from the point of view of their participation in the competition, we selected 19 questions from the total. For this selection, two criteria were adopted: (i) the questions are contextualized in the participation of the students in the competition and (ii) each question points to one of three dimensions – students' views (V_i), students' emotional dispositions (E_i), and students' perception of their competence (C_i). In addition, the organization of the questions into categories led to consider six main thematic domains:

1. Adherence to beyond-school mathematics.
2. Fun and enjoyment experienced.
3. Ways of solving mathematical problems.
4. Solving and expressing mathematical problems.
5. Self-improvement experienced.
6. Mathematical performance and knowledge.

The 19 questions are presented (Table 17.2) as they were organized and coded along the three dimensions considered.

It is important to note that this coding should not be considered as an absolute way of untying the three dimensions. The distribution of the questions by the dimensions considered in the three-dimensional model is tricky in that the separation of the three dimensions is not trivial; on the contrary, they intertwine in many ways as previously pointed out. In short, it is not possible to precisely establish the boundaries between emotions, beliefs, and perceived competence.

In the categories adopted, we may observe the prevalence of the emotional disposition items, especially when compared with the weight of the perceived competence items. In justifying this, we recall that the data that are being analysed mainly focus on the participants' progress and experience during the qualifying phase. This phase undertakes an eminently formative character and can be seen as a preparatory phase, sometimes considered by the participants themselves as training for the final, which involves a greater competitive edge.

By assuming the investigation of students' attitudes from the point of view of their participation in the competitions, we are admitting that there is an implicit perception of self-competence to engage in the moderately demanding qualifying phase. The students' perception of their competence at the outset may also be related to the possibility of help-seeking either from family members or teachers and also from the organization itself that provides feedback and allows reformulating any submitted solution. In virtue of this, the question of perceived competence is placed on certain elements of the qualification phase, namely, on the diverse ways of approaching and solving problems, on the ability to communicate and explain the problem-solving process, and on the confidence in dealing with mathematical ideas. Furthermore, this dimension is apparent in the questions that refer to getting qualified to the final and also to reaching the top places of the competition.

Table 17.2 Coding of the dimensions considered in the questionnaire design

1. Adherence to beyond-school mathematics		
1	*E1*	It is nice to participate in the competition
2	*E2*	I have fun with participating in the competition
3	*V1*	My participation in the competition increases my interest in mathematics
2. Fun and enjoyment in solving problems		
4	*E3*	I feel the problems of the competition are fun
5	*E4*	It pleases me the type of problems proposed in the competition
6	*E5*	It is nice to get the problems of the competition right
3. Ways of solving problems		
7	*V2*	I think that the problems of the competition incite me to think
8	*V3*	I think that the problems of the competition can be solved in several ways
9	*C1*	I may try different ways in solving a problem of the competition
4. Solving and expressing problems		
10	*V4*	I think that the problems of the competition incite me to explain
11	*C2*	In solving a problem of the competition, I expect to be able to explain my solution
12	*E6*	In solving a problem of the competition, I wish that my solution is easy to follow
13	*E7*	It pleases me to discuss the problems of the competition with other people
5. Self-improvement experienced		
14	*V5*	My participation in the competition helps me to be better at mathematics
15	*C3*	My participation in the competition makes me more confident in mathematics
16	*E8*	It pleases me to learn more mathematics in the competition
6. Mathematical performance and knowledge		
17	*C4*	It is important to be good at mathematics to participate in the competition
18	*V6*	I think that the problems of the competition require a lot of mathematics knowledge
19	*C5*	It is important to be good at mathematics to reach the final of the competition

18.4.3 Brief Description of the Participants

From the data collected through the questionnaire, we obtained an elucidative image of the participants in these competitions. Of the 350 young respondents, about 55% are female and 45% are male. Their ages range from 10 to 15 years. In fact, students attending the levels of schooling involved in these competitions generally range between 10 and 14, so the presence of students over the age of 14 may mean that they are students with some retention. This situation reinforces the contention that the competitions are inclusive, seeking to involve all students. From the graph (Fig. 18.4), we see that young students aged 11 and 12 have the most significant participation in the competitions. It is also evident that the pupils' participation declines as their age increases. The reasons for fewer individuals to participate as they progress through the years of schooling may be related to a number of factors. Some students say they are not available to participate because in seventh and 8th grades they have to cope with a larger number of school subjects and, consequently, with a greater workload at school. Other students also find it difficult to reconcile their various extracurricular activities, namely, sports, with their participation in the SUBs.

The comparison between the graphs of Figs. 18.4 and 18.5 reveals a clear compatibility, since the 10–11-year-olds are those who attend the fifth grade where the highest percentage of participants (48.9%) is recorded. And, as said before, the participation declines as the level of schooling increases, which for many young people means an increase in school work. Frequently, the organizing team of the competitions gets e-mail messages from participants who justify their delay in sending the solution to a problem due to the extra time dedicated to the preparation for assessments at school.

Regarding school performance, the data show the presence of students with different levels of performance, in tune with the concept of an inclusive mathematical competition. In fact, the graph of Fig. 18.6 reveals that although the percentages of participants with the lowest performance levels – level 1 (2.3%, corresponding to eight students) and level 2 (about 6.6%) – are low, the percentage of students with

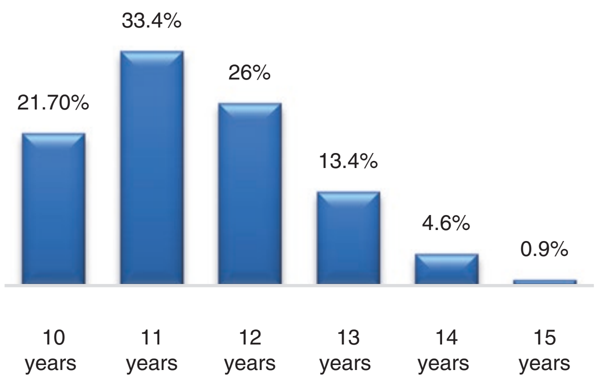


Fig. 18.4 Distribution of the respondents by age

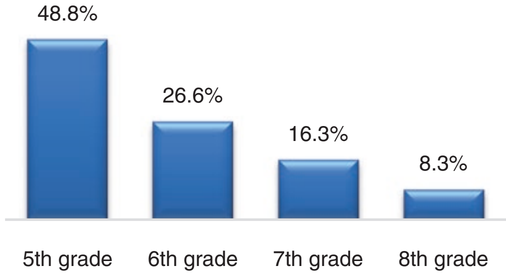


Fig. 18.5 Distribution of the respondents by school grade

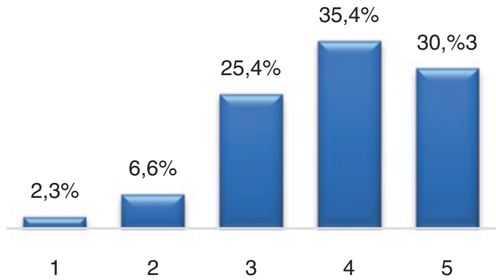


Fig. 18.6 Distribution of the respondents by school performance level (1–5)

level 3 (25.4%) has a significant expression, which shows that the competition is not only or preferentially directed to the students with a high level of school performance. The highest percentage refers to participants with performance level 4 (35.4%), followed by the percentage of participants with level 5, which is 30.3%. Thus, it is possible to recognize a clear heterogeneity in the participants, being noteworthy the fact that there is a reasonable percentage of students who are average achievers and also noticeable that the younger ones are more predisposed to engage in the SUB12 competition, which addresses grades 5 and 6.

18.5 Students' Attitudes in the Context of Their Participation in the Mathematical Competitions SUB12 and SUB14

For the presentation and analysis of the data on the attitudes of the students participating in the competitions, we will cover the six thematic areas described above. In each of these areas, the results of the questionnaire responses are given on frequency charts, and their interpretation will be corroborated and substantiated through the selection of interview excerpts and examples of e-mail messages sent by participants during several editions of the two competitions.

18.5.1 Adherence to Beyond-School Mathematics

Concerning the *adherence* of the participants to these mathematical competitions that take place beyond school, the data reveal an overwhelming majority of answers (90%) agreeing with the assertion that it is nice to participate in the competition (Fig. 18.7). This information shows that the students have a very positive overall attitude towards their participation in this mathematical competition of an inclusive nature. A closer examination of the data reveals that this broad attitude is more precisely described as different dimensions of this attitude are detailed.

When observing the data on the fun that is involved in that participation (Fig. 18.8), we note that the affirmative answers also stand out with a percentage of about 75%; even so, that result does not replicate the numbers obtained on being nice to participate. Furthermore, if we consider the increase in interest in mathematics related to the students' participation (Fig. 18.9), we find that the number of concordant answers shows a small increase (being close to 80%), compared to the fun experienced in the participation.

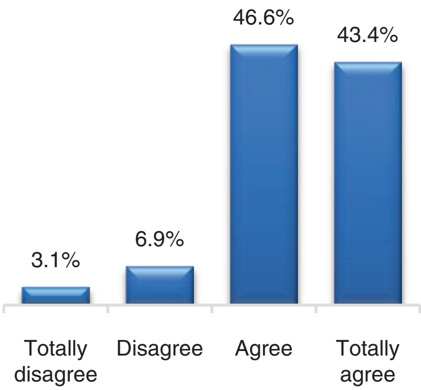


Fig. 18.7 It is nice to participate in the competition

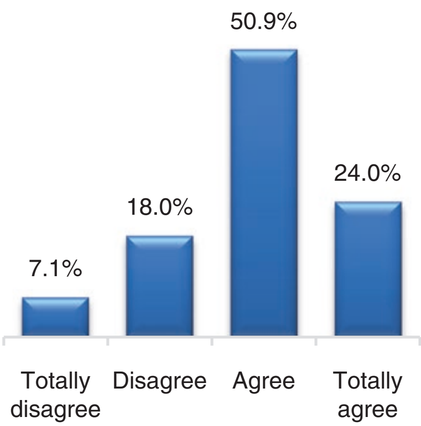
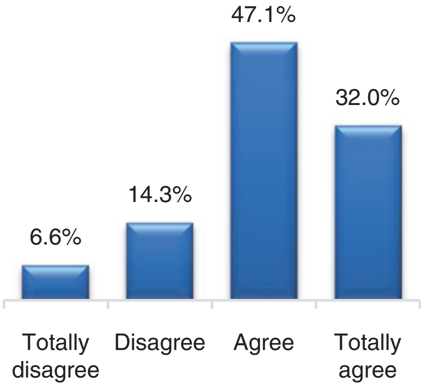


Fig. 18.8 I have fun with participating in the competition

Fig. 18.9 My participation in the competition increases my interest in mathematics



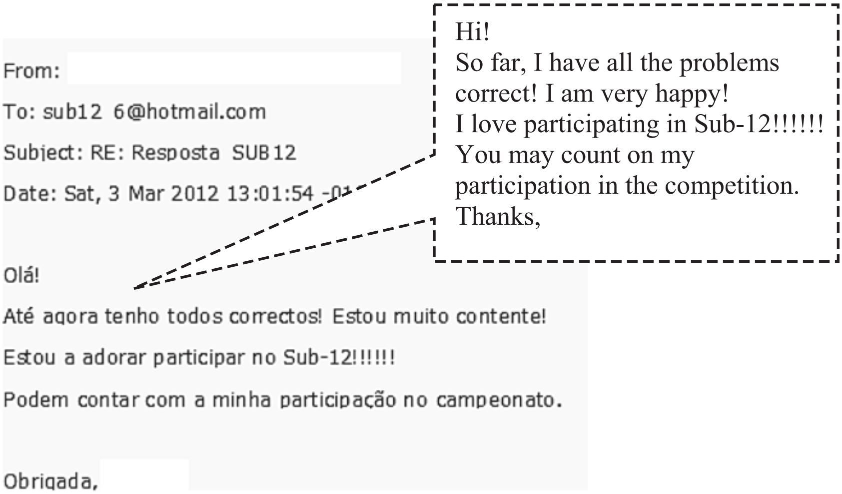


Fig. 18.10 Message from a girl in sixth grade, stating that she likes to participate and, at the same time, revealing her satisfaction for her correct answers

This apparently indicates that the *nice* experience of participation is not limited to the *fun* that students have with it. Likewise, the emotional atmosphere of having a nice experience does not quite match the increase in *interest* for mathematics, although the view of getting more interested seems to be quite discernible in the answers of the respondents.

The strong adherence to the competition is characterized by high levels of emotional disposition to engage in the competition that suggest an intense affective link to the experience of participation. Such indicators also appear in the words of some participants, either in interviews or in their e-mail messages. For example, Figs. 18.10 and 18.11 show some of the comments and emotional expressions sent by the participants in the e-mails in which they submitted their solutions to problems and convey some of their reactions to facts of the qualifying phase.

Fig. 18.11 Message from another participant that expresses her fondness for the competition

18.5.2 Fun and Enjoyment in Solving Problems

The results referring to the category of fun and enjoyment in problem solving show one side of the participation in the competition that is directly related to the type of problems proposed. The results (Figs. 18.12 and 18.13) show that most students find that the problems of the competition are fun (about 80%) and they enjoy solving that kind of mathematical challenges (about 80%). This satisfaction is also quite evident in the words of one of the participants (Fig. 18.14) who, in one of his e-mails containing the resolution to a problem, manifests his enjoyment for certain problems that are their favourite.

In addition to this very favourable emotional disposition about the moderate challenges that constitute a trademark of the competition, another aspect clearly demonstrated by the respondents refers to the fact that they find it very pleasant to be able to get the problems right (about 90%) (Fig. 18.15). This result reinforces the idea that it is possible to engage students in enthusiastically solving problems, in particular by proposing them interesting mathematical challenges that are within their reach.

As can be seen from the utterances of two of the interviewed participants, getting the correct answer to a problem is a goal that stimulates them and makes them feel good about themselves, sometimes seeming a magical experience that appears to be very rewarding.

> We are challenged to solve a problem and when we get it, it is...It makes us feel good about ourselves. We know that we can, and then we try the next one and we get it. If we cannot solve it straight away, then we will succeed on the second or third attempt. (Excerpt from an interview to a participant in SUB14)
>
> What excites me is to get the right solution to the problem. For me it is a kind of magic…. It seems magic when I discover the solutions to the problems. (Excerpt from an interview to a participant in SUB12)

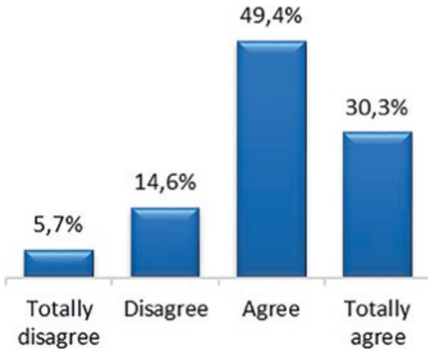


Fig. 18.12 I feel the problems are fun

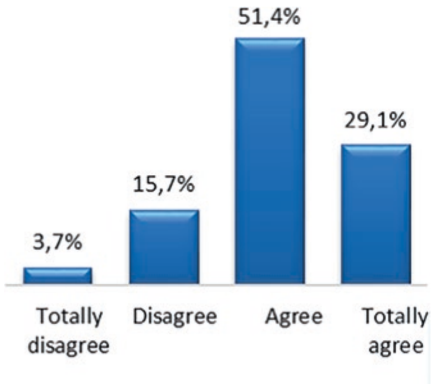


Fig. 18.13 It pleases me the type of problems

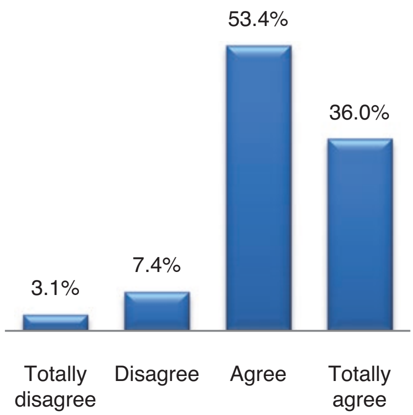
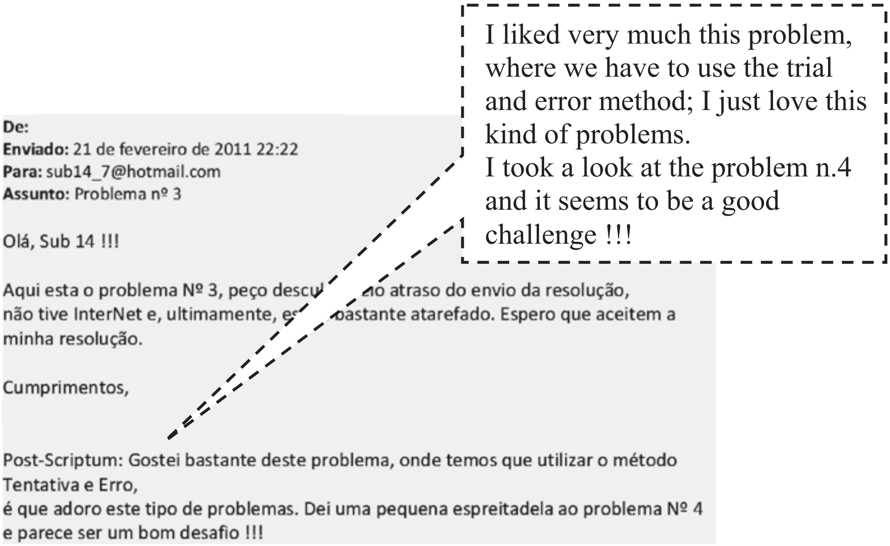


Fig. 18.14 It is nice to get the problems right


I liked very much this problem, where we have to use the trial and error method; I just love this kind of problems.
I took a look at the problem n.4 and it seems to be a good challenge !!!

De:
Enviado: 21 de fevereiro de 2011 22:22
Para: sub14_7@hotmail.com
Assunto: Problema nº 3

Olá, Sub 14 !!!

Aqui esta o problema Nº 3, peço descul ao atraso do envio da resolução,
não tive InterNet e, ultimamente, es bastante atarefado. Espero que aceitem a
minha resolução.

Cumprimentos,

Post-Scriptum: Gostei bastante deste problema, onde temos que utilizar o método
Tentativa e Erro,
é que adoro este tipo de problemas. Dei uma pequena espreitadela ao problema Nº 4
e parece ser um bom desafio !!!

Fig. 18.15 E-mail message from a boy from seventh grade stating that the problems please him and that he finds them challenging

18.5.3 Ways of Solving Problems

There is almost unanimity (about 97%) in the belief that the problems of the competition incite students to think (Fig. 18.16). This reveals that although the competitions are inclusive and the challenges are of moderate difficulty, they are not seen by the participants as routine and as having an immediate answer. In addition, we have already seen that the participants like the problems proposed and get involved in their resolution, so the fact that they are led to think is something that seems to be linked to their liking of the problems presented. Regarding the answers on the existence of different ways of solving the problems, a large majority (92%) also agrees with this idea about the problems, as shown in Fig. 18.17.

It is important to distinguish between the belief that problems can be solved in different ways and the emotional disposition of feeling comfortable in trying different ways to find a solution. There is clear evidence that the number of students with a willingness to try different ways of approaching a problem is actually lower. In the graph of Fig. 18.18, we can see that about 40% of the participants never or only sometimes tried different ways to solve a problem. This may relate to the fact that there are students with different mathematical backgrounds in the competition. It may as well suggest the existence of less open and flexible students in facing problem situations as an effect of the inclusive character of the competitions.

In any event, for many participating students, this relationship with the problems proposed in the competitions is permeated by a strong affective disposition that makes them fascinated by the possibility of exploring different ways to find the solutions to the challenges; even when seemingly more difficult, this is an activity

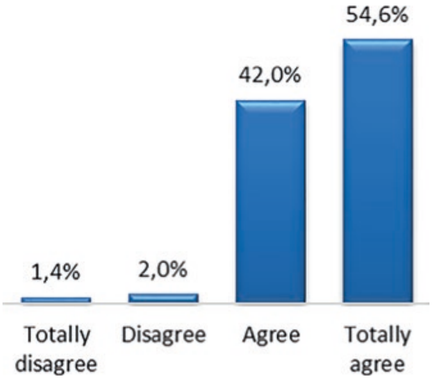


Fig. 18.16 I think they incite me to think

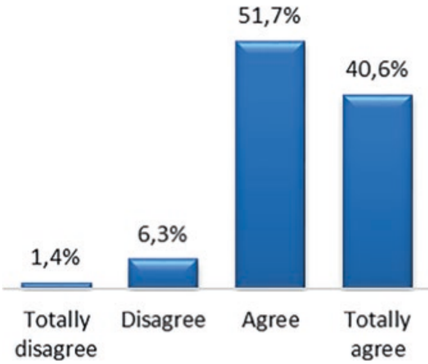


Fig. 18.17 They can be solved in several ways

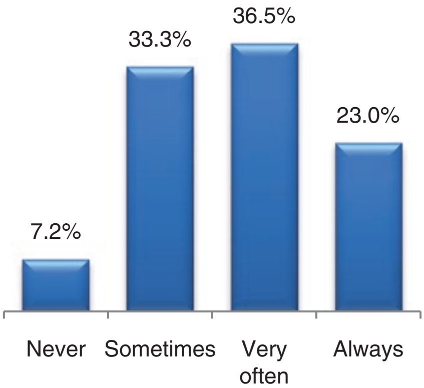


Fig. 18.18 I may try different ways to solve the problem

that gives them pleasure and that eventually leads to success. The following excerpt is illustrative of this kind of suggested fascination:

> I am fascinated by the fact that a problem has several possible ways to be solved...more than one. A seemingly difficult problem that I cannot immediately solve, after rereading the statement and thinking a little, ends up being possible to solve. (Excerpt from an interview to a participant in SUB14)

18.5.4 Solving and Expressing Problems

One of the ten characteristics that define the dynamics of the SUB12 and SUB14 is that of enhancing and encouraging mathematics communication associated with the problem-solving activity. This is well denoted by the fact that all the problems posted on the website are accompanied by a reminder, as seen in Fig. 18.19.

The solving and expressing process is an indispensable requirement for any solution to be considered correct. This requirement is clearly recognized by the participants who in their responses to the questionnaire were quite convergent on the idea that the problems incite them to explain. Almost 95% of the participants (Fig. 18.20) reveal this view on solving the problems of the competitions.

Despite the high percentage of participants who are aware of the importance and requirement of explaining the problem-solving process, the number of respondents who feels able to do so, always or very often, is much lower. Only about 60% of the respondents (Fig. 18.21) feel that they are usually able to provide an explanation of their solution that seems actually good to them. We realize from the part of several participants in the competition their effort and concern to find a good way to explain their problem-solving process. Some participants even expressed this feeling in their e-mail messages, stating that they had tried to find the best way to expose their reasoning even though some believed they had not totally succeeded (Fig. 18.22).

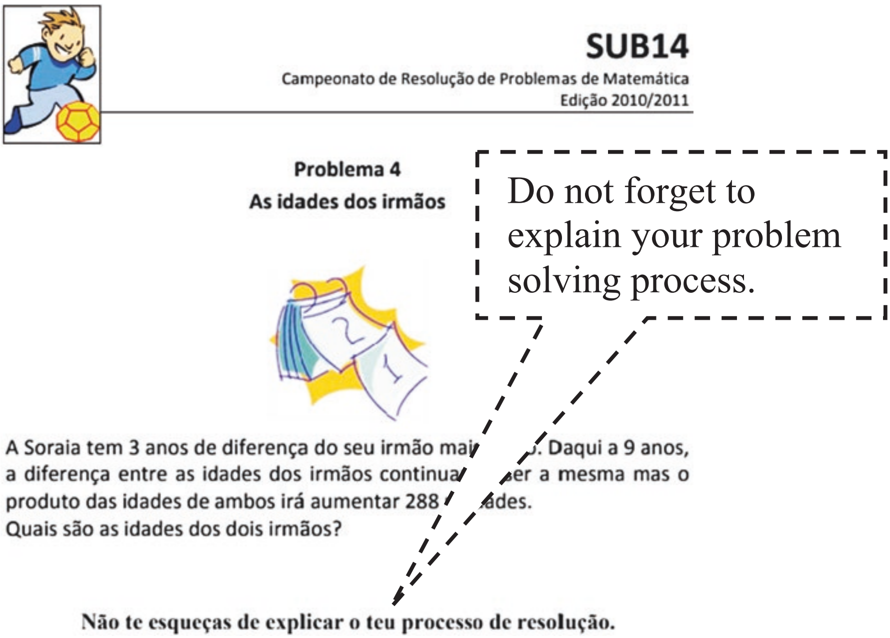


Fig. 18.19 Statement of a problem posted on the SUB14 webpage in one of the editions

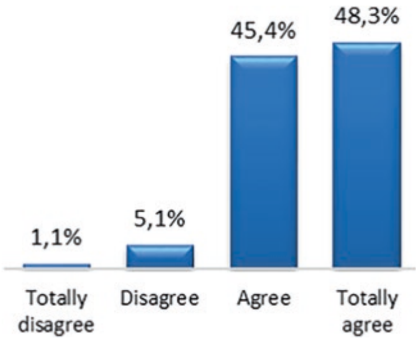


Fig. 18.20 I think they incite me to explain

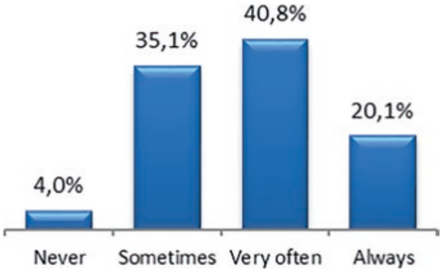
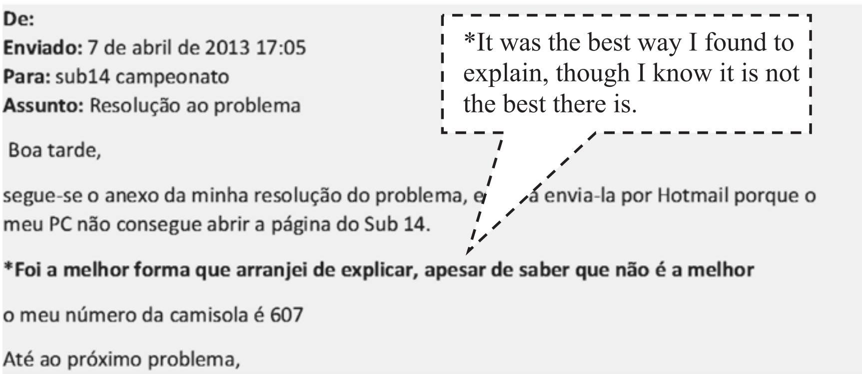


Fig. 18.21 I expect to be able to explain



De:
Enviado: 7 de abril de 2013 17:05
Para: sub14 campeonato
Assunto: Resolução ao problema

Boa tarde,

segue-se o anexo da minha resolução do problema, e á envia-la por Hotmail porque o meu PC não consegue abrir a página do Sub 14.

***Foi a melhor forma que arranjei de explicar, apesar de saber que não é a melhor**

o meu número da camisola é 607

Até ao próximo problema,

Fig. 18.22 E-mail from a participant in SUB14 referring to his way of explaining the problem-solving process

One of the ex-participants who was interviewed acknowledged that at the beginning of his participation, he was not used to explain the problem-solving process and then described how he struggled to achieve and improve his way of explaining. He pointed out that the desire to communicate his solution in a way that satisfied him impelled, in many cases, his search for other "paths" to arrive at the solution.

> In the beginning, what I did was describing; I described all the steps. It was boring to be describing. But it was one of the biggest challenges: 'How do I explain the way that I got

> there?' This was important because there were times when I was thinking, and thinking – 'This way is hard for me to explain, I have to find another way'. So I would a try different path because the first one was difficult to describe. (Excerpt from an interview with an ex-participant in SUB12 & SUB14)

The concern with being able to explain is often referred by several interviewees. The motivation they reveal to produce an explanation of the solution process arises along with the desire that the solution becomes easy to understand by others, particularly by the members of the organization who will be reading it and sending feedback. One participant's statement shows such concern in the way she paid attention to the process of explaining her solution.

> If I quickly get to the right answer but cannot explain it, I do not feel very happy about it, because the answer is not the most important thing. If we do not explain how we get there, the answer is worthless. (…) I make sure that I have things well organized so that the people who will read my solution to the problem and see if it is correct will understand it. (Excerpt from an interview to a participant in SUB12)

More than 90% of the respondents express their wish that their explanation is easy to understand (Fig. 18.23). The comparison between the graphs of Figs. 18.21 and 18.23 shows a certain difference between the perception of being able to explain and the emotional disposition to provide a good explanation. This difference seems to reflect the idea expressed by some participants that despite their effort and their intention to do the best, their forms of expression and communication were not always the ones they considered ideal ("the best there is"). It is important to emphasize that coupled with a belief that participation in the competitions prompts the explanation of the resolution, we can see the participants' true perception of competence about what they are capable of doing and also their emotional disposition to reach the best final product.

Another aspect reported in Fig. 18.24 is directly related to one of the characteristics of the competitions, which is the possibility they offer the participants to discuss and get the support of others, such as parents, family, teachers, friends, and colleagues. About 78% of the respondents appreciate this possibility. For some students, this is a favourable condition as it allows them to see different ways of solving a given challenge and to compare various proposals coming from people with different mathematical and life experiences.

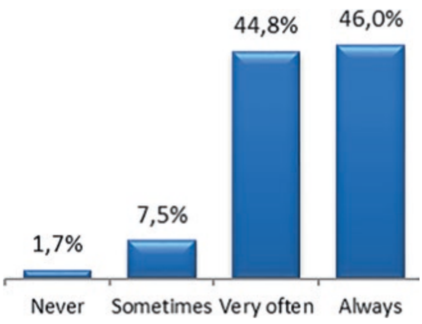


Fig. 18.23 I wish that my solution is easy to follow

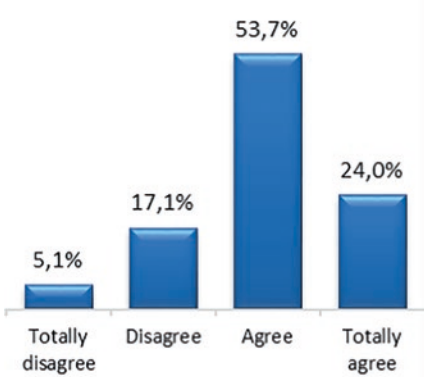


Fig. 18.24 It pleases me to discuss the problems with other people

The appreciation for discussing problems with others is well documented in an interview with a SUB12 participant who describes how she had been exchanging ideas with various family members and teachers of different subjects and compared their different strategies in solving a problem.

> I really enjoyed all the problems. I really liked this one, because it was the first one, but I have been enjoying them all.... But the Problem 2 was fun because I was solving it in a tutorial class and it was my Music teacher who helped me with it.

> In the Problem 7, of the washed towels, I was solving it at home and my mother said that you could not count the towels that were washed but my father said that you had to count the washed towels. Then I took it to my grandfather, and he got a result that was very different from ours. In my solution, I obtained 80, so I took it to school and the Mathematics Club teacher solved it too and she said it was 80. (Excerpt from an interview to a participant in SUB12)

18.5.5 Self-Improvement Experienced

The expressed view that participation in the competitions makes the participants better and more confident in mathematics is very clear in the answers of a large majority of respondents. In both items (Figs. 18.25 and 18.26), around 80% of the respondents seem to hold that belief.

Linked to this belief of becoming better in mathematics and to the feeling of self-confidence, we may find an emotional disposition concerning the satisfaction with learning more mathematics. About 86% (Fig. 18.27) of the respondents show their pleasure with learning mathematics through their participation in the competitions. For example, to some of the students, the experience of successfully solving mathematical problems is a motivating factor for dealing with subsequent new problems. It means, therefore, a favourable mind-set to face new challenges. For other students, the problem-solving experience is felt as a way to develop mathematical thinking and to explore several strategies and mathematical topics that are present in the different types of problems proposed in the competitions.

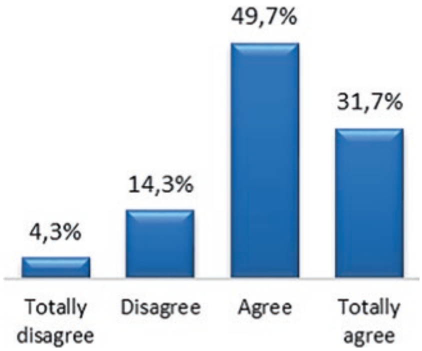


Fig. 18.25 My participation in the competition helps me to be better at mathematics

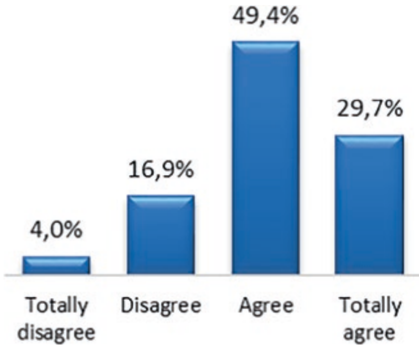


Fig. 18.26 My participation in the competition makes me feel more confident

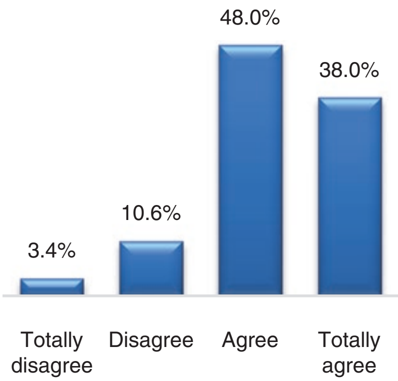


Fig. 18.27 It pleases me to learn more mathematics in the competition

Through the interviews we have carried out, we obtained statements from some participants who expressed not only their conviction of self-improvement but also their pleasure with learning mathematics in the context of the competitions. The words of a participant in one of the interviews clearly show the strong interconnection between the emotional dimension and her perception of competence. Those words convey an awareness of increased competence in problem solving as she gets renewed success in each of the proposed problems and, at the same time, the pleasure of reaching the solution, which fuels her will to continue to be challenged with new problems.

> The fact that I have solved so many problems, always gives me motivation to solve more, because I feel that I am able to solve problems. Perhaps it makes me believe in myself, that I can solve more problems. It helps me because I get to the problems with a different mindset. (…) And it helps me to see myself as a person who can overcome the problems, the difficulties, and if I get it that helps me to solve the next ones. (Excerpt from an interview to a participant in SUB14)

Another participant highlights the nature of the problems and their diversity as a factor that has enabled her to undertake a deep and diversified learning.

> The SUB12 is basically a competition where...we solve a lot of problems, which are also very, very deep. We develop a lot of reasoning and we can learn new ways of solving problems. We also learn a variety. For example, it is varied, it has algebra, or geometry, or other things. (Excerpt from an interview to a participant in SUB12)

18.5.6 Mathematical Performance and Knowledge

The view that it is important to be a good achiever in mathematics to participate in the competitions gets high agreement. The results show that more than 70% of respondents agree with this idea (Fig. 18.28). However, when questioned about the mathematical knowledge required to solve the problems proposed in the competitions, we found a decrease in the percentage of participants who considered that much mathematical knowledge was necessary to solve them (about 58%) (Fig. 18.29). This difference between the participants' beliefs about the advantage of being a high achiever and the knowledge that is actually required to solve the challenges proposed shows once again that the problems posed do not require more than the mathematical knowledge that most students have. This result also reflects one of the characteristics that define these competitions – they are, by nature,

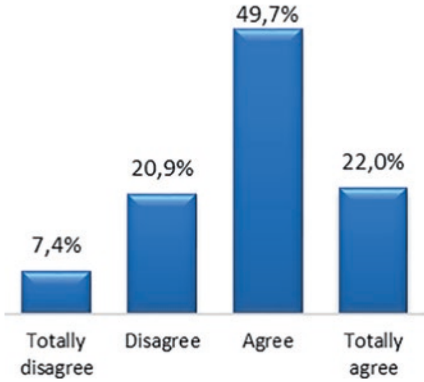


Fig. 18.28 It is important to be good at mathematics to participate in the competition

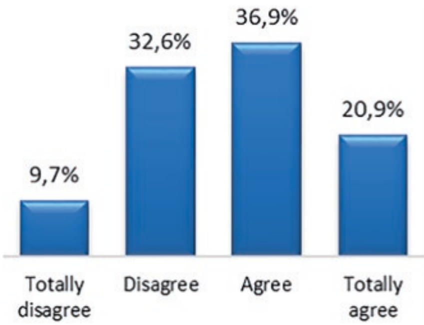


Fig. 18.29 I think that the problems of the competition require a lot of mathematics to be solved

independent of the school curriculum, that is, problems are not designed to fit specific curricular topics.

For some participants, problem solving is not only limited to mathematical knowledge; it is also very important to know how to think mathematically in order to solve them. The opinion of a participant in SUB12 about the need for mathematical knowledge to solve the problems proposed was as follows:

> There were some (problems) that were more to use reasoning and such...which did not take much (mathematical) knowledge but there were others where it was necessary. (Excerpt from an interview with a participant in SUB12)

Another participant seems to believe that the problems were within the reach of all students and that solving them is not only dependent on mathematical knowledge but also on certain mental habits. Despite the idea that problems can be solved by all students, in his words, there emerges the belief that there are students with greater intrinsic ability to solve problems.

> (…) Because, if students think hard about the problems, if they read things carefully, then they can solve them. Anyway, there are also people with more ability than others. (Excerpt from an interview with a participant in SUB12)

In relation to the struggle for the first three places in the competition final, there seems to be more agreement on the need for a good mathematical knowledge (Fig. 18.30).The results of the questionnaire clearly show that about 87% of the respondents consider it important to have a good mathematical knowledge to obtain one of the first three places. This view is related to another characteristic of these competitions, which is that the true competitive spirit is restricted to the final phase. In fact, at that stage, the qualified students solve the problems individually, without any help and in limited time, besides being at stake the dispute of the first three awards.

In many of the students' e-mails, expressions of rejoicing arise from the realization that they have been qualified to the final. In certain cases, the participants also reveal a certain anticipation of a new round of the competition that they believe to be more demanding (Fig. 18.31).

In general, participants consider the qualifying phase as training for the final competition. Thinking, explaining, and calculating, for example, are skills that

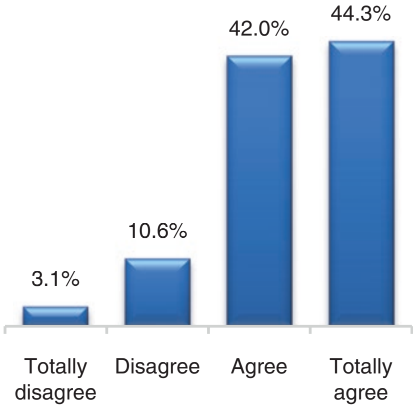


Fig. 18.30 It is important to be good at mathematics to get to the top places in the final

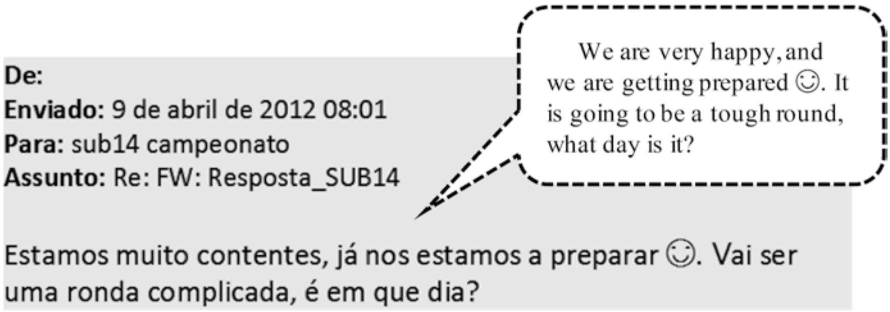


Fig. 18.31 Message from a student who got the confirmation that he qualified for the final

some participants believe to have developed over the 6 months of the qualifying phase and which they feel will be indispensable for the final.

> With the SUB's I was training, because I had to calculate, I had to learn, to think, and to explain my reasoning well, especially for the Final. (Excerpt from an interview with a participant in SUB14)

Regarding the need for mathematical knowledge to solve the problems proposed in the final, participants are aware that they require some amount of knowledge:

> We needed mathematics to solve them [the problems of the Final]. Yes, we had that notion. (Excerpt from an interview with a participant in SUB14)

In addition to the emphasis given by the participants to the relevance of mathematical knowledge at the final, other aspects were also highlighted by the students who were qualified, such as the reward of their dedication and commitment and the pride of being part of those who like to be challenged:

> Reaching the final is very good. I am pleased because the work I have done over the months, solving mathematical problems, working on them, was very good. So, I am in the final. And it gives me satisfaction, not because it is important to win but because going to the final allows me to participate and solve even more difficult problems amongst the best ones as well. (Excerpt from an interview with a participant in the Final of SUB14)

In the statement above, it is clear that reaching the final of the competition is a meaningful achievement to the student; it also highlights her perception of being confident in addressing possibly even more challenging problems.

18.6 Final Comments

In this final section, we set out to draw some conclusions that may support our understanding of students' attitudes towards problem solving and mathematics in the context of the mathematical competitions SUB12 and SUB14.

One of the clear results of the study is, of course, a very positive overall attitude towards participation in those inclusive competitions. Being a quite likely result, the several data obtained nevertheless allow a more refined and more consistent meaning to be given to this global attitude.

We may start by focusing on the affective dimension and observing that the fact that it is pleasant to participate in the competition, which is well evidenced by the answers to the questionnaires and statements of the students, shows that this liking cannot be simply explained by the fun side that the participation entails for the young participants. Apparently, this liking of participating is very much associated with a satisfaction with the type of problems proposed and the fact that they are felt as appealing. Another quite shared emotional element is related to the pleasure of being able to get the correct answer, that is, to overcome the mathematical challenges proposed. This adds to the circumstance that students are glad to be able to explain their resolutions satisfactorily, that they like to be allowed to discuss the problems with other people, and also that they feel that they are learning more mathematics.

Thus, the emotional dimension is distributed through several aspects of the competition, which refer to its intrinsic inclusive nature, to the type of problems proposed, and to the encouragement of expressing the process that leads to the solution of a problem.

With regard to the way students express their views, it may be pointed out that their interest in mathematics increases while at the same time students perceive that the problems can be solved in several different ways and incite them to think, and they find themselves impelled to explain how they got the solution. Another relevant fact relates to their awareness that the problems do not require a great extent of mathematical knowledge, leading us to suppose that they feel the problems as being at their reach.

Finally, with regard to the dimension of their perceived competence, we observe that students are aware of their competence, both in solving problems and in expressing the solution process, and also in the degree of knowledge they consider at play in the two phases: the qualifying and the final. Briefly, most students report that it is feasible to come up with various ways of tackling a problem, although a considerable percentage of students admit to having less dexterity and flexibility to actually do so. Similarly, there is a range of students who are not completely sure of being able to perfectly explain their solution to problems. This fact is corroborated by statements from some students who feel somewhat insecure about what they are able to achieve in this respect. In addition, there is evidence that many students believe that their learning and struggling to explain the problem-solving process also help them to become better at mathematics. Furthermore, the degree of mathematical knowledge they find necessary to participate in the competition is seen as relevant, although the preponderance of that knowledge is more noticeable when it comes to reaching the most demanding level of the competition, which is the final.

In view of the results, it is time to summarize some of the characteristics of the attitudes of the students participating in SUB12 and SUB14. At this point, we again underline the very close interaction between (i) participation, (ii) problem solving, and (iii) mathematics.

- The competition represents a scenario of participation in a mathematical activity beyond school that is globally enjoyable for the students, thus representing a *positive emotional connection* with mathematical challenges and mathematics.
- This overall pleasant scenario has *favourable emotional aspects* such as the appeal they encounter in the challenges posed but also in other aspects such as the pleasure of obtaining the correct solution, the willingness to express the problem solving in an interesting and right way, and the chance to discuss the challenges with different people in their daily school and home environments.
- The pleasant scenario also has *relevant cognitive aspects*, namely, an increased interest in mathematics, as well as a realization that problem solving involves reasoning, mathematical thinking, and several possible and legitimate ways of approaching challenging mathematical situations.
- Among the cognitive aspects to be highlighted, it is apparent that students value the search for different strategies and for ways of conveying their reasoning and problem-solving processes; they also *believe on improving their ability* to solve and express and on becoming better at mathematics; for some students, the qualification phase, which involves several months of work in solving challenges, is synonymous with preparation and prolonged learning.
- Concerning the way in which their *self-competence is revealed and nurtured*, we find that participation in the competition increases students' confidence in mathematics and problem solving, making them aware of their skills and problem-solving competence.
- In general, students seem to feel challenged and have an interest and pleasure in solving the challenges, even when they do not always achieve what they consider to be the highest possible level; there is therefore a widespread awareness that students assess themselves in terms of their competence, showing a certain *perception that they are competent to successfully participate* in the qualification phase; the students also seem to presume that the final phase is more demanding, but they show a positive expectation in their participation, underlining, for example, the opportunity to be competing with the best ones.

- Students recognize the importance of mathematical knowledge in the context of their participation but indicate that there are important skills involved in solving the problems that go beyond-school mathematical knowledge.

The three-dimensional model that we have adopted in this research confirms the idea of a very close interaction between the students' emotional disposition and the dimensions related to beliefs and self-competence. The results seem to reinforce the importance of developing students' positive attitudes by promoting "(…) a vision of mathematics based on processes more than products, to encourage students to shift the idea of success from the production of correct answers to the enactment of meaningful thinking processes (…)" (Di Martino & Zan, 2011, p. 481).

From our point of view, the mathematical competitions SUB12 and SUB14 provide fulfilling experiences to the young participants (in each of the three dimensions of the model adopted), which are linked to the will to solve and express mathematical problems, to a favourable view about mathematics, and to a perception of improvement. Among the various operational characteristics of the competitions, the emphasis placed on the selection of rich but moderately demanding problems contributes to an engaged participation with positive effects on students' relationship with problem solving and mathematics.

References

Amado, N., Carreira, S., & Ferreira, R. (2016). *Afeto em competições matemáticas inclusivas. A relação dos jovens e suas famílias com a resolução de problemas*. Autêntica, Brazil: Belo Horizonte ISBN 978-85-513-0007-7.

Applebaum, M., & Leikin, R. (2014). Mathematical challenge in the eyes of the beholder: Mathematics teachers' views. *Canadian Journal of Science, Mathematics and Technology Education, 14*(4), 388–463.

Barbeau, E. (2009). Introduction. In E. Barbeau & P. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI Study* (pp. 1–9). New York: Springer.

Barbeau, E., & Taylor, P. (Eds.). (2009). *Challenging mathematics in and beyond the classroom: The 16th ICMI study*. New York: Springer.

Carreira, S., Jones, K., Amado, N., Jacinto, H., & Nobre, S. (2016). *Youngsters solving mathematical problems with technology: The results and implications of the problem@web project*. New York: Springer. https://doi.org/10.1007/978-3-319-24910-0

Charles, R., & Lester, F. (1982). *Teaching problem solving: What, why and how*. Palo Alto, CA: Dale Seymour.

Di Martino, P., & Zan, R. (2010). Me and Maths': Towards a definition of attitudes grounded on students' narratives. *Journal of Mathematics Teacher Education, 13*(1), 27–48.

Di Martino, P., & Zan, R. (2011). Attitude towards mathematics: A bridge between beliefs and emotions. *ZDM – Mathematics Education, 43*, 471–482.

Di Martino, P., & Zan, R. (2015). The construct of attitude in mathematics education. In B. Pepin & B. Roesken-Winter (Eds.), *From beliefs to dynamic affect systems in mathematics education (advances in mathematics education)* (pp. 51–72). Springer International Publishing. https://doi.org/10.1007/978-3-319-06808-4_3

Eligio, U. (2017). An overview of the growth and trends of current research on emotions and mathematics. In U. Eligio (Ed.), *Understanding emotions in mathematical thinking and learning* (pp. 3–41). London: Elsevier. https://doi.org/10.1016/B978-0-12-802218-4.00001-7

Eric, C. (2011). Primary 6 students' attitudes towards mathematical problem-solving in a problem-based learning setting. *The Mathematics Educator, 13*(1), 15–31.

Freiman, V., & Applebaum, M. (2011). Online mathematical competitions: Using virtual marathon to challenge promising students and to develop their persistence. *Canadian Journal of Science, Mathematics and Technology Education, 11*(1), 55–66.

Freiman, V., & Lirette-Pitre, N. (2009). Building a virtual learning community of problem solvers: Example of CASMI community. *ZDM – Mathematics Education, 41*, 245–256. https://doi.org/10.1007/s11858-008-0118-8

Freiman, V., & Manuel, D. (2015). Relating students' perceptions of interest and difficulty to the richness of mathematical problems posted on the CAMI website. *Quadrante, XXIV*(2), 61–84.

Freiman, V., & Véniza, N. (2006). Challenging virtual mathematical environments: The case of the CAMI project. In *Pre-conference paper of the study conference for ICMI study 16 – Challenging mathematics in and beyond the classroom* [Retrieved from http://www.amt.edu.au/ icmis16pcanfreiman.pdf]

Freiman, V., Vezina, N., & Gandaho, I. (2005). New Brunswick pre-service teachers communicate with schoolchildren about mathematical problems: CAMI project. *ZDM – Mathematics Education, 37*(3), 178–189.

Hannula, M. (2012). Exploring new dimensions of mathematics-related affect: Embodied and social theories. *Research in Mathematics Education, 14*(2), 137–161. https://doi.org/10.1080/14794802.2012.694281

Hannula, M., Di Martino, P., Pantziara, M., Zhang, Q., Morselli, F., Heyd-Metzuyanim, E., et al. (2016). Attitudes, beliefs, motivation,and identity in mathematics education. In *Attitudes, beliefs, motivation and identity in mathematics education. ICME-13 Topical Surveys*. Cham: Springer. https://doi.org/10.1007/978-3-319-32811-9

Kenderov, P., Rejali, A., Bussi, M. G., Pandelieva, V., Richter, K., Maschietto, M., et al. (2009). Challenges beyond the classroom – Sources and organizational issues. In E. J. Barbeau & P. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI study* (pp. 53–96). New York: Springer.

Koichu, B., & Andzans, A. (2009). Mathematical creativity and giftedness in out-of-school activities. In R. A. Berman & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 286–307). Rotterdam: Sense Publishers.

Leikin, R. (2004). Towards high quality geometrical tasks: Reformulation of a proof problem. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th international conference for the psychology of mathematics education* (Vol. 3, pp. 209–216). Bergen, Norway: PME.

Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education – CERME-5* (pp. 2330–2339). Larnaca, Cyprus: University of Cyprus and ERME.

Leikin, R. (2014). Challenging mathematics with multiple solution tasks and mathematical investigations in geometry. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 59–80). Dordrecht, the Netherlands: Springer.

Manuel, D. (2010). *Étude de la créativité mathématique dans les solutions aux problèmes proposés dans la communauté virtuelle CASMI*. (Master Thesis), Université de Moncton, Moncton, NB, Canada.

Mata, M. L., Monteiro, V., & Peixoto, F. (2012). Attitudes towards mathematics: Effects of individual, motivational, and social support factors. *Child Development Research*, 2012, Article ID 876028, 10 pages. https://doi.org/10.1155/2012/876028.

McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.

OECD. (2013). *PISA 2012 results: Ready to learn: Students' engagement, drive and self-beliefs (volume III)*. PISA, OECD Publishing. https://doi.org/10.1787/9789264201170-en
Polya, G. (1973). *How to solve it*. Princeton, NJ: Princeton University Press.
Protasov, V., Applebaum, M., Karp, A., Kasuba, R., Sossinsky, A., Barbeau, E., et al. (2009). Challenging problems: Mathematical contents and sources. In E. J. Barbeau & P. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI study* (pp. 11–51). New York: Springer.
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
Stockton, J. C. (2012). Mathematical competitions in Hungary: Promoting a tradition of excellence & creativity. *The Mathematics Enthusiast, 9*(1–2), 37–58.
Taylor, P. (2006). *Challenging mathematics and its role in the learning process*. [Retrieved from: http://www.amt.edu.au/pdf/icmis16ptaylor.pdf].
Wedege, T., & Skott, J. (2007). Potential for change of views in the mathematics classroom? In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the fifth congress of the European Society for Research in mathematics education* (pp. 389–398). Larnaca, Cyprus: University of Cyprus and ERME.

Chapter 19
Roles of Aesthetics and Affect in Mathematical Problem-Solving

Norma Presmeg

19.1 Introduction

Both aesthetics and affect are under-researched domains in mathematics teaching and learning (Sinclair, 2004, 2008). Yet there is ample evidence, from the personal experience of teachers of mathematics, of the motivational effect of an "Aha!" moment experienced by a student in solving a mathematical problem and of the pleasure that such an experience affords. And research mathematicians frequently speak of the aesthetic elements involved in their work, for instance, of the beauty of an elegant proof (Burton, 1999). Thus, it would appear that aesthetic sense is a significant issue in the learning and doing of mathematics and worthy of further research attention. Especially, as in the case of the nonroutine problem-solving involved in the Problem@Web Project, when students are not under compulsion to solve the mathematics problems, what is it that gives them the motivation to do so and to persist in the face of difficulty in order to reach a solution? My goal in this chapter is to tease out some of the significant aspects of both aesthetic sense and affective issues in the solving of nonroutine mathematical problems and to illustrate these aspects from relevant research projects.

19.2 A Vignette

I start with a vignette of a high school teacher who was particularly aware of the aesthetic elements of mathematics and who stressed these in his teaching. This teacher was especially adept at helping his students to make connections in their

N. Presmeg (✉)
Illinois State University, Normal, IL, USA

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_19

learning of mathematics, between different topics, but also in linking aesthetic and cognitive processes.

In my original fieldwork (Presmeg, 1985), Mr. Blue (pseudonym – in most cases participants were identified by their favourite colours) was a high school mathematics teacher who did not feel the need to use imagery or inscriptions in solving the problems on my test for preference for visuality in mathematics (called the Mathematical Processing Instrument, MPI). In fact, his *mathematical visuality* score on this test was only 3 out of a possible 36, the most nonvisual score of all of the 13 teachers in my study. Yet he used many of the classroom aspects that had been identified in the literature and tested in my research as being facilitative of visual thinking in mathematics classrooms (Presmeg, 1991). Of 12 such classroom aspects, he was observed to use 7 during the year in which I observed his lessons. This teaching visuality score placed him squarely in the middle group of teachers (as one of four teachers in this group), between those of the nonvisual group (four teachers) and the visual group (five teachers) according to the visuality of their teaching. It turned out that *mathematical visuality* (as measured using the MPI) and *teaching visuality* were only weakly correlated in this sample of teachers (Spearman's rho = 0.404, not significant) – a result that made sense because good teachers know when their students need more visual thinking than they do. In the classes of these 13 teachers, of the 54 senior students who preferred to think visually according to my test, those who performed well in their final school-leaving examinations in mathematics were in the classes of teachers in the middle group, contrary to the common sense notion that visualizers would do best with visual teachers.

Commenting in an interview on some of the rich characteristics that I had identified in his lessons – in which he stressed abstraction and generalization in addition to introducing and encouraging visual inscriptions – Mr. Blue spoke as follows:

> Mr. Blue [with excitement]: You've got to be careful sometimes I think; you've got to be careful making, bringing things too, from the abstract, to too concrete. Then it's that way forever, then everything is like that. You've got to be careful with that because sometimes you must remember that our abstractness carries us to flights of imagination of where we can go with it. And that's what I would like them as often to see here, when we do something, this is another possible way of doing this problem; more algebraically, what you can do with it.

For Mr. Blue, algebra was often the vehicle of abstraction. It was noteworthy that for Mr. Blue, the aesthetic elements resided in the mathematics itself and not in some external addendum introduced in the classroom to "make the mathematics interesting" – which would imply that mathematics is intrinsically boring! In his teaching, Mr. Blue frequently expressed his own pleasure in the beauty of mathematics. Indeed, he expressed his feelings often not only towards mathematics but also towards his boys (he taught in a boys' school). In a trigonometry lesson, he spoke with his students about errors that some of them were making:

> Mr. Blue: Don't just square things, and suddenly they disappear into space. … And then of course I was really saddened by this: now let me say this to you. Don't do this anymore. Now you know better than that in this room. You cannot take the square root of individual what?

> Boys: Terms.
>
> Mr. Blue: Terms. … Don't force it! Maths just won't be forced. That's the beauty of it, that's its beauty: where it stands strong against this forcing things into it that don't have any place for it at all. It must go on the way it always has gone on.

The way that Mr. Blue encouraged metacognition was also apparent in an algebra lesson on change of base of logarithms, with the same class. The problem under discussion, which had been done by the boys in a test, involved a quadratic equation in base three logarithms:

$$(\log_3 x)^2 - 10\log_3 x + 9 = 0$$

> Mr. Blue: So this would be the fastest way: factorize. You can do the change of base with tens, you can get it, it will be right, when you've finished [but it is slow]. … We could put a y in for log to the base 3 of x, couldn't we? Then factorize. … The whole thing in higher grade is to think in patterns, and relate the patterns of the former work received. And you get bigger and bigger problems. If you look at this one now, how many ideas were in this problem? This idea was a log idea, this turns into a quadratic idea, this turns into factorization, this turns into exponentials to get the answer. All in one problem. That's what you must start getting used to.

The connections between domains that Mr. Blue was helping his boys to identify in this lesson are a topic that is relevant to the beauty of mathematical form.

Mr. Blue's references to the beauty of mathematical abstraction have resonance in the writings of great classical thinkers and mathematicians:

> Aristotle: "The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful."
>
> Bertrand Russell: Mathematics possesses "a supreme beauty … capable of a stern perfection such as only the greatest art can show."
>
> Hardy: "There is no place in this world for ugly mathematics."
>
> However, references to beauty by mathematicians are often in lieu of addressing aesthetic notions, which might have a connotation of fuzziness to mathematicians. (Sinclair, 2004, p. 263)

19.3 Some Theoretical Formulations of "the Aesthetic"

In two thorough and thoughtful papers, Sinclair (2004, 2008) discusses current theoretical formulations of the aesthetic in the arts and the sciences, including mathematics. Aesthetics falls under the umbrella of axiology, which is concerned with judgements of value, in classical philosophy (Sinclair, 2008). Axiology embraces qualities of being in balance, having value, and being worthy and proper; thus, aesthetics involves more than artistic taste and style; aesthetics concerns what, how, and why something is valued. Sinclair (2004) points out that there are three groups of aesthetic responses. In relation to the focus of this chapter, these may play distinct roles in mathematical problem-solving, as follows:

- *Evaluative*: These aesthetic responses ground value judgements. Some theoretical formulations in this mode centre aesthetic qualities in the solution or mathematical proof being evaluated, e.g. elegance or symmetry in the solution itself. This formulation is the objectivist view. Then an external agent may do the evaluating, which presupposes certain agreed-upon criteria for judging the worth of the solution. Other formulations place more emphasis on the problem-solver's thought processes as the locus of aesthetic properties, as in the following group.
- *Generative*: Aesthetics in this view generates new ideas and insights in the thought processes of someone solving a mathematical problem, which could not be derived by logic alone. Several of the solutions given by young students to problems in the mathematical competitions SUB12 and SUB14 involved in the Problem@Web Project (Jacinto, Carreira, & Amado, 2011) illustrate this aspect. I return to one of these insightful solutions later in this chapter.
- *Motivational*: These responses attract mathematicians to certain problems or fields of mathematics. Equally, such aesthetic concerns may provide the incentive to persist in the face of difficulty. This aspect is particularly germane to the situation in the mathematical competitions SUB12 and SUB14, in which participation is optimal, and some of the problems may be challenging and go beyond the mathematics that participants have studied in their classes.

What is the role of affective states and reactions in regard to aesthetic concerns in problem-solving?

19.3.1 Distinctions Between Aesthetics and Affect

Aesthetics and affect are intimately connected, but different authors have provided differing interpretations of the nature of these connections. In an important seminal formulation, Goldin (2000) explicitly linked affective and cognitive representations in his model of problem-solving competence. For Goldin, affect is a tetrahedral construct comprising beliefs, attitudes, emotional states, and finally, the axiological dimension of values, ethics, and morals. Thus for him the aesthetic, as relating to the latter dimension, falls under the umbrella of affect. Sinclair (2008) argues convincingly that it should rather be the other way around: she points out that the problem-solver becomes alert to aesthetic responses *through* affective states, given that feelings and emotions rely on sensory perception. "From the inquiry viewpoint, the aesthetic functions is a non-logical form of knowing, which aligns itself much more with cognition (broadly viewed) than with affect" (p. 10). She continues as follows:

> In other words, the aesthetic and the affective domains each *function* differently in the problem solving process: the aesthetic draws the attention of the perceiver to a phenomenon (a pattern, a relationship, a contradiction), while the affective can bring these perceptions to the conscious attention of the perceiver. (p. 11, italics in original)

Aesthetic responses thus link with emotions, attitudes, beliefs, and values, but they have a sociocultural component (Sinclair, 2004): they may exhibit historical variation, and they may be culturally specific.

Sinclair (2008) points out that the aesthetic is not confined to artistic experience. Based on Dewey's formulation, an experience has aesthetic quality "whenever there is coalescence into an immediately enjoyed qualitative unity of meanings and values, drawn from previous experience and present circumstances" (p. 6). Such experiences are aesthetic in that they combine emotion, satisfaction, and understanding. They can happen "while appreciating art, fixing a car, having dinner, or solving a mathematical problem" (ibid.).

Both aesthetics and affect are inescapably connected to the cognitive domain in mathematical problem-solving, as I illustrate in the next section using data from a research project in which graduate students were asked to solve nonroutine mathematical problems (Presmeg & Balderas-Cañas, 2001). This project confirmed that generative and motivational aesthetic responses may evoke many emotions, including tension, curiosity, bewilderment, frustration, and loss, as well as more positive states. Frustration and loss may contribute to ultimate excitement and satisfaction; however, it is necessary for the problem-solver to be able to handle the negative states in order to move past them. Further, *mathematical self-identity* is involved. In analysing the graduate students' varied responses, we found it useful to use the notion of affective pathways developed by Goldin (2000). It is significant that recent theoretical formulations highlight the dynamic nature of such pathways, as illustrated in the title of a recently edited book, *From Beliefs to Dynamic Affect Systems in Mathematics Education* (Pepin & Roeskin-Winter, 2014).

19.4 Illustrations from Empirical Research: Aesthetics, Visualization, and Affect in Graduate Students' Mathematical Problem-Solving

Affective pathways are "sequences of (local) emotional reactions that interact with cognitive configurations in problem solving. Such pathways provide solvers with useful information, favouring the learning process and suggesting heuristic problem-solving strategies" (Gómez-Chacón, 2012, p. 104). However, such pathways may have positive or negative consequences for problem-solving. The following are two contrasting pathways identified by Goldin (2000).

- *Affective pathway 1 (enabling):* curiosity → puzzlement → bewilderment → encouragement → pleasure → elation → satisfaction → global structures (specific representational schemata, general self-concept structures)
- *Affective pathway 2 (constraining):* curiosity → puzzlement → bewilderment → frustration → anxiety → fear/despair → global structures (self-/mathematics-/science-/technology-hatred)

These affective pathways were useful in the analysis of systems of cognition, affect, and aesthetic sense in graduate students' solving of nonroutine mathematical problems.

As another potentially important component of aesthetics, visual imagery used in mathematics is often of a personal nature, related with cognition and belief systems, but also laden with affect (Goldin, 2000; Presmeg, 1997). Such imagery also may enable or constrain mathematical problem-solving (Presmeg, 1985): it is inextricably interwoven with emotional states that carry meaning, intertwined with language, cognition, and self-image. The significance of *meta-affect* (affect about affect: Goldin, 2000) was brought out in the analysis of the cases that follow. For instance, Presmeg and Balderas-Cañas (2001) describe the case of Ms. Blue,[1] an elementary school teacher, whose meta-affect so paralysed her thinking that she was not able even to enter the first sense-making stage of solving three nonroutine problems presented to this group of graduate students.

Ms. Blue was one of the four graduate students interviewed individually in this research project. There were two tape-recorded interviews with each of these students, but some of the results of only the first of these interviews are presented here. In the first interview, three nonroutine problems were presented in sequence, and the student was asked to "think aloud" or to describe his or her thinking process in retrospect. The three problems (taken from Part C of Presmeg's 1985 MPI) were as follows (Fig. 19.1):

Ms. Blue represents an extreme case in which negative affect inhibited her problem-solving to such an extent that her thinking "froze up", and she was unable to proceed at all with any of the problems, despite a "strong background" in mathematics, in the sense that she had completed successfully several graduate mathematics courses. Early in the interview, it became apparent that she considered these problems too difficult for her to solve: "I *hate* these problems!" she exclaimed with passion. She drew diagrams and wrote formulas, but these were unproductive:

C-1: A boy walks from home to school in 30 minutes, and his brother takes 40 minutes. His brother left 5 minutes before he did. In how many minutes will he overtake his brother?

C-2: An older brother said to a younger, "Give me eight walnuts, then I will have twice as many as you do." But the younger brother said to the older one, "You give me eight walnuts, then we will have an equal number." How many walnuts did each have?

C-3: A train passes a telegraph pole in a quarter of a minute and in three quarters of a minute; it passes completely through a tunnel 540 meters long. What is the train's speed in meters per minute, and its length in meters?

Fig. 19.1 Three problems used in the first interview

[1] Ms. Blue and Mr. Blue are unrelated. They were asked to choose their favourite colour as a pseudonym, in both cases.

"Actually I want to visualize it with a picture, but the picture never really works with the numbers". She made futile attempts to understand the conditions of all the problems, and she could not proceed with any of them. Her meta-affect seemed to be connected with past experiences. She seemed paralysed by fear of the problems, which she later confirmed, and attributed to unpleasant experiences in the past. Goldin (2000) summed up this situation as follows: "If there is already a history of unfavourable affect …, then the anxiety and fear that evoke avoidance and denial processes can occur almost as soon as a problem is posed" (p. 218). The affective pathway that is illustrated in the case of Ms. Blue is the constraining one:

- *Affective pathway 2 (constraining):* [curiosity and puzzlement were not displayed] bewilderment → frustration → anxiety → fear/despair → global structures.

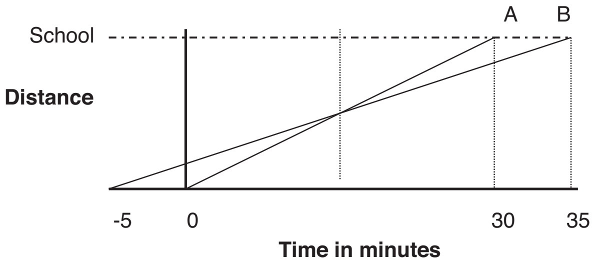
The history of negative affect in Ms. Blue's case resulted in fear/despair and "mathematics-hatred" responses in the very first stages of problem-solving. There was also evidence of negative affect concerning her self-identity as a mathematician.

Continuing with empirical data from the same project, three further cases of nonroutine problem-solving are presented, each of which gives evidence of a more productive affective pathway than that of Ms. Blue. In these three cases, despite difficulties, the students were able to persist and arrive at solutions to the problems by various methods.

An enabling affective pathway was experienced in the case of Mr. Silver, who was a teacher of college-level mathematics. His solution processes could be analysed in terms of all four "moments" of our framework, namely, preparing to solve, processes of solution, conclusion of these processes, and checking. However, the complexity of these processes helped us to become aware of the cognitive demands of these problems: the solution processes were not straightforward, as is illustrated in Mr. Silver's solution to problem C-3 about the boy and his brother walking to school at different rates (Fig. 19.1). He first drew two lines for the boy and his brother, called A and B, but he was unsure if the lines represented distance or time, and whether negative time was acceptable. After some puzzling, he drew a two-dimensional diagram (Fig. 19.2).

The diagram marked the end of the preparation phase, and it would have been easy to conclude that the boy overtook his brother halfway. But for Mr. Silver, the

Fig. 19.2 Mr. Silver's diagram for problem C-3



symmetry was obscured by the cognitive demands of the two starting points, and he resorted to algebra, working with the rate formula. He returned to the diagram in the checking and hindsight phases, but his main confidence lay in his algebraic solution processes, which he used immediately in the other two problems. He expressed metacognition, but not meta-affect. However, his persistence, confidence, and initial puzzlement imply the presence of affect.

Two further solutions of C-3, those of Mr. Gold and Mr. Green, illustrate varied solution processes, which however still yielded enabling affective pathways. Mr. Gold also taught college-level mathematics courses. His metacognition was strong (but not meta-affect). He was confident and enthusiastic, despite moments of doubt and uncertainty in certain moments of the processes. For C-3 he went straight to algebra, with a diagram. Still in the preparation phase, and thinking aloud, he assigned "x over 30" as A's velocity. Thinking of the brother, B, he struck an obstacle:

> I'd wish to check this. An easy part … distance … Let's put one mile in here, for the distance from home to school. So if it takes this brother 40 to walk a mile … 40 is two-thirds of an hour … he's walking at 60 miles per hour.

Mr. Gold appeared to see nothing unusual in this unrealistic hypothesis! After a long pause, he started afresh, using his diagram, and succeeded after complicated algebraic processing: "The diagram helps me to visualize in my mind, because it seemed chaotic, the processes in my mind. I had a diagram to relate to all these equations".

The case of Mr. Green illustrates how meta-affect can in fact hinder the finding of a solution. Of the three college mathematics teachers, Mr. Green's meta-affect was most noticeable: "Oh I don't know why I am so nervous. I've never been studied before like that (nervous laugh)!" In his problem-solving affect, he appeared to enter the negative pathway, experiencing frustration and anxiety at times during the solving of both C-3 and C-6. Referring to C-3 he exclaimed: "Oh man, it [algebra] doesn't work today! Should be easy!" Each time, he broke free from this negative pattern, experiencing encouragement and satisfaction as he solved the problems successfully, using combinations of diagram, algebra, and imagery. After several tentative diagrams, he came up with the metaphor of a clock face for C-3 (Fig. 19.3);

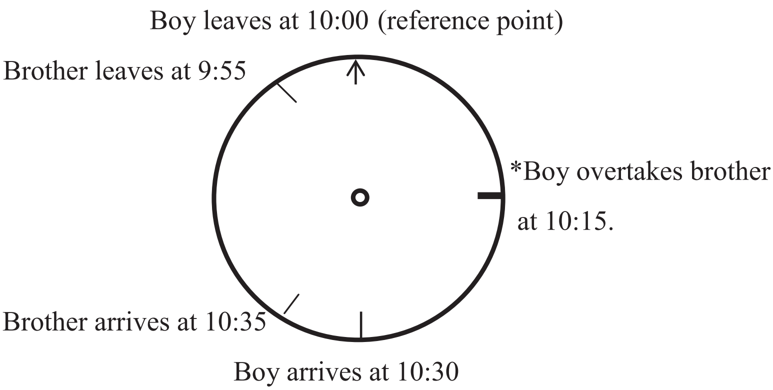


Fig. 19.3 Mr. Green's metaphor: a unified clock face image

however, it was algebraic equations that gave him a solution. Even then, there was confusion about whether his answer of 20 min applied to the boy or to his brother. He did not see the symmetric solution that the clock provided.

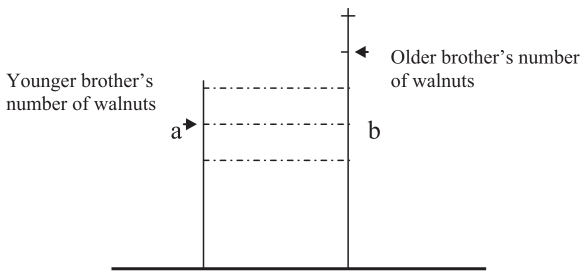
Problem C-4 (the walnuts problem – see Fig. 19.1) was solved relatively quickly and without much difficulty by all three college teachers, once the cognitive obstacle (for making equations) of giving and receiving walnuts was overcome. None of them drew a diagram, although Mr. Silver and Mr. Green reported imagery. The problem could be solved visually (which is why it was included in the MPI), but the visual solution is relatively complex (Fig. 19.4), and algebraic solutions were used in all three cases (e.g. as in Fig. 19.5).

The enabling affective pathway could be applied to all three cases, those of Mr. Silver, Mr. Gold, and Mr. Green:

- *Affective pathway 1 (enabling):* curiosity → puzzlement → bewilderment → encouragement → pleasure → elation → satisfaction → global structures (specific representational schemata, general self-concept structures).

All three were college-level mathematics instructors, with positive meta-affect possibilities, despite the hindering effect of Mr. Green's emotions about his affect.

C-4. Solution 2:I drew a diagram to represent the number of walnuts.



From the conditions of the problem, the top half of line b is divided into 4 equal parts, each representing 8 walnuts. Thus 7 of these parts represents the older brother's number of walnuts and 5 parts represents the younger brother's. Thus the younger brother had 40 and the older brother 56 walnuts.

Fig. 19.4 A visual solution to problem C-4

C-4. Solution 1: I used symbols and equations, e.g.,
Let the younger brother have x walnuts and the older, y walnuts.

$$y + 8 = 2(x - 8)$$

and $y - 8 = x + 8.$

Solve simultaneously: $x = 40$ and $y = 56.$

The younger brother has 40 walnuts and the older 56 walnuts.

Fig. 19.5 A solution to problem C-4 using symbols and equations

It is possible that their college teaching may have influenced their preference for algebraic solutions. Despite various forms of puzzlement and bewilderment, all three experienced elation and satisfaction on solving the problems successfully, with intact positive self-concept.

19.4.1 The Roles of Visualization with Regard to Aesthetics and Affect

In the previous section, summarizing the affective pathways of four graduate students, in the sense-making phase, both Mr. Silver (Fig. 19.2) and Mr. Green (Fig. 19.3) drew effective diagrams for problem C-3. In both cases, the solution could have been immediately inferred from the symmetry of the visual forms – but in neither case was this aspect apparent, both students resorted to algebraic equations to solve the problem. However, the presence of the diagram gave them a "rock" on which to anchor their algebraic reasoning, which had the positive effect of leading to enabling affective states, despite initial bewilderment or discouragement. Visual aspects of aesthetic sense in problem-solving could be examined further. In several other instances in my research, the patterning aspects of aesthetic sense resulted in metaphoric thinking that had the potential to be fruitful and to overcome some of the negative affective states, such as frustration, that threatened to derail the problem-solving processes. Mr. Green's example of a clock metaphor that was ultimately fruitful resonates with the work of Carreira (2001) in connection with mathematical models. Table 19.1 shows the extent of diagram and imagery use (by self-report) of the four graduate students in my research.

As is evident from Table 19.1, problems C-3 and C-6 prompted the use of a diagram by all four students, although such diagrams did not always provide an immediate solution or method of proceeding. Problem C-6, the train in tunnel problem (Fig. 19.1), is an interesting instance of a problem in which an aesthetic sense of symmetry could result in creative visual solutions in concrete or more abstract ways. Here is the text of the problem once more: "C-6. A train passes a telegraph pole in ¼ minute, and in ¾ minute it passes completely through a tunnel 540 metres long. What is the train's speed in metres per minute and its length in metres?" The cognitive complexity of this problem seemed to reside in synchronizing the pole and the tunnel. All three college teachers resorted to algebraic

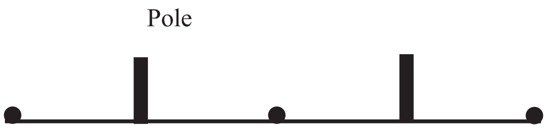
Table 19.1 Analysis of use of visualization by four graduate students

Problem number	Ms. Blue		Mr. Silver		Mr. Gold		Mr. Green	
	Diagram	Imagery	Diagram	Imagery	Diagram	Imagery	Diagram	Imagery
C-3	√		√		√		√	√
C-4	X	√	X	√	X	X	X	√
C-6	√		√	√	√	√	√	√

solutions, despite their diagrams. However, a creative visual solution to the problem is as follows (Fig. 19.6).

In my original research (Presmeg, 1985), some of the participants drew a picture of the train, complete with engine and coaches and smoke emerging from the stack, in a diagram that elaborated on the abstract one in Fig. 19.6. The concrete details seemed unnecessary. However, Susana Carreira (personal email communication, 2014; Carreira, 2015) pointed out that with today's available technology, it is easy to add concrete details to the picture, as she demonstrated (Fig. 19.7). Such details could be important resources in students' sense-making and could enhance their positive affect and incentive to solve the problem. Unlike the algebraic process, the visual method yields an instant solution.

The examples of students' solutions to problems from the SUB12 and SUB14 competitions give ample evidence of the power of effective visual solutions in this context too (Jacinto, Amado, & Carreira, 2009; Jacinto et al., 2011). The aesthetic function of illustration is pointed out by Jacinto et al. (2009), and its link with positive affect is also suggested (Jacinto & Carreira, 2012). The level of difficulty of problems presented is a factor in this regard, with optimal results when the level of difficulty is neither too low nor too high but moderate (Carreira, Tomás Ferreira, & Amado, 2013).



The first segment of the journey is where the train is before it enters the tunnel. Each segment takes the train a quarter minute. It takes the front of the train half a minute to go the length of the tunnel. Thus, the speed of the train is 540 x 2 = 1080 metres/minute, and the length of the train is half the length of the tunnel, i.e., 270 metres.

Fig. 19.6 A creative visual solution to the train in tunnel problem

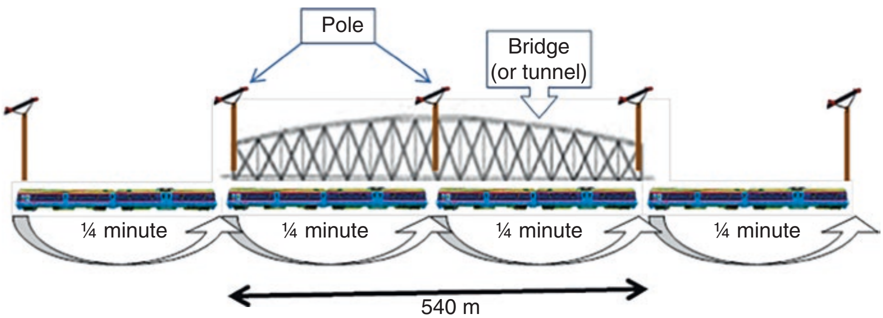


Fig. 19.7 Pattern imagery and concrete imagery: Susana's creative concrete solution

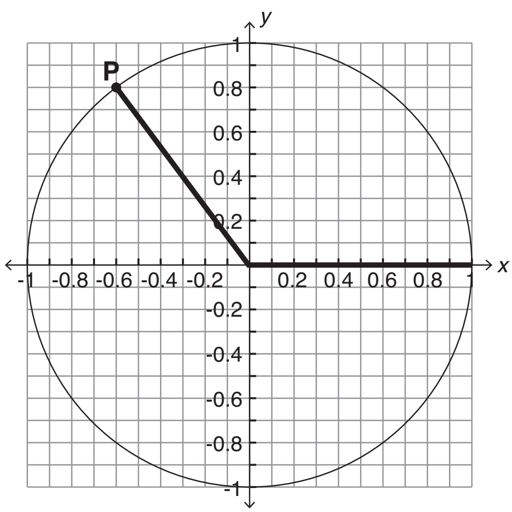
The following section provides evidence of the significance of another aspect in mathematical problem-solving, namely, the role of connected knowledge.

19.5 Aesthetic Effects of Connections in Mathematical Problem-Solving

Connections, patterns, and metaphors are recurring themes in my research that related to affect and to aesthetics (Presmeg, 1992, 1998, 2006a, 2006b). I present a brief vignette from the case of Sam (Presmeg, 2006b), whose initial compartmentalization prevented him from solving a trigonometric problem (Fig. 19.8) involving an angle in the second quadrant; I report his elation when he finally connected the unit circle and the graph of a sine function. Sam was chosen as one of three students (from 27 in this class of prospective elementary school teachers) to be interviewed, because of his strong abductive thinking (Peirce, 1998) in class discourse.

Sam knew the right triangle definitions of the trigonometric ratios (using the mnemonic SOH CAH TOA). He called the rotation angle theta and marked its supplement, the reference angle in the second quadrant. He dropped perpendiculars to the *x*- and *y*-axes and joined P to the circle's intersection with the *y*-axis (Fig. 19.9). He identified the sine of the *reference angle*, the acute angle, as having a value of 0.8.

> Sam: That's point 8, yeah. But over here, let's see, [...] the sine of the whole angle is one.
> Interviewer: The sine now of the obtuse angle?



This graph shows an angle. Give the approximate value of the sine of the angle.

Fig. 19.8 Third question in the preliminary interview (from Brown, 2005)

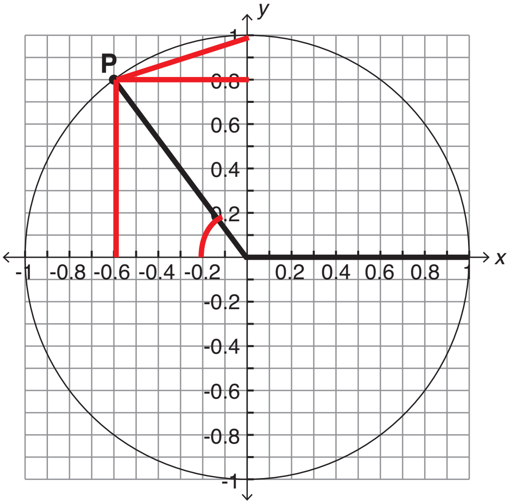


Fig. 19.9 Sam's inscriptions on the unit circle

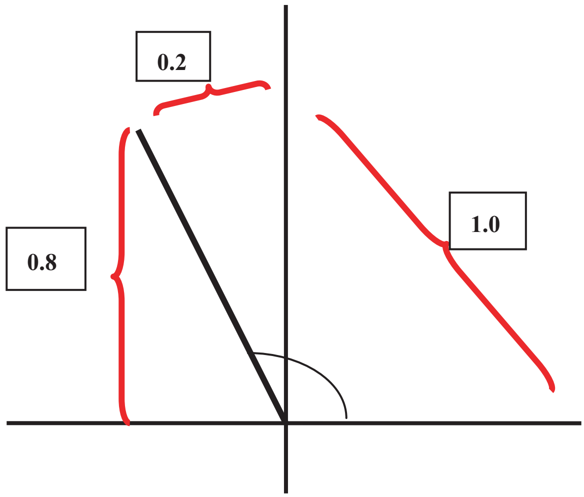


Fig. 19.10 A reconstruction of Sam's imagery

Sam: So this would be point 2 [pointing to the arc between P and the y axis].

Sam appeared to have an image resembling that in Fig. 19.10.

Sam: So I'd say negative one point two … I dunno.

Interviewer (I): How did you get one point two?

Sam: So, the sine of this angle is one.

I: The ninety-degree angle?

Sam: Yes, so this is one. And then this is … let's see, this point is … negative point six, point eight.

I: Oh, I see. You're figuring out the coordinates?

Sam: I was just thinking of a unit circle. And with coordinates …'cause now like, the sine of this angle here [indicating point of intersection of circle and y axis] is, the cosine

> zero, the sine one. […] And then it goes, that's 90, which, it still stays positive though, so … one point two, because this is point two.

Sam then explained negative and positive values in the quadrants using coordinates of points. After the interviewer told him that the correct answer for the value of the sine of theta is point eight, he persisted:

> Sam: But if this … if the sine of this angle here is one*, how can a bigger angle be less*?
> I: Ah, that's a good question. Do you know what a sine graph looks like?
> Sam: Yeah.
> I: Can you draw me one? Can you put values in there?

Sam drew the sine graph for one revolution and inserted appropriate radian measures on the θ-axis and 1 and −1 on the *y*-axis.

> I: There you go! Now you just said, how can it be less, if it's [the angle is] bigger that 90?
> Sam: Yeah, it, it's not … [following the curve with his finger].
> I: So it goes down again.
> Sam: So that spot is the same here. Yeah! [He marks symmetrical points on the sine curve on either side of π/2.]

Sam seemed elated to make this connection between the sine of an angle as defined in a unit circle and the sinusoid graph. His case illustrates the intricate detail involved in the aesthetics of personal connections and how these intertwine with affect and with cognition. In the hurly-burly of mathematics classroom experience, it would be extremely difficult for a teacher to identify such delicate detail. However, teachers could benefit from an awareness of the value of connections in regard to positive affect and of the enhanced learning that accompanies such aesthetic elements.

19.6 A Strikingly Aesthetic Solution to a SUB14 Problem

One final example is drawn from the students' solutions to problems in the SUB14 competition (Jacinto et al., 2011). Leonor's work powerfully illustrates the *generative* aspect of aesthetics in mathematics. Leonor was solving the following problem:

Problem 10: Cat and Mouse
A hungry cat surprises a mouse. Immediately the mouse starts running and the cat follows in pursuit. When the mouse starts its escape, it has a lead of 88 mouse's little steps on the attacker. It turns out that 2 steps of the cat are equivalent, in distance, to 12 little steps of the mouse. Moreover, while the mouse takes 10 little steps, the cat takes 3 steps. How many steps must the cat take to catch the mouse?

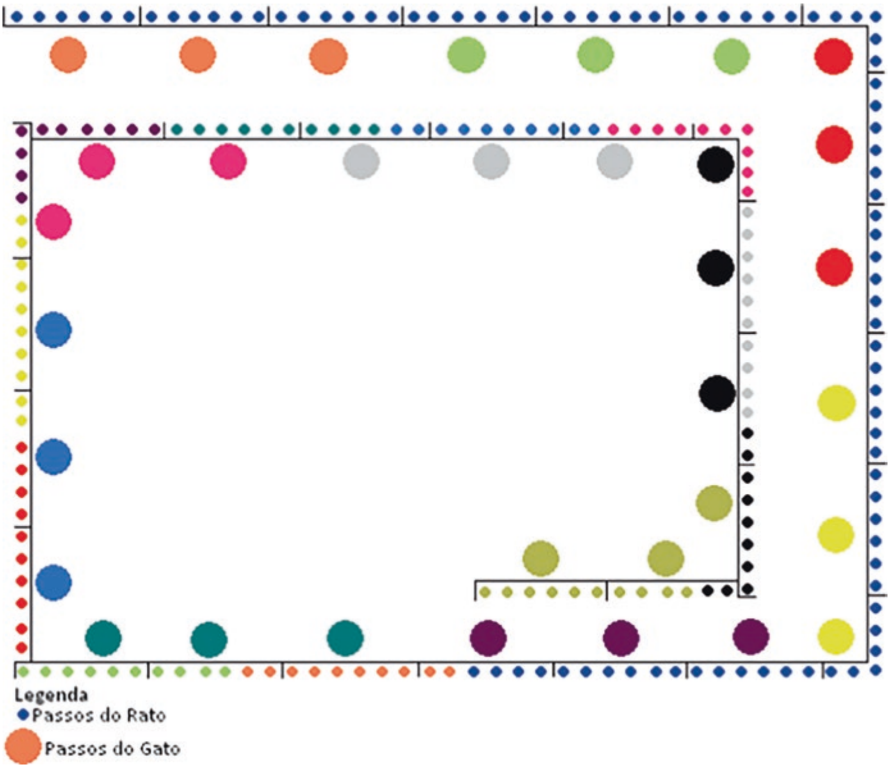


Fig. 19.11 Leonor's solution to the cat and mouse problem

Leonor devised a visual solution to the problem (Fig. 19.11), which provides an example of the potential of such solutions to cut through the complexity of relationships in mathematical problem-solving.

Leonor's solution effectively synchronizes distance and time in a computer representation using colour coding and "chunking" of steps of a cat and mouse. The colour coding enables her to impose a *time* element on the chunking of six little mouse steps for the *distance* of one cat step. The originality and power of Leonor's solution is evident: her visual representation serves both to *organize* the very complex distance and time relationships that are given and to *enable a solution* that is elegant and certain. Before examining Leonor's solution, I tried to solve the problem myself, using algebraic equations – and I struggled to reconcile the two ratios given, those of 12 mouse steps being equivalent (in distance) to 2 cat steps and of 10 mouse steps being equivalent (in time) to 3 cat steps. And the mouse had a head start of 88 little steps! Leonor's visual solution does not eliminate the need for reasoning: *logic* is certainly involved in her analysis of the 6:1 distance ratio, and in the superposition of a colourful 10:3 time ratio. This use of strong logic in an effective visual solution is consonant with the theoretical position I adopt (following Krutetskii, 1976) that strong logic is involved in all effective mathematical thinking, whether visual or nonvisual: visualization and logic are not opposed but are orthogonal in this model (Nardi, 2014; Presmeg, 2006a, 2006b). My further research through the decades showed again and again the effectiveness of alternating or combining visual modes and nonvisual reasoning in solving mathematical problems at all levels, from elementary to college-level mathematics.

19.7 Motivational Effects of Aesthetics

In this chapter, I provided some illustrations of the power of aesthetic elements to configure solutions in mathematical problem-solving. From Mr. Blue's promotion of the "beauty of abstraction", through the positive and negative affective pathways traversed by four graduate students in solving nonroutine mathematical problems, through Sam's bewilderment in linking two trigonometric domains, to the generative creativity of Leonor's visual solution to a SUB14 problem, it is evident that aesthetics and affect are significant factors. Recurring themes were the pitfalls of compartmentalization and the need for connections, the power of metaphors and visual inscriptions in both the generative and motivational aesthetic responses, and in some instances the openness to various kinds of solution, which encourages a playful and exploratory stance. Many of these elements were observable in students' work in the Problem@Web Project, along with evidence that the open use of technology provides enhanced opportunities for visual inscriptions. Once again, here is the question with which this chapter opened:

> When students are not under compulsion to solve the mathematics problems, what is it that gives them the motivation to do so, and to persist in the face of difficulty in order to reach a solution?

Some answers to this question are presented in the next subsection.

19.7.1 Features of SUB12 and SUB14 Problem-Solving that Lead to Positive Affect and Motivation

Summarizing the aspects of the Problem@Web Project that are effective in engaging students in the problem-solving activities, the following are significant factors:

- Entering solutions for the competitions is optional. This aspect lessens the pressure, thereby opening the way to better engage aesthetic responses that are generational.
- The organizers are open to all effective solutions, thereby encouraging creativity and playful exploration.
- There is relative lack of time pressure, which provides space for the role of playing.

- If there is puzzlement or bewilderment, there is the option of engaging others in the solution process to help, thereby turning a potentially negative affective pathway into a positive one.
- The facilities provided by technology make colourful, aesthetically pleasing drawings relatively easy to construct.

Thus, the process can be pleasurable and exciting. But what exactly gives participants the *motivation* to persist in the face of difficulties, in order to reach a solution to a problem? The "motivational aesthetic" identified by Sinclair (2004) is relevant. As she claims:

> The motivational aesthetic does not operate merely as an eye-catching device; neither does it provide merely the psychological support needed to struggle through a problem. Rather, it is central to the very process that enables the mathematician to deliberately produce qualitatively derived hypotheses: It initiates an action-guiding hypothesis. (p. 277)

Thus, the motivational aesthetic not only has the potential to suggest potentially fruitful ways of proceeding, but it also is adequate to sustain persistence in the face of difficulty in solving nonroutine problems (as in the Sub12 and Sub14 competitions).

Sinclair (2001) was concerned to operationalize the theoretical formulations of "the aesthetic" in order to investigate how these issues could be used to enhance classroom learning of mathematics. Thus, she developed a "colour calculator" for an aesthetically rich learning environment for use in research with 15 grade 8 students learning about rational and irrational numbers: colour patterns came from repeating decimals, e.g. 1/7, enabling learners to experiment – leading to aesthetic experiences and positive affect. The aim of her project was to encourage free problem-solving associated with aesthetic elements, as middle school students worked with fraction patterns in individual interviews. Sinclair's research was specifically targeted to aesthetic elements, affect, and motivation in the mathematics learning of young students. She identified not only the difficulties associated with research on these "ephemeral" topics, but also their importance, and their potential to make a difference in encouraging students to continue with the subject of mathematics.

19.8 Conclusion: Some Possible Further Directions for Research

Sinclair's research and the manifestations of persistence by students in the Problem@Web Project suggest that positive emotions generated under the umbrella of aesthetic experiences not only have a strong motivational effect in the solving of nonroutine mathematical problems but may also encourage students to persist in the broader context of mathematics classes they are currently taking and in the desire to continue with mathematics in the future. These two aspects, namely, students' enhanced learning in their current mathematics classes and motivation to continue studying mathematics in further mathematics courses, are strong conjectures in need of further research and empirical evidence. The Problem@Web Project provides a fruitful setting for investigating both aspects in further research.

Another research question that is suggested is the extent to which teachers might implement some of the characteristics of the mathematical competitions SUB12 and SUB14 in their classroom teaching. For instance, openness to all effective solutions to nonroutine problems of a moderate level of difficulty, relative lack of time pressure, and the possibility of help-seeking in the face of difficulty are all elements that are possible for teachers to implement either in class or in projects completed at home. Whether such elements result in aesthetic enjoyment, positive emotions, and enhanced motivation to engage in mathematics is a research focus that could be addressed using both qualitative and quantitative methodologies. The delicate details of students' thinking and affect and their interconnections, as suggested in some of the research reported in this chapter, hint that mixed methodologies might be fruitful in this regard, in order to investigate not only *whether* such effects exist and how extensive they are but also why enhanced learning and motivation may occur.

The role of technology in this regard is another whole aspect, which is being fruitfully addressed in the Problem@Web Project. Sinclair (2001, 2004, 2008) is one researcher amongst many others who are engaged in addressing the ways in which technology use not only changes the nature of learning mathematics but also has potential to enhance the aesthetic elements – both generative and motivational – in such learning. There is much more to investigate in ways that the affordances of technology enhance the mathematical aesthetic experience of students; however, it is already clear that aesthetic and affective issues and their interrelationships with cognition are significant research topics in mathematics education that should not be neglected.

References

Brown, S. (2005). *The trigonometric connection: Students' understanding of sine and cosine* (Unpublished Ph.D. dissertation). Illinois State University, Normal, IL.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics. *Educational Studies in Mathematics, 37*(2), 121–143.

Carreira, S. (2001). Where there's a model, there's a metaphor: Metaphorical thinking in students' understanding of a mathematical model. *Mathematical Thinking and Learning, 3*(4), 261–287.

Carreira, S. (2015). Mathematical problem solving beyond school: Digital tools and students' mathematical representations. In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 93–113). Cham, Switzerland: Springer International Publishing.

Carreira, S., Tomás Ferreira, R. A., & Amado, N. (2013). Young students solving challenging mathematical problems in an inclusive competition: Enjoyment vis-à-vis help-seeking. In B. Ubuz, Ç. Hasser, & M. A. Mariotti (Eds.), *Proceedings of the 8th annual congress of the European Society for Research in Mathematics Education* (pp. 1289–1298). Manavgat – Side, Antalya, Turkey: Middle East Technical University.

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning, 2*, 209–219.

Gómez-Chacón, I. (2012). Affective pathways and interactive visualization in the context of technological and professional knowledge. *Nordic Studies in Mathematics Education, 17*(3–4), 101–118.

Jacinto, H., Amado, N., & Carreira, S. (2009). Internet and mathematical activity within the frame of "Sub 14". In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the sixth congress of the European Society for Research in Mathematics Education* (pp. 1221–1230). Lyon, France: Institut National de Recherche Pédagogique.

Jacinto, H., & Carreira, S. (2012). *Problem solving in and beyond the classroom: Perspectives and products from participants in a web-based mathematical competition*. Paper presented at the 12th International Congress on Mathematical Education, 8–15 July 2012, Seoul, South Korea.

Jacinto, H., Carreira, S., & Amado, N. (2011). Home technologies: How do they shape beyond-school mathematical problem solving activity. *Proceedings of the 10th International Congress for Technology in Mathematics Teaching*, July 2011, Portsmouth, UK.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Nardi, E. (2014). Reflections on visualization in mathematics and in mathematics education. In M. Fried & T. Dreyfus (Eds.), *Mathematics and mathematics education: Searching for common ground* (pp. 193–220). New York: Springer.

Peirce, C. S. (1998). *The essential Peirce. Volume 2*. Bloomington, IN: Indiana University Press Edited by the Peirce Edition Project.

Pepin, B., & Roeskin-Winter, B. (Eds.). (2014). *From beliefs to dynamic affect systems in mathematics education*. New York: Springer.

Presmeg, N. C. (1985). *The role of visually mediated processes in high school mathematics: A classroom investigation* (Unpublished Ph.D. dissertation). University of Cambridge.

Presmeg, N. C. (1991). Classroom aspects which influence use of visual imagery in high school mathematics. In F. Furinghetti (Ed.), *Proceedings of the 15th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 191–198). Assisi, Italy: PME.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23*(6), 595–610.

Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 299–312). Mahwah, NJ: Lawrence Erlbaum Associates.

Presmeg, N. C. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior, 17*(1), 25–32.

Presmeg, N. C. (2006a). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Rotterdam: Sense Publishers.

Presmeg, N. C. (2006b). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. In J. Navotna, H. Moraova, M. Kratna, & N. Stehlikova (Eds.), *Proceedings of the 30th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19–34). Prague: PME.

Presmeg, N. C., & Balderas-Cañas, P. E. (2001). Visualization and affect in nonroutine problem solving. *Mathematical Thinking and Learning, 3*(4), 289–313.

Sinclair, N. (2001). The aesthetic is relevant. *For the Learning of Mathematics, 21*(1), 25–32.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning, 6*(3), 261–284.

Sinclair, N. (2008). Aesthetics as a liberating force in mathematics education. *ZDM: The International Journal of Mathematics Education*. https://doi.org/10.1007/s11858-008-0132-x. Accessed 25 Jan 2014.

Chapter 20
On Choice, Collaboration and Closeness in Problem Solving: Aesthetic Experiences of Pre-service Teachers

Nathalie Sinclair and Annette Rouleau

20.1 The Role of the Aesthetics in Mathematics Thinking and Learning

This section is divided into two parts. In the first, we review research related to the aesthetic dimension of mathematical activity as it is done by professional mathematicians. In the second, we focus on research related to the aesthetic dimension of mathematical activity as it is done by students and teachers. We note that almost all of this research has relied on post hoc descriptions made by those engaged in mathematical activity, which is not surprising given how difficult it can be to observe aesthetic responses during the process of mathematical problem solving. We will return to this issue when we discuss our own methods of research.

20.1.1 Focus on Mathematicians

A substantial part of the early writing on the mathematical aesthetic was concerned with determining the aesthetic merits of different mathematical proofs and theorems. The mathematician G.H. Hardy (1940) took an objective view of aesthetics and proposed criteria that could be used to characterise beautiful or elegant results, such as depth and significance, unexpectedness, inevitability and economy. Of course, these criteria could prove to be quite subjective. Indeed, in the early 1990s, David Wells ran a survey in the *Mathematics Intelligencer* showing that the aesthetic metric of mathematical theorems was highly subjective. In the large number of responses he received, from eminent mathematicians around the world, who were

N. Sinclair · A. Rouleau (✉)
Simon Fraser University, Burnaby, BC, Canada
e-mail: nathsinc@sfu.ca; arouleau@sfu.ca

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_20

asked to rate the beauty of 22 theorems, Wells (1990) found that many factors were at play in evaluating the aesthetic merit of these theorems: field of interest; preferences for certain mathematical entities such as problems, proofs or theorems; past experiences or associations with particular theorems; and even mood. He also found that aesthetic judgements change over time, something that Netz (2009) has made very clear as well in his study of the style of Archimedean mathematics.

The inferences made by Wells correspond to a contextualist view of aesthetic appreciation and are summed up by this respondent: "beauty, even in mathematics, depends upon historical and cultural contexts, and therefore tends to elude numerical interpretation" (p. 39). Indeed, the history of mathematics shows how new mathematical ideas, such as irrational numbers, non-differentiable but continuous functions and even homotopy theory, often inspire negative aesthetic responses before they become accepted within the community and eventually judged in positive aesthetic terms. In a similar vein, Michael Polanyi's (1958) ground-breaking "post-critical" examination of the non-explicit dimension of scientific knowledge argues that mathematical knowledge cannot be divorced from personal and communal values and commitments. His concept of indwelling, which is the process through which scientific knowledge is developed, is aesthetic inasmuch as it is about knowing through our bodies, through transforming our bodies in relation to other (sometimes inanimate) bodies (see also Chandrasekharan, 2014). The non-explicitness of this knowing has a family resemblance to intuition, which we will return to later on.

In addition to Wells' study, another empirical investigation into mathematicians' aesthetics was conducted by Burton (2004), who interviewed 70 mathematicians from a wide range of fields. Based on her analysis, she proposed an epistemological model of "mathematician's coming to know", which includes the aesthetic as one of five categories (the others being its recognition of different approaches, its person- and cultural-social relatedness, its nurturing of insight and intuition and its connectivities). The aesthetic category was restricted to mathematicians' sense of whether or not beauty played a role in their creation or appreciation of mathematics. Our sense is that some of the other categories may also have aesthetic dimensions as well. For example, the nurturing of insight evokes Poincaré (1908/1956) and Hadamard's (1945) sense in which the aesthetic acted as a guide for mathematicians. While problem solving, for example, Hadamard wrote specifically about the "sense of beauty" (p. 130) that informs the mathematician that "such a direction of investigation is worth following; we feel that the question *in itself* deserves interest" (p. 127; *italics in original*). It also relates to Polanyi's non-explicit ways of knowing that arise out of dwelling within mathematics.

In addition, the category of person- and cultural-social relatedness, which Burton set in opposition to a more Platonic epistemological view, may also have an aesthetic dimension. This can be seen in the writing of mathematician William Thurston (1995), who claims that "the social setting is extremely important […] We are inspired by other people, we seek appreciation by other people, and we like to help other people solve their mathematical problems" (p. 11). This relates to the notions of taste and style that von Neumann (1947) wrote about wherein certain mathematicians are seen as the arbiters of taste, deciding which programmes should be developed, which problems should be solved or which areas of mathematics are the most significant and/or interesting. This can be seen in contemporary mathematics through the influence of mathematicians such as Alexander Grothendieck and Robert Langlands.

By acknowledging the contextual nature of the aesthetic, inquiry shifts from what is beautiful or ugly in mathematics to what is the function of value judgements in mathematics. In this vein, drawing on pragmatic concern for the role of the aesthetic in mathematics and on interviews with contemporary mathematicians, Sinclair (2004) has identified three distinct roles the aesthetic plays in mathematical inquiry—where the term "aesthetic" is used to designate judgements of value. The most recognized and public of the three roles of the aesthetic is the evaluative; it concerns the aesthetic nature of mathematical entities and is involved in judgements about the beauty, elegance and significance of entities such as proofs and theorems. The generative role of the aesthetic is a guiding one and involves nonpropositional modes of reasoning used in the process of inquiry and that is often expressed through affective responses to felt patterns. Lastly, the motivational role refers to the aesthetic responses that attract mathematicians to (or repel them from) certain problems or certain fields of mathematics. Tymoczko (1993) sees this as the crux of the aesthetic function in mathematics: of the infinitely many true propositions one could state in mathematics, only a very few attract attention.

We cite these three roles of the aesthetic, and especially the motivational one, because it relates closely to the context of our study in which we invited pre-service teachers to consider choice in relation to their problem solving. Although they were not posing their own problems, they were offered a set of problems from which they could choose to report on, and we think these choices can provide insight into significant aesthetic dimensions of problem solving in schools, both by teachers and by students.

A final aspect of the aesthetic that is less discussed in the literature relates to the experiences that mathematicians report having while doing mathematics, experiences that are often described in emotional terms as well as cognitive ones and that Dewey (1934) described as aesthetic experiences. We are intrigued by Rota's (1997) assertion that beauty and elegance are words used by mathematicians that are seen as being more safe and objective ways for them to express their own aesthetic experiences:

> Mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding acknowledgement of the fuzziness of this phenomenon […]. (pp. 132–133)

This statement links the aesthetic and the affective as highly intertwined, with the affective often providing physical evidence of an aesthetic response, be it a smile, a frown or a more consuming feeling of "Oh!" when one perceives a connection or sees something in a new way. Silver and Metzger (1989), in their study of research mathematicians, also support this view: "decisions or evaluations based on aesthetic considerations are often made because the problem solver 'feels' he or she should do so because he or she is satisfied or dissatisfied with a method or result" (p. 70). Positive or negative feelings can arise from a perception, or an awareness, of something being worthwhile, important or interesting. In other words, the aesthetic and the affective domains each function differently in the problem-solving process: the aesthetic draws the attention of the perceiver to a phenomenon (a pattern, a relationship, a contradiction), while the affective can bring these perceptions to the conscious attention of the perceiver.

The aesthetic can also be tightly coupled with the cognitive. At one level, and perhaps a rather cerebral one, we might talk about perceptions of simplicity, structure, conciseness and lucidity. However, Johnson's (2007) view of aesthetics offers a more visceral link, not dissimilar from that of Polanyi. For example, the work of Radford (2003), which focuses on the direct connection between bodily movement and abstract mathematical conceptualisations, has a strong affinity with this aesthetic approach. In particular, Radford shows how the rhythmic utterances of students are used to construct meaning for algebraic patterns they are studying—the rhythm, and not the actual words, acts as semiotic markers of generalization. Others might respond to the visual or the tactile, instead of the rhythmic. We see embodiment as being closely intertwined with intuition as well. As Rotman (2008) has argued, "If one accepts the embodied—metaphorical and gestural—origins of mathematical thought, then mathematical intuitions becomes explicable in principle as the unarticulated apprehension of precisely their embodiment" (p. 38). Intuition thus becomes a "felt connection to [the] body" (p. 38). Indeed, as the mathematician André Weil suggests, the aesthetic pleasure in mathematics may be that of reconnecting to the body: "nothing gives more pleasure to the researcher [than] these obscure analogies, these murky reflections of one theory in another, these furtive caresses, these inexplicable tiffs" (1992, p. 52).

20.1.2 Focus on Learners and Teachers

For Poincaré, mathematicians were the only ones to possess the "special aesthetic sensibility" that was capable of generating productive ideas in the mathematician's unconscious mind. The early research in education by Dreyfus and Eisenberg (1986) reified this view in that they found that when students were asked which solution to the problem of how many times the digit 7 appears in the numbers from 0 to 99,999, they concluded they lacked aesthetic sensibility because they did not show the same aesthetic appreciation as expert mathematicians. This led the authors to infer that novice mathematicians may not be able to develop a mathematical aesthetic.

Papert (1978) challenges this elitist view. He argues that mathematicians are not the only ones to possess the "special aesthetic sensibility" and shows how non-mathematicians can also be guided towards correct mathematical ideas through appeal to aesthetic considerations. The aesthetic responses of these non-mathematicians had little to do with Hardy's (1940) objective criteria of depth and significance or even of surprise. Instead, they involved emotional reactions to an equation's form and structure—a desire to get rid of a square root sign or to place the important variable in a prominent position. Subsequently, another empirical work has shown that, when provided with inquiry opportunities in rich environments, students do indeed use aesthetic values in choosing problems, generating conjectures and evaluating their solutions (Eberle, 2014; Lehrer, 2008; Sinclair, 2001). These values sometimes, but not always, overlap with canonically mathematical aesthetic values such as fruitfulness, visual appeal and surprise. In working with university-level students, Brown (1973) reported on a preference for "genealogical solutions", in that students tended to prefer their own solutions to those of mathematicians—perhaps because their "messy" solution encoded the parts of the problem-solving process that contribute to their understanding. For Brown, a difference in aesthetic preference does not entail a lack of aesthetic sensibility.

Similar work, focusing on the problem-posing phase of inquiry, has shown that pre-service elementary teachers can use aesthetic values to pose more interesting mathematical problems (Crespo & Sinclair, 2008). However, little is known about the aesthetic aspects of how pre-service teachers (or students or even teachers) *choose* problems to solve or what they *value* in a problem they have solved.

In terms of aesthetic experiences, it goes without saying that students tend to describe mathematics in very negative terms that relate more to anxiety, fear and detachment and then to the kinds of pleasure and enlightenment to which Weil and Rota refer. In Buerk's (1982) study of "able women" who avoided mathematics, but who she invited to participate in some problem-solving experiences in which the problems were open-ended and the women could collaborate, one of the women reflected that "[…] this wandering has helped me to think through the values that define my own character as well as mathematics. This seems to me an important step in repairing the flawed 'relationships' I have with math" (p. 24). We cite this research because it highlights the negative experience many adults have of mathematics and also draws attention to the way in which gaining positive experiences can change one's values and relationship to mathematics. Having a relationship with mathematics is not just about liking it or being good at it but also about a felt connection. Indeed, drawing on Sinclair's (2004) tripartite model in which the generative role of the aesthetic was related to "getting a feel for" and "establishing intimacy", it may be that such a relationship is necessary in the problem-solving process. DeBellis and Goldin (1999) have also written about the importance of intimacy in the problem-solving process and, in particular, how developing intimacy requires time and care. While these authors view intimacy as part of the affective domain, we also see it as having an aesthetic component inasmuch as intimacy with mathematics implies a sensory knowledge of mathematics as well as a valuing of that sensory knowledge.

20.2 Theoretical Framing of the Aesthetic

In our study, we investigate the aesthetic aspects of pre-service teachers' problem-solving experiences as described in their journal entries. In line with much of the empirical work cited above, we take a contextual and democratic view of the aesthetic, instead of seeing it as an objective judgement to be made by a cultural elite. We also acknowledge both the social and affective dimensions of the aesthetic, seeing it not just as an axiological process in which values are created or adopted but also as a quality of human experience.

For our primary theoretical lens, we will use Pimm's (2006) pragmatic formulation of the aesthetic, which does not constrain aesthetic responses to ones typically associated with mathematics, such as elegance and brevity, but that is consonant with known aesthetic aspects of research mathematics. Pimm writes:

> Aesthetic considerations concern *what* to attend to (the problems, elements, objects), *how* to attend to them (the means, principles, techniques, methods) and *why* they are worth attending to (in pursuit of the beautiful, the good, the right, the useful, the ideal, the perfect or, simply, the true). (p. 160; *italics in original*)

When the pre-service teachers are explaining the choice of problems to write about in their journals, we will look for instances of the *what*, in terms of the aspects of the problem or the problem situation that they attended to; the *how*, in terms of processes they attend to in the solving of their problems; and the *why*, in terms of why the reasons they offer for why the problem was worth attending to. We know that mathematicians might choose to work on a given problem because of its visual appeal or its "unexpected simplicity" (Penrose, 1974, p. 267), but we know very little about what pre-service teachers attend to when considering a mathematical problem. In her analysis of the ways in which a middle school teacher attended to the aesthetic, Sinclair (2008) identifies three examples related to the *what*. Only one seemed to elicit a response from the students, which was when the teacher drew attention to the value of arbitrariness in mathematics when introducing the notion of a mathematical relation. Instead of providing examples using familiar numbers or relations, he pointed out that -23.79 could be related to a million or to π, which made the students laugh. Although this example is not specific to problem posing or problem solving, it does highlight the way in which aesthetic considerations are not necessarily framed in terms of aesthetic words such as beauty and elegance. They are, instead, about the different kinds of values that can be associated to certain things—and in this case, there is a valuing of the arbitrary and the unusual.

In terms of the *how*, some mathematicians have written about preferring certain methods over others, which might include geometric approaches to solving a problem (see Le Lionnais, 1948/1986) or negative aesthetic reactions to certain techniques, such as computer-based ones (Montano, 2014). An example of *how* for the teacher in Sinclair's study is when the teacher described algebra as suitable for working with relations that have patterns (such as a set of collinear points), but not for working with a random set of points. Algebra is thus being described as a method or technique for working with certain kinds of patterns but not others.

In terms of *why*, there is certainly evidence of truth, beauty and usefulness as being worthy of attention amongst mathematicians. Amongst pre-service teachers, there is some evidence that student accessibility makes problems worth attending to (Crespo & Sinclair, 2008) when they are asked to pose "good" problems. Indeed, it is reasonable to expect that pre-service teachers will have educational concerns in mind when thinking about what is worth attending to—unlike mathematicians. Brown (1973) showed that some students preferred their solutions over the more elegant ones of mathematicians because they valued solutions that were their own, perhaps out of pride or perhaps because such solutions better reflected their understandings. Similarly, Sinclair (2008) describes an example of when the teacher in her study offered the textbook definition of a relation and then offered her own definition. As Sinclair writes, "He invited the students to consider his definition precisely because it is his, thus suggesting that people may have individual understandings of mathematical concepts and that it is acceptable to work with definitions that are personally useful or relevant" (p. 32). Instead of being in pursuit of the perfect or the true or the beautiful, the teacher's invitation exemplifies how attending to the *why* can be in pursuit of the personal (it's mine!).

Pimm's articulation of aesthetic considerations does not make explicit the more affective, social and embodied aspects of the aesthetic that we have highlighted in the previous section. We can, however, elaborate his articulation in order to make these aspects more overt. In particular, we can include affective experiences in the *why* category in that pre-service teachers (as well as students and mathematicians) might consider a problem worth attending to if it has provided them with some kind of pleasure or enlightenment to which Rota refers or the "felt connection to the body" about which Weil writes.

20.3 Investigating Pre-service Teachers' Aesthetic Responses

In this section, we provide some background on the goals of the course in which the pre-service teachers, who participated in our study, were enrolled and the nature of the data that we collected for this study. We also outline our methods of analysis.

20.3.1 Context of the Research

The data for this study was obtained from 66 pre-service teachers enrolled in two sections of an elementary mathematics method course taught by the second author. The design of the course was shaped by a strong desire to transform the way the pre-service teachers felt about mathematics as well as their own ability to do mathematics. Accordingly, the course's 11 lessons were developed around rich problem-solving tasks that offered multiple approaches to solving and were frequently open-ended. Collaboration was fostered through the use of visibly random grouping and vertical surfaces, such as a whiteboard or chalkboard (see Liljedahl, 2016). The intent was for the pre-service teachers to leave the course with a newfound appreciation for the teaching and learning of mathematics that they would hopefully take with them into their future classrooms.

Throughout the course, the pre-service teachers were engaged in solving a number of problems. Every four-hour-long lesson provided the opportunity to work on several problems, with the final problem of each day designated as a problem they could use for a journaling assignment. The original intent was to have the pre-service teachers select three from a list of nine problems and write about their experience in solving the problem. To eliminate pedagogical tensions around subjective evaluation of personal journal writing, evaluation of the assignment was limited to objective measures of completion, e.g. the pre-service teacher had written about three problems and described their problem-solving process and why they chose to write about the problem. The task outline was as follows:

> Throughout the course you will be working collaboratively in-class on many mathematics problems. Your efforts for these problems are to be kept in a portfolio, from which you will choose three for submission. However, the intent isn't just to write out the problem and the solution. You must provide a narrative of the problem-solving process that you and your group went through: What were you feeling, thinking, doing? What did you try first? What did you do when you felt unsure? Frustrated? What did you like? Dislike? Why did you decide to include this particular problem?

Thus, instead of asking the pre-service teachers to choose a particular problem to try to solve, we asked them to choose problems that they wanted to write about. This methodological decision enabled us to gain insight not only into the motivational role of the mathematical aesthetic but also into the overall aesthetic experiences of the pre-service teachers, which we conjectured might involve social and emotional dimensions as well.

By the end of the very first lesson, however, the pre-service teachers were already inquiring whether they could choose to write about *any* of the problems they encountered in class, not just the final problem of the day. These students had already had a powerful experience with a problem they had worked on during the lesson and wished to chronicle that experience for the assignment. This was unexpected yet not unwelcome. Having no strong attachment to the original nine problems other than a desire for organization and the personal belief that they were "richer" problems, an in-the-moment decision was made to open up the assignment to include any problems worked on in class. This resulted in 24 problems from which the pre-service teachers could choose, and in the end, 18 of those 24 were written about in the problem journals. In what follows we describe in some detail two of the problems the pre-service teachers encountered, which are representative of the two kinds of problems that were offered. The first is from the original list of nine designated problems, and the second is an example of a problem that the students asked permission to include in their journal.

The most commonly chronicled problem-solving experience was a place value problem given during the second lesson. On the original list of 9 problems, 36 of the pre-service teachers chose to journal about this problem. Its instructions were simple: *How many times does the digit "7" appear in the numbers from 0 to 1000?* Dreyfus and Eisenberg (1986) use a variation of this problem to showcase relevant features of aesthetic thought in mathematics problem solving. While the problem fit well with the day's lesson on place value, its value in an elementary mathematics methods course was twofold. First, it is important for pre-service teachers to have mathematical experiences that are similar to those they intend to provide for their students. Telling pre-service teachers about the benefits of problem-solving activities in a classroom is far less effective than having them undergo the experience personally. "Coming to know something is not a spectator sport although numerous textbooks, especially in mathematics, and traditional modes of instruction may give that impression" (Brown & Walter, 1983, p. 2). Secondly, it is an effective problem for encouraging collaboration.

It was apparent from the submissions that the problem responsible for changing the assignment parameters was "the tax collector". Not included on the original list of nine problems, it was the second problem of the first lesson and the first time the pre-service teachers were required to work collaboratively. Nineteen of the sixty-six students chose to journal about this problem for their final submission. The problem is a simple task that required the pre-service teachers to revisit their prior knowledge about factors. Presented as a game, the premise of the tax collector is that you are given a series of paychecks, from $1 to $48. You can choose any paycheck to keep. Once you choose, the tax collector gets all paychecks remaining that are factors of the number you chose. The tax collector must receive payment after every move. If you have no moves that give the tax collector a paycheck, then the game is over and the tax collector gets all the remaining paychecks. The goal is to beat the tax collector. Again, the intent behind using this problem was twofold: to provide the pre-service teachers with an early experience in mathematical problem solving that was engaging *and* collaborative, both of which were unusual for the majority of the pre-service teachers.

20.3.2 Method of Analysis

Our data corpus was comprised of 66 problem-solving journals that described 198 problem-solving experiences (three for each pre-service teacher). In their responses to the prompts, the pre-service teachers provided a variety of reasons for choosing the three problems to journal about. Some reasons were very succinct, such as "because it was challenging", while others were more elaborate and involved the broader context such as the particular dynamic of the group or the relation of the problem to something outside the classroom. Our first round of analysis involved reading through the journals in order to discern instances of Pimm's (2006) *what*, *how* and *why* formulation of the aesthetic. Since the pre-service teachers were not just writing about the strictly mathematical aspects of the problems, for which Pimm's formulation would apply directly, but also about their problem-solving process as a whole, we decided to slightly modify Pimm's three categories as follows:

the *what* focused on aspects of the problems identified by the pre-service teachers; the *how* focused on the process of solving the problem; and the *why* addressed the reasons given to explain the worthiness of the problem.

In addition to our specific focus on the choice of problem, we also noticed as we read through the journals that the pre-service teachers wrote about aspects of their experience that were not strictly focused on the problems they had chosen but were more broad. We thus attended specifically to their aesthetic responses, which we identified through their use of words such as enjoy, feel, surprise and excited. Guided in part by our review of the literature, and in particular of the writings of mathematicians such as Rota, Thurston and Weil, we noticed that the aesthetic responses clustered in two areas, which we categorised as collaboration and personal closeness with mathematics. These became our themes, and using NVivo coding software, we then recoded the collected data accordingly.

20.4 Choice, Collaboration and Closeness

This section comprises three subsections. In the first, we describe the results of our analysis of the pre-service teachers' descriptions of their reasons for choosing to journal about their three problems. In the second and third, we present our analysis of data relating to collaboration and closeness, respectively.

20.4.1 Aesthetic Considerations in Choosing Which Problem to Write About

Following Pimm (2006), we report on each type of aesthetic consideration that we found in the pre-service teachers' explanations of the three problems that they chose to write about in their journals. We found several different explanations of *what* to attend to that were similar to what has been reported by mathematicians, as well as some that are quite distinct. We begin with the former set of explanations.

The most frequently mentioned aspect of the problems the pre-service teachers chose to report on was that it was sufficiently difficult, that is, not too easy but not too hard: "challenging but attainable" as one participant said, or in the words of another, "it was just enough of a challenge to be kept intrigued". Several participants also mentioned choosing problems that they found interesting, such as one participant who described a problem as being "an interesting mind exercise". Another aspect of the problem that was mentioned related to it somehow being surprising; one participant wrote that the problem was "so lopsided I find that it intrigues me" and another said "I had never heard of palindromes in mathematics". These surprising problems often provoked a desire to understand how the problem worked: "I wanted to really get to understand it". Finally, several pre-service teachers wrote about the problem being somehow immediately accessible, either because they could see a pattern (the number pattern was "accessible" to me) or how to solve it ("I was quickly able to come up with a strategy") or because it corresponded well to a personal strength: "[it] was directed to my skills and my way of thinking".

In terms of aspects of the problem that we have not seen reported by mathematicians, we found two different kinds of explanations. Some pre-service teachers wrote about a particular aspect of the problem that they could relate to, such as the problem involving money or a connection to literacy, or a problem they had seen before or a problem that reminded the participant of a family member. Another aspect of the problem frequently mentioned by the pre-service teachers was that the problem did not have to be solved in any specific way. This is quite an interesting aspect of the *what* category because it would be taken for granted in the context of mathematicians, who try to solve problems in any way they can and not just by using a strategy or concept they have just learned! The pre-service teachers may not have seen right away that the problem had different solutions, but their instructor had made it clear to them.

The *how* of what the pre-service teachers attended to was evident in the processes they wrote about in their journals. Many chose problems in which they had the experience of working collaboratively, an atypical experience for the majority of them: "this was the first problem where I understood the power of group work" wrote one participant, while another elaborated that "I never experienced this [collaboration] type of learning during my educational journey in math classes". Others chose to write about problems in which the process of finding the answer was key: "fun, creative and collaborative" was one participant's description of solving, while another stressed the importance of finding the answer successfully "in our own way". Still others chose problems because the way the problem was solved was appealing: either through "the power of 'guess and check' and trial and error" or because "it felt like we were on the right track". This idea of being on a path forward arose repeatedly with one participant describing it as "it feels like there is a constant progression", while another related her enjoyment of the process of solving to that of "going for a nice, long walk".

Workspace was an intriguing aspect of the pre-service teachers' *how* experience of problem solving that perhaps differs from mathematicians' accounts. Some chose problems that they experienced not only collaboratively but while working at vertical surfaces. One participant mentioned enjoying solving "*all* of the problems we did in class" and went on to add that it is "due to the use of vertical surfaces". Vertical surfaces in the classroom were simply whiteboards or chalkboards at which groups of three or four pre-service teachers stood when engaged in problem solving. The vertical surfaces allowed them to "share ideas freely" and to persevere, with one participant suggesting that a vertical surface "makes it easy to erase everything and start again when you need a new perspective". Many pre-service teachers also mentioned the connection between the vertical surfaces and being "actively engaged" with one of the participants writing:

> I also loved the vertical surfaces. I found more of us were engaged this way as we weren't sitting in chairs looking at our own paper but instead all eyes were on one large chalk board. Even our body language felt more engaging. I found myself leaning in, open minded and not as closed off if we were to be sitting and slouched over. This was new to me, but I remember already feeling the power of vertical surfaces even after just one day.

This idea of working collaboratively at a vertical surface (such as a blackboard or whiteboard) is not novel for mathematicians who, as Menz (2015) has documented, typically need to think through diagrams and written symbols that cannot easily be communicated through speech only. It was, however, outside the norm of school mathematics for the pre-service teachers, whose collective experience in problem solving was almost exclusively working individually with paper and pencil at a desk.

We turn next to *why* the pre-service teachers felt the problem they chose was worth attending to. Frequently mentioned was that the problem offered a moment of insight. Referred to as an "aha" moment in their journals, one participant went on to describe the experience as "cathartic". Affective responses often played a role in deciding a problem's worthiness with participants using words like "enthusiastic", "excited" and even "feeling warm and fuzzy inside" to explain why they chose to include a particular problem. Related to this was the overall joy experienced in solving the problem. "Because I honestly just really enjoyed it", wrote one pre-service teacher, while another wrote of the "pure enjoyment I got out of trying to solve [it]". Personal insights were another factor in the *why* behind choosing a problem with one participant reaching the realization that "our past influences shape our approaches" and another selecting a problem that was a "representation of my growth". Connected to this were those pre-service teachers who chose problems that allowed them to "continue the learning outside of the classroom". One participant wrote, "I have never in my life been so excited to show my friends and family something I learned in school".

This preference for problems that extend the learning beyond the actual moment of solving is one that resonates with the mathematicians' attraction to fruitfulness, that is, to the valuing of a problem, solution or concept because of its potential for generating further mathematical ideas. In this case, however, the fruitfulness was specific to the generating of future learning. It seems that many of the pre-service teachers viewed the problems from the dual perspectives of learner and teacher, as we see in one participant who wrote, "It showed me the importance of quite a few things in regards to learning and teaching". Yet another wrote about the importance of "striving to teach math with the ease and confidence that you demonstrated". Viewing the problems with this duality was evident in many of the journal entries, in which participants wrote about liking the "in-class applications" or "classroom potential" of a problem. Focusing even more specifically on the individuals within those classrooms, still others wrote of a desire to "impress future students" with a particular problem or because "I think students would really enjoy participating in this problem". This suggests that part of the aesthetic appeal of a problem for pre-service teachers lies in its applicability for use in their future classrooms.

20.4.2 The Social Nature of Aesthetic Experience

The image of a mathematician as a lone individual working privately was negated by Burton (1998), whose research on mathematicians revealed the prevalence of, and the preference for, working collaboratively. More recently, as mentioned above, Menz (2015) has documented the way in which mathematicians work together, often at a blackboard, to understand the nature of the problem, build shared meanings around the objects and relations at play and communicate through diagrams, gestures and words.

Unfortunately, the negative stereotype persists in school mathematics and is perpetuated in classrooms where mathematics is most often pursued as an individual endeavour. This was true for the majority of the pre-service teachers, many of whom were experiencing collaborative problem solving in mathematics for the very first time. As one pre-service teacher wrote, "It was my first time experiencing group work in math class, engaging in discussions with peers, and working together to solve a mathematical problem. Solving the problem was fun; I was actively engaged the whole time". In all, 29 pre-service teachers connected this positive experience of working together to an increased enjoyment and appreciation of mathematics. We see this as an aesthetic response that arose from the social nature of their problem-solving experiences. The pleasure was not just about being in a group and not just about solving a problem but instead about coming to know mathematics together.

For many of the pre-service teachers, the experience of working collaboratively resulted in a stronger, more positive relationship with mathematics. One wrote about the group experience as "making me think deeply about finding a strategy" and "being pushed 'mathematically'". Still another wrote:

> Throughout the process, I have seen myself evolve in how I view math. I can attribute this to the way in which I have been learning math. Working through these problems as a group have, for the most part, allowed me to feel supported through the process and as a result stay engaged. I have been prompted to extend my thinking, and further my learning as a result.

This linking of the affective and cognitive domains—in the mentioning of feeling support and also extending his/her thinking—resulted in an aesthetic response that we associate with the social dimension of Sinclair's (2004) motivational role of the aesthetic. As with mathematicians, these pre-service teachers write about wanting to work on problems not just because the problems are interesting or important but because of the social complicity.

Working collaboratively was a pleasurable experience that the pre-service teachers saw as strengthening their mathematical skills. This, in turn, led to an even stronger desire to collaborate and an increased affinity for problem solving. Indeed, another pre-service teacher wrote of this recursive motivational effect of collaboration on problem solving stating:

> As the term went on, I noticed myself begin to take more risks as I realized the benefits to the group and myself of sharing diverse ways of looking at problems and talking my ideas through with others.

The taking of risks seems of utmost interest in this last quote, in part because it seems to be this affective comfort zone in which “ponderings, what ifs, it seems to me thats, it feels as thoughs” can occur, which Burton (1999, p. 30) describes as being helpful in nurturing intuition.

Still others wrote of how working collaboratively allowed them to solve problems they felt were impossible. “It took all of us to test our theories and explain our thought processes to reach the final answer”, wrote one pre-service teacher, while another mentioned, “If I did not have a partner there to double-check my work and my numbers, I would not have been able to solve the problem”. The pleasure in their unexpected success suggests an aesthetic response evoked by collaboration. In a similar comment, another pre-service teacher describes herself as “delighted by how clever a problem it really is” after a successful collaborative experience with an “impossible” problem. Here we clearly see the impact of the social dimension on the motivational role of the aesthetic and the overall stimulating effect on the pre-service teachers.

That is not to say that all the collaborative experiences were positive. Of the four pre-service teachers who shared negative experiences of problem-solving, one felt uncomfortable in several group situations, while the other three spoke of a single unpleasant occurrence. The first pre-service teacher related her discomfort to her own perceived lack of mathematical ability but adds:

> Looking back I realize that it was silly of me to let my fear of embarrassment in front of my peers to hold me back from expressing my ideas, but in a situation with people you barely know it can be hard to let your inhibitions go.

Two of the other three pre-service teachers shared an experience of being part of a group whose approach to the problem was difficult to understand. They felt excluded. One wrote, “I ended up in a group that felt I could not contribute to and that I felt extremely disengaged in”. The fourth pre-service teacher’s journal entry stemmed from a group experience in which all the group members chose to work sitting together at one table. He noted that they all immediately began working individually, writing, “I think that the work at the “horizontal” spaces set conditions for atomized work and encouraged us to get set in our ways”. He later added, “Instead of working as a group, we were working in a group”. This telling phrase perhaps encapsulates the experience of all four of these pre-service teachers. Being in a social situation, which lacked collaboration, negatively affected their aesthetic experience of problem solving.

Interestingly, this negative experience was felt by several other pre-service teachers. However, rather than having the experience of problem solving in two different kinds of social situations, these pre-service teachers contrasted their collaborative experience in problem solving with working alone:

> After experiencing working on these problems in a group, both with random peers and my friends, I thought it would be interesting to experience the different emotions and feelings of doing one of these questions independently, as many students in math class are required to do. I must say that many of the emotions I felt during this process were not positive and this was a great reminder of how isolating math can be if you let it.

The feeling of isolation here is related to the social context, but it may also affect the sense of alienation from mathematics itself. In contrast, the development of social relations may also promote the development of relations with mathematics, which in turn may well be at the heart of the intuitions on which the generative role of the aesthetic depend.

Yet another remarked on an unsuccessful attempt to solve a problem alone at home:

> There was no one motivating me to continue, or pushing my thinking or presenting new ideas regarding how to solve it. At this point I started really appreciating math done in a collaborative setting because I had never given up before in one of my groups during class.

This intertwining of the affective and aesthetic domains suggests that collaboration itself became less of a way of "doing" mathematics and more of a way of experiencing mathematics. It was through collaboration that aesthetic motivation was possible for these pre-service teachers. In addition, we conjecture that the safe, shared space that was created also provided fertile grounds for the generative role of the aesthetic, inasmuch as the pre-service teachers also developed their relations *with* mathematics. In the next section, we turn to this theme more directly in examining instances in which the pre-service teachers described a sense of rapprochement.

20.4.3 Getting Close to the Mathematics

In reading the literature on the aesthetic dimension of mathematical activity, there were a few instances in which mathematicians described a kind of felt connection to mathematics, which we associated with Sinclair's (2004) generative role of the aesthetic but also to the notion of intimacy and relationship—both of which we see as establishing some kind of closeness with mathematics. We identified 26 instances in which the pre-service teachers evoked closeness. By far the most common theme in these 26 instances was how the problem-solving experiences helped the teachers learn something about themselves, which we see as closeness in that the mathematics is inserting itself in the pre-service teacher's relationship with herself. There were 20 such instances. For example, one pre-service teacher wrote:

> I learned a lot about myself in group situations, and my own thoughts and opinions about my own personal math abilities. I feel as though I grew a great deal as a math learner through these problems; I started off very shy and never wanted to share my thoughts because I was always doubting myself, to becoming really comfortable and very sure of myself!

This comment could be read in affective terms, as a change in confidence, for example. It could also be read in terms of the pre-service teacher's identity as a mathematics learner. We see it as an example of aesthetic closeness in parts because it combines affective and cognitive dimensions but also because of the sense of a new relationship with mathematics that shifts the way she sees her own mathematical abilities. Another pre-service teacher also wrote about increased confidence, especially in a group: "Despite my original shy nature with math, I even found myself picking up the chalk first which made me proud because I was not as afraid to do math in front of others".

A similar comment, which could also be seen as being related to identity, was made by a different pre-service teacher:

> I never really though myself one to be affected by the thought of "satisfying" numbers, but there is something just so pleasing about symmetry in this regard…

There is a shift in this pre-service teacher's perception of herself, as someone who could find a mathematical object "satisfying", but also a new closeness to number because of the pleasure of symmetry. This response alludes to the more traditional aesthetic category of symmetry, but it is not the detached criterion of beauty that G.H. Hardy or Bertrand Russell might talk about. Instead, it is wrapped up in a new relationship with herself, once again, as someone who might respond positively to particular mathematical objects. Another pre-service teacher expressed a similar feeling, writing: "To be honest, I was surprised by my feeling of joy and excitement from a math problem!"

A different type of closeness that was articulated by the pre-service teachers connects very closely with the generative role of the aesthetic. We only found six such responses, but they each mentioned something about a "feeling" or "inkling" or "sense of pattern" that they had when working on the problem. One pre-service teacher wrote "I remember being a bit in between these two feelings, like I was on the edge of my seat knowing some epiphany was about to happen but still not knowing what to expect". This is not about the "aha" experience, but about the feeling leading up to it, when you are anticipating such an experience. It seems in line with Weil's description of "murky reflections", "furtive caresses" and "inexplicable tiffs"—when you *feel* you know before you actually know.

20.5 Discussion and Conclusion

Our goal in this chapter has been to examine the aesthetic responses of pre-service teachers engaged in open and collaborative problem solving. Given the particularity of their contexts, as students in a course but also future teachers, we expected their aesthetic experiences to diverge from those reported by and about expert mathematicians. Given the methodological challenges of capturing aesthetic responses in the course of problem-solving, we chose to study our phenomenon of interest by asking the pre-service teachers to describe their reasons for choosing certain problems (out of the many they encountered over the course of the semester) to include in their journals. The focus on choice, we hoped, would give rise to axiological considerations, especially in relation to the aesthetic values at play.

In analysing the pre-service teachers' journal responses, we focused on the three types of aesthetic considerations put forward by Pimm (2006), which involve what is being attended to, how it is being attended to and why. Given our context, we adapted these three types of considerations as follows: the *what* focused on aspects of the problems identified by the pre-service teachers; the how focused on the process of solving the problem; and the why addressed the reasons given to explain the worthiness of the problem. In relation to the *what*, the pre-service teachers referred to two considerations that have been found in the research on mathematicians' aesthetic: the problem should be adequately difficult and it should be surprising. The pre-service teachers also spoke of two other considerations: the problem should have some kind of personal connection and it should have multiple solution methods. In relation to the *how*, the most important consideration was collaboration, as a method of doing mathematics. The pre-service teachers also wrote about attending to the particular methods they found appealing such as "guess and check". One very important consideration was the workspace, which involved the use of vertical surfaces. Finally, in relation to the *why*, the pre-service teachers wrote about valuing problems because they provided insight and enjoyment, which is something the research literature suggests they share with mathematicians. They also referred to the fruitfulness of a problem in terms of their perception that the problem could be used in their own classrooms, in the future.

In reading the pre-service teachers' responses, we were also struck by two themes that have been mentioned in the literature on the practices of mathematicians but that have not typically been featured in discussions of the mathematical aesthetic. One theme had to do with the social context in which mathematical problem solving can occur and the concomitant affective and cognitive implications of such activity. We also argued that the collaborative opportunities may have occasioned opportunities for intuitive ways of thinking that have been linked with the generative role of the aesthetic. This relates also to the second theme of closeness, in which we saw evidence of the pre-service teachers expressing a new relation to mathematics. The closeness was also evident in their writing about having a "feeling" for mathematical ideas—not solutions to problems but intuitions about how the problem could be solved.

Although we have highlighted the types of problems that the pre-service teachers were given the opportunity to work on and the assignment they were given as part of the assessment of their work, we have said very little about the role of the teacher in creating an environment in which the various aesthetic responses we identified could occur. Encouraging collaboration certainly helped, but the pre-service teachers also reported feeling safe in the classroom, which would enable them to take risks in their problem solving. The teacher repeated over and over again that she was interested in the process, which also likely mitigated any feelings of anxiety the pre-service teachers might have had.

Overall, we have found that the pre-service teachers wrote about having many aesthetic responses that are similar to those reported by mathematicians. They also wrote about some types of aesthetic experiences that are more pertinent to their roles both as students and future teachers. If the role of the teacher is to enculturate students into mathematics, then we see it as essential for them to develop an awareness of the aesthetic nature of mathematical activity, but we also underscore the importance of helping them develop an aesthetic awareness that is relevant to the teaching and learning of mathematics.

References

Brown, S. (1973). Mathematics and humanistic themes: Sum considerations. *Educational Theory, 23*(3), 191–214.

Brown, S., & Walter, M. I. (1983). *The art of problem posing*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Buerk, D. (1982). An experience with some able women who avoid mathematics. *For the Learning of Mathematics, 3*(2), 19–24.

Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics. *Educational Studies in Mathematics, 37*(2), 121–143.

Burton, L. (1999). Why is intuition so important in mathematics but missing from mathematics education? *For the Learning of Mathematics, 19*(3), 27–32.

Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Chandrasekharan, S. (2014). Becoming knowledge: Cognitive and neural mechanisms that support scientific intuition. In L. Osbeck & B. Held (Eds.), *Rational Intuition: Philosophical Roots, Scientific Investigations* (pp. 307–337). New York, NY: Cambridge University Press.

Crespo, S., & Sinclair, N. (2008). What can it mean to pose a 'good' problem? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*(5), 395–415.

DeBellis, V., & Goldin, G. (1999). Aspects of affect: Mathematical intimacy, mathematical integrity. In O. Zaslovsky (Ed.), *Proceedings of the 23rd International Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249–256). Haifa, Israel: PME.

Dewey, J. (1934). *Art as experience*. New York, NY: Perigree.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics, 6*(1), 2–10.

Eberle, S. (2014). The role of children's mathematical aesthetics: The case of tessellations. *The Journal of Mathematics Behavior, 35*, 129–143.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hardy, G. H. (1940). *A mathematician's apology*. Cambridge, UK: Cambridge University Press.

Johnson, M. (2007). *The meaning of the body: Aesthetics of human understanding*. Chicago, IL: The University of Chicago Press.

Le Lionnais, F. (1948/1986). 'La beauté en mathématiques. In F. Le Lionnais (Ed.), *Les Grands Courants de la Pensée Mathématique* (pp. 437–465). Paris, France: Editions Rivages.

Lehrer, R. (2008). *Developing a culture of inquiry in an urban sixth grade classroom*. Presentation at the American Education Research Association conference, New York. 27 Mar 2008.

Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem solving. In P. Felmer, W. Van Dooren, & J. Kilpatrick (Eds.) *Posing and Solving Mathematical Problems* (pp. 361-386). New York, NY: Springer.

Menz, P. (2015). *Unfolding of diagramming and gesturing between mathematics graduate student and supervisor during research meetings* (Ph.D dissertation). Department of Mathematics, Simon Fraser University.

Montano, U. (2014). *Explaining beauty in mathematics: An aesthetic theory of mathematics*. Cham, Switzerland: Springer.

Netz, R. (2009). *Ludic proof: Greek mathematics and the Alexandrian aesthetic*. New York, NY: Cambridge University Press.

Papert, S. (1978). The mathematical unconscious. In J. Wechsler (Ed.), *On Aesthetics and Science* (pp. 105–120). Boston, MA: Birkhäuser.

Penrose, R. (1974). The role of aesthetics in pure and applied mathematical research. *The Bulletin of the Institute of Mathematics and its Applications, 10*(7/8), 266–271.

Pimm, D. (2006). Drawing on the image in mathematics and art. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity* (pp. 160–189). New York, NY: Springer.

Poincaré, H. (1908/1956). Mathematical creation. In J. Newman (Ed.), *The World of Mathematics* (Vol. 4, pp. 2041–2050). New York, NY: Simon and Schuster.

Polanyi, M. (1958). *Personal knowledge: Towards a post-critical philosophy*. New York, NY: Harper & Row.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning, 5*(1), 37–70.

Rota, G. C. (1997). *Indiscrete thoughts*. Boston, MA: Birkhauser.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham, NC: Duke University Press.

Silver, E., & Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D. McLeod & V. Adams (Eds.), *Affect and Mathematical Problem Solving* (pp. 59–74). New York, NY: Springer-Verlag.

Sinclair, N. (2001). The aesthetic "is" relevant. *For the Learning of Mathematics, 21*(1), 25–32.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning, 6*(3), 261–284.

Sinclair, N. (2008). Attending to the aesthetic in the mathematics classroom. *For the Learning of Mathematics, 28*(1), 29–35.

Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics, 15*(1), 29–37.

Tymoczko, T. (1993). Value judgments in mathematics: Can we treat mathematics as an art? In A. White (Ed.), *Essays in Humanistic Mathematics* (pp. 67–77). Washington, DC: The Mathematical Association of America.

Von Neumann, J. (1947). The mathematician. In R. Heywood (Ed.), The *Works of the Mind* (pp. 180–196). Chicago, IL: University of Chicago Press.

Weil, A. (1992). *The apprenticeship of a mathematician*. Basel, Switzerland: Birkhaüser.

Wells, D. (1990). Are these the most beautiful? *The Mathematical Intelligencer, 12*(3), 37–41.

Chapter 21
Facebook and WebQuests as Tools for Engagement with Mathematical Problem-Solving: The Emotions Experienced in the Math@XXI Gymkhana

Rosa Antónia Tomás Ferreira and Marli Duffles D. Moreira

Inclusive competitions have been receiving increasing attention, both from the educational community and the general public (Kenderov et al., 2009; Stockton, 2012). Such competitions are aimed at all students, regardless of their school achievement. This means that although students deal with (mathematical) challenges, they perceive those challenges as attainable and related to their daily lives which, in turn, favour the search for a solution.

Inclusive competitions are typically available in beyond the classroom contexts. Given the challenging environment that usually surrounds them, such competitions stimulate student participation and foster positive affect towards learning mathematics, namely, regarding motivation, interest, and pleasure (Freiman & Véniza, 2006). Furthermore, if collaboration also occurs, students' learning of mathematics is promoted, including those who have more difficulties (Barbeau & Taylor, 2009).

The social interaction that emerges during the participation in inclusive competitions promotes a context of cultural relationships that favours each and every student's individual development. Several studies have shown that similar activities beyond the classroom contribute to developing a positive attitude towards school and towards mathematics (Barbeau & Taylor, 2009; Freiman & Applebaum, 2011). One might argue that students' willingness to participate in a mathematical competition beyond the classroom signals their predisposition to enjoy mathematics and face it in a positive manner; however, when a large number of students are involved

R. A. Tomás Ferreira (✉)
Departamento de Matemática, Faculdade de Ciências, Universidade do Porto & Centro de Matemática da Universidade do Porto, Porto, Portugal
e-mail: rferreir@fc.up.pt

M. D. D. Moreira
Departamento de Matemática, Universidade Federal de Viçosa, Viçosa, Brazil
e-mail: marliddmoreira@ufv.br

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_21

in such after-school activities, one must also look at the characteristics of those activities that promote such engagement and positive attitudes.

Math@XXI (Moreira, 2016) is a school mathematical competition of inclusive nature under the format of a gymkhana. This pedagogical intervention was developed in a public school in Northern Portugal, in a beyond the classroom context. Aimed at students in the third cycle of basic education (aged 12–15 years old, 7th through 9th grade levels), the gymkhana included four digital tournaments (WebQuests) and one on-site final tournament without Internet usage.

Approximately 155 students organized themselves, freely, in 13 teams of 11 participants each and one team of 12. Students came from eight 7th grade classes and five 8th grade classes. The teams were quite heterogeneous in terms of their elements' grade levels and school mathematics achievement. Six mathematics teachers of the school supported the teams' work. Each team named itself.

From the beginning, Facebook was included in the Math@XXI design as a means for facilitating the communication amongst the elements of each team and between them and the facilitator (second author). As such, 14 closed groups were set up, one per team, the facilitator being the sole common element in those groups. Facebook was also intended to be the vehicle to publish the WebQuests that constituted the digital part of the competition. During such digital tournaments, students were encouraged to ask for help to the gymkhana's facilitator or the teachers involved whenever they felt the need. The facilitator was present at the school at least once a week in order to provide support to the gymkhana's development.

Math@XXI lasted for 5 months, starting in January 2015 with the publication of the first WebQuest on the closed groups in Facebook. The WebQuests were common to all the teams and were designed around four integrating themes: (i) triangle, (ii) number 7, (iii) infinity, and (iv) mode. Figure 21.1 depicts how each of these themes was proposed to the participants in Math@XXI. As shown, there was a pattern in the design of the WebQuests – for each aggregating theme, the participants were asked to complete tasks that pushed towards establishing different kinds of connections: with the history of mathematics, with distinct areas within mathematics, with other sciences, and with daily life. The tasks challenged students to solve problems and engage in mathematical explorations (Ponte, 2005). As a final product of each digital tournament, the teams were asked to produce an issue of a journal, in digital or printed format, presenting their responses to the tasks posed in the WebQuests.

The facilitator scored all teams for their work throughout the five tournaments, in a total of 240 points, half of which equally distributed by the four WebQuests. Scoring was a way to *give a kick* to the competition. In truth, the real competitive part of Math@XXI was the final tournament. The digital tournaments had the major goal of pushing participants to engage with mathematics, find new connections within mathematics and with other areas, including daily life, and develop an appreciation for mathematics. Throughout these tournaments, the scores and the feedback given (on-site and via Facebook) helped the participants in identifying what they needed to improve in their next journal issue. The teams' work on the WebQuests was assessed based on the following criteria, which were of public knowledge:

Tournaments	Connections	Tasks
First WebQuest **Theme: TRIANGLE** (January 12, 2015)	History	Pythagoras, his ideas and accomplishments
	Mathematics	Pythagoras' theorem
	Geography	Circumcentre of an imaginary triangle formed by three extreme points in Portugal
	Daily life	Total area of a roof
Second WebQuest **Theme: NUMBER 7** (February 13, 2015)	History	Thales, his ideas and accomplishments
	Mathematics	Prime numbers
	Physics	Rainbow
	Daily life	Distances in maps
Third WebQuest **Theme: INFINITY** (March 13, 2015)	History	Georg Cantor, his ideas and accomplishments
	Mathematics	Golden ration and irrational numbers
	Natural sciences	Water
	Daily life	Medication in "infinite" dosages
Fourth WebQuest **Theme: MODE** (April 20, 2015)	History	Blaise Pascal and Pierre de Fermat, their ideas and accomplishments
	Mathematics	Mean, median, and mode
	Statistics	Data handling
	Daily life	Results of a football championship
Final tournament (on-site) **Without Internet** (May 29, 2015)	Group A	About a topic in history of mathematics
	Group B	About a problem of daily life
	Bonus task	Crosswords about the themes addressed in the digital tournaments

Fig. 21.1 Organization of the five tournaments of Math@XXI

correctness of the answers to the tasks, clarity and correctness of the written texts, cooperation amongst all team members, and creativity in the solution of the tasks.

The final tournament took place on a Friday afternoon, in an after-school schedule, involving 105 students working in their respective teams, 12 in total. This tournament allowed no Internet usage nor help request. Given the weight of this tournament's score in the final score of the competition, the partial ranking of the digital tournaments could be drastically changed after the final tournament. Though the tasks of the final tournament addressed the same themes of the WebQuests and were of the same nature – mathematical problems and explorations – the participants' working conditions were significantly different. This could alter the participants' emotional states, but, at the same time, it could challenge the participants' attitudes that they had been developing while engaged in the competition. In particular, the self-confidence and collaborative skills that the students developed during the digital tournaments were driving forces for the final tournament.

The closing event of the Math@XXI gymkhana occurred on June 8, 2015. All students of the 12 teams which participated in the final tournament were given medals. The two teams with the highest final scores received a prize: a 1-day guided tour in Porto (second largest city of the country, about 50 km away from the school site), including a visit to the Faculty of Sciences of the University of Porto.

In this chapter, we present the partial results of a wider research study (Moreira, 2016) that was developed based on Math@XXI. Following a design-based research methodological approach, the Math@XXI intervention was grounded on the ideas of mathematical enculturation and activity theory, with an emphasis on the interrelationship between affect and cognition. Our aim is to describe the emotions experienced by students in Math@XXI and discuss the potential of Facebook and WebQuests to support the development of students' global, positive, affect towards mathematics.

21.1 Mathematical Enculturation and Activity

The intervention Math@XXI was designed based on Bishop's (1991, 2002) notion of *Mathematical Enculturation* and on Leontiev's *Activity Theory* (Leontiev, 1978). Both authors emphasize that learning is a sociocultural phenomenon that is developed through each individual's appropriation of objects of the human culture, historically constructed. In this section, we briefly present those two theoretical perspectives (since they framed the whole Math@XXI project, though they were not relevant to the data analysis herein presented), pointing to how we perceive their interrelationship in the context of Math@XXI.

According to Bishop (1991, 2002), mathematics is a cultural good that should be available to all students through a process of enculturation. Mathematical enculturation is a social and deliberate process of engagement with mathematics, involving the appropriation of objects of the mathematical culture that has been constructed throughout human history: "each new generation reconstructs the meanings of the old [generation], and thus 'culture' is no fixed collection of objects" (Bishop, 1991, pp. 152–153).

Mathematical enculturation is an interpersonal process: when students interact in small groups, they share meanings and disagreements, while they appropriate mathematical objects to which they give a personal meaning. Hence, Bishop (1991) has urged a change in how mathematics education has been and still is perceived: from a technical focus that emphasizes doing to a cultural core that privileges understanding, that is, "to promote the transformation from 'technique', *a way of doing*, to 'meaning', *a way of knowing*" (p. 124 [first italics added]).

According to Leontiev (1978), human activity is social in its essence, and it promotes a dialectical relationship between the external reality and the structure of the consciousness. The author sustains it is through activity that the appropriation of the objective world is achieved and that activity mediates between the subject and the object. Thus, the process of appropriation of the objects of human culture is the outcome of an effective activity of the individual on those objects.

Learning is an activity that encompasses needs and motives, and it occurs through actions and interactions with a given culture. Such activity involves thinking and doing, two actions that feed into each other. Leontiev (1978) emphasizes the essential role of the individual's activity to the process of appropriation of knowledge. This demands a fundamental attitude on the student's part towards knowledge: the student must be the main character of the pedagogical scene, not a passive element whose role is to receive ready-to-use (pieces of) knowledge.

All human activity is mediated by tools. Math@XXI is perceived as a cognitive tool, that is, a technology (in a broad sense) that enables (mediates) students' meaningful and active learning, as they construct knowledge, communicate, collaborate, and reflect. The teams (subjects), through their activity, accessed the mathematical culture (objects), promoting the development of a process of mathematical enculturation. The social mediation occurred amongst the students who collaboratively solved the tasks, within their teams and with the help of the gymkhana's facilitator and their teachers. Facebook and WebQuests played the role of mediation tools.

WebQuests allow for the creation of a stimulating learning environment in which the following three elements interact in a mutual way: the student (subject), the technology (tool), and the mathematical culture (object). Thus, WebQuests work as mediating tools between students and knowledge, in line with the premises of Leontiev's (1978) Activity Theory. They

> …are activities of guided research, mainly on the Web, based on authentic, challenging tasks. Therefore, they facilitate learning, whether individually or in groups. These are, in fact, their main foundations and the ingredients able to motivate students, and to engage them, actively, in their own learning at school. (Costa & Carvalho, 2006, p. 10)

There is a general format for the structure of a WebQuest which was followed in Math@XXI (Costa & Carvalho, 2006). The *introduction* presented the overarching themes of the WebQuests and provided general guidelines for the students about the tasks they would be invited to complete. These *tasks* were described in detail in a second tab. The *process* tab gave students the outline for the various steps of the work they were to engage with, as well as some orientations to complete their work. The tab for *resources* offered students a list of references that should be consulted in order to complete the tasks (all resources were available on the Web). The scoring criteria for each proposed activity were made explicit in the *assessment* tab. Finally, the *conclusion* tab asked for a summary of the work that was developed and a reflection about what was learnt.

Costa and Carvalho (2006) advocate that using WebQuests as a pedagogical strategy helps in developing some "essential socio-cognitive competencies that are indispensable for an individual's development in [today's] society of knowledge" (p. 12). In particular, WebQuests support students in developing several skills such as research, communication, collaboration, and social participation.

WebQuests may become significant tools for mediating the teaching and learning process, in which several aspects concur towards meaningful learning (of mathematics): students' activity is at the core of the teaching and learning process, tasks allow students to get involved with complex knowledge, and collaborative work is encouraged (Costa & Carvalho, 2006). In fact, when WebQuests are properly designed and structured, they engage students in modes of work that move away from what they are typically called to do at school.

Facebook is a social network of wide usage by youngsters in their daily life, having potential as a pedagogical tool and as a means of communication within the school scenario. As such, it might promote networks of digital interactions and collaborative activity (Allegretti, Hessel, Hardagh, & Silva, 2012), since it widens physical and time barriers, and it allows multiple formats of communication (written word, figures, digital objects – photos, videos, etc.). In addition, Facebook stimulates the expression of emotions – for example, it allows the use of emoticons which are aimed at sharing emotional states in figurative format.

Facebook offers a friendly and dynamical environment, which invites students to share and express ideas and emotions in a spontaneous way. Its flexibility makes it possible to engage in this process in different formats: as an individual user, as a member of a closed group, or in an open setting – the whole community, including the school. As a pedagogical resource, "Facebook allows for a personal and a collective experience with Mathematics, its objects, processes and characters, acting as a motivational element to learning" (Moreira, 2016, p. 85).

The first four tournaments of Math@XXI were based on WebQuests. And each WebQuest asked the students to collaboratively edit a journal which should include the solutions to the tasks of the WebQuests. This product of students' work intended to reinforce students' authorship, autonomy, and responsibility for their own learning. The WebQuests were posted on Facebook, in each team's closed group, which contributed to giving students a sense of belonging (a symbolic integration inside the group, the team) and to stimulating them to organize themselves and work collaboratively. The whole design of the digital tournaments of Math@XXI was thought to move away from the usual work of students in school settings, even in activities that go beyond the classroom. In particular, the themes of the WebQuests and respective tasks were independent from the curriculum, in the sense that they were not chosen to fit specific curricular topics. Thus, Math@XXI was expected to offer students an innovative experience of learning.

In the design of Math@XXI, the resource to closed groups in Facebook intended to use this digital tool as a means of communication and exchange of information between the members of each team and the gymkhana's facilitator and amongst the team's elements as well. In the first case, Facebook was thought to be the prime vehicle for disclosing the WebQuests of Math@XXI and for answering any queries regarding the work that the teams were expected to complete. In the second case, students could share their learning, and, due to the affordances of Facebook, they could also freely express their emotions and other affective states. As such, "Facebook was thought to offer a virtual environment to developing collaborative learning, with the participation and involvement of all students pursuing a common goal: winning the Math@XXI gymkhana as a team" (Moreira, 2016, p. 86).

21.2 Affect in Mathematical Learning

The relationship between affect and cognition has sparked considerable interest amongst researchers for a long time. Initially, affective factors were associated with causes for the effects on cognition. "Mathematics education, however, need not necessarily draw cognition and affect together by means of causal links alone. (…) affect, far from being the 'other' of thinking, is a part of it. Affect influences thinking, just as thinking influences affect. The two interact" (Walshaw & Brown, 2012, p. 186). Hannula (2002) corroborates this idea:

> Emotion and cognition are seen as two complementary aspects of mind. These two have some phenomenological differences that make it reasonable to separate them. Cognition is neuron-based information processing, whereas emotions include other physiological reactions, too. However, this splitting of mind into emotion and cognition is only an analytical tool, and the interaction between the two is so intense, that neither can be fully understood in separation from the other. Rather, emotion and cognition are seen as two sides of the same coin. (p. 27)

In addition, affect has a social dimension: "Far from being located inside people's brains, affect is widely dispersed; it involves individuals and their relations in and with the world" (Walshaw & Brown, 2012, p. 187). Mathematics is a sociohistorical product, a part of the cultural legacy of many different civilizations; it "is a product of the human mind and the possibilities of human rational thinking" (Maasz & Schloeglmann, 2006, pp. 3–4). Thus, in their learning experiences, students should explore the connections of mathematics with life and the culture of various people, helping them developing a positive affect towards the discipline and the skill of using mathematics to read the world (D'Ambrosio, 2009).

Goldin (2002) emphasizes the central role affect plays in the mathematical activity, pointing the interaction between affect and cognition and its influence, which might be positive or negative, on students' learning:

> When individuals are doing mathematics, the affective system is not merely auxiliary to cognition – it is central. However affect as a representational system is *intertwined with cognitive representation*. Affective configurations can stand for, evoke, enhance or subdue, and otherwise interact with cognitive configurations in highly context-dependent ways. The very metaphors used in thinking may carry positive or negative affect. (p. 60)

Nowadays, research in neuroscience has shed more light into this complex endeavour of better understanding the role of affect in learning. According to Damásio (2011), mind and body act in an interconnected way. People respond to physical and sociocultural stimuli of the environment with the whole of their being, thus condemning the separation of cognitive and emotional processes in education.

According to McLeod (1992), an unavoidable reference in the literature on affect in mathematics education, the affective domain may be seen as built on three facets – beliefs, attitudes, and emotions. These facets may be used to "describe a wide range of affective responses to mathematics" (p. 578), but they are very difficult to separate and differentiate, perhaps due to the invisible nature of affect (Leder, Pehkonnen, & Törner, 2002). Nevertheless, we can analyse them by looking at their stability and intensity.

While beliefs and attitudes are usually stable, emotions are prone to change quickly (McLeod, 1992). Beliefs, attitudes, and emotions

> also vary in the level of intensity of the affects that they describe, increasing in intensity from "cold" beliefs about mathematics to "cool" attitudes related to liking or disliking mathematics to "hot" emotional reactions to the frustrations of [not having success in] solving nonroutine problems. (p. 578)

Thus, emotions can be seen to rest at the stronger end of an intensity continuum, with beliefs at the opposite end and attitudes in between. In addition, whereas emotions "may appear and disappear rather quickly" (McLeod, 1992, p. 578), beliefs take a long time to develop.

In this chapter, we assume the basic descriptors of affect (beliefs, attitudes, and emotions) in the sense of Gómez-Chacón (2003). Mathematical beliefs "are one of the components of the subjective knowledge about mathematics and its teaching and learning that is implicit in the individual" (p. 20). Students' beliefs are grounded not only on their experiences with mathematics but also on their sociocultural contexts. The idea that mathematics is based on formulas and rules is an example of a (typical) belief.

Attitude is a positive or negative predisposition "which determines one's personal intentions and influences one's behavior" (Gómez-Chacón, 2003, p. 21). Disliking doing geometrical proofs is an example of a (negative) attitude; a positive attitude is, for instance, being curious about fractals. Emotions "are organized responses that go beyond the boundaries of psychological systems, including the physiological, the cognitive, the motivational, and the experiential system" (p. 22). Feeling happy and frustrated with the solution process of a problem are examples of positive and negative emotions, respectively. In the following section, we pay special attention to the affective dimension of emotions, given the stated goal of this chapter. We looked for students' expression of emotions while engaged in the WebQuests of Math@XXI and in the Facebook closed groups that were created for each participating team and the gymkhana's facilitator.

21.2.1 Emotions

Despite the numerous approaches to studying emotions and the various definitions that have come to the fore, there is large consensus about certain aspects of this construct. For example, emotions connect to personal goals in the sense that "they code information about progress towards goals and possible blockages, as well as suggest strategies for overcoming obstacles" (Hannula, 2006, p. 219). Emotions also involve physiological reactions – for instance, the release of adrenaline in the blood is a physiological response to an arousal (Hannula, 2002).

Zan, Brown, Evans, and Hannula (2006) suggest that "emotions affect cognitive processing in several ways: they bias attention and activate action tendencies" (p. 6), indicating a significant influence of this affective dimension in the learning process. Although emotions are always present in human experiences, they become observable only when they are intense. This invisibility makes it quite challenging to study the role of emotions in the complex ecology of teaching and learning.

Emotions are unstable (McLeod, 1992) and influenced by several factors. One important factor of change "is the cognitive (possibly unconscious) analysis of the situation with respect to one's goals" (Hannula, 2002, p. 28). De Bellis and Goldin (2006) also highlight this relationship between emotions and cognition. For them, emotions designate affective states that are clearly accessible to consciousness in association with precise cognitive contents since we know the motive for fear, shame, joy, etc. Students are usually aware of their emotions, and they are able to identify their typical emotional reactions in different situations. Hannula suggests that "proceeding towards goals induces positive emotions while obstacles that block the progress may induce anger, fear, sadness or other unpleasant emotions" (p. 29).

In the view of Goetz, Frenzel, Hall, and Pekrun (2008), "positive emotions play a pivotal role within educational settings" (p. 10). The most commonly found emotions, in the context of mathematical learning and achievement, are enjoyment, anxiety, anger, and boredom (Frenzel, Pekrun, & Goetz, 2006). All these emotions, amongst others, could be triggered during the participation in Math@XXI, as it is the case with any innovative experience.

There is a direct relationship between academic emotions in mathematics and learning and achievement (Kleine, Goetz, Pekrun, & Hall, 2005). Academic emotions can be classified according to two criteria: valence and activation. "While valence is considered as a bipolar dimension (positive vs. negative), activation is understood as unipolar in nature, and indicates the extent to which a given emotion is activating" (p. 221). Amongst positive emotions, enjoyment, pride, and hope are seen as activating emotions and relief and relaxation as deactivating emotions; amongst negative emotions, anxiety, anger, and shame/fault are seen as activating emotions and boredom and hopelessness as deactivating emotions. Kleine et al. (2005) alert that one cannot assume that positive emotions have a positive effect on learning and that negative emotions have a negative effect on learning.

Some emotions are favourable to learning while others hinder this process. Some examples of the latter emotions include

> persistent fear and confusion, foreboding, resignation, prolonged uncertainty, lack of self-confidence (which leads to withdrawl and alienation), and boredom. Since emotions are contagious, favourable emotions to learning should be promoted: comfort experiences, good humour, feelings of pleasure and fun, in articulation with a sense of challenge and persistency, states of acceptance and ambition, mystery, and curiosity. (Neves & Carvalho, 2006, p. 206)

Math@XXI strived to prevent such unfavourable emotions from emerging in the course of the gymkhana: the supporting teachers and the facilitator were always available to provide help to the participants, whenever they asked for it (asking for help was not only allowed but also encouraged in Math@XXI) during all WebQuests; the students could only participate in teams, benefiting from working collaboratively; and the facilitator offered constant enthusiastic encouragement to keeping engaged and completing the tasks, either on-site or via Facebook or the WebQuests themselves.

21.2.2 Local and Global Affect

Beliefs, attitudes, and emotions interact with each other in a cyclic way, during the learning process. Students' experiences when engaged in mathematical activities give rise to emotions which, in turn, influence their attitudes and beliefs and both influence new attitudes and emotions (Gómez-Chacón, 2003). All these elements constitute forces that should be intentionally activated in the mathematics education of youngsters: "If the goal is to improve the teaching and learning of mathematics, it seems convenient to take into consideration students' and teachers' affective factors. Emotions, attitudes, and beliefs act as driving forces of mathematical activity" (p. 24).

Gómez-Chacón (2000, 2003) distinguishes two possible approaches to analysing the role of affect in mathematics learning. Local affect is related to the affective state that a student experiences while solving a specific mathematical task; global affect is associated with the affective individual structure that a student constructs upon his/her experiences in prior mathematical activities. Thus, local affect has to do with the "representation of the information dealing with the emotional reactions which affect the conscious processing all through" (Gómez-Chacón, 2000, p. 151), while global affect "has to do with the socio-cultural influences on the individual and the ways in which this information is internalised, and shapes his belief structure" (p. 151).

Gómez-Chacón (2000) cautions to the need of combining both approaches in understanding the role of affect in mathematics learning. If we wish to "understand students' affective relations with mathematics, it is not enough to observe and know the states of change of feeling or emotional reactions (…) and detect cognitive processes associated with positive or negative emotions" (p. 166) when solving a specific task, that is, it is not enough to look at the local affect. We must go further and consider "the affective dimension of the student in relation to mathematics in more complex scenarios (global affect), which allows us to contextualise the emotional reactions in the social reality that produces them" (p. 166).

For example, when we observe a student working on a mathematical task and notice that he seems curious and confident, we identify a positive local affective state (local affect) of that student. When a student shows enthusiasm and self-confidence regarding mathematical activities in general, this reveals a positive identity that has been constructed throughout his schooling trajectory in mathematics (global affect). Goldin's (2002) perspectives resonate with those of Gómez-Chacón

(2000). Affect in mathematics learning may be approached locally, looking at transient affect, which is connected to a specific context, or globally, looking at long-term and multi-contextualized affect.

Damásio's (2011) ideas on emotions and cognition resonate with what Gómez-Chacón (2000) advocates about the importance of looking at affect globally:

> Knowing the relevance of emotions in reasoning processes does not mean that reason is less important than emotions, that reason should be relegated to the background, or that it should be less cultivated. On the contrary, when we look at the broad function of emotions, it is possible to increase its positive effects and reduce its negative potential. (Damásio, 2011, p. 314)

Both the Math@XXI gymkhana and the whole research endeavour developed around this competition (Moreira, 2016) were grounded in the premise that students' success in learning mathematics is directly related to developing positive affect towards mathematics. Adopting Gómez-Chacón's (2003) broad notion of affect – "a vast category of feelings and humour (moods) that are generally considered as something different from pure cognition" (p. 20) – we looked at students' expression of emotions in the context of their participation in the Math@XXI gymkhana and how the work in the WebQuests and the use of Facebook contributed to developing a global positive affect towards mathematics.

21.3 Methodological Procedures

The research that was carried out around Math@XXI (Moreira, 2016) was conducted under an overarching design-based research (DBR) methodological approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). As Cobb et al. (2003) put it, the DBR approach has "both a pragmatic bent – 'engineering' particular forms of learning – and a theoretical orientation – developing domain specific theories by systematically studying those forms of learning and the means of supporting them" (p. 9).

Math@XXI was designed with the aim of developing an educational product (mediating tool), within the complex ecology of the process of teaching and learning mathematics. This intervention was grounded on the theoretical principles of Bishop's (1991, 2002) Mathematical Enculturation, Leontiev's (1978) Activity Theory, and the interplay between cognition and affect in mathematics learning (Gómez-Chacón, 2003). Each of the four digital tournaments worked as a cycle of iteration: the following WebQuest was prepared and developed taking into account the feedback received from the participants in the gymkhana as well as their supporting teachers. In addition, the whole intervention successively adapted itself to the context and its circumstances, meeting the needs of the participants. Thus, while promoting change in educational practices (Wang & Hannafin, 2005), we sought to understand how, when, and why educational innovations work (Design-Based Research Collective (DBRC), 2003).

Data were collected from various sources. We gathered all posts placed in the Facebook murals of the participating teams, which were posted by the participants as well as the gymkhana's facilitator. We looked at how Facebook was used in the heart of the gymkhana, with a focus on how it provided a means to express emotions. Many of these posts contain pictures which were taken by the facilitator, either to share instances of the mathematical culture or to document important moments of the competition (such as the submission of the journals, participants' collaborative work, meetings with the facilitator, etc.).

The end products of the competition, the journals of the four WebQuests, were also gathered and analysed, looking for evidences of the participants' emotional states. At the end of each WebQuest, participants completed a small, anonymous, and individual on-line survey aimed at gathering information on how they were feeling the competition. Most questions were closed, but the two last ones asked them to elaborate on what they were enjoying the most and the least regarding their work on Math@XXI. The fourth and final survey included more closed items but also open questions. As in the previous surveys, the options for the Likert scale were (i) I totally agree, (ii) I partially agree, and (iii) I totally disagree. In this final survey, we searched for a more global perspective on the emotions felt throughout the competition, as well as for information regarding the participants' views about mathematics and the mathematical learning triggered by their participation in the gymkhana. The response rates were 47%, 47%, 50%, and 34%, respectively. The surveys were completed by the participants of the teams that were still engaged in the competition (the two teams that did not submit any journal did not complete these surveys).

We used content analysis and statistical procedures to analyse the data collected. In the next section, we analyse the participants' emotional engagement with the WebQuests and their expressed emotions on Facebook.

21.4 Expressing Emotions in the Math@XXI Gymkhana

During the Math@XXI gymkhana, there were many moments in which students expressed various emotions. The journals that were constructed, as end products of the WebQuests, contained several elements that revealed, direct or indirectly, the students' emotional states. The murals of the Facebook closed groups of all teams were full of posts in which the students and the gymkhana's facilitator expressed various kinds of emotions, both positive and negative. And this was reflected in the students' answers to the on-line surveys.

At the end of each WebQuest, students were invited to respond to an on-line survey. These surveys shared some of the items, but we also looked for different kinds of information in each of them. In all surveys, participants were asked to indicate their degree of agreement with four sentences: (i) *The tasks you completed on Math@XXI were challenging and interesting*; (ii) *Participating in Math@XXI is being fun*; (iii) *The tasks in Math@XXI contributed to my learning of Mathematics*;

and (iv) *My experience in Math@XXI has increased my enjoyment of Mathematics*. In general, students expressed a very positive assessment of the gymkhana and quite stable throughout the four digital tournaments, as shown in Fig. 21.2.

The last survey, which called for a global assessment of the whole experience of participating in Math@XXI, reinforced the positive appreciation of the gymkhana that was made after each digital tournament. Using NVivo software, considering the word as a unit of analysis, we generated a cloud of words relative to what the participants enjoyed the most in Math@XXI. The two most frequent words were "everything" and "I enjoyed", suggesting a high degree of satisfaction with the gymkhana (Fig. 21.3). Interestingly, when participants were asked to indicate what they enjoyed the least in the gymkhana, the most common response was "nothing"!

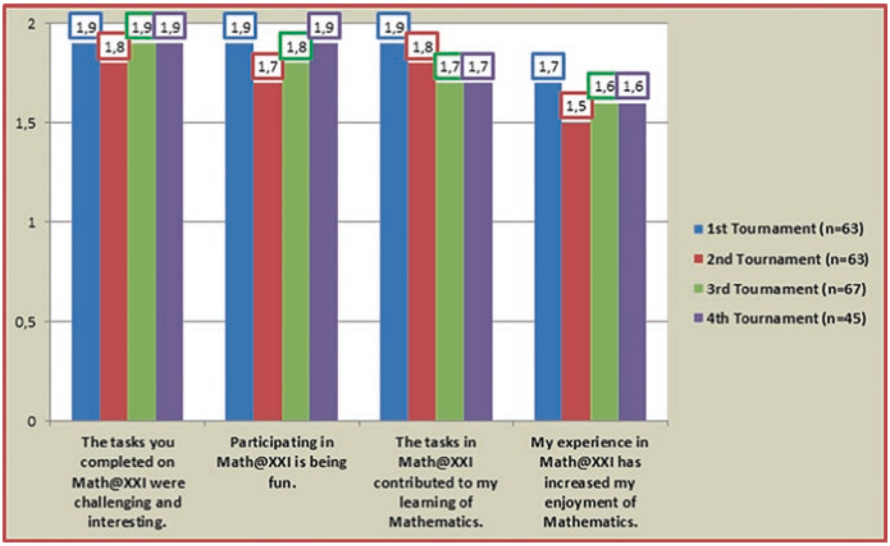


Fig. 21.2 Participants' assessment of Math@XXI after each digital tournament

Fig. 21.3 Cloud of words about what students enjoyed the most in Math@XXI

More than 60% of the students said that what they had enjoyed the most was having “the opportunity to work as a team” and “to learn more about mathematics” and “know new things”. Several students appreciated “having found things about mathematics that, if it weren’t the gymkhana we wouldn’t even guess they were connected to mathematics”. Math@XXI allowed “to face mathematics in a funnier [interesting] way”. In fact, some “were amazed at what [they] had learned” in the competition. Math@XXI also seemed to have contributed to a better relationship with mathematics: “I liked everything, learning much about mathematics”, “I loved everything, it is impossible not to enjoy everything we went through”, “[Math@XXI] is a good competition to learn more about mathematics and, above all, to learn enjoying mathematics”. In the following sections, we will describe how emotions emerged in the Math@XXI gymkhana, looking at both the WebQuests and the Facebook as means to express emotions.

21.4.1 The Emotional Engagement with the WebQuests

As we will see next, students’ engagement with the tasks of the WebQuests was always very high. They organized themselves to complete the tasks, found the time to meet and work collaboratively on the WebQuests, and worried about submitting their journals, the end products of the WebQuests, with an interesting and careful graphic look.

The direct expression of emotions in the productions derived from students’ work on the WebQuests was not very frequent. Yet, in some cases, it was possible to *observe* or infer students’ emotional states. In general, those emotions were positive and activating, in the sense of Kleine et al. (2005).

For example, the pages of some journals suggested feelings of affection towards mathematics. In Fig. 21.4, we can see the first page of the third journal edited by the GENIOUS OF MATH team (left) and a page of another journal of this team (right). In both cases, the students drew a heart as a symbol for affection within a sentence or a greeting. Though one might argue that drawing a heart does not mean loving mathematics, the fact is that these students spontaneously decided to express such a feeling in the cover page of their journal.

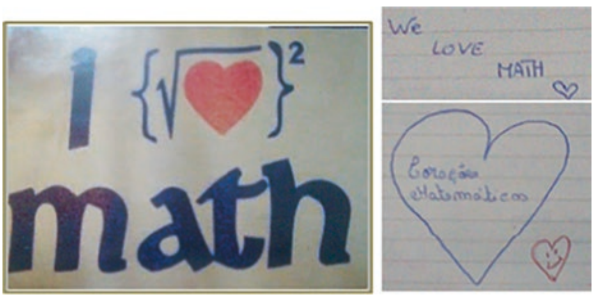
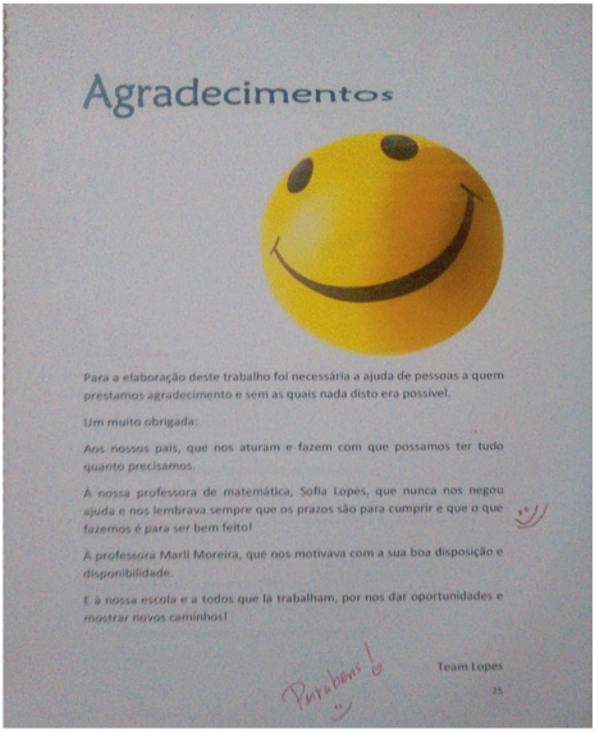


Fig. 21.4 Pages of journals edited by the GENIOUS OF MATH team


Agradecimentos

Para a elaboração deste trabalho foi necessária a ajuda de pessoas a quem prestamos agradecimento e sem as quais nada disto era possível.

Um muito obrigada:

Aos nossos pais, que nos aturam e fazem com que possamos ter tudo quanto precisamos.

À nossa professora de matemática, Sofia Lopes, que nunca nos negou ajuda e nos lembrava sempre que os prazos são para cumprir e que o que fazemos é para ser bem feito!

À professora Marli Moreira, que nos motivava com a sua boa disposição e disponibilidade.

E à nossa escola e a todos que lá trabalham, por nos dar oportunidades e mostrar novos caminhos!

Parabéns!

Team Lopes

25

Fig. 21.5 A thank you page included in the journal of the TEAM LOPES team

In one of its journals, TEAM LOPES decided to include a page to thank those who contributed to their accomplishment (Fig. 21.5). These students thanked their parents, their mathematics teacher, the gymkhana's facilitator, and the school and its staff for their help, guidance, support, and encouragement. The use of an emoticon conveys the students' emotion of gratitude expressed in their thank you note.

In these examples, the expression of emotions was direct and explicit. However, this was not the most frequent case. Indeed, we could only infer students' emotions while engaged in Math@XXI from their work in the WebQuests. The fact that the journals, the products of the WebQuests, were elaborated with time and care by the teams withdraws the possibilities of expressing emotional states in a spontaneous way. Thus, WebQuests were not the best vehicle to express emotions. Nevertheless, we could find some indirect evidences of the presence of emotions in students' work on the WebQuests. For instance, we could *find* emotions in some drawings the students made while working on the WebQuests.

At the end of each WebQuest, students were challenged to solve a bonus task. In general, these tasks stimulated students' imagination. For example, the first WebQuest asked students to create an artistic object of any kind (drawing, painting, sculpture, poetry, etc.) having the triangle as its inspiring element. Figure 21.6 shows the production of the PROSTUDENTS team (on the left)

Fig. 21.6 Drawings of the PROSTUDENTS, the 7SETE, and the OS MERCENÁRIOS teams in response to the bonus task of the first WebQuest

Fig. 21.7 Illustrations made by the MARRETAS team for their second and third journals, respectively

and those of the 7SETE (on the right, above) and the OS MERCENÁRIOS team (on the right, below).

Figure 21.7 presents two illustrations that the MARRETAS team used in two different journals. These drawings represented the mathematical objects "seven" (on the left) and "infinity" (on the right), which were the themes for the second and third WebQuests, respectively. Such drawings reveal students' proximity to the mathematical objects in a relaxed state of mind. In addition, it suggests an approximation of mathematics to students' daily lives. All of these elements favour learning.

Though indirectly, the drawings depicted in the previous figures transmit positive emotions. Happiness seems to emerge in Fig. 21.6, and fun is the emotional state triggered by the giraffe (Fig. 21.7). In this figure, the symbol for infinity is presented in an embellished way, suggesting students were feeling the beauty in mathematics. Such emotions are conveyed indirectly, but they are activating, positive emotions, thus contributing to a positive global affect towards mathematics.

As we have seen earlier, students were generally enthusiastic about participating in the gymkhana and about engaging in its activities. Occasionally, it was difficult to get organized and start working or to find the time to work collaboratively. However, students' interest on Math@XXI and enthusiasm about the tasks pushed them to overcome these difficulties and to actually gather in school and work together. Sometimes, the whole team was gathered around a table; other times, students worked divided in smaller groups. Figure 21.8 illustrates how the members of the 7SETE team worked throughout the gymkhana and their group spirit (the bottom right picture on Fig. 21.8 was taken by the facilitator on the students' request).

All teams showed interest and engagement with Math@XXI. Even those which did not complete the WebQuests or which dropped out of the competition gathered at the school library to work on the WebQuests. In fact, TEAM LOPES submitted the journal corresponding to the first WebQuest only, and the CRAQUES DA MATEMÁTICA, the MATEMATIX, and the OS MATRAQUILHOS teams did not submit any journal. However, as part of the inclusive character of Math@XXI, they were all admitted to the final tournament. The two latter teams did not show up in the final tournament.

In Fig. 21.9, we can see students of CRAQUES DA MATEMÁTICA team (on the left) and several elements of the MATEMATIX team (on the right) gathered around a computer, working on the tasks. Hence, we can witness students' interest and engagement with the WebQuests of Math@XXI. These positive attitudes were associated with emotions of happiness and enthusiasm, reflected in these students' joyful facial expressions, whether while working in groups or simply standing for a

Fig. 21.8 Different moments of work of the 7SETE team

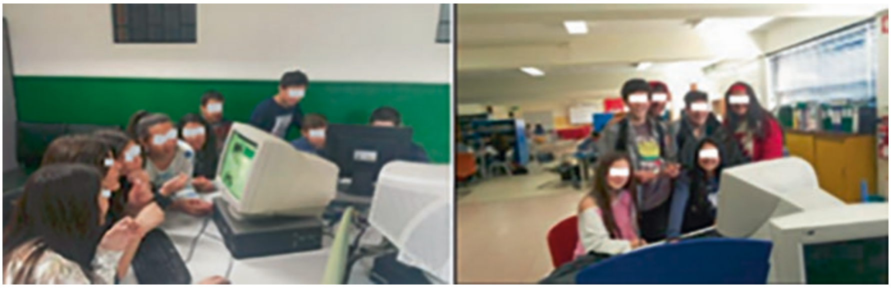


Fig. 21.9 The interest of CRAQUES DA MATEMÁTICA and MATEMATIX students

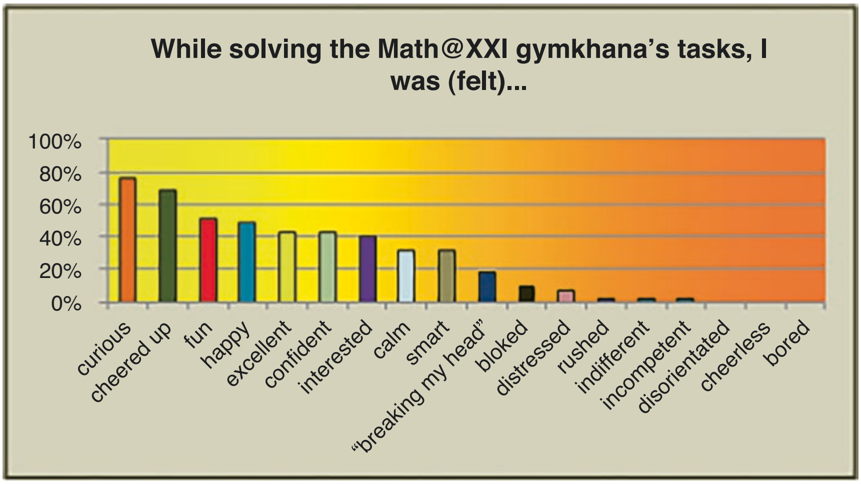


Fig. 21.10 Students' emotional states while engaged with the tasks of the WebQuests

Kodak moment. Thus, even the students of those teams which did not succeed much in the digital part of the Math@XXI gymkhana expressed positive and activating emotions (Kleine et al., 2005).

At the end of the fourth digital tournament, the on-line survey included a question (*While solving the Math@XXI gymkhana's tasks, I was (felt)…*) in which students should choose as many options as they felt necessary about their emotional states while engaged in the WebQuests' tasks. Given the question posed, as well as its timing, all options related to global affect, in Gómez-Chacon's (2000, 2003) terms. The emotional states that received a higher number of choices were (i) curious (75,6%), (ii) cheered up (68,9%), and (iii) fun (51,1%). All these positive and activating emotional states favour students' mathematical learning and aid the construction of a positive relationship with mathematics. The options for bored, cheerless, and disoriented, which expressed negative and deactivating emotions, did not get any mark (Fig. 21.10).

21.4.2 The Emotional Murals of Facebook

The presence of digital communication, via Facebook, was evident throughout the Math@XXI gymkhana. This communication was made in various formats and with different purposes. Many posts, amongst the students and with the gymkhana's facilitator, were registered at times that went much beyond the usual schedule of school life. Here, we use the term posts to refer to all communications which Facebook identifies as units, with a specific date and time; one single post may contain references to several events. During Math@XXI, Facebook was also used as a channel for private messages between the students and the facilitator, as well as amongst themselves. There were also several communication processes between the personal profiles of individual students. Such messages were not considered as data sources.

By looking at all eligible Facebook communication situations of all teams, in their closed groups with the facilitator, during the course of the gymkhana, we broke down the teams' posts into (research) units, which were later regrouped, by analogy, into categories. Five categories of Facebook usage, as a communication artefact within Math@XXI, emerged from the data: (i) publication of the gymkhana's tasks (the WebQuests); (ii) dissemination of mathematical culture; (iii) weekly records of various moments and information; (iv) manifestation of students' appropriation of mathematical objects; and (v) expression of emotions (Moreira, 2016; Moreira & Tomás Ferreira, 2016). In this chapter we look at this latter usage of Facebook in the context of Math@XXI, which was one of the most significant usages of Facebook.

The participating students and the facilitator expressed their emotions in different situations of the Math@XXI gymkhana. In fact, unlike the relatively scarce expression of emotions in the WebQuests (i.e. in the journals edited by the participating teams), the Facebook murals of all teams contained several posts in which, in a much more explicit way, students expressed their emotions. All emotions were expressed freely and spontaneously. We must, though, bear in mind that the facilitator might have had some influence in those students' emotional manifestations in the sense that her posts always had an emotional charge.

Most expressed emotions were activating emotions, both positive and negative with a prevalence of the former. For example, the members of the MARRETAS team showed *joy* and *pride* for their achievements within the gymkhana (Fig. 21.11). Almost all members of the team placed likes in this post, the expression "We are the greatest…", followed by several emoticons and expression of agreement attest the presence of such positive and activating emotions.

In Fig. 21.12, the members of the OS MERCENÁRIOS team exhibited *joy* and *satisfaction* for participating of Math@XXI. This picture was taken after one of the many meetings with the gymkhana's facilitator; in particular, the OS MERCENÁRIOS asked the facilitator to photograph the moment in which they submitted one of their journals. These students were happy about their achievements and confident in succeeding in the competition. Such emotional states that we also have mentioned in the previous section (see Figs. 21.8 and 21.9) were recorded in the Facebook murals of all the teams, by the facilitator.

Fig. 21.11 Expression of joy and pride in the mural of the MARRETAS team

Fig. 21.12 Expression of joy and satisfaction for participating in Math@ XXI

Figure 21.13 reveals feelings of *cheerfulness*. Indeed, a member of the 7SETE team indicated "feeling cheerful" due to Pythagoras! In fact, the students in this team were all 7th graders, and Pythagoras (his life and legacy) and the Pythagorean theorem were totally new themes for them. Apparently, the work completed within the gymkhana helped the student in widening his knowledge of mathematics and in improving his grades in a history test. And the same happened with other team mates.

The expression of joyful emotions occurred frequently associated with other activating emotions. We have seen an example of joy associated with pride in Fig. 21.11 and with satisfaction in Fig. 21.12. Figure 21.14 exhibits the manifesta-

Fig. 21.13 Expression of cheerfulness in the mural of the 7SETE team

Fig. 21.14 Expression of activating emotions in the mural of the GENIOUS OF MATH team

tion of *joy*, *fun*, and *anxiety*. A student from the GENIOUS OF MATH team, in a Facebook dialogue with the gymkhana's facilitator, expresses joy for the work already completed, which she refers being fun, as well as some anxiety (in the sense of expectancy) towards the following tournament of the competition. Both emotions are activating, but joy is a positive one and anxiety a negative emotion.

The gymkhana's facilitator published several posts in Facebook to stimulate students' curiosity towards the mathematics around them and important mathematical characters as well. Figure 21.15 depicts an excerpt of the dialogue that followed one of those publications in the mural of TEAM LOPES. Students engaged in a playful dialogue around how they easily found the answer for the question posed: "Who is this man?", exhibiting *fun*, a playful emotional state.

Sometimes, *anxiety* was the main emotional state evident in the posts and dialogues in Facebook. For example, in a dialogue with the gymkhana's facilitator (Fig. 21.16), a member of the MARRETAS team showed her anxiety about knowing her team's scores in the WebQuests. Though being a negative emotion, its activating character reveals the students' interest in the work accomplished in the gymkhana. A negative, activating emotion is associated with a positive attitude.

Fig. 21.15 Playful emotional states in the Facebook mural of TEAM LOPES

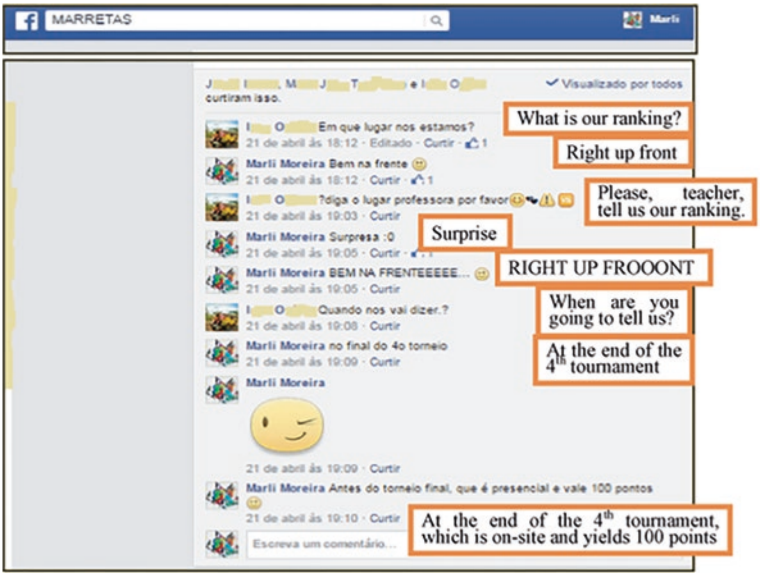


Fig. 21.16 Expression of anxiety in the mural of the MARRETAS team

Perseverance is another positive attitude towards learning mathematics. This attitude was also found in the realm of emotional states expressed in the teams' Facebook murals, as we can see in Fig. 21.17, in which an element of the GENIOUS OF MATH team reveals her, and her team's, perseverance in completing the tasks.

We did not observe the manifestation of negative emotions, which could hinder the participation in the gymkhana, prejudice the mathematical enculturation process

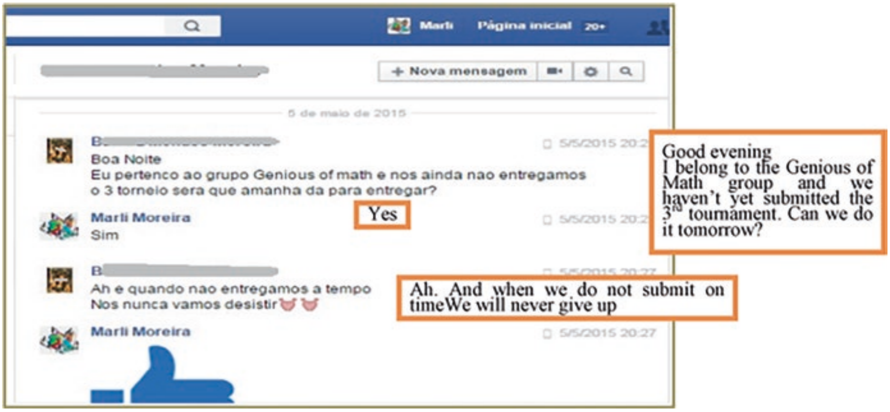


Fig. 21.17 Expression of interest and perseverance in the mural of the GENIOUS OF MATH team

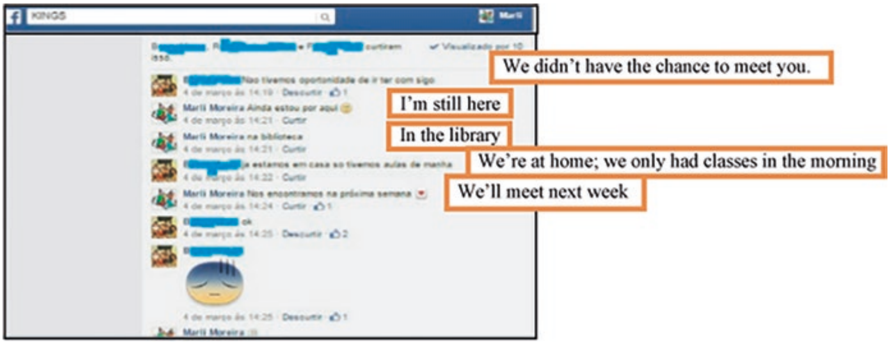


Fig. 21.18 Expression of discontentment in the mural of the KINGS team

of the students, or suggest boredom, helplessness, or rejection of the gymkhana's activities. All instances of negative emotions were associated with students' interest in Math@XXI and engagement or commitment with the tasks. The expressions of anxiety in Figs. 21.14 and 21.16 were all related to students' interest in knowing their scores or to students' enthusiasm in engaging in the following challenges.

Also associated with interest and commitment, there were manifestations of *discontentment* in the Facebook murals of some teams. For example, in Fig. 21.18, the KINGS excuse themselves for failing to have met the gymkhana's facilitator, expressing their dissatisfaction for this failure, both in written form and resorting to emoticons. Again, though negative, discontentment was an activating emotion.

In fact, students' *interest* and *commitment* towards Math@XXI, which we have already highlighted in the previous section, were also recorded in many Facebook posts by the gymkhana's facilitator. For example, in Fig. 21.19, we can observe two Facebook posts revealing students' engagement with the tasks proposed in the

Fig. 21.19 Students' engagement with the tasks proposed in the WebQuests

WebQuests. On the left, some elements of the GENIOUS OF MATH team work together in a task, using their notes and the computer; on the right, the facilitator stresses, with an image, the importance of paying attention to the work that is being done. Thus, Facebook also served as a testimony of students' engagement with the mathematics behind the WebQuests, as well as a witness of the students' mathematics enculturation.

As it can be seen in many of the presented evidences, very frequently, students resorted to emoticons and other figurative symbols that Facebook makes available to its users. Smiling, sad, happy, or cheerful faces, as well as likes and other symbols were often used to express emotions in a pictorial way. We must bear in mind that the usage of such tools, which are part of the affordances of Facebook, is made freely, spontaneously, by the participants in Math@XXI. It is their choice to communicate their emotions.

In Fig. 21.20, we can see the use of such elements to express emotional states. In the same post, some elements of the 7SETE team used gestures to express themselves, manifesting their joy for handing in their journal to the gymkhana's facilitator at the school. Here, Facebook allowed for the combination of several means to express emotions.

Fig. 21.20 Expression of emotions by different means in the mural of the 7SETE team

21.5 Facebook and WebQuests: A Promising Association in the Promotion of Positive Global Affect Towards Mathematics

Facebook was intended to be used as a virtual space for collaborative learning where the main theoretical pillars of Mathematical Enculturation (Bishop, 1991), Activity Theory (Leontiev, 1978), and the interplay between cognition and affect (Damásio, 2011; Gómez-Chacón, 2003) informed and sustained the design of the Math@XXI gymkhana. Associated with the WebQuests that constituted the digital part of this competition, Facebook contributed to the development of several abilities, such as communication, sharing, and collaborative work (Moreira, 2016), which are essential for coping with the demands of the society of the twenty-first century (Allegretti et al., 2012; Costa & Carvalho, 2006).

Technology is transparent to today's youth. Youngsters are used to acting and interacting with many technological devices and platforms, and social networks are no exception. Thus, associating Facebook, which most students utilize in their daily, personal communications, with their work on the WebQuests of Math@XXI, favoured the inclusion of mathematics as a topic of their on-line conversations. This, in turn, made the mathematical culture become nearer their lives.

Indeed, students' opinion about the use of Facebook and WebQuests in the Math@XXI gymkhana was largely favourable. At the end of the competition, they were invited to respond, individually, to express their degree of agreement with two sentences: (S1) *Being able to communicate with my colleagues and the teacher/facilitator through Facebook in the Math@XXI gymkhana was nice*, and (S2) *I enjoyed learning and completing tasks through the digital environment (WebQuests) of the Math@XXI gymkhana*. Figure 21.21 summarizes the results of this part of the survey, which suggest students were highly satisfied with the affordances of both the WebQuests and Facebook, within Math@XXI.

Facebook played different roles in supporting the development of Math@XXI. In a prior study, the least significant role of Facebook was that of exhibiting manifestations of students' appropriation of the mathematical objects that were at the core of the four WebQuests of Math@XXI (Moreira & Tomás Ferreira, 2016). Not surprisingly, those manifestations were necessarily feeble; in fact, phenomena are much more observable in other data sources, namely, the answers to the final tournament and the journals that were the end products of the WebQuests, than in the posts found in the closed groups of the participating teams. Similarly, the expression of emotions was much more present in the Facebook murals of the participating teams, rather than in the productions resulting from the WebQuests. In fact, there were some direct expressions of emotions in the WebQuests, such as the appreciation for mathematics reflected in the cover pages of the journals. But most of the emotions found in students' work on the WebQuests were indirect. Students' interest and commitment towards their work were the most *observable* attitudes, in association with joy and enthusiasm for that work, even on the part of the teams that scored poorly in the digital part of the gymkhana or that dropped out the competition. The characteristics of these two technological tools, WebQuests and Facebook, seem to be related to the spontaneous expression of emotions and to the ways in which those emotions are conveyed. The social dimension of Facebook seems to have favoured the expression of emotions, and it evidenced the social experience of participating in Math@XXI. Further research is needed to better understand how technological tools such as WebQuests and Facebook, in similar and distinct contexts as that of Math@XXI, can promote positive emotions towards mathematics and mathematical activity and make those emotions visible.

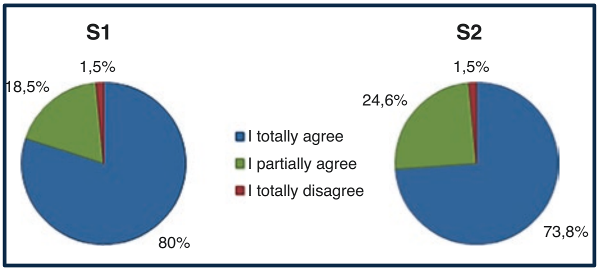


Fig. 21.21 Students' opinions about the use of Facebook and WebQuests in Math@XXI

Using the murals of their Facebook closed groups, students expressed their emotions while participating in Math@XXI in many forms (using direct speech or the tools made available by Facebook, such as emoticons and other figurative symbols) and in many moments. The vast majority of shared emotions were activating emotions (Kleine et al., 2005), like joy, pride, and cheerfulness, but not all expressed emotions were positive. For example, several students expressed some anxiety in finding the time to complete the tasks or in knowing the scores of the competition. Others expressed discontentment for failing their commitments. Nevertheless, such negative emotions revealed students' interest, engagement, and enthusiasm in participating of Math@XXI; thus, though negative, they favoured learning (Neves & Carvalho, 2006).

In many instances, it was possible to observe students' local affective positive states, in Gómez-Chacón's (2003) terms: when they were enthusiastically engaged in completing the WebQuests,[1] when they manifested interest in fulfilling the requirements, and when they showed joy and pride in handing in their journals to the gymkhana's facilitator. Due to limitations in the instruments that were used, the evidences for participants' development of a positive local affect towards mathematics were not very strong – future investigations should look more carefully at how such local affect is developed and, in particular, how technological tools like WebQuests and Facebook favour that development. In this chapter, we have analysed only a small part of the data collected in the context of Math@XXI, which mainly pointed out evidences of the contribution of Math@XXI for students' construction of a positive global affect (Gómez-Chacón, 2003) towards mathematics and its learning.

In particular, the association of WebQuests and Facebook in this gymkhana, with the valences of each one of these two technological tools, showed promise in supporting that construction. Facebook benefits from instantaneity and interactivity, two characteristics that lack in the WebQuests but which favour the expression of emotions. In turn, WebQuests provide an environment that triggers positive and activating emotions, thus contributing to constructing positive global affect towards mathematics. The students' significant usage of Facebook during Math@XXI, their enthusiasm in participating in all activities of the gymkhana (not only in the WebQuests but also in the final tournament), and the several instances in which the students evidenced the necessary self-confidence to succeed in the activities proposed support the claim that the association of Facebook and the WebQuests in the gymkhana helped students in developing a positive global affect towards mathematics. Yet, again, further research is needed, in similar and distinct contexts of that of Math@XXI, to provide stronger evidences for the aforementioned claim and to better understand the processes of developing positive global affect towards mathematics.

[1] The WebQuests may be found in https://sites.google.com/site/matematicxxi/home, https://sites.google.com/site/matematicxxi2/home, https://sites.google.com/site/matematicxxi3/ home, and https://sites.google.com/site/matematicxxi4/home.

Acknowledgement The first author was partially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds (FEDER), under the partnership agreement PT2020. The second author was partially supported by CAPES/Brazil, grant no. 8783/13-6.

References

Allegretti, S., Hessel, A., Hardagh, C., & Silva, J. (2012). Aprendizagem nas redes sociais virtuais: Opotencial da conectividade em dois cenários. *Revista Cet, 1*(2), 53–60 Available at http://revistacontemporaneidadeeducacaoetecnologia02.wordpress.com/24-2/

Barbeau, E., & Taylor, P. (Eds.). (2009). *Challenging mathematics in and beyond the classroom. The 16th ICMI Study*. New York: Springer.

Bishop, A. J. (1991). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bishop, A. J. (2002). Mathematical acculturation, cultural conflict, and transition. In G. de Abreu, A. J. Bishop, & N. C. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 193–212). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.

Costa, F. A., & Carvalho, A. A. A. (2006). WebQuests: Oportunidades para alunos e professores. In A. A. A. Carvalho (Org.) (Ed.), *Actas do Encontro sobre WebQuest* (pp. 8–25). Braga: CIEd.

D'Ambrosio, U. (2009). *Educação Matemática: Da teoria à prática (17a edição)*. Campinas: Papirus.

Damásio, A. (2011). *O erro de Descartes: Emoção, razão e cérebro humano*. Lisboa: Temas e Debates.

De Bellis, V. A., & Goldin, G. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63*(2), 131–147.

Design-Based Research Collective – DBRC. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5–8.

Freiman, V., & Applebaum, M. (2011). Online mathematical competition: Using virtual marathon to challenge promising students and to develop their persistence. *Canadian Journal of Science, Mathematics and Technology Education, 11*(1), 55–66.

Freiman, V., & Véniza, N. (2006). *Challenging virtual mathematical environments: The case of the CAMI Project*. Pre-conference paper of the study conference for ICMI study 16 – Challenging mathematics in and beyond the classroom. Available at http://www.amt.edu.au/icmis16pcan-freiman.pdf

Frenzel, A. C., Pekrun, R., & Goetz, T. (2006). Perceived learning environment and students' emotional experiences: A multilevel analysis of mathematics classrooms. *Learning and Instruction, 17*, 478–493.

Goetz, T., Frenzel, A. C., Hall, N. C., & Pekrun, R. (2008). Antecedents of academic emotions: Testing the internal/external frame of reference model for academic enjoyment. *Contemporary Educational Psychology, 33*, 9–33.

Goldin, G. A. (2002). Affect, meta-affect and mathematical belief structures. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 59–72). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Gómez-Chacón, I. (2000). Affective influences in the knowledge of mathematics. *Educational Studies in Mathematics, 43*, 149–168.

Gómez-Chacón, I. (2003). *Matemática emocional: Os afetos na aprendizagem matemática*. Porto Alegre: Artmed.

Hannula, M. (2002). Attitudes towards mathematics: Emotions, expectations, and values. *Educational Studies in Mathematics, 49*, 25–46.

Hannula, M. (2006). Affect in mathematical thinking and learning: Towards integration of emotion, motivation, and cognition. In J. Maasz & W. Schlöglmann (Eds.), *New mathematics education research and practice* (pp. 209–232). Rotterdam: Sense Publishers.

Kenderov, P., Rejali, A., Bussi, M. G., Pandelieva, V., Richter, K., Maschietto, M., et al. (2009). Challenges beyond the classroom – sources and organizational issues. In E. J. Barbeau & P. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI Study* (pp. 53–96). New York: Springer.

Kleine, M., Goetz, T., Pekrun, R., & Hall, N. (2005). The structure of students' emotions experienced during a mathematical achievement test. *ZDM, 37*(3), 221–225.

Leder, G. C., Pehkonnen, E., & Törner, G. (2002). Setting the scene. In G. C. Leder, E. Pehkonnen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 1–10). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Leontiev, A. N. (1978). *O desenvolvimento do psiquismo*. São Paulo: Editora Moraes.

Maasz, J., & Schloeglmann, W. (Eds.). (2006). *New mathematics education research and practice*. Dordrecht, The Netherlands: Sense Publishers.

McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research in mathematics education, teaching and learning* (pp. 575–596). New York: Macmillan.

Moreira, M. (2016). *Matemátic@XXI: Conexões surpreendentes*. Portugal: Doctoral dissertation, University of Porto.

Moreira, M., & Tomás Ferreira, R. A. (2016). A comunicação matemática com recurso ao Facebook: A experiência na gincana escolar Matemátic@XXI. In M. H. Martinho, R. A. Tomás Ferreira, I. Vale, & H. Guimarães (Eds.), *Atas do XXVII Seminário de Investigação em Educação Matemática* (pp. 249–273). Porto: APM.

Neves, M., & Carvalho, C. A. (2006). A importância da afetividade na aprendizagem da matemática em contexto escolar: Um estudo de caso com alunos do 8° ano. *Análise Psicológica, 24*(2), 201–215.

Ponte, J. P. (2005). Gestão curricular em matemática. In Grupo de Trabalho em Investigação (GTI) (Org.) (Ed.), *O professor e o desenvolvimento curricular* (pp. 11–34). Lisboa: Associação de Professores de Matemática.

Stockton, J. C. (2012). Mathematical competitions in Hungary: Promoting a tradition of excellence & creativity. *The Mathematics Enthusiast, 9*(1–2), 37–58.

Walshaw, M., & Brown, T. (2012). Affective productions of mathematical experience. *Educational Studies in Mathematics, 80*(1–2), 185–199.

Wang, F., & Hannafin, M. J. (2005). Design-based research and technology-enhanced learning environments. *Educational Technology Research & Development, 53*(4), 5–23.

Zan, R., Brown, L., Evans, J., & Hannula, M. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics, 63*(2), 113–121.

Chapter 22
On the Edges of Flow: Student Problem-Solving Behavior

Peter Liljedahl

22.1 Flow and the Optimal Experience

In the early 1970s, Mihály Csíkszentmihályi became interested in studying, what he referred to as, the *optimal experience* (1990, 1996, 1998):

> a state in which people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it. (Csíkszentmihályi, 1990, p. 4)

The optimal experience is something we are all familiar with. It is that moment where we are so focused and so absorbed in an activity that we lose all track of time, we are un-distractible, and we are consumed by the enjoyment of the activity. As educators we have glimpses of this in our teaching and value it when we see it.

Csíkszentmihályi, in his pursuit to understand the optimal experience, studied this phenomenon across a wide and diverse set of contexts (1990, 1996, 1998). In particular, he looked at the phenomenon among musicians, artists, mathematicians, scientists, and athletes. Out of this research emerged a set of elements common to every such experience (Csíkszentmihályi, 1990):

1. There are clear goals every step of the way.
2. There is immediate feedback to one's actions.
3. There is a balance between challenges and skills.
4. Action and awareness are merged.
5. Distractions are excluded from consciousness.
6. There is no worry of failure.
7. Self-consciousness disappears.
8. The sense of time becomes distorted.

P. Liljedahl (✉)
Simon Fraser University, Burnaby, BC, Canada
e-mail: liljedahl@sfu.ca

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_22

9. The activity becomes an end in itself.

The last six elements on this list are characteristics of the internal experience of the doer. That is, in describing an optimal experience, a doer would claim that their sense of time had become distorted, that they were not easily distracted, and that they were not worried about failure. They would also describe a state in which their awareness of their actions faded from their attention and, as such, they were not self-conscious about what they were doing. Finally, they would say that the value in the process was in the doing – that the activity becomes an end in itself.

In contrast, the first three elements on this list can be seen as characteristics external to the doer, existing in the environment of the activity, and crucial to occasioning of the optimal experience. The doer must be in an environment wherein there are clear goals and immediate feedback and there is a balance between the challenge of the activity and the abilities of the doer.

This balance between challenge and ability is central to Csíkszentmihályi's analysis of the optimal experience (1990, 1996, 1998) and comes into sharp focus when we consider the consequences of having an imbalance in this system. For example, if the challenge of the activity far exceeds a person's ability, they are likely to experience a feeling of frustration. Conversely, if their ability far exceeds the challenge offered by the activity, they are apt to become bored.[1] When there is a balance in this system, a state of, what Csíkszentmihályi refers to as, *flow* is created (see Fig. 22.1). Flow is, in brief, the term Csíkszentmihályi used to encapsulate the essence of optimal experience and the nine aforementioned elements into a single emotional-cognitive construct.

22.1.1 Flow in Mathematics Education

Flow is one of the only ways for us, as mathematics education researchers, to talk productively about the phenomenon of engagement. The nine aforementioned elements of flow give us not only a vocabulary for talking about aspects of the subjective personal experience of engagement, but it also gives us a way to think about the potential environments that occasion engagement in our classrooms.

Having said that, not much empirical work has been done at the intersection of flow and mathematics education. Williams (2001) used Csíkszentmihályi's idea of flow and applied it to a specific instance of problem-solving that she refers to as discovered complexity. Discovered complexity is a state that occurs when a problem-solver, or a group of problem-solvers, encounters complexities that were not evident at the onset of the task and are within their zone of proximal development (Vygotsky, 1978). This occurs when the solver(s) "spontaneously formulate a question (intellectual challenge) that is resolved as they work with unfamiliar mathematical ideas" (p. 378). Such an encounter will capture, and hold, the engagement

[1] It is important to note that *frustration* and *boredom* are names that Csíkszentmihályi (1990, 1996, 1998) assigned to these states where there is an imbalance between ability and challenge. As such, for Csíkszentmihályi, *frustration* and *boredom* are defined by these imbalances.

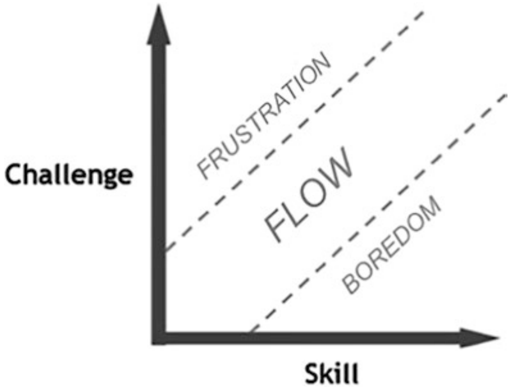


Fig. 22.1 Graphical representation of the balance between challenge and skill

of the problem-solver(s) in a way that satisfies the conditions of flow. What Williams' framework describes is the deep engagement that is sometimes observed in students working on a problem-solving task during a single problem-solving session.

Extending this work, I used the notion of flow to look at situations of engagement extended over several days or weeks wherein students return to the same task, again and again, until a problem is solved (Liljedahl, 2006). The results of this work showed that although flow was present in each of the discrete problem-solving encounters, what allowed the engagement to sustain itself across multiple encounters was a series of discovered complexities in each session linking together to form what I referred to as a *chain of discovery*.

22.1.2 Flow as a Framework for Describing Teaching

In prior research (Liljedahl, 2016a), I looked at the practices of two teachers through the lens of flow[2] in general and their ability to set clear goals, provide instant feedback, and maintain a balance between challenge and skill in particular. From this a number of conclusions emerged. First, thinking about flow as existing in that balance between skill and challenge, as represented in Fig. 22.1, obfuscates the fact that this is not a static relationship. Flow is not the range of fixed ability-challenge pairings wherein the difference between skill and challenge is within some acceptable range. Flow is, in fact, a dynamic process. As students engage in an activity, their skills will, invariably, improve. In order for these students to stay in flow, the challenge of the task must similarly increase (see Fig. 22.2).

In a mathematics classroom, these timely increases of challenge often fall to the teacher. But this is not without obstacles. For example, if a student's skill

[2] I am not the first to do this. Schmidt et al. (1996) used flow as a framework for investigating mathematics and science teaching across six countries – France, Japan, Norway, Spain, Switzerland, and the United States – between 1991 and 1995.

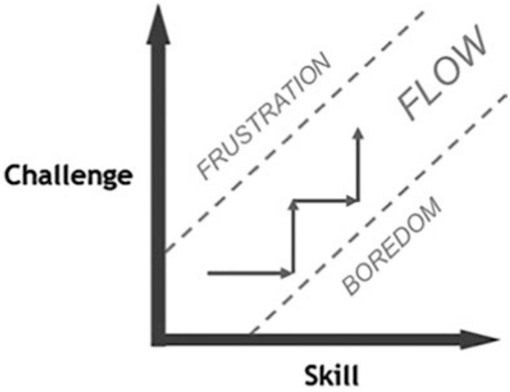


Fig. 22.2 Graphical representation of the balance between challenge and skill as a dynamic process

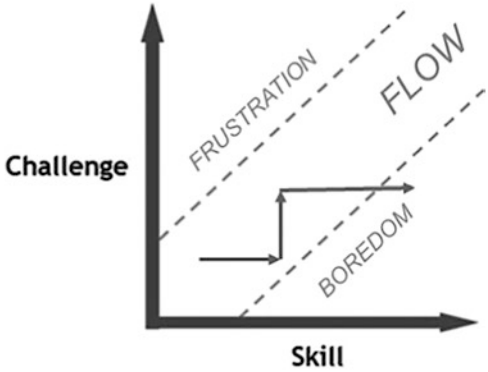


Fig. 22.3 Too fast increase in skill

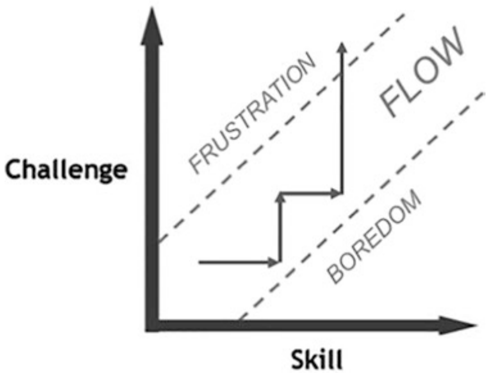


Fig. 22.4 Too great increase in challenge

increases either too quickly or too covertly for the teacher to notice, that student may slip into a state of boredom (see Fig. 22.3). Likewise, when the teacher does increase the challenge, if that increase is too great, the student may become frustrated (see Fig. 22.4).

This leads to the second conclusion from the aforementioned research (Liljedahl, 2016a). How teachers manage these situations of boredom and frustration is important. In the study one of the teachers managed such situations synchronously, either giving extensions or hints to the class as a whole, usually after three groups finished, or she got three of the same questions, respectively. For most groups the timing of these hints and extensions was off and not helpful in maintaining flow. The second teacher, however, managed these situations asynchronously, dealing with groups individually as they got stuck or completed a problem. Student engagement in the second teacher's class was visibly higher as he was maintaining flow through the constant and timely maintenance of the balance between ability and complexity for each group. In short, timing matters.

22.1.3 The Next Stage

Given the importance of timing, the reality is that the teacher is not always going to be able to get to every student or every group just as an imbalance between challenge and skill occurs. In the research presented in this chapter, I look more closely at this phenomenon in general and how students cope autonomously with an imbalance between their skills and the challenge of the task at hand.

22.2 Methodology

To answer this question, I observed student collaborative and individual problem-solving work within two carefully selected classrooms. In what follows I describe this setting as well as the methods I use to capture and analyze the data.

22.2.1 The Problem-Solving Setting

To get at the answers to this question, I needed to observe students in settings that were natural to them and where their work was visible. To this end I strategically selected two classrooms belonging to two different teachers – Cameron and Charmaine. Both of these teachers conducted their classrooms according to a teaching framework called *building thinking classrooms* (Liljedahl, 2014, 2016b, 2016c).[3]

This framework is predicated on a desire to design "a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and

[3] The second teacher in the prior research study (Liljedahl, 2016a) was also teaching according to this framework.

constructing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly" (Liljedahl, 2016b, p. 364).

My earlier empirical work on the design of such spaces emerged a collection of nine elements that both describes a thinking classroom and offers a prescriptive framework for teachers to building such a classroom. For both Cameron and Charmaine, five of these elements are particularly salient for describing their classroom norms (Yackel & Rasmussen, 2002).

1. At the beginning of every class, students are assigned to a visibly random group (Liljedahl, 2014, 2016b, 2016c) of two to four students. These groups will work together on assigned problem-solving tasks for the duration of the lesson.
2. Once in groups, the lessons begin with the assignment of tasks to be solved. In the beginning of the school year, these tasks are highly engaging, non-curricular, collaborative tasks that drive students to want to talk with each other as they try to solve them (Liljedahl, 2008). After a period of time (usually 1–2 weeks), these are gradually replaced with curricular problem-solving tasks that permeate the entirety of the lesson and emerge rich mathematics (Schoenfeld, 1985) that can be linked to the curriculum content to be "taught" that day.[4]
3. The work on these aforementioned problem-solving activities is done with groups standing on vertical nonpermanent surfaces such as whiteboards, blackboards, or windows (Liljedahl, 2016b, 2016c). This makes visible all work being done, not just to the teacher but to the groups doing the work. To facilitate discussion, there is only one felt pen or piece of chalk per group.
4. Throughout this work, student engagement is maintained through the teacher's judicious and timely use of hints and extensions (Liljedahl, 2016a, 2016b). Csíkszentmihályi's theory of flow (1990, 1996, 1998) is the framework for thinking about this. Hints and extensions need to be given so as to keep students in a perfect balance between the challenge of the current task and their abilities in working on it.
5. At some point within this sequence of tasks, the teacher will pull the students together to debrief what they have been doing. At this time the teacher will either go over one or more of the students' solutions or work through a new problem together with the class as a whole. This helps reify the work the students have been doing and is timed so that every group is able to participate in discussion and benefit from the reification. In the *building thinking classroom* framework, this activity is called *leveling to the bottom* (Liljedahl, 2016b).[5]

[4] Although the curricula tasks are simply questions from the textbook, I characterize them as problem-solving tasks because they are often new to the students, present something that is problematic for them, and often cause them to be stuck. However, as will be seen in the presentation of results, in some cases the tasks become rudimentary for the students.

[5] The term *leveling* originally comes from Schoenfled (1985) and refers to that moment when a teacher will go over the solution to a problem or exercise students have been working on. *Leveling to the bottom* specifies when that leveling is to occur.

Both Cameron's and Charmaine's classrooms are guided by these principles of teaching. This is not to say that their classrooms are identical. Charmaine, for example, makes regular use of non-curricular tasks for the beginning part of her lesson (two to three times a week). She finds these motivate the students. Charmaine *levels to the bottom* with about 25–30 min left in class after which she assigns five to eight questions for students to check their understanding if they wish. Generally, the students take advantage of the time allotted (usually 20 min) for this to complete them in class. About half of the students do so in self-identified groups working at a board. The other half of the class completes these sitting in their desks either on their own or in small self-determined groups.

Cameron, on the other hand, tends to use only curricular problems. And rather than assign practice questions, he just keeps the students working on progressively harder curricular problems in their random groups for the whole period. He feels that this is sufficient practice for the students and feels no need to assign further work for them to do. Cameron tends to *level to the bottom* twice in every lesson – once after all groups have gotten through two to three problems and once at the very end of the lesson.

For the purposes of the research presented here, both of these classrooms offered the affordances for me to easily observe students working within and an environment designed to occasion flow. The teachers were both managing engagement through the timely use of hints and extensions to maintain a balance between the challenge of an activity and the ability of each group. The student work was visible, and there was enough autonomy afforded in the room that the students would need to take action when they found themselves in a situation where challenge and ability may be out of balance.

22.2.2 The Data

Data for the research presented in this classroom were collected in Cameron's grade 12 precalculus class and Charmaine's grade 11 precalculus class.[6] Each class was visited five times over a 7-week period in the middle of the second semester.[7]

When doing the aforementioned research on teachers' actions vis-à-vis their efforts to occasion flow (Liljedahl, 2016a), I used video recorded data. This was a

[6] In the province where this research was conducted, these are the academic streams of mathematics laddering toward university-level calculus. Grade 11 students are typically 16 or 17 years old, and grade 12 students are typically 17 or 18 years old.

[7] In the province where this research was done, schools are either semestered or linear. Linear schools have students taking eight classes on a rotation schedule over the course of the whole year. Semestered schools have students taking the same four classes every day for the first half of the school year (first semester) and another four classes every day for the second half of the school year (second semester).

natural fit as I had only one subject to track. For the research presented here, however, video turned out to be prohibitive for two reasons. First, as I was looking for very specific situations within the classroom, I needed to be constantly monitoring student activity around the room. Trying to do so through a digital video recorder narrowed my field of view too much to quickly scan what was emerging around the room. As such, I used instead a variant of the methodology of noticing (Jacobs, Lamb, & Philipp, 2010; Mason, 2002, 2011; van Es, 2011). Rather than scanning the room for emergent phenomena, however, I was looking for one of two very particular phenomena – situations where a group of individual student's abilities exceeded the challenge of the task (see Fig. 22.3) and the challenge of the task exceeded the ability of the group or individual student (see Fig. 22.4). I call these *moments of imbalance*.

Once a *moment of imbalance* was identified, I would focus in on that group or that individual. I initially tried video recording these instances of imbalance, but this was too intrusive for the students. I learned early that these instances are very delicate and the slightest disturbance would collapse the moment. Instead, I would observe silently, taking detailed field notes and occasional photographs – both of which the students were used to having their teachers do as a regular part of their teaching. When these moments seemed to be waning, I would then conduct short in-the-moment interviews.

There is an efficiency in this methodology in that all of the data that was generated was relevant to the phenomenon of interest.

All three of these forms of data (field notes, photographs, and audio recordings) were accomplished using an iPad app called Notability™. This app not only allows for the simultaneous taking of handwritten notes, photographs, and audio recordings but synchronizes the artifacts of notes and photographs with the audio track. That is, upon playback the selection of any one note or photo will cue the audio to the instance when those artifacts were created.

22.2.3 Analysis of Data

Csíkszentmihályi's theory of flow (1990, 1996, 1998) in general and the imbalance between student ability and task challenge in particular were central to the identification of moments of interest. The analysis of these moments, however, was focused more on students' actions and reactions in these moments of imbalance. To this end, the data were also analyzed using analytic deduction (Patton, 2002). That is, I was looking for the emergence of codes through a constant comparative method in general and the emergence of similar behaviors under similar moments of imbalance in particular.

22.3 Results and Analysis

From this analysis a series of six nuanced moments of imbalance emerged, each marked by a different type of student action or reaction. In what follows I present cases exemplifying each of these moments as well as some general comments about similar such cases. These six nuanced moments of imbalanced are broken into two main categories – moment wherein the skill of the students exceeds the challenge of the task at hand and moments wherein the challenge of the task at hand exceeds the skills of the students.

22.3.1 When Skills Exceed Challenge

There were three distinct types of reactions by individual or groups of students when faced with a situation wherein their skills exceeded the challenge of the task at hand. Contrary to Csíkszentmihályi's theory of flow (1990, 1996, 1998), however, not all of these resulted in boredom.

22.3.1.1 The Case of Quitting

Within the data there were several moments wherein students quit. A portion of these cases were a result of the students being bored with the activity at hand. To exemplify this, I present the case of Mikaela and Allison, two students in Charmaine's class.

As mentioned, at the end of each lesson, Charmain assigned five to eight questions for the students to work on with whomever they wished and on whatever surface they wished. During the five lessons I observed in Charmaine's class, I noticed that during this phase of the lesson about half of the students chose to work at their desks and half of the student chose to work on the vertical spaces. At each location, a few students worked on their own, while the rest worked in groups of two or three. At the end of one of these lessons I observed in Charmaine's class, my attention was drawn to Mikaela and Allison.

These two girls had not been in the same group during the random grouping during the main part of the lesson, but for this portion of the class, they had chosen to work together, and they had chosen to do so sitting down at their desks. What drew my attention to them was that they seemed to be off task, animatedly talking about something not related to mathematics. As I start to attend to them, I notice that they had finished the first two questions assigned to them. After some period of non-mathematical discussion Mikaela says, "Ok. Let's do the next one."

The two girls looked at the textbook they were sharing, wrote down the third question, and then independently solved it quickly and without difficulty. They compared answers and then resumed their previous off task conversation. I lingered behind them waiting to see if one or the other would prompt the other to do the next question. After 10 min without a return to the mathematics assigned, I decide to interview the pair.

Researcher	I notice you are not working on the assigned questions. What's up?
Mikaela	We did some of them.
Researcher	I saw that. I noticed that you did two very quickly. Took a little break from the math and then went back and did another one. I was sort of waiting to see if you would get back to it.
Allison	This stuff is easy. I'll finish it at home on my own.
Mikaela	Its actually too easy. I don't even think I will bother finishing it at home.
Allison	…Yeah. I probably won't either.
Researcher	It's easy? Is that why you stopped working on it?
Mikaela	Yeah.
Researcher	I saw you two work together before at the end of class. I don't recall seeing you two giving up before.
Allison	We aren't giving up. Sometimes we don't finish all the questions because they are hard and we run out of time. But these are easy.
Researcher	What makes them easy?
Mikaela	They just are. The first three are exactly the same and we could do them no problem.

In my conversation with Charmaine after class, she confirmed for me that these girls often work together but was surprised when I told her about my conversation with them.

Charmaine	Allison and Mikaela always work together at the end of class and they are usually very diligent. I'm surprised that they were off task. That's not their style.

During the ten lessons I observed in Cameron's and Charmaine's classes, I only managed to capture three other instances that I would say fall into the same category – quitting because the students were bored by seemingly too easy collection of tasks. Each of these cases occurred in Charmaine's class, occurred during the last phase of the class where students worked on questions out of the text, involved students who were sitting down, and involved students who were not normally off task. And in each case, the students perceived the questions they were working on to either be easy or redundant. I say perceived because in the case of Mikaela and Allison, for example, although the first three questions were easy and redundant, the next three questions were not redundant and were quite a bit more challenging.

22.3.1.2 The Case of Seeking Increased Challenge

Quitting out of boredom was not the only reaction to a situation where the skills of a group or of an individual exceeded the challenge of the task at hand. Some students opted, instead, to autonomously seek increased challenge. To exemplify this, I look at a case from Cameron's class captured while students were working at the whiteboards in randomly assigned groups. During this part of the lesson, Cameron was circulating the room helping groups that were stuck (or had made a mistake) and giving more challenging questions to groups that were done. Cameron is very deliberate about how he does this. Before a group can get his help or the next question, he engages the group in conversation to assess where the group's thinking is at – both as a collective and individually. This takes time, and sometimes groups that are done are left waiting.

During one of my visits to his class, my attention was drawn to a group of three boys – Carl, Ameer, and Colton. What I had noticed about them was that these three boys were smoothly moving through all of the questions in Cameron's repertoire and they were doing so without Cameron having once visited them to give them the next question. What they were doing was pulling the questions from the visible work of groups that Cameron had visited and given the next question to. I watched them do this for 30 min during which I began to discern the modus operandi of the group.

In essence, this group of boys used the visible work of others around them to not only pull new questions but to also check solutions to questions they had already solved. Sometimes they did this remotely, just through observation. Other times, especially when answers didn't match, they engaged their peers in discussions.

After 30 min of watching them work like this, I decide to ask them about what they were doing.

Researcher	So, I notice that you guys are now on question 5 and your teacher has not visited you once. How are you getting your questions?
Ameer	We just look around and see what the next question is and do that one.
Researcher	What would your teacher say about that?
Carl	Um…he'd probably want to check to see that we got the previous one before giving us the next one…
Ameer	…but we are doing that.
Researcher	Why don't you just wait for your teacher to get here and give you the next question?
Carl	We're on a roll. And sometimes we have to wait a long time.
Researcher	Do you realize that you are doing the problems out of sequence from the order your teacher is giving them?
Colton	Oh really? That's probably why some were so hard.

It turns out that this was a very common behavior in both Cameron's and Charmaine's classrooms. Rather than wait for their teacher to give them the next questions, groups were opting, instead, to move on their own. For the most part, they did this just by pulling the next question from groups that were ahead of them. This was facilitated by the visible nature of the work afforded by the vertical non-permanent surfaces that the classes were working on.

Related to this behavior, but much less common, was the phenomenon of students creating their own extensions to problems they had been working on. In the ten lessons I observed, I only saw this happen twice. In both situations the groups made a change to the problem they had just solved. In one of these cases, the group did this every time they were waiting for the teacher. It could be argued that this is a form of problem posing (Brown & Walters, 1983), but my sense was that the students were more trying to anticipate the teacher's next type of question than pursue a curiosity.

22.3.1.3 The Case of Tolerance in the Face of the Mundane

An altogether different reaction to being tasked with doing easy and redundant questions is to just do them – without quitting and without seeking to increase the challenge. I observed such behavior in the case of Jennifer, who always worked at her desk on her own at the end of Charmaine's lessons.

What drew my attention to Jennifer was that she seemed to be moving through the questions with great easy, never asking for help. Upon closer inspection I also noticed that, in addition to the assigned questions, she was also completing questions that were not assigned. Toward the end of one of the class, I asked her about this.

Researcher	I have been watching you while I have been here. I notice that you always do a lot of questions. Can you tell me about that?
Jennifer	Yeah. I like to do a lot of questions. It's good practice. It's how I learn.
Researcher	So, are you looking for harder and harder questions to challenge yourself?
Jennifer	Not really. I just do all of them. So, if the teacher asks us to do 4a, I will also do 4bc and d and so on.
Researcher	Do you find them easy?
Jennifer	Yeah.
Researcher	How many do you do?
Jennifer	I just work the whole time at the end of class and then for maybe an hour at home.

Jennifer uses practice as a way to ensure that she is learning the content of the day. Unlike the students who quit in the face of boredom, Jennifer seems to be perfectly content working on questions that she considers to be easy for long periods of time. Her tolerance for the mundane is high.

In my time in Charmaine's class, I saw two other girls who I suspect were very much like Jennifer in their approach to learning and their tolerance for the mundane.

These girls also worked alone in their desks in the last part of every lesson. I also observed similar behaviors in a pair of boys working at a whiteboard toward the end of one of Charmaine's lessons.

From my conversations with Charmaine, I learned that these boys, Kirk and Philip, always worked together on a whiteboard to complete the end of class assignment. In the third lesson, I observed I saw them exhibiting much of the same behavior I had seen in Jennifer and the other two girls. That is, they were doing questions beyond what was assigned.

Researcher	I don't think you were asked to do that question?
Kirk	We know. Sometimes it is good to check that you really know what you are doing by doing a few more just like it.
Researcher	Do you always do that?
Kirk	Not always. Usually only if we are not sure.

Like Jennifer, Kirk and Philip are staying within equally challenging tasks to build up their understanding of the mathematics. Unlike Jennifer, however, they seem to be doing so as a way to continue to build their understanding, as opposed to just practicing.

22.3.2 When Challenge Exceeds Skills

There were three distinct types of reactions by individual or groups of students when faced with a situation wherein the challenge of the task at hand exceeded their skills. According to Csíkszentmihályi's theory of flow (1990, 1996, 1998), such an imbalance should result in frustration. This was rarely the case, however.

22.3.2.1 The Case of Quitting

Of course, frustration is a likely result of such an imbalance, and as such, I was on the lookout for such a reaction. I found it in three students in Cameron's classroom work on the first question of the day.

Sometimes at the beginning of a lesson, both Cameron and Charmaine begin with a small lesson or a short example. But they both claim that they do this rarely and only in situations where the lesson pertains to entirely new content. As a result, most lessons begin with groups being tasked with the solving of a problem. Often, these problems are similar to questions from a previous lesson (not necessarily the lesson immediately before) or a small extension from a question encountered previously. During one of my visits to Cameron's class, I observed a group of three students – Shannon, Katrina, and Robert – who seemed to be lost. After about 3 min, they became quite exasperated and quit.

Researcher	I have been watching your group for a bit and I notice that you aren't working?
Robert	We gave up. This question is stupid.
Katrina	We tried, but we weren't getting anywhere. So we gave up.
Researcher	What do you think the problem is?
Shannon	This question is too hard.
Robert	…too hard. We don't get it.
Katrina	And the teacher hasn't come over to help us.
Researcher	What kind of help are you looking for?
Shannon	You know, a hint or something.
Researcher	What would a hint do for you?
Shannon	Help us understand the question.
Katrina	…or remind us a little bit about how to do it.

For this group the question they have been asked to solve exceeded their abilities, and without any help from the teacher they gave up. Interestingly, the type of help they were seeking was either to reduce the complexity of the task (*understand the question*) or increase their ability (*remind us* of what we have done in the past).

Surprisingly, in the ten lessons I observed, I only managed to capture a total of four instances of a group giving up. In each of these cases, there was visible exasperation and disengagement present as well as a marked lack of progress on the task. And in each of these cases, the groups claimed that the task was too difficult for them.

I also managed to capture two cases of individual students, working on questions at the end of one of Charmaine's lessons, giving up. These were harder to capture as there were many instances of students not working during this autonomous time. As expected, during such freedom some students will choose to not do anything. Most of these students told me that they were taking a break or planning to work on these questions at home. Only two students admitted to me that the questions were too difficult and they were giving up, with one of these stating, "I'll just do them at home with my tutor.".

22.3.2.2 The Case of Getting Help

A much more common reaction to facing too great challenge was for students in both classes to seek help. What this looked like, however, was much more subtle than simply calling on the teacher. Like the case of students autonomously seeking increased challenge, many groups who were stuck sought help from the groups around them. This is nicely exemplified by the random group of Mikaela, Lena, and Michael working on a question in Charmaine's class.

I watched this particular group for an entire class. What stands out from these observations was how much they interacted with the groups around them, both passively and actively. This interaction fell into two main categories – checking answers and getting ideas.

Checking answers involved passively looking around the room and seeing if any other groups had arrived at the same answer as them. This became much more active if they saw an answer that differed from theirs. This happened twice during the lesson I observed. In the first instance, they had a quick conversation with the group next to them. This resulted in their neighboring group changing their answer. In the second instance, it involved them crossing the classroom to another group and having a lengthy discussion and then together redoing the problem as a group of six and arriving at an answer that they were all happy with.

Similarly, getting ideas passively involved one or more members of the group looking around the room at other groups' work. Interestingly, Lena did this for every problem regardless if they were stuck or not. A much more active approach was used on their last question together. For this task, they were stuck for quite a while, and Lena's scanning of the class had not helped the group move forward. At this point Mikaela walked over to Allison's group[8] and asked her if she knew what to do. Allison's group was making good progress and was confident about the direction they were going in, but Mikaela didn't understand what they were doing. Eventually, Lena and Michael joined Mikaela in trying to understand what was happening. Michael quickly caught on and urged both Lena and Mikaela to return to their station where he explained it to them.

After 40 min of observing this group, I interviewed them.

Researcher	I notice that you have been moving about the room a bit. Why?
Michael	Oh. We were just stuck so we went over there to get some ideas.
Researcher	Did it help?
Michael	Oh yeah. We got it now.
Mikaela	Michael got it. It took me a little longer, but I'm good now.
Researcher	You were also moving around a little bit earlier in the class?
Mikaela	Oh, you mean when we were checking answers? Yeah, we thought we were doing something wrong, but we were good.
Researcher	Lena, you like to look around a lot.
Lena	I do? What do you mean?
Researcher	You know, when Michael was working on the board you look around a lot at the other groups.
Lena	Right. I am just making sure we are on the right track.

This sort of behavior was endemic in both classrooms with too many occurrences for me to track. The vertical work spaces facilitated the ability for groups to passively check their answers and get ideas. The random groups created the porosity (Liljedahl, 2014) that made the more active interactions and movement of ideas possible. Together it meant that groups were never helpless if they were stuck – thereby avoiding becoming frustrated.

[8] Recall that Mikaela and Allison regularly worked together when given the opportunity.

22.3.2.3 The Case of Perseverance in the Face of Challenge

But not all groups sought help when they were stuck. In the ten lessons I observed, I captured a few instances where a group or an individual opted to not seek help, neither passively nor actively, from the groups around them. One such group was a pair of boys working on the assignments at the end of one of Charmaine's classes. These boys, Oliver and Connor, worked persistently on one of the questions for 15 min without making progress. Even when Charmaine approached them, they resisted her offering of help. At the end of class, I asked them about this question.

Researcher	Question #5 was a tough one, huh?
Oliver	Yeah, that one took us a while.
Connor	In the end it wasn't that hard though. We were just missing something.
Researcher	Oh really. How did you figure it out?
Connor	We just kept at it and then we saw it.
Researcher	I noticed that your teacher came over to help. Did she help you?
Oliver	No, we wouldn't let her. We knew we knew how to do it and we wanted to figure it out ourselves.

I observed similar behavior in Stephanie, who worked on the assignment by herself at a whiteboard. She spent 10 min on the same question before moving on to other questions and then returned several times to try it again. In the end she never did solve it in class, but she never sought any form of help from either the people around her or the teacher. I spoke to her as the bell rang.

Researcher	Did you ever get it?
Stephanie	7a? Not yet. I'll work on it at home.
Researcher	Until you get it?
Stephanie	Until I get it.
Researcher	What if you don't. Will you get some help?
Stephanie	I always get it eventually.

Stephanie showed great perseverance with this task. From the interview it seems like this is a normal occurrence that she is comfortable with. Her confidence in that she will eventually solve it indicates that she is willing to persevere for long periods of time.

There were no such occurrences in Cameron's class. I suspect this is because there were no times in Cameron's lesson where the students were not immersed in an environment saturated with potential help afforded by the vertical spaces and random groups. This is not to say that the students in Cameron's class were not capable of such perseverance but only that Charmaine's class offered an opportunity for me to observe such perseverance.

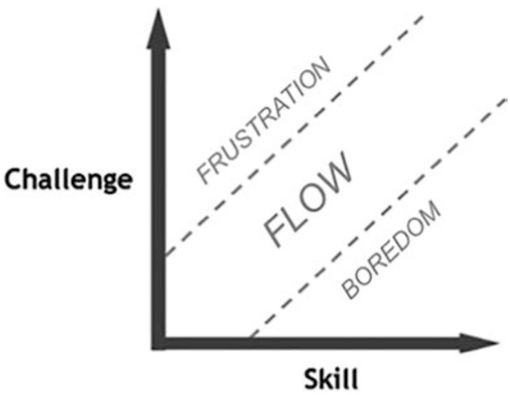
22.4 Discussion

The aforementioned six nuanced moments of imbalance show that for different individuals and different groups the transitions from flow to boredom or frustration have variable immediacy. Mikaela and Allison became bored and got off task as soon as their groups abilities exceeded the challenge of the task at hand. Similarly, Shannon, Katrina, and Robert became frustrated and gave up as soon as the challenge exceeded their ability. For these two groups and the groups and individuals who reacted similarly to an imbalance between ability and challenge, Csíkszentmihályi's original representation of flow holds (see Fig. 22.5).

For Jennifer and Stephanie, this transition was not as abrupt. Jennifer spent long periods of time within a space where her ability far exceeded the challenge posed by the tasks she was working on without getting off task or quitting. The groups and individuals like Jennifer had, what I have come to call, a *tolerance* for the mundane that prevented them from sliding into boredom. Likewise, Stephanie worked persistently without giving up on a task that presented a too great challenge for her ability. Groups and individuals who demonstrated the same tenacity had, what I have come to call, *perseverance* in the face of challenge that prevented them from becoming frustrated. Taken together, these two cases, and the cases like them, indicate that for some students the boundary between flow and boredom and frustration is not as thin as Csíkszentmihályi's (1990, 1996, 1998) theory of flow would imply (see Fig. 22.6).

Other students used the buffer created by perseverance and tolerance to avoid frustration or boredom as they sought to correct the imbalance between skill and challenge that they were experiencing. Carl, Ameer, and Colton used the groups around them to passively and actively check their own answers and to seek out more challenging tasks when they were done. Similarly, Mikaela, Lena, and Michael used the groups around them to both passively and actively access help when they were stuck. These groups, and the groups and individuals like them, managed to autonomously maintain the balance between challenge and ability. When their ability was

Fig. 22.5 Csíkszentmihályi's representation of the balance between challenge and skill



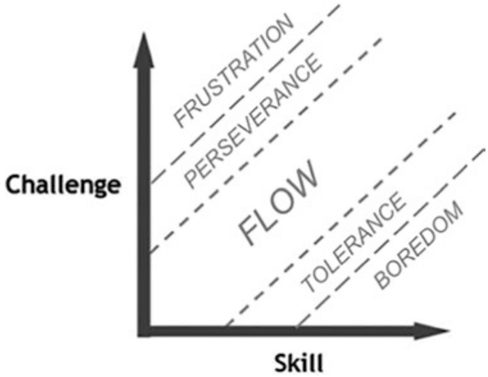


Fig. 22.6 Modified representation of the balance between challenge and skill

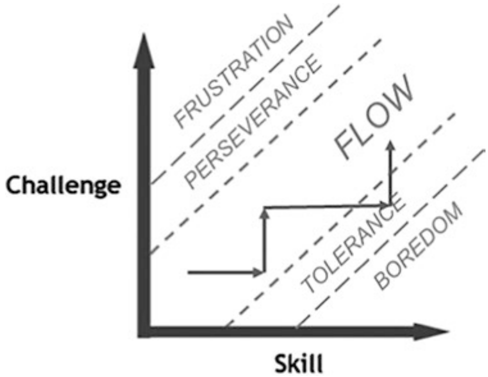


Fig. 22.7 Reaction to a too great ability

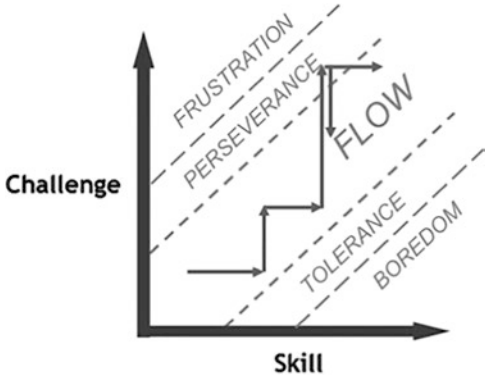


Fig. 22.8 Reaction to a too great challenge

too great, they autonomously sought to increase the challenge (see Fig. 22.7), and when the challenge was becoming too great, they autonomously sought to increase their ability or decrease the challenge (see Fig. 22.8).

To what degree these groups had tolerance or perseverance was not evident from the data as they too quickly managed to right the balance between skill and challenge. What was evident, however, was that when there was an imbalance, these groups did not quit out of boredom or give up out of frustration. The highly visible and collaborative environments created by the use of vertical nonpermanent surfaces and visibly random groups, no doubt, facilitated the management of this balance.

22.5 Conclusions

I began the research presented here as an extension of research wherein I used Csíkszentmihályi's theory of flow (1990, 1996, 1998) as a lens for looking at how teachers used hints and extensions to create and maintain engagement in their classrooms. This prior research showed that flow, as articulated by Csíkszentmihályi (1990, 1996, 1998), is a usefull lens for describing effective and ineffective teachers' actions and reactions to student work.

In the research presented here, however, Csíkszentmihályi's theory of flow (1990, 1996, 1998) was insufficient for predicting the majority of student reactions when faced with an imbalance between challenge and skill. Although some students did quit out of boredom and frustration, very few did so. Instead, they showed resilience to these kinds of imbalances in the form of either perseverance in the face of challenge or tolerance in the face of the mundane. And they often used this resilience as a buffer while they autonomously corrected these imbalances by actively and passively seeking help or increased challenges.

To a great degree, this resilience was facilitated by the collaborative and visible structures created by the participating teachers' adherence to the building thinking classrooms framework of teaching (Liljedahl, 2014, 2016b, 2016c). These structures filled the space with opportunities to either access help or increase the challenge as needed.

References

Brown, S., & Walter, M. (1983). *The art of problem posing*. Philadelphia: The Franklin Institute Press.

Csíkszentmihályi, M. (1990). *Flow: The psychology of optimal experience*. New York: Harper and Row.

Csíkszentmihályi, M. (1996). *Creativity: Flow and the psychology of discovery and invention*. New York: Harper Perennial.

Csíkszentmihályi, M. (1998). *Finding flow: The psychology of engagement with everyday life*. New York: Basic Books.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.

Liljedahl, P. (2006). Learning elementary number theory through a chain of discovery: Preservice teachers' encounters with pentominoes. In R. Zazkis & S. Campbell (Eds.), *Number theory in mathematics education: Perspectives and prospects* (pp. 141–172). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Liljedahl, P. (2008). *The AHA! Experience: Mathematical Contexts, Pedagogical Implications*. Saarbrücken, Germany: VDM Verlag.
Liljedahl, P. (2014). The affordances of using visibly random groups in a mathematics classroom. In Y. Li, E. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 127–144). New York: Springer.
Liljedahl, P. (2016a). Flow: A framework for discussing teaching. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 203–210). Szeged, Hungary: PME.
Liljedahl, P. (2016b). Building thinking classrooms: Conditions for problem solving. In P. Felmer, J. Kilpatrick, & E. Pekhonen (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). New York: Springer.
Liljedahl, P. (2016c). *Classroom practices for supporting problem solving*. Paper Presented in TSG 2119 (Problem Solving) at ICME 13. Hamburg, Germany.
Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.
Mason, J. (2011). Noticing: Roots and branches. In M. G. Sherin, V. Jacobs, & R. Philipp (Eds.), *Mathematics teacher noticing* (pp. 35–50). New York: Routledge.
Patton, M. Q. (2002). *Qualitative research and evaluation methods*. Thousand Oaks, CA: Sage.
Schmidt, W., Jorde, D., Cogan, L., Barrier, E., Gonzalo, I., Moser, U., et al. (1996). *Characterizing pedagogical flow: An investigation of mathematics and science teaching in six countries*. Dordrecht, The Netherlands: Kluwer.
Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
van Es, E. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. Jacobs, & R. Philipp (Eds.), *Mathematics teacher noticing* (pp. 134–151). New York: Routledge.
Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
Williams, G. (2001). Sustained overlap in zones of proximal development? If so, how did that happen? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th international conference for Psychology of Mathematics Education* (Vol. 1, p. 378). Utrecht, The Netherlands: PME.
Yackel, E., & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 313–330). London: Kluwer Academic Publishing.

Chapter 23
Positioning and Emotions in Tackling Algebra Problems with Technology: The Case of High- and Middle-Achieving Students

Wajeeh Daher, Osama Swidan, and Amani Masarwa

23.1 Introduction

Affective aspects in learning and teaching have become a growing area in educational research, due to its relationships with other aspects of students' learning. In particular, in the mathematics classroom, affective aspects have a mutual relationship with the cognitive aspect. This puts the study of the affective aspect of students' learning on the agenda of mathematics education research. In the present chapter, we study students' emotions in context. More specifically, we present our insights about students' emotions in relation to their positioning when they engaged in solving problems involving the quadratic function as a product of two linear functions using dynamic software. This issue of students' affect has not been widely studied. Moreover, in this study, we intend to broaden the insights we have gained in our previous research, in which we examined students' positioning and emotions in one group learning geometry (Daher, Swidan, & Shahbari, 2015). Following that study, we wondered whether and how emotions and positioning interplay among high-achieving and middle-achieving students when they learn algebraic topics with a dynamic digital tool. This chapter intends to examine this issue using the discursive positioning and emotions framework developed by

W. Daher (✉)
Al-Qasemi Academic College of Education, Baqa al-Gharbiyye, Israel

An-Najah National University, Nablus, Palestine
e-mail: wajeehdaher@najah.edu

O. Swidan
Ben-Gurion University, Beersheba, Israel

A. Masarwa
Al-Qasemi Academic College of Education, Baqa al-Gharbiyye, Israel

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_23

Evans, Morgan and Tsatsaroni (2006). Using this framework enables us to characterize students' positioning and related emotions in groups of high- and middle-achieving students during mathematical problem-solving and to compare the positionings and emotions in the two different groups.

23.2 Literature Review

23.2.1 Emotions in Mathematics Education

Hannula (2015) points to the role of affective elements in mathematical problem-solving as being widely acknowledged, starting from Pólya (1957) who addressed the necessity to consider emotions as affecting the problem-solving process. Beginning from the 1980s of the twentieth century, researchers considered affect as a significant component of students' mathematical problem-solving (e.g. Goldin, 1988; McLeod, 1988; Schoenfeld, 1985). This consideration of affective elements in mathematical problem-solving was supported by studies from other fields of research such as psychology (e.g. Salovey & Mayer, 1990) and neuroscience (e.g. Damasio, 1996). These studies influenced how we look at the role of affective elements in education, so they turned to have mutual relationships with other elements in mathematics education, such as the cognitive and the social elements.

Emotion is one of the fundamental elements of the affective aspect (Hannula, 2004). Hannula (2004) describes the aspects that have reached agreement regarding emotions. First, emotions are connected to personal goals. Second, emotions involve physiological reactions. Third, emotions are functional regarding human coping and adaptation. Moreover, emotions, when managed appropriately, become a potential for effective thinking rather than a disturbance to this thinking (Antognazza, Di Martino, Pellandini, & Sbaragli, 2015; Salovey & Mayer, 1990).

Researchers suggest different approaches to study students' emotions. Op't Eynde (2004) suggests a socio-constructivist perspective and a situated approach for the study of emotions in mathematics education. Brown and Reid (2004) suggest emotional orientations and somatic markers for that study. Roth and Radford (2011) suggest a cultural-historical framework to study students' emotions during learning mathematics. Evans, Morgan, and Tsatsaroni (2006) suggest discursive positioning as the context in which students' emotions are studied when learning mathematics. In the present research, we intend to use the discursive positioning framework to explore students' positioning and emotions in groups of high-achieving students and groups of middle-achieving students. Our interest in the framework is based on our belief that emotions are discursively and socially based.

23.2.2 The Discursive Positioning Framework for Studying Students' Emotions

Evans et al. (2006) describe the characteristics of the discursive positioning framework for studying students' emotions. This framework assumes that meaning making occurs in social practices, using language and other semiotic resources. The interaction related to the social practices has an emotional dimension that helps maintaining social identity. Moreover, empirical data in this framework is seen as text, the analysis of which demands attention to its context(s). This analysis entails a combination of structural and textual phases that each informs the other.

The structural analysis considers the positions available to or claimed by the participants. Analysis of the available or claimed positions displays the ways emotions occur to the participants. In more detail, positions are associated with power in relation to others, as well as with differing values within the discourse, which creates spaces within which emotion may arise. Usually, there is more than one available position for a participant, either within a single discourse or several competing discourses. In this case, potential for conflict between positions may result in emotionally charged positions.

Evans, Morgan, and Tsatsaroni, in their writings about discursive analysis, describe the positionings considered in the structural analysis: helper and seeker of help (helper positioned more powerfully), collaborator and solitary worker, leader of activity and follower of directions (the latter less powerful), evaluator and evaluated and insider and outsider.

The textual analysis considers the exchange of meanings. This phase has two functions (Evans, 2006): (a) showing how positionings in social interactions are actually taken up by the participants and (b) providing indicators of emotional experience. Furthermore, in the textual analysis, indicators of interpersonal relationship and emotional experience are considered (Tsatsaroni, Evans, & Morgan, 2007). This analysis has two stages. In the first stage, the focus is to identify the interpersonal aspects of the text that establish the positions of the participants. Indicators at this stage include reference to self and others, reference to valued statuses (e.g. claiming understanding or correctness), modality (indicating degrees of (un)certainty), hidden agency (e.g. passive voice) or repetition. The second stage of the textual analysis attends to (a) indicators of emotional experience generally understood/used within the (sub)culture: direct verbal expression (e.g. "I feel anxious"); the use of particular metaphors (e.g. claiming to be "coasting"); emphasis by words, gesture, intonation or repetition (indicating strong feelings); and body language (e.g. facial expression or blushing) and (b) indicators suggested by psychoanalytic theory, as indicators of defences against strong emotions like anxiety, or conflicts between positionings (as "Freudian slips"), surprising error in problem-solving, behaving strangely (as laughing nervously) and denial (e.g. of anxiety).

23.3 Research Rationale and Goals

In spite of the acknowledgement of the role of affective aspects in mathematical education in general and mathematical problem-solving in particular, research related to this aspect is still not widespread (Antognazza et al., 2015). This is especially true when talking about emotions as a significant factor in problem-solving. We intend to study this issue using the two phases of the discursive framework developed by Evans et al. (2006). In more detail, we intend, in the present chapter, to analyse the positionings taken by ninth grade students and their related emotions when working collaboratively, with the help of GeoGebra, on problems involving the quadratic function as a product of two linear functions. Doing so, we will compare high-achieving and middle-achieving students' positioning and emotions. This will shed light on the influence of knowledge on the two constructs. Thus, the present chapter will give us insight regarding different affective and social issues related to students' learning of mathematics.

23.3.1 Research Questions

1. How are positionings taken up by high-achieving and middle-achieving ninth grade students, working in group so to solve problems involving the quadratic function as a product of two linear functions, in the presence of technology?
2. How are high-achieving and middle-achieving ninth grade students' emotions associated with the positionings that they claim, when using technology to tackle problems involving the quadratic function as a product of two linear functions?

23.4 Methodology

23.4.1 Research Setting and Participants

We analysed in the present study the affective aspect of the learning of six groups of grade 9 students, aged between 14 and 15 years old. The students in three groups were high-achieving students, while the students in the other three groups were middle-achieving students. All the participating students had not work with GeoGebra before, and they were introduced to it in 2 h' time. Furthermore, the students had learned some issues in the topic of the quadratic function (e.g. whether the function has maximum or minimum, how to find the vertex of the function and the domain in which the function is increasing or decreasing), but not the quadratic function as a product of two linear functions. The third author taught the six groups in a middle school in Israel. She did that in the frame of after-school lessons as mathematics enrichment. Before doing that, she took the permission from the students' parents and the school headmaster.

23.4.2 Data Collecting and Analysing Tools

We collected our data using observations of the learning of the six groups. We also conducted interviews with their members. Every group's learning was videoed, and at the end of each lesson, the three students in each group were interviewed individually regarding their positionings and emotions during the work on algebra problems with technology. We analysed the two types of collected data using the discursive analysis framework presented above. Moreover, we combined the analyses of the data collected by the two tools. For examples, on analysing students' positioning and emotions using the framework, refer to the various writings of Evans, Morgan and Tsatsaroni. The findings section in this chapter also sheds light on this method.

The Task

The six groups of seventh grade students worked with a sequence of activities, all related to the quadratic function as a product of two linear functions. The following is an example on these activities.

In the same coordinate system, we want to draw the three functions: $y = x$, $y = x + 2$ and $y = x(x + 2)$.

What are the algebraic characteristics of the linear function:$y = x$?
What are the graphical characteristics of the linear function:$y = x$?
What are the algebraic characteristics of the linear function: $y = x + 2$?
What are the graphical characteristics of the linear function:$y = x + 2$?
What are the algebraic characteristics of the quadratic function: $y = x(x + 2)$?
What are the graphical characteristics of the quadratic function:$y = x(x + 2)$?
What are the similarities and the differences between the characteristics of the two above linear functions and the characteristics of the quadratic function?

Note: Algebraic characteristics are related to the parameters of the equation, while the graphical characteristics are related to the intersection points with the axes, increasing or decreasing of the function, etc.

23.4.3 Findings

The present chapter aimed at characterizing groups of students' positioning and emotions when using technology to tackle algebra problems. A second aim was to distinguish between these characteristics in high-achieving and middle-achieving groups of students. Analysing students' positioning and related emotions in high- and middle-achieving groups of students, we found that mainly the students had the following positionings: leader, collaborator, evaluator, seeker of help and outsider. We will describe how the students in the high-achieving and the middle-achieving groups claimed each of the positionings and experienced their emotions in each positioning. Doing so, we will address the following aspects of learning that the positioning related to behavioural, cognitive, metacognitive, social and linguistic.

The emotional aspect of learning will be considered in light of the positioning taken. Furthermore, addressing the different aspects of learning, we will describe the similarities and differences in the regularities colouring the middle-achieving and high-achieving group members' claiming of positioning and the related emotions.

23.4.4 Collaborator's Functioning

The different groups, high-achieving as well as middle-achieving, utilized collaboration to learn the conceptions related to the quadratic function as a product of two linear functions. This is exhibited in that generally the members of each of the participating groups claimed the collaborator positioning to pursue, with the help of the activity and the mathematical software, their learning of the quadratic function. This claiming resulted in making the group's members enthusiastically understanding the appropriate mathematical relations. Learning enthusiastically made the group members enjoy their learning as they said in the interview. For example, Asis said in the interview: "Working collaboratively made us enthusiastic to perform the activities. This made us enjoy our learning to a great degree". Thus, the group collaboration helped to make the students' emotions concerning their mathematical learning positive ones.

To claim the collaborator positioning, the group members were involved with behavioural processes (working with GeoGebra), social processes (class discussions) as well as cognitive processes (mathematical reasoning). These three types of processes not only helped the group members claim the positioning of the collaborator and actualized this positioning but, at the same time, supported their attempts, as described above, to perceive the conceptions related to the quadratic function as a product of two linear functions. In the interview, the students associated their behavioural processes, i.e. working with GeoGebra, with positive and negative emotions: enjoyment of their work when the software helped them to solve the mathematical problem and frustration when finding difficulty to operate the software. In the interview, Sana expressed her enjoyment of working with GeoGebra: "GeoGebra is a fantastic program; it allows us to see relationships between functions. I really enjoyed working with GeoGebra".

Excerpt 23.1 shows the members of one middle-achieving group as they claim the collaborator positioning.

Excerpt 23.1 Claiming the collaborator positioning

A1	Sana	We need to find the intersection with x for the three functions.
A2	Amal	What's the first function?
A3	Asis	$3x - 2$
A4	Sana	[drew the first function in GeoGebra]
A5	Amal	What's the second function?
A6	Asis	$2x + 3$

A7	Sana	[drew the second function in GeoGebra]
A8	Asis	The third function is $(3x - 2)(2x + 3)$.
A9	Sana	[drew the third function] Let's find the intersection points with x [she seemed enthusiastic and enjoying the activity].

Excerpt 23.1 shows how the group members claimed the collaborator's positioning. This positioning is related here to the behavioural aspect of the group's learning. This aspect is expressed by the students' action with the GeoGebra software (e.g. A4, A7 and A9). However, this positioning is also concerned with the metacognitive aspect. The utterances of Sana in lines A1 and A9 and Amal in lines A2 and A5 suggest that they were regulating the processes of how to solve the mathematical problem. Claiming the collaborator positioning, through working with GeoGebra, made the group members enthusiastic and enjoy their learning [A9].

23.4.5 Leader's Functioning

The leader's positioning in the high- and middle-achieving groups was utilized to direct the learning of the group. Moreover, the leader worked to advance the group learning towards perceiving the conceptions related to the quadratic function as a product of two linear functions, including the solution of the related mathematical problems.

Leaders in the high-achieving as well as in the middle-achieving groups claimed their positioning through carrying out processes related to the different aspects of learning, mainly the cognitive, metacognitive, meta-emotional, social and linguistic aspects. Below, we elaborate on these processes.

The group leader's cognitive functioning was actualized through demonstrating knowledge during carrying out the mathematical activity. See, for example, excerpt 23.2, where Fairouz, a leader in a middle-achieving group, argues regarding the conditions of drawing a quadratic function:

Excerpt 23.2 Cognitive activity of the leader

B1	Salim	What do you think the similarity is between two linear functions and their product?
B2	Noura	You mean graphically?
B3	Fairouz	We only need to know the intersection points of the two linear functions with the x-axis in order to know the intersection points of the resulting quadratic function.

The group leader's metacognitive functioning was actualized through asking questions during the group learning as means to decide upon the method of solving a problem. For example, Sema, a leader in a high-achieving group, asked the teacher:

“Shouldn’t we use the previous solution method?” In the interview, Sema said she asked the question in order to decide what solution method was most appropriate for solving the mathematical problem. Moreover, she reported that this made her more confident to suggest how to proceed in solving the task and, as a result, to be content of her role in the group.

The group leader’s meta-emotional functioning was actualized through trying to change the negative mood of the group when encountering a difficulty. For example, Alaa, a leader in a middle-achieving group, tried to lessen the anxiety of group (C3). Moreover, this meta-emotional functioning contributed to the positive emotions of all the group members, as it converted their negative emotions into neutral or positive ones as content. In the interview, Alaa said: “the group got anxious because we did not get what we expected. Simplifying the product of the linear functions again, we arrived at the mistake. This changed the negative mood of the group and made us content”.

Excerpt 23.3 Meta-emotional activity of the leader

C1	Siham	The intersection points of the quadratic function are not as the intersection points of the linear functions [she seemed anxious].
C2	Amin	[murmured something inaudible, looking anxious too]
C3	Alaa	Don’t worry. It’s O.K. Sure we made a mistake. Let’s read again our solution to find it.

The group leader’s social functioning was actualized through answering other members’ questions, asking questions and requesting actions from the group members to keep the group learning going. For example, Sema, a leader in a high-achieving group, requested Lina to work with GeoGebra and Huda to write in order to organize the work of the group [see excerpt 23.4]. In the interview, Sema said: “I made my requests from the other members, so that the group could succeed in solving the mathematical problem”.

Regarding the linguistic aspect of the leader’s functioning, the leaders in the two types of groups used the first person plural pronoun to talk about the mathematical actions that they needed to perform, which showed them as collaborators with the other members of the group. This happened in the high-achieving groups more than in the middle-achieving ones. For example, in excerpt 23.4 that describes the beginning of a high-achieving group working with activity 1, Lina and Sema who claimed the positioning of the leader used the first person plural pronoun to talk about the mathematical actions that the group needed to perform (D1-D5). On the other hand, the middle-achieving group in excerpt 23.5 did not use first person plural pronoun numerously as the high-achieving one in excerpt 23.4.

Furthermore, generally speaking, the leader’s functioning resulted in different emotions, but generally speaking, this functioning resulted in enjoying the activity by the leader during the successful solution process of the mathematical problem; frustration, when not being able to find a way for solving the activity; and content when finally solving the activity.

23.4.6 Alternating Between Positions: Leaders, Evaluators, Seekers of Help, Outsiders and Collaborators

The groups differed in alternating among positions, where by alternating we mean interchanging repeatedly and regularly with one another in positions. At the beginning of the activity, members in the high-achieving groups claimed different positions that included seekers of help and outsiders. In excerpt 23.4, the group consisted of Lina, Sema and Huda, where Lina claimed first the leadership of the group, asking group to start solving the activity (D1). She did that showing confidence, pointing at the activity text and holding the mouse in order to work with GeoGebra as a tool for solving the activity (D1). At the same time, Huda was shaking her pen, pointing with tension at the question (D2), which showed her as requesting help from the other group members. Sema seemed quiet, reading the activity text with concentration that indicated her intention to understand the activity. She did that, not paying attention to the rest of the group members (D2), which showed she was a momentary outsider to the group learning.

Excerpt 23.4 Alternating between positions at the beginning of the activity

D1	Lina	[holding the mouse, saying loudly with confidence] Let's start.
D2	Huda and Sema	[Huda seems tense, pointing at the first question of the activity, while Sema seems concentrating quietly on the activity text]
D3	Sema	The question says we should draw the three functions: $y = x$, $y = x + 2$ *and* $y = x(x + 2)$ in the same coordinate system
D4	Lina	[draws the three functions in GeoGebra]
D5	Sema	A moment please to see what the question requests [..3..] we need to write the parameters a and b.
D6	Lina	[Lina held the pen in confidence and intended to write the answer of the question on the copybook that was in front of Huda].
D7	Sema	[talking to Lina] Look, would you work with the mouse and Huda writes?
D8	Huda	[took the pen from Lina, saying in a low voice] a? what does she mean by a?
D9	Sema	*a* is the coefficient of *x*.

Excerpt 23.4 shows how the leadership alternated from Lina to Sema, where both of them claimed the leadership by initiation. Further, Sema claimed her leadership also by taking decisions regarding the group learning (D5) and knowledge (D9).

As the activity proceeded, the group members kept alternating among positions that were confined to leaders, evaluators and collaborators. This claiming of the advanced positions was enabled due to the members' knowledge possessed generally by all the group members. This knowledge, associated with the cognitive aspect of learning, enabled the group members to lead the solving process of the mathematical problem and to function as evaluators of each other's mathematical actions performed during the solving process. Moreover, the knowledge of the group members supported their inclusion in the group learning, so being outsider to the group learning in the high-achieving groups happened only at the beginning of the activity and as a result of not controlling the group's problem-solving, due to unfamiliarity with the problem situation. Being insiders, the members of the high-achieving groups enjoyed their learning, as Huda said in the interview: "I enjoyed being active the whole solution process".

In comparison with the high-achieving groups, members in the middle-achieving groups also alternated, at the beginning of the activity, between different positions that included leaders, seekers of help, collaborators and outsiders. This alternating was due to carrying out behavioural, cognitive and metacognitive processes to pursue the group's learning. More specifically, the alternating happened in working with GeoGebra, which is a behavioural process, demonstrating knowledge which is a cognitive process, and planning which is a metacognitive process.

As the activity proceeded, the discursive positioning in the middle-achieving groups was not similar to that in the high-achieving groups. This was presented in the middle-achieving group's leader claiming her position almost during the whole learning of the group, while the other members alternated among the different positions mentioned above. This claiming of leadership was due, as mentioned above, to the leader's attention to the cognitive, social, metacognitive and meta-emotional aspects of the group's learning.

The positions claimed by the members of the high-achieving groups by means of knowledge affected their emotions throughout the learning process. Having the knowledge to solve the mathematical problem resulted in the group members having more positive emotions (as content and calmness) than negative emotions (as anxiety, tension and confusion). The members in these groups got anxious when they stopped to lead the solution process. In that case, they responded in one of three ways: discussing the solution, working with GeoGebra and approaching the teacher. Getting help from one of the three previous resources and returning to lead the solution process lessened or stopped the negative emotions of the group members. In the interview, the members of the high-achieving groups described their content with their problem-solving more than did the members of the middle-achieving groups. Sema, a member in a high-achieving group, said she was content from the whole process of problem-solving, though they encountered difficulties throughout this process. On the other hand, Salim, a member of a middle-achieving group, said he did not enjoy the problem-solving in the entire time, but he was content at the end because the group managed to solve the problem.

23.4.7 Encountering Difficulties and Its Effect on Group Members' Emotions

Upon encountering a difficulty that was not easy to solve in the high-achieving group, the group members approached the teacher requesting her help or the teacher identified their need for help and approached them to discuss the difficulty. In the discursive situations characterized by encountering difficulties, the group members of the high-achieving groups turned into seekers of help, while their teacher turned into their leader in overcoming the difficulties.

The members of the middle-achieving groups, due to the lack of appropriate previous knowledge, whether in content knowledge or in the technological knowledge, encountered more difficulties in claiming the positions of leaders and collaborators during the solution process of the mathematical problem than the high-achieving groups. This led to their experiencing more negative emotions than the high-achieving groups. In addition, the members of the middle-achieving groups started the activity being anxious because they could not claim the positioning of leading the activity or, at least, behaving as collaborators. Moreover, the members of the high-achieving groups experienced calmness in the same manner and extent, while the members of the middle-achieving groups experienced calmness in two ways. First, they experienced calmness in relation to the leader's calmness. In other words, the members of the middle-achieving groups experienced calmness when their leader experienced it and experienced anxiety and confusion also when their leader experienced it. On the other hand, the collaborators in the middle-achieving groups experienced frequently the loss of control when (1) working with GeoGebra, related to the behavioural aspect; (2) trying to remember the characteristics of the linear functions, related to the cognitive aspect; or (3) trying to explore the characteristics of the quadratic functions, related mainly to the cognitive aspect.

Working with a dynamic technological tool, the members of the two types of groups encountered at the beginning difficulties that relate to the behavioural aspect of the group's learning, where the middle-achieving groups encountered more difficulties. Salim, Noura and Fairouz were the members of a middle-achieving group. They wanted to draw the function $(2x - 9)(3x - 4)$ in GeoGebra [E1] but found difficulty doing that due to not writing correctly appropriate number of brackets [E2-E5].

Excerpt 23.5 Students' difficulties in working with GeoGebra and related emotions

E1	Fairouz	We should write $3x - 4$ multiplied by $2x - 9$. [Noura started to write the expressions]
E2	Salim	Perhaps the brackets can be put afterwards, wait Noura, wait, it keeps moving.
E3	Fairouz	Write it from the beginning.

E4	Salim	No, no [he takes the mouse from Noura who seemed annoyed by the act of Salim. Salim works on GeoGebra] O.K. Now write it again. [Noura puts her hand on her mouth with boredom]
E5	Fairouz	[Fairouz worked on GeoGebra, and then she said with annoyance] Something wrong with the brackets.
E6	Teacher	Don't get anxious. Brackets, at the beginning, are tricky. Everything will get O.K.

Excerpt 23.5 shows some of the difficulties encountered by the middle-achieving groups, as a result of their behavioural functioning, specifically when working with the technological tool. Fairouz, in the interview, said they felt out of control and thus frustrated for not being able to draw the graph of the function
$f(x) = (2x - 9)(3x - 4)$ easily in GeoGebra.

Salim pointed at the teacher's interference as supporting them in having back control over their work with GeoGebra and returning to be satisfied with their work on the mathematical activity.

Encountering difficulties in approaching the quadratic function as a product of two linear functions not only influenced students positioning and emotions, as described above, but also coloured the linguistic aspect of their learning. Specifically, it coloured their use of pronouns. For example, when the students got anxious for not being able to proceed with the carrying out of the activity, they approached another group member using a singular first or second person pronoun or no pronoun at all. This is the case in Excerpt 23.6, where a middle-achieving group encountered difficulties simplifying an algebraic expression.

Excerpt 23.6 Having difficulty in simplifying an algebraic expression

F1	Siham	We want to draw the quadratic function $y = (2x - 9)(-x - 4)$.
F2	Amin	Multiply first the brackets.
F3	Alaa	$(-x - 4)(2x - 9) = -2x^2 + 9x - 8x + 36$, Now we compute +9 − 8.
F4	Amin	−17
F5	Siham	Minus 8 plus 9.
F6	Amin	Minus 17, plus
F7	Alaa	What?
F8	Amin	Minus one or plus one.
F9	Alaa	What?
F10	Siham	Plus 1.
F11	Alaa	Write $-2x^2 + x - 36$.

Excerpt 23.6 shows that in time of difficulty that constrains the group's sense of control, and thus produces anxiety, singular pronouns or no pronouns are used.

23.5 Discussion

Research on students' emotions in mathematics learning is growing (e.g. Antognazza et al., 2015; Daher, 2011; Hannula, 2004). In the present chapter, we present our insights about students' emotions in relation to their positioning when they solved problems involving the quadratic function as a product of two linear functions using dynamic software. To elaborate more, the present chapter had two aims: first, characterizing students' positioning and emotions when tackling algebra problems with technology, and, second, distinguishing between these characteristics in high-achieving and middle-achieving groups of students. The research findings indicated that to claim the collaborator positioning, the high- and middle-achieving group members were involved with behavioural processes (working with GeoGebra), social processes (class discussions) as well as cognitive processes (mathematical reasoning). These processes helped them reach their learning goal, thus resulting in positive emotions. It could be said that the collaborator position was associated with positive emotions, though negative emotions were experienced when having difficulty in solving the mathematical problem, i.e. when the collaborators experienced difficulties in arriving at the learning goal.

To claim the leader positioning, the high- and middle-achieving group members were involved with different types of learning processes, but their functioning was distinguished from the other group members by their performance of metacognitive and meta-emotional processes. These processes helped plan, monitor, evaluate and take decisions regarding the group learning, especially in time of difficulty in arriving at the learning goals. At the same time, these processes helped maintain the leader positioning (Black, Soto, & Spurlin, 2016) for they affected critically the group learning including their emotions. In this case, not only positioning influenced emotions, but emotions influenced positions too.

In addition, the leader metacognitive functioning was actualized by asking questions as means to decide upon the method of solving a problem. This decision-making could be looked at as a social process (Vroom & Jago, 1974) with the goal to advance the group learning. Moreover, it seems that critical thinking skills, actualized in decision-making, were needed to claim the leader's positioning. Furthermore, the goal of the leader meta-emotional functioning was to change the negative mood of the group when encountering a difficulty. This meta-emotional functioning, expressed in the leader being attentive to the members' emotions, is also pointed at as one function of the successful leader who motivates the members' work (Leithwood, Louis, Anderson, & Wahlstrom, 2004). So, we argue that the leader positioning was claimed by paying attention to different aspects of the group learning. Of those, the metacognitive and meta-emotional processes are of critical importance to the leader's functioning (Zaccaro, 2007).

The alternating between positions in the two types of groups, which occurred at the beginning of the activity, happened through performing different types of processes. More specifically, this altering happened through working with GeoGebra which is a behavioural process, demonstrating knowledge which is a cognitive process and planning which is a metacognitive process. Moreover, in maintaining the leader's positioning, knowledge demonstration; a cognitive process, and knowledge evaluation; a metacognitive process, were needed. Thus, to maintain the leader's positioning, two types of processes were needed: "regular processes" and "meta" processes.

Furthermore, knowledge being possessed by all the members of the high-achieving groups resulted in the changing of leaders in these groups. At the same time, it resulted in the fixation of leaders in the middle-achieving groups. It could be argued that possessing knowledge also helped make evaluations of other members' performance. Thus the cognitive aspect of students' learning, more than any other aspect, was behind the difference between claiming positioning in the two types of groups. This aspect is emphasized by Evans et al. (2006) as affecting who is in authority in the group and, as a result, as affecting students' experiencing of emotions.

Students encountered sometimes difficulties in learning the quadratic function as a product of linear functions. This encounter not only influenced students' positioning and emotions but also their linguistic use of pronouns. This was expressed in their use of singular pronouns or no pronoun at all when getting anxious for not being able to proceed with the carrying out of the activity. It could be claimed that when experiencing negative emotions, students lack the feeling of the social climate.

23.6 Conclusions

High- and middle-achieving groups of students utilized GeoGebra to tackle problems involving the quadratic function as a product of two linear functions. Doing so, they showed similarities as well as differences regarding their positioning and related emotions. These similarities and differences could be related to the different aspects of learning expressed in the learning processes performed by the two types of groups in order to achieve their learning goal. These learning processes could be categorized as "regular" processes, consisting of cognitive, social and behavioural processes, and as "meta" processes, consisting mainly of metacognitive and meta-emotional processes. The two main types of learning processes were performed by the students to claim their positioning, where leaders were concerned more than the other group members in performing the meta processes, more specifically the metacognitive and meta-emotional processes.

Knowledge demonstration and evaluation were behind the difference between the high-achieving and the middle-achieving groups regarding the claiming of leadership in the group. Generally, all the group members in the high-achieving groups could demonstrate and evaluate knowledge, so the high-achieving group's members could alternate in claiming the leader positioning.

Students' emotions were related to their positioning and the success or difficulty to proceed in the problem-solving activity as group leaders or collaborators.

The middle-achieving groups experienced more difficulties in solving the mathematical problems than the middle-achieving groups, and thus the leaders of those groups as well as the collaborating members experienced more negative emotions than the high-achieving groups.

One limitation of the study is that it did not consider positioning and emotions in low-achieving groups of students. Future research is needed to explore this issue.

References

Antognazza, D., Di Martino, P., Pellandini, A., & Sbaragli, S. (2015). The flow of emotions in primary school problem solving. In K. Krainer & N. Vondrova (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)* (pp. 1116–1122). Prague, Czech Republic: European Society for Research in Mathematics Education.

Black, H., Soto, L., & Spurlin, S. (2016). Thinking about thinking about leadership: Metacognitive ability and leader developmental readiness. *New Directions for Student Leadership, 2016*(149), 85–95.

Brown, L., & Reid, D. (2004). Emotional orientations and somatic markers: Implications for mathematics education. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the twenty-eighth conference of the International Group for the Psychology of mathematics education* (Vol. 1, pp. 123–126). Bergen, Norway: PME.

Daher, W. (2011). Learning mathematics in the mobile phone environment: students' emotions. *Journal of Interactive Learning Research, 22*(3), 357–378.

Daher, W., Swidan, O., & Shahbari, J. (2015). Discursive positionings and emotions in a small group's learning of geometric definitions. In K. Krainer & N. Vondrova (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)* (pp. 1160–1166). Prague, Czech Republic: European Society for Research in Mathematics Education.

Damasio, A. (1996). *Descartes' error: Emotion, reason and the human brain*. New York: Avon Books.

Evans, J. (2006). Affect and emotion in mathematical thinking and learning. In J. Maasz & W. Schloeglmann (Eds.), *New mathematics education research and practice* (pp. 233–256). Rotterdam, The Netherlands: Sense Publishers.

Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics: Affect in Mathematics Education: Exploring Theoretical Frameworks, A PME Special Issue, 63*(2), 209–226.

Goldin, G. A. (1988). Affective representation and mathematical problem solving. In M. J. Behr, C. B. Lacampagne, & M. M. Wheeler (Eds.), *Proceedings of the 10th annual meeting of PME-NA* (pp. 1–7). DeKalb, IL: Northern Illinois University, Department of Mathematics.

Hannula, M. (2004). Affect in mathematics education – Exploring theoretical frameworks. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference* (Vol. 1, pp. 125–154). Bergen, Norway: Bergen University College.

Hannula, M. S. (2015). Emotions in problem solving. In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 269–288). Cham, Switzerland: Springer.

Leithwood, K., Louis, K., Anderson, S., & Wahlstrom, K. (2004). *How leadership influences student learning*. New York: The Wallace Foundation.

McLeod, D. B. (1988). Affective issues in mathematical problem solving: Some theoretical considerations. *Journal for Research in Mathematics Education, 19*, 134–141.

Op't Eynde, P. (2004). A socio-constructivist perspective on the study of affect in mathematics education. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of mathematics education* (Vol. 1, pp. 118–122). Bergen, Norway: Bergen University College.

Pólya, G. (1957). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Roth, W. M., & Radford, L. (2011). *A cultural historical perspective on teaching and learning*. Rotterdam, The Netherlands: Sense Publishers.

Salovey, P., & Mayer, J. D. (1990). Emotional intelligence. *Imagination, Cognition and Personality, 9*, 185–211.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. San Diego, CA: Academic Press.

Tsatsaroni, A., Evans, J., & Morgan, C. (2007). Pedagogic discourse, positioning and emotion: Illustrations from school mathematics. *Review of Science, Mathematics and ICT Education, 1*(1), 83–105.

Vroom, V., & Jago, A. (1974). Decision making as a social process. *Decision Sciences, 5*, 743–755.

Zaccaro, S. J. (2007). Trait-bsed perspectives of leadership. *American Psychologist, 62*(1), 6–16.

Chapter 24
Affect and Aesthetics in Mathematical Problem-Solving: New Trends, Methodologies, Results and Critical Aspects

Pietro Di Martino

Research on mathematics-related affect has grown exponentially in the last decades, as showed by recently published overviews (Hannula et al., 2016; Hannula, Pantziara & Di Martino, 2018).

The beginning of the *modern* interest towards affect in mathematics education research is around the 1980s when, exactly in order to understand and interpret the students' failures in problem-solving activities, different researchers underlined the need to go beyond a purely cognitive approach. The main focus is the failure in problem-solving activities of students who seem to have the required cognitive resources.

In the volume *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (Silver, 1985a), Silver included affect within the issues he considered need to be addressed in the development of the research on mathematical problem-solving:

> The first issue may appear to be out of place in the current climate of cognitive research on human problem solving [...] However, it is precisely because of the current climate that the issue is so important [...] Anyone who has worked with students in mathematics classes knows that affective considerations have a substantial face validity and such weak research support deserves to be studied in more detail. The time has to come to renew our efforts to uncover the affective/cognitive links in problem solving. (Silver, 1985b, p. 253–354)

In the same volume, McLeod affirmed:

> Limiting one's research perspective to the purely cognitive seems acceptable for those interested mainly in the performance of machines; however, researchers who are interested in human performance need to go beyond the purely cognitive if their theories and investigations are to be important for problem-solving in mathematics classrooms. (McLeod, 1985, p. 268)

P. Di Martino (✉)
Department of Mathematics, Università di Pisa, Pisa, Italy
e-mail: pietro.di.martino@unipi.it

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_24

Four years later, the volume *Affect and Mathematical Problem Solving: A New Perspective* (McLeod & Adams, 1989) was published. In the volume, for the first time in the field of mathematics education, the affective factors were taken into account to analyse students' behaviour and difficulties *internal* to a specific mathematical activity: problem-solving. In some sense, the publication of *Affect and Mathematical Problem Solving: A New Perspective* represented the beginning of a new era in mathematics education, despite that – as Mayer (1982, p. 36) claims – it "raises interesting questions but provides few answers".

Our community has come a long way since the end of the 1980s: different answers have been found to the questions posed in *Affect and Mathematical Problem Solving: A New Perspective*, and new challenges in the field of affect emerged. The present volume is proof of that new relevant issues, approaches, methodologies and results concerning affect and problem-solving are presented and described (see Table 24.1).

I will try to discuss briefly and schematically all these aspects, also highlighting some theoretical and methodological critical issues that arise.

Two of the six chapters of the section affect and aesthetics in mathematical problem-solving concern aesthetic issues. They offer a really interesting picture of the related research, discussing the role of the aesthetic in mathematics education and the relationship between aesthetic and affect.

Concerning this latter issue, essentially we can recognize two theoretical approaches:

1. Aesthetic is seen as a part of affect. According to Goldin (2000), affect has a tetrahedral nature, it includes the three dimensions (emotions, beliefs and attitudes) considered by McLeod in the very famous 1992 Handbook (McLeod, 1992) and a fourth dimension related to values. In this theoretical model, aesthetic falls in this fourth dimension.

Table 24.1 Synthesis of chapters in Sect. 3: affect and aesthetics in mathematical problem-solving

Chapter	Focuses	Sample	Methodology
18	Attitude, mathematical competition, problem-solving	Grade 5, 6, 7, 8 students	Mixed: interviews, questionnaires, e-mails,
19	Aesthetic, affect, problem-solving	Graduate students	Qualitative: interviews
20	Aesthetic, problem-solving	Primary preservice teachers	Qualitative: journals
21	Emotions, mathematical competition, problem-solving	Students of the third cycle of basic education	Mixed: students' mathematical products, journals, Facebook posts, online survey
22	Engagement, problem-solving	Grade 11 and grade 12 students	Qualitative: direct observation
23	Emotions, technology, problem-solving	Grade 9 students	Qualitative: video recorded data, interviews

2. Aesthetic and affect are closely intertwined but different domains with different functions. According to Sinclair (2008), the aesthetics functions as a nonlogical form of knowing, drawing the attention of the perceiver to a phenomenon, while the affective can active the awareness of these perceptions.

Sinclair and Rouleau stress another dichotomy in the research on aesthetic: aesthetic can be seen as an objective and absolute judgement reflecting the view of a cultural elite, or researchers can assume a contextual and *democratic* view of the aesthetic. This latter approach appears to be more significant for mathematics education: the classroom has to be the environment where the judgement (also the aesthetic one) is negotiated and shared.

The two chapters discuss the role of aesthetic in the problem-solving process, starting from the role of aesthetic in the work of research mathematicians. This approach is complex because – as Sinclair and Rouleau underline – few mathematicians write about their problem-solving processes, and still fewer explicit aesthetic (or affective) considerations.

This interesting approach is based on an *epistemological* assumption: if something is relevant in the work of professional mathematicians, then it can (have to?) be relevant in mathematics education. This assumption has also driven relevant research in the field of affect, for example, Liljedahl research about the Aha! experience (Liljedahl, 2008).

Anyway, the researcher needs to explicit why (and subsequently how) they suppose to use their knowledge of the mathematicians' problem-solving experiences for promoting aesthetic and affective responses in the mathematics classroom. Concerning aesthetic, the mathematicians' experience reveals the connection between aesthetic considerations and decisions or evaluations. Since decisions and evaluations are crucial in the problem-solving process, and the development of the problem-solving competence is a crucial goal of mathematics instruction, it appears relevant to promote aesthetic responses in the teaching of mathematics. Moreover, to promote aesthetic experiences can be a teaching strategy in repairing a negative relationship with mathematics.

The attention on aesthetic issues in mathematics education is therefore fully justified. How to promote aesthetic responses in the students' mathematical experiences is still an outstanding issue.

Presmeg and Sinclair and Rouleau discuss some possibilities to promote aesthetic responses in problem-solving activities (both with students and with preservice teachers). All the authors stress how the level of difficulty of the problem is a key factor: it is crucial that the problem appears neither too easy nor too hard.

This observation appears to be strictly related to the idea of flow introduced by Liljedahl in his chapter. According to the theory developed by Csíkszentmihályi (1990), flow is a state in which people are so involved in an activity that nothing else seems to matter. Using Peter's words, flow is one of the only ways for mathematics education researchers to talk productively about engagement.

Flow is possible only when there is a balance between challenge and individual's ability. Therefore, the theory of flow is useful for interpreting student reactions when faced with an imbalance between challenge and their skill.

Liljedahl's approach is particularly interesting because it is not only theoretical but also directly linked to practice. In particular, he discusses the flow is facilitated by specific problem-solving settings, for example, when a classroom is conducted according to the building thinking classrooms framework of teaching (Liljedahl, 2016).

The theory of flow is particularly stimulating because it describes the balance between challenge and ability as a dynamic process where teachers play a crucial role. As a matter of fact, through appropriate feedbacks, teachers have to manage two different *dangerous* situations: when challenge exceeds skills (frustration) and when skills exceed challenge (boredom).

In this volume, also the chapter developed by Daher, Swidan and Masarwa is focused on the dynamics aspects of affect. In particular, scholars compare positioning and emotions in learning algebra with technology of high- and middle-achieving students.

This kind of studies is particularly relevant but also complex from a methodological point of view.

They are relevant because on the one hand studies that focus on the dynamics of affective states in classroom are still rare (Hannula, 2012); on the other hand, further evidence of the dynamic progression of motivational and emotional states through the problem-solving process is needed (Lewis, 2017).

They are complex because the data are collected through direct observation of students during problem-solving activities, and direct observation is confronted with the *issue of circularity*. Lester was the first to discuss this issue in relation to the research on beliefs, but his considerations are easily generalizable to other affective constructs:

> For researchers to claim that students behave in a particular manner because of their beliefs and then infer the students' beliefs from how they behave involves circular reasoning. *The reasoning goes something like this: Question: How do you know that students' beliefs influence how they do mathematics? Answer: Because in our study students did mathematics in a certain way. Question: But how do you know that the students'beliefs contributed to this behavior? Answer: Because they would not have behaved this way if they did not hold these beliefs.* (Lester, 2002, p. 348)

We have discussed the issue of circularity related to methodologies based on direct observation, but more in general, for their nature, it is difficult to infer affective states or traits. If it is true that affective constructs are not directly observable, it is also true that individuals themselves are often not conscious of these processes (Hannula, Pantziara & Di Martino, 2018): studies based on self-report are not however without their methodological critical aspects.

Chapters 18 and 21 of this book describe studies based on different kinds of self-report: some of these self-reports (e.g. the students' posts on Facebook) are particularly original, and I believe they deserve a special attention for the future of the interpretative research about affect.

The two studies are focused on the potential of mathematical problem-solving competitions for promoting positive emotions and attitudes towards mathematics. The main goal of these competitions is affective: they are seen as way of motivating participants to engage with mathematics and therefore of developing appreciation for mathematics.

As already mentioned, these studies are based on the collection of self-reports before, during and after the competition.

On the one hand, these studies are interesting because they monitor the students' evolution in the period. On the other hand, the comparison between initial and final self-report – particularly relevant in order to monitor the evolution of attitude towards mathematics – has a clear problematic: typically the final self-report is filled by a minority of participants, and this minority is likely representative of those for whom the competition was a good experience.

Methodological issues a part, these competitions certainly have interesting features. They are inclusive: the students' participation is voluntary, and the problems have to be intended for all. But they are also detached by the official curriculum: there is more choice for the selection of the problems. In this frame, it is easier to develop realistic mathematically rich problems having more connections with other areas.

These features certainly can contribute to develop a positive attitude towards this kind of mathematical competition. It remains unexplored and problematic if and how these experiences (out of school) can have effects on the students' attitude towards mathematics at school.

In conclusion, the six chapters of the section affect and aesthetics in mathematical problem-solving of the present volume offer an interesting and diverse contribute. They deal with significant issues, offering new theoretical approaches and new methodologies but also highlighting new critical aspects in the research about affect, aesthetic and problem-solving.

I want to replicate the hope that Silver stated in his introduction to the volume *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (Silver, 1985a, p. ix): "If this volume leads to an increased understanding of the nature of past problem-solving research, or an increased appreciation of the potential benefits of cross-fertilization of ideas among workers in different fields, or a renewed enthusiasm for attacking some of the underrepresented themes and issues from previous research, then it will have served its purpose".

References

Csíkszentmihályi, M. (1990). *Flow: The psychology of optimal experience*. New York: Harper and Row.

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning, 2*, 209–219.

Hannula, M. (2012). Exploring new dimensions of mathematics-related affect: Embodied and social theories. *Research in Mathematics Education, 14*(2), 137–161.

Hannula, M., Di Martino, P., Pantziara, M., Zhang, Q., Morselli, F., Heyd-Metzuyanim, E., et al. (2016). *Attitudes, beliefs, motivation and identity in mathematics education: An overview of the field and future directions*. Cham, Switzerland: Springer International.

Hannula, M., Pantziara, M., & Di Martino, P. (2018). Affect and mathematical thinking: Exploring developments, trends, and future directions. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education – Twenty years of communication, cooperation and collaboration in Europe*, *New Perspectives on Research in Mathematics Education – ERME series* (Vol. 1, pp. 128–141). London, New York: Routledge.

Lester, F. (2002). Implications of research on students' beliefs for classroom practice. In G. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 345–353). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Lewis, G. (2017). Emotions and problem solving by prospective primary school teachers. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education*. Dublin, Ireland: DCU Institute of Education & ERME.

Liljedahl, P. (2008). *The AHA! Experience: Mathematical contexts, pedagogical implications*. Saarbrücken, Germany: Dr. Müller.

Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem solving. In P. Felmer, J. Kilpatrick, & E. Pekhonen (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 361–386). New York: Springer.

Mayer, R. (1982). The psychology of mathematical problem solving. In F. L. Lester & J. Garofalo (Eds.) Mathematical problem solving. Issues in research. Philadelphia: The Franklin Institute Press.

McLeod, D. (1985). Affective issues in research on teaching mathematical problem solving. In Silver E. (Ed.) Teaching and learning mathematical problem solving: multiple research perspectives (pp. 267–279). Hillsdale, NJ: Erlbaum.

McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research in mathematics education, teaching and learning* (pp. 575–596). New Yok: Macmillan.

McLeod, D., & Adams, V. M. (Eds.). (1989). *Affect and mathematical problem solving*. New York: Springer Verlag.

Silver, E. (Ed.). (1985a). *Teaching and learning mathematical problem solving: Multiple research perspectives*. Hillsdale, MI: Lawrence Erlbaum Associates.

Silver, E. (1985b). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: multiple research perspectives* (pp. 247–266). Hillsdale, MI: Lawrence Erlbaum Associates.

Sinclair, N. (2008). Aesthetics as a liberating force in mathematics education. *ZDM: The International Journal of Mathematics Education, 41*, 45–60.

Part IV
Bringing Together Technology, Creativity, and Affect and Aesthetics in Mathematical Problem Solving

Chapter 25
A-Musing with Chess and the *Eight Queens* Math Puzzle: Looking for Connections Between Problem-Solving, Technology, Creativity, and Affect in Mathematics Education

Viktor Freiman

> *The crisscrossing of a checked pattern may give some people a "nervous feeling", but when a recreational mathematician sees a checkerboard floor [their] mind leaps happily towards puzzle possibilities.*
>
> —Martin Gardner

25.1 Introduction

It is probably not often that students in mathematics have a chance to deal with tasks that require more than one mathematics lesson to solve, or a whole day, or a month, or a year. Similarly, students in mathematics may not experience problems that cannot be solved or are completely open-ended. This raises the issue of how students' curiosity and perseverance can be fostered to pursue investigating one particular mathematical problem over a long period of time or to move beyond the solutions already found while looking for new strategies or other possible solutions. Likewise, there is the issue of ways to push the boundary of students' creativity beyond questions formulated by someone else so that they become active problem seekers and posers. Such endeavours are likely to involve teaching them to take risks by attempting unknown tasks and helping them in mindset growth (Yeager & Dweck, 2012) while promoting resilience facing difficult challenges.

There exists a category of mathematical problems which are very well-known (some have a long history), which are being used as part of recreational activities

V. Freiman (✉)
Université de Moncton, Moncton, NB, Canada
e-mail: viktor.freiman@umoncton.ca

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9_25

(mathematics for fun) yet contain (or lead to) very complex and advanced mathematics theories; some remain at the heart of modern debates and even unsolved. Being explored in depth by the creative work of generations of mathematicians and amateurs of mathematics, they still trigger the interest of today's scientists, educators, and ordinary people who like intellectual challenges and enjoy looking for answers to difficult questions.

Map colouring, Konigsberg bridges, Goldbach's conjecture, Collatz conjecture, Tour of the Knight on a Chessboard, and *Eight Queens* puzzle are very popular examples of this type of problems. They have drawn attention of famous mathematicians of the past (such as Gauss and Euler) and of modern times (including Erdős and Tao) who, among many others, attempted (or are still attempting) solving them, thus pushing boundaries of our knowledge even further. Moreover, some of the problems are recently overtaken by computer scientists and programmers who enrich them with novel and creative solutions using technology. What is most intriguing is that all these problems can be tackled (at least in some adapted forms) by very young learners, even at the primary level.

When teaching a challenging mathematics curriculum to kindergarten and primary students (Freiman, 2006, 2018), I noticed a particular value of the above-mentioned problems in that they are quite simple in terms of initial access but potentially very rich in the mathematics involved. For instance, one of the Grade 3 problems from the textbook *Défi mathématique* 3 (1981) is the famous *Eight Queens* chess puzzle. In asking students to first put six queens, then seven queens, and then eight queens on the 8 × 8 chessboard in a way that no queen was crossing the path on any other, all the students started working enthusiastically. They were easily able to find the solution with six queens and many, after some trials and error, succeeded with seven queens. Yet, adding the eighth one seemed to be a big challenge, not only for young learners. The question that arises is why this problem is so challenging and how it can be solved.

In fact, already the initial problem with six queens allows students to see that a steplike (following the knight jump) disposition of queens turns out to be a plausible strategy: a1, b3, c5, d7, with addition of e2 and f4, gives a correct solution. The students are then puzzled about why the same strategy does not work for eight queens. Students quickly see that after placing first four queens (a1, b3, c5, and d7), then adding two more (e2 and f4), the placing of two further queens turns out to be difficult, and, in the end, impossible. They then wonder what to do or how to (re-)organize their search for a solution. When finally a solution is found, new questions may arise such as the number of different solutions that might exist, or what happens if the chessboard is not 8 × 8 but 7 × 7 or 9 × 9; how would such changes impact on the number (set) of solutions?

Another example of such challenges, accessible to all, is so-called Sudoku puzzles, which have become very popular in recent times and can be found in recreational sections of many modern newspapers almost on a daily basis (e.g. one Sudoku puzzle was published on 15 May 2018 in a local New Brunswick, Canada, French Acadian newspaper, L'Acadie Nouvelle, https://www.pressreader.com/canada/acadie-nouvelle/20180515/281861529150574). When pointing to many people's passion in solving Sudoku puzzles, Crook (2009) asks why interest in solving them continues if today's computer technology allows for simply putting the numbers into an app and getting an almost instant solution. Recreational by nature, problems like a Sudoku puzzle and the *Eight Queens* chessboard puzzle not only have interesting historical roots demonstrating how the collective creative scientific mind was affected by seemingly simplistic entry-level rules that constantly show multiple deep connections to serious mathematical theories, they also attract today's mathematicians and computer scientists who look for unknown paths and novel ways of investigation. For the amateur, and for educators who want to instil love for mathematics and pleasure from taking on challenges in their students, such puzzles remain, as is mentioned by Crook (2009), sources both of enjoyment and opportunity for (hopefully productive) struggle by posing a 'serious mental challenge' and by giving the solvers 'an encouragement to persist' (p. 460). In this sense, Danesi (2009) argues that "satisfaction that comes from solving a puzzle seems to provide 'relief' from the large-scale puzzles inherent in everyday life'[1], an observation that might explain the phenomenon the author labels 'puzzle crazes"[2], referring to a surprising growth in popularity which was also remarkable around the chessboard puzzles in different moments of human history and which I briefly outline in the next section.

25.2 170 Years Ago: It All Starts with a Chess Puzzle

The history of the *Eight Queens* puzzle is believed to have started in 1848 when there was a first publication in a *Berliner Chess Magazine* (Schachzeitung). It reads in German as follows:

> Wie viele Steine mit der Wirksamkeit der Dame können auf das im Uebrigen leere Brett in der Art aufgestellt werden, dass keiner den andern angreift und deckt, und wie müssen sie aufgestellt werden?
>
> (Free English translation: How many stones representing a Queen can be placed on an otherwise empty chessboard in such a way that no one attacks or covers any other one, and how can they be placed?)

While the editors (referring to an anonymous author – 'Schachfreund') think the solutions are not very difficult ('allerdings nicht sehr schwierig') to find, they still think that collecting readers' voices will be a pleasant endeavour ('angenehme Pflicht').

While its author was anonymous, many sources attribute the problem to the German chess player Max Bezzel. Few solutions were submitted to the magazine in the following year, and, overall, about 40 of them were reported between 1849 and 1854. It turns out that the problem was republished (or independently published) in June 1850 in the *Leipziger Illustrated Magazine* with reference, this time, to

[1] https://www.psychologytoday.com/us/blog/brain-workout/200906/the-appeal-sudoku.

[2] Idem.

Dr. Nauck who was long time considered as the one who was the first to publish it and even to suggest it to Gauss. In fact, the issue dated 1 June 1850 of the magazine (p. 352) gives a short article entitled ‘Eine in dass Gebiet der Mathematik fallende Aufgabe von Herrn Dr. Nauck in Schleusingen’ (One problem from the field of mathematics given by Dr. Nauck from Schleusingen). The text of the problem reads as follows:

> Man kann acht Schachfiguren, von jede den Rang einer Koenigin hat, auf dem Schachbrete so aufstellen, dass keine von einer anderen geshlagen warden kann.
>
> (In English: It is possible to put eight chess pieces, each of them representing a queen, on a chessboard, so no one can capture any other one.)

Nauck comments that, given many existing solutions, it would be interesting to find in which boxes the queens can be placed if two of them are already on the board: in boxes B4 and D5 (the rule remains the same: no queen can capture any other queen).

Nauck also mentions that it is often thought (and I saw it many times with young children trying the puzzle) that if seven queens are already placed, it would be easy to add one more. At the end of the notice, Nauck indicates that he has found 60 different solutions while working on the problem in a mathematical way and discovering some interesting rules. Finally, the article says that “a happy Doctor will be pleased to provide his solution to friends of chess or of mathematics who will be interesting in the problem” (idem, p. 352).

In the September of the same year (1850), Nauck published all solutions of the problem and corrected himself regarding the number of solutions (92 rather than 60). He also demonstrates an ingenious way of recording the solutions using digits (instead of letters) for the rows on the chessboard and thus discovering many interesting patterns in numbers representing positions of queens that helped him in his search for different solutions.

25.3 Gauss and Schumacher[3] Jump in: Search for Initial Solutions

While the puzzle can be analysed by chess players and by mathematicians from their own perspectives and goals, this was yet another opportunity in linking chess and mathematics (something which was not new in the middle of the nineteenth century). In fact, already a century before, at least two famous chess puzzles were studied in-depth by Euler, one of which is finding a path for a Knight on a chessboard to visit each box on the board exactly once and to return to the same box. The second one was a similar problem to the one of eight queens but for eight rooks (Euler, 1759).

[3] An exchange of letters between a mathematician C. F. Gauss and an astronomer H. C. Schumacher was published by C. A. F. Peters in 1860 (C. A. F. Peters (Hrsg.) Briher. Erster Sechster Band. Altona, Druck von Gustav Esch).

It is therefore not surprising to learn that Gauss, who is said to be an active reader of the *Illustrated Magazine*, got interested in Nauck's challenge and, according to several authors, without investing all his time, made a number of attempts at working on the problem, each time sharing his findings by exchanging letters with his friend the astronomer, Schumacher. The first letter of the series (dated 1 September 1850) refers to the June 1st issue of the magazine and to the puzzle as being 'similar to one of finding the Knight Paths on the chessboard ('Rösselsprung'). Along with the text of the problem, Gauss mentions that Nauck reports having first successfully found queens on b5 and d4 squares (Fig. 25.1) before finding a total of 60 solutions. On his side Gauss (mentioning that it is easy to find the way to put seven queens but difficult to see how to add one more), after spending some time on the puzzle and trying different possibilities of specific cases, said he could easily find different solutions, and their total number is actually more than 60. Namely, he has found 76 solutions. After making this comment, Gauss concludes the letter with other matters (not related to the problem). In his response (dated 4 September 1850), Schumacher makes some comments of his own about the problem, saying that it is very interesting but appears to be more difficult than seems to be on a first sight. He asks if he can share it (without mentioning Gauss) with his friend Staunton, an editor of Chess Chronicle magazine. He also asks whether 76 solutions are those to the original problem or its particular case with queens on b5 and d4. He also says that sometimes small masterpieces of mind challenges can be found in chess magazines (he mentioned three of them, *Chess Chronicle*, *la Régence*, and *Berliner Schachzeitung*), and now knowing Gauss's interest for them, he would have the pleasure to share such puzzles with him next time.

The exchange between the two scientists continues with Gauss's letter of 12 September 1859 where he confirms that the case with queens on b5 and d4 gives only one solution, but he also mentions that he needs to correct his initial number of solution bringing it down to 72 (from 76). He provides more details about his way of handling the problem: finding nine completely different solutions, each generating eight positions that can be obtained by the symmetries of the chessboard (using reflection and rotations, even by looking on the reverse of the sheet of paper). Yet

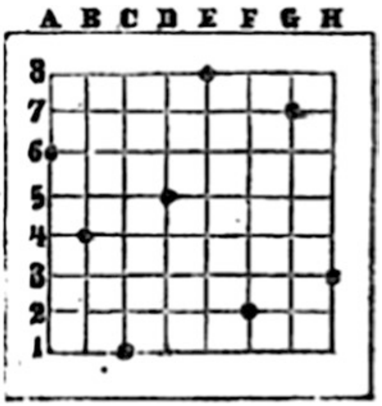


Fig. 25.1 One solution with queens on B4 and D5 shared by Gauss with Schumacher

Gauss says he believes that some symmetrical solutions could exist which would bring eight solutions down to four, two, or even one. He thought he had found one of them (with 4 solutions but after having sent the letter, he realized the solution was incorrect; therefore there are 72 of them instead of 76). He concludes the letter by saying that it is plausible to imagine that the symmetrical solution does not exist.

From the subsequent response by Schumacher, it becomes clear that Gauss succeeded to trigger in Schumacher a great interest in the problem so that he attempted to provide a more detailed analysis of how the solutions (or more precisely, their number) can be found. For Schumacher, the positioning of queens in the columns a and b would be critical in determining a total number of solutions since he believed that once queens are correctly placed in the two columns, there would be only one original (without taking symmetries into account) solution. Moreover, he argues that of eight squares in the column, only the first four (a1, a2, a3, and a4) matter, because of the 180° rotation. Now, he is not quite sure how to handle the corners. Hence, depending on counting a1 (with other six options) or not (so, five options with a2, a3, a4), Schumacher estimates either $21 \times 8 = 168$ or $15 \times 8 = 120$. Schumacher admits of perhaps counting, mistakenly, solutions that do not exist. While considering nine as base solutions suggested by Gauss, he says he would be thankful if Gauss could teach him a correct way to handle the problem. Later in the same letter, Schumacher returns to the problem insisting that his conclusions are based on the conjecture that any disposition of queens in columns a and b produce only one solution, a conjecture which might not be correct he once again admits.

The discussion goes further with Gauss providing much more detailed analysis of the problem in which he gives six examples of positions (Fig. 25.2) which illustrate some important discoveries he had made, the first of which is finding an example of a correct symmetric solution (number 1). This one has only 4 symmetries, instead of 8.

Furthermore, Gauss takes up two assumptions made by Schumacher, one that for all placements of two queens in the columns a and b one solution exist, and the second that this existing solution is the only one. Indeed, Schumacher was right to argue that having both conjectures right, one could find six more. This gives 42 possible solutions. Then 42 divided by 2 (for reflections) would give a total of $21 \times 8 = 168$ solutions.

However, both assumptions are, in fact, incorrect. The first one is disproved by position 4 (no solution for queens a1 and b3). Positions 2 and 3 illustrate the cases where the second assumption does not work; here there are two distinct solutions with two queens in a2 and b5.

After having invalidated Schumacher's conjectures, Gauss makes reference to a recent correction made by Nauck in the September 1850 issue of *Illustrated Magazine* that brought the number of solutions to 92 as being produced by 11 distinct positions with 8 symmetries and one with 4 symmetries. While stating that the proof that 92 is a maximum number of solutions was not provided by Nauck, Gauss uses his way of recording solutions using numbers (digits) to designate rows and columns thus 'arithmetizing' the problem in a quest for a systematic (almost automatic) producing of all solutions which would satisfy two conditions: (1) the

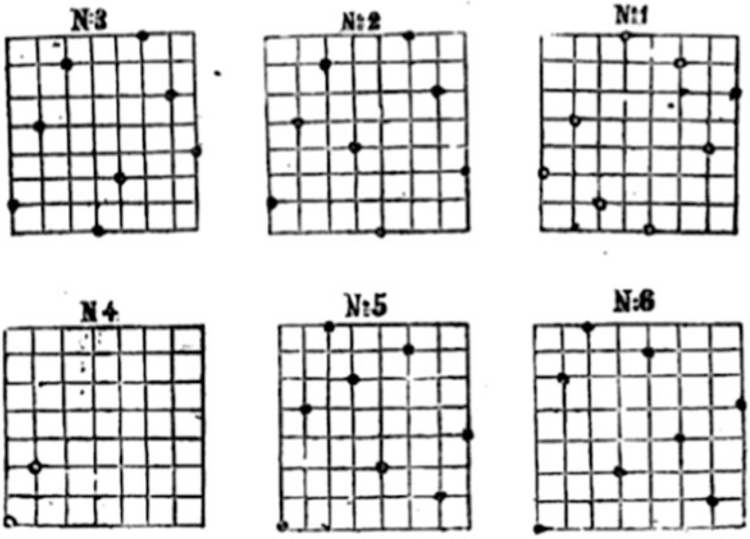


Fig. 25.2 Six solutions given by Gauss

sums of the digits designating a column plus corresponding row with the queen are all different; and (2) the sums of the digits designating rows are added to the numbers of columns reordered in a decreasing order are also different.

Hence, Gauss gives, as example of this procedure (algorithm, using today's terminology), the sums for the solution 15863724 (position 5), as following:

1 2 3 4 5 6 7 8
1 5 8 6 3 7 2 4

The sums are 2, 7, 11, 10, 8, 13, 9, 12 – all different.
And in decreasing order:

8 7 6 5 4 3 2 1
1 5 8 6 3 7 2 4

The sums here are 9, 12, 14, 11, 7, 10, 4, 5 – also all different.

Gauss also shows how these two rules can help in a systematic search for solutions (which makes it very easy, with the investment of '1–2 hours of time') (idem, p. 119):

One can begin, for instance, with 1 and 3. What could be next? There are only four options for the next number (row): 5, 6, 7, or 8. Let's try 5 in the third place (column). Then, for the fourth place, 4 and 6 are to be excluded (4 + 4 = 8, the sum which already exists in the column 3 (3 + 5 = 8) or, in decreasing order, 6 + 5 = 11 is the same as 5 + 6 for the column 3. If none of the numbers work, then one could go back to the beginning and continue trying other candidates; this method would eventually lead to finding all solutions. Gauss adds that using a grid for noticing possibilities would help to make verification faster.

In a step that adds finesse to his analysis, Gauss connects the puzzle to the field of complex numbers by reformulating its text as finding all complex numbers in a form a + bi in a such a way that (1) a and b have values 1–8; (2) all values of a are different, same for values of b; (3) the sums $a + b$ are different; and (4) the differences $a - b$ are also different. According to Gauss, this wraps up the problem in 'its most elegant way' (idem, p. 120).

By connecting the example of the *Eight Queens* problem to the discussion of mathematical problem-solving, technology, creativity, and affect, several lessons can be learned from this story of correspondence between two scientists.

25.4 Lessons We Learn from the History

There are several lessons to learn from this historical example regarding how technology, creativity, and affect ignite and shape a collective interest in solving mathematical problems.

First, the publication of a problem in a magazine not specifically devoted either to chess (which was the case of Bezzel's) or to mathematics. It was presented by Nauck as a chess puzzle of a recreational nature, thus addressing to a wide readership. Yet eventually it triggered the interest of not only of chess amateurs but also those who love mathematics.

Second, there is the noticeable and particular attention (interest) that the problem drew in Gauss who spent considerable time analysing it.

Third, there was an obvious desire (or even a need) to share common passion with the friend. It is also remarquable to see how this shared interest initiated a rich discussion (at least an exchange of comments) between them.

Fourth, even though Schumacher was not a mathematician (but rather an astronomer), Gauss used his interest in chess as a motivational 'hook'; so the first call was not a mathematical one but rather from the chess context.

Fifth, the discussion moved from the initial representation (and clarification) of the problem, through the first mutual (not necessary successful) attempts at finding solutions (or more solutions) to succeeding finally in bringing (at least at first glance) a certain degree of a shared understanding of the problem without necessarily coming to a conclusion (that leaves space for eventual continuation; an open-ended process).

Sixth, the solution process is interesting in terms of patterns of creativity as it provides an interplay of different representations (diagrams, letters, numbers), a use of a variety of mathematical concepts (symmetries, arithmetic operations, complex numbers) and strategies (estimations, systematic trial and error), as well as numerous remarks that demonstrate affective attachment to the problem (recognizing errors, admitting possible lack of detail or proof, while, at the same time, recognizing the problem as 'interesting', 'easy to handle', and 'elegant' with a connection to complex numbers (idem., p. 120).

Seventh, the way of finding solutions indicated by Gauss (with reference to Nauck), while showing the approach to the search for solutions, is clearly inductive and procedural (and thus technological).

In what follows, my further analysis of the future development on this problem provides more details of each of the aspects that can be identified from the initial steps of its history. The problem was posed more than 170 years ago and still keeps many scientists from different fields (mathematics, computer science, and artificial intellect) busy with its (still) intriguing complexity. In terms of its complexity, I do not mean the original version of the problem as it was initially published by Bezzel in 1848, or investigated by Nauck, Gauss, and Schumacher. I mean its continuous generalizations which generate new types of questioning, new ways of looking for creative solutions, use of new (twentieth and twenty-first century) technological affordances, and the sheading of light on the affective features of problem-solving. The latter, as a focus of analysis, can involve the pleasure of challenge and the joy of working hard on trying to advance on solutions, as development, which enhances perception of an ever-increasing beauty of the fireworks of creative human endeavour.

25.5 Chess Puzzles as a Part of Recreational Problems in Mathematics: Rich Source for Pleasure and Challenge

After having been extensively studied by mathematicians in the second part of the nineteenth century, the problem of eight queens has become a (almost necessary) part of collections of recreational mathematical problems. Namely, in 1852, a French mathematician Eugène Lionnet, who founded, in 1848, a philotechnical association (Association philotechnique, https://www.philotechnique.org/) in Paris, France, to provide training for adults, has included the *Eight Queens* problem from the magazine for candidates for the examination (Journal des candidats aux écoles polytechnique et normale) (Question 251, Lionnet, 1852, pp. 114–115).

Lionnet used the mathematical context of placing the first eight (natural) numbers in a row in such a way that the difference between any two of them would be unequal to the difference of their ranks in the row. He asks how many of such combinations exist while mentioning that the problem provides, as a direct consequence, all the solutions to the problem of putting eight queens on a chessboard without having any of them attacking the other seven. Without providing detailed solutions, it is mentioned in the same book (p. 148) that, in response to Lionnet's problem, M. (Monsieur) Koralek, using empirical methods, found 92 solutions. Some 17 years later, the problem (also authored by Lionnet, 1869, Question 963, p. 560) was posed for any number (n) of squares on a chessboard (in arithmetical form, it is a search for numbers 1, 2, 3, …, *n*, satisfying the above-mentioned conditions). Giusto Bellavitis, an Italian mathematician (1803–1880), found 12 basic (essentially different) solutions for question 251 posed by Lionnet in the third edition of the works of the French Science Academy (Terza rivista di alcuni articoli dei comptes rendu della Accademia delle sienze di Francia, 1861, p. 41).

In 1862, a Russian mathematician and chess player, Carl Jaenisch (Carl Ferdinand von Jaenisch), has published a book devoted to mathematical analysis of the game of chess in which he provides a detailed solution to the *Eight Queens* problem (von Jaenisch, 1862). Jaenisch's work, in general, and his solution of the queens puzzle, as a "troublesome and difficult task" (Lehman, 1862, p. LXXXVII), were praised during The Chess Congress in 1862 in a commentary by Lehman who valued "a rare combination of great mathematical knowledge' with 'great proficiency in Chess" (idem).

The work of Cretaine (1865) on the problems of the Knight tour (extensively studied by Euler, among others) brought a very interesting and creative geometric representation of the *Eight Queens* problem. By cutting the square by 2 diagonals into 8 rectangular triangles, Cretaine noticed that inside of each triangle there are 10 parts, a complete square or half of it (from 16 squares divided by 2 by diagonals; for a total of 80 parts) (Fig. 25.3). Beyond a very original way of searching for solutions, Cretaine (1865) used a particular mnemonic sentence, for example, one shown on the position below: 'Ma CHère anNa Fait La QuéTe' (there is no particular sense in this sentence) to help a better orientation in different spatial configurations which lead to different solutions.

While presenting a detailed analysis (including presenting an ingenious technological tool) given by Cretaine is beyond the scope of this chapter, he makes an important observation regarding the affective side of his work. Namely, he says that the first trial is not always successful since there is the work of turning the position around in different sense, but with some practice, especially making connections between the sides of the square and mnemonic text, the orientation becomes clearer and more rapidly accessible. The second comment is also interesting since Cretaine mentions that his work on the problem had a more recreational (than purely mathematical, VF) nature with the goal to surprise those who, over the previous 15 years, had tried to find all solutions to the puzzle. In this, he recognized the importance of the results of many mathematicians who worked on the problem during this period and whose findings he also presents in his text (the reference is made particularly to the solutions obtained by Solvyns and Jaenisch). For the purpose of this chapter, and the complex relationship between affect, technology, creativity, and mathematical problem-solving, it is the comment about recreational nature of Cretaine's work that prompts the need to look closely at this aspect of mathematics and its interesting connections to chess.

The literature on recreational mathematics shows that it has rich historical roots and, according to some authors, is not dissociable from mathematics itself. It is a form of mathematical problem-solving in a creative way that can bring much pleasure to professional mathematicians but also to everyone who loves mathematics and its challenges when presented in a context of games, puzzles, and other forms

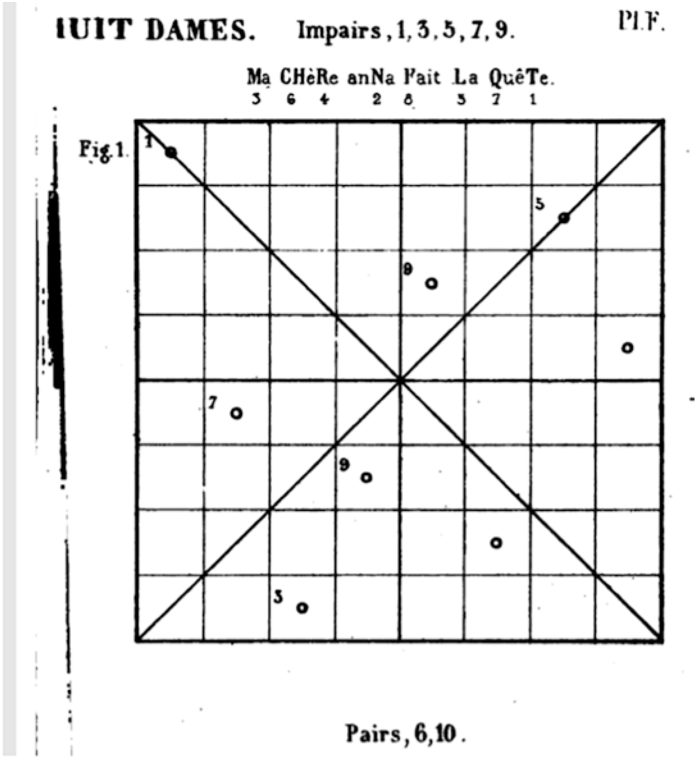


Fig. 25.3 Cretaine's (1865) representation of solutions: cutting chessboard into triangles and using mnemonic techniques for checking out solutions

of intellectual entertainment. Recreational problems can be found in very ancient cultures, in the Middle Ages, and in more recent times. For example, the first printed book published by Bachet in 1612 contains several problems identified, already, in the title as 'pleasant' and 'delectable' (Bachet & de Méziriac, 1884).

Not being limited to the potential impact on solvers, the problems from Bachet's book, as well as the author's commentaries with the analysis of possible solutions and strategies (such as producing magic squares), were sources of the development of mathematical ideas and theories and its further popularization. Yet, the extending of puzzles is not limited to the growth of mathematical, or more generally scientific, thought; it also had a great impact on the spiritual life of earlier mathematicians thus extending the affective value of such problems in terms of appreciation of beauty and magical power.

Among the problems that have its firm place as classics of recreational mathematics, there are different chess puzzles. One of them, known as Chess Knight Tour, seems to have drawn the attention of several mathematicians early in the eighteenth century (Stewart, 1997). Euler (1766; English translation, 2017) reports getting involved in the solution of the puzzle in a context of a chess game where someone suggested a problem of the knight visiting all squares of the chessboard but never landing twice on the same square. To play the game, it was decided to use chips placed on each square, and someone playing the game was challenged to collect all chips by removing them one by one once they are found on a square visited by the knight.

The participants were convinced the game was an easy one and were surprised at not being able to complete the tour, despite of several attempts. The person who offered the challenge knew the problem and had to show the solution to the group; yet, Euler admits that it was difficult to follow the path and remember the order of squares on it. Euler himself says that he had to make several tries to get at least one solution, but this was difficult to repeat. One solution that Euler gives in his paper presents a path which begins in a corner of the board but admits it provides little help when the path was to begin from a different square.

Euler then analyses the challenge in more depth, finds some patterns that help in doing a more systematic search, yet a number of solutions seems to be too high to get a more precise analysis. Hence, Euler decides to look into simpler cases; for example, ones with 4-, 9-, 16-, and 25-squares board. While the first three sizes of board lead to a negative answer that a complete Knight tour does not exist, the last one (with 5×5 board) gives a possibility to discover some patterns. As an example, one of the solutions which allows for an intriguing pattern based on 180° symmetry meaning that two symmetric numbers (like ones in the opposite diagonal corners) always make a total of 26 (Fig. 25.4):

Besides the appreciation of the beauty of the mathematical pattern on this board, it helped Euler in finding new solutions derived from this one (http://eulerarchive.maa.org/docs/originals/E309.pdf, p. 334).

On a wave of increasing popularity of puzzles and logical games at the end of the nineteenth century, both chess puzzles and their numerous variations and generalizations have found their firm place in many recreational mathematics books as part of their collections of challenging problems. A famous book of this kind was a recreational mathematics book by Lucas (1891) that is discussed in more in detail in the next section.

Fig. 25.4 Euler's solution of the Knight's Paths puzzle: looking for number patterns

23	18	5	10	25
6	11	24	19	14
17	22	13	4	9
12	7	2	15	20
1	16	21	8	3

25.6 The *Eight Queens* Puzzle as Part of a Collection of Recreational Mathematics Problems

Before going in more depth into Lucas's analysis of the *Eight Queens* puzzle, it is worth noting a few comments from the preface of the book which justifies its value as the following: (1) the author being proud of finishing his first book as product of his spirit which creates a first affective attachment to it as for a father to his first child; (2) even if a first look at the title of the book could lead to a conclusion that it is just a game or amusement, it is still plausible to expect that some knowledge of science of number going beyond a mediocre level is necessary to take full benefit of the explanations given in the book and also get some deeper insight into small inventions shared by the author; and (3) it is also a question of the utility of such a book, question that arises in many (also scientific) contexts; yet Lucas grounds his beliefs of so-to say 'utility' of his work into the perspective of empowerment (power of knowledge):

> En disséquant les résultats du calcul, en créant des abstractions, en les combinant, variant, tourmentant, le géomètre fortifie le raisonnement et acquiert sur cet universel outil un pouvoir infini. (page VIII)
>
> (English loose translation): in collecting results of calculations, in creating abstraction, in combining both, in modifying, in mixing up, the geometer solidifies reasoning and acquire an infinite power over this universal tool.

Even more interesting are the final remarks made by Lucas (idem) in which he emphasizes the socio-affective components of his work as conversation with an unknown friend. This is a very important aspect to the author who already sees the fruits of recognition being the peers who express their empathies and also of collaborations that were initiated following the first edition of the book.

By bridging the past and modern issues of mathematics and natural philosophy, in the introduction to Volume 1 of his book, Lucas expresses a need for a new geometry, one of the situations (Géométrie de situation, or Geometria situs) to which he connects the *Eight Queens* puzzle. This geometry, which had been envisioned by Leibnitz, would allow calculations to be done on different groups of numbers according to their place. As an example, Lucas cited Euler's work on puzzles (as already mentioned above), one of which was related to the Knight's Tours on a chessboard and the second a magic square (a predecessor of Sudoku). Lucas mentions that, without providing a solution to the Knight Tour Puzzle, Euler succeeded in describing a method to generate solutions. It appears that the problem attracted many mathematicians who made important progresses. Among such progresses, Lucas cites the success in finding a Knight Tour which, being represented by numbers, forms a magic square. As noted above in the section about the exchange between Gauss and Schumacher, Gauss had identified the *Eight Queens* as having an analogy to the Knight Tour.

While engaging the reader in a pleasure of analysing the *Eight Queens* puzzle, Lucas begins with the historical account with reference to the correspondence between Gauss and Schumacher (mistakenly thought as being originally posed by

Nauck to Gauss; this error, along with ones committed by some other authors who even thought it to be firstly posed by Gauss, the whole matter is analysed in depth by Campbell, 1977). By making reference to Lionnet and Belavitis, the authors mentioned above, Lucas informs the readers about the contributions made by Glaisher (1874) and Parmentier (1884) while focusing in more detail on the methods suggested by Gűnther (1874) and de La Noë.

The first method (by Gűnther) is illustrated by an example of 5 × 5 matrix ingenious representation where each case is labelled by a letter (a–f) and an index (1–5) in a way that the same letters are aligned parallel to one diagonal and the same indices are aligned parallel to the other diagonal (p. 70) (Fig. 25.5):

Then, according to the method, one can make a list of five terms of the matrix which do not contain the two elements either from the same row or from the same column. This obtains 120 solutions of the five-rook puzzle (this has the same conditions as for five queens but applied to the rooks, according to the rules of moving a rook). The next step is to eliminate the records containing the same letters or the same indices which would leave the remaining combinations presenting all solutions of the *Five Queens* puzzle. For chessboards larger than 5 × 5 (such as 6 × 6, 7 × 7, or 8 × 8), an algebraic method from the theory of determinants could be helpful yet seems to be hardly applicable to even bigger boards (9 × 9 or 10 × 10).

Another ingenious method given by de La Noë was also discussed by Lucas. This time, representational creativity led to decomposing the chessboard into concentric squares (Fig. 25.6):

By placing the first queen (a) in the middle square, for example, one can notice that it attacks 28 cases, the same number as the number of squares in the outer stripe. Then, one could try to put as many queens as possible in the second stripe eventually finding two symmetric dyads (b, c) and (b1, c1). Applying the same rule to the third stripe, squares (d) and (e) can be found. Yet, no more squares remain to complete the puzzle. An interesting observation is shared by Lucas; in fact, no solution exists with queens simultaneously placed in the first two squares, a pattern which apparently holds for all chessboards with an even number of cases, up to 8 × 8, but not for one of 10 × 10.

Lucas concludes his analysis with a rather recreational note by giving a list of 12 basic solutions along with mnemonic procedures of how to memorize them; for

Fig. 25.5 Gűnther's method using determinants

$$\begin{vmatrix} a_1 & f_2 & g_3 & h_4 & i_5 \\ b_2 & a_3 & f_4 & g_5 & h_6 \\ c_3 & b_4 & a_5 & f_6 & g_7 \\ d_4 & c_5 & b_6 & a_7 & f_8 \\ e_5 & d_6 & c_7 & b_8 & a_9 \end{vmatrix}$$

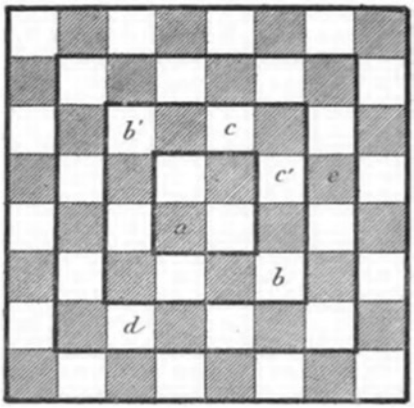


Fig. 25.6 de La Noë's method of searching for solutions

{ *C'est difficile si tu veux que huit cadrent.*
{ *sept deux six trois un quatre huit cinq.*

72631485
61528374
58417263

Fig. 25.7 Lucas: vertical pattern in verifying solutions

instance, for the first 3 solutions (Fig. 25.7), he gives for the first one a mnemonic sentence:

Then, one can obtain solutions 2 and 3 by moving each queen one row down thus subtracting 1 from each of 8 digits (in the case of a number 1 – queen is moved to the row 8) (Fig. 25.8).

For the solutions 4, 5, and 6, the method would be moving all queens one rank towards the right (e.g. 3, the last digit in the solution 3 would be the first in the solution 4, and so on).

In the twentieth century, the queens puzzle continues to occupy an important place in recreational mathematical books, each bringing something particularly amazing. For example, in the first decade, the Russian mathematician Ignatiev (1908) gives a complete list of 92 solutions (p. 147) and then provides an interesting method of construction of the table (a loose summary from Russian):

> Put a queen on the lower possible square in the first column; then the same for the second column, same for the next one (always referring to the position of a queen in the closest column on the left and taking the lowest possible row). This can be repeated until no more move is possible. Then we can go back to the previous column, move the queen one, two, three … squares up, then continue the same procedure.

35841726
46152837
57263148

Fig. 25.8 Lucas: horizontal pattern in verifying solutions

The 1940 edition of the Ball's book (first edition dated 1892, revised by Coxter 1938, reprinted 1940) has an interesting reference to Guenther's work with determinants arguing that the method would not be possible to apply to a bigger chessboard: for the 8 × 8 one, the determinant would contain 8! = 40,320 terms – impossible to handle unless a wiser method applies, for example, one given by Glaisher (1874); if all solutions for the n×n board are known, then for some types, solutions can be obtained for $(n + 1) \times (n + 1)$ board; afterwards, all other solutions can be easily produced (p. 166).

The more the collective search for the solutions of the *Eight Queens* (or a more generalized N-queens puzzle), as well as its endless modifications and extensions, moved towards the second part of the twentieth century and continued on into the twenty-first century, the more its computational or algorithmic nature was drawing attention to the point the puzzle has become an (almost) mandatory part of computer scientists' activities. In the next section is a brief survey of developments from hand-made calculations to computing algorithms, along with other affordances of digital technology.

25.7 From Hand-Made Calculations to Sophisticated Computing Algorithms

One of the most popular methods of algorithmizing the search for solutions for an n × n board is called 'backtracking'. This principle can be more easily shown on a 4 × 4 board, as it was done, among others, by Knuth (1968, p. 42) (Fig. 25.9):

The principle is based on a 'search tree' methodology which can be explained as follows:

1. Starting from an empty board, first place a queen in column one. There are four positions for the queen, which correspond to the four children at the root of the tree.
2. In column one, first try the queen in row 1, then row 2, etc., from left to right of the tree.
3. After successfully placing a queen in some column, then proceed to the next column and recursively try again. If stuck at some column k, then backtrack to the previous column $k - 1$, and again try to advance the queen in column $k - 1$ from its current position.
4. All nodes at level n represent solutions; for $n = 4$ there are two solutions.

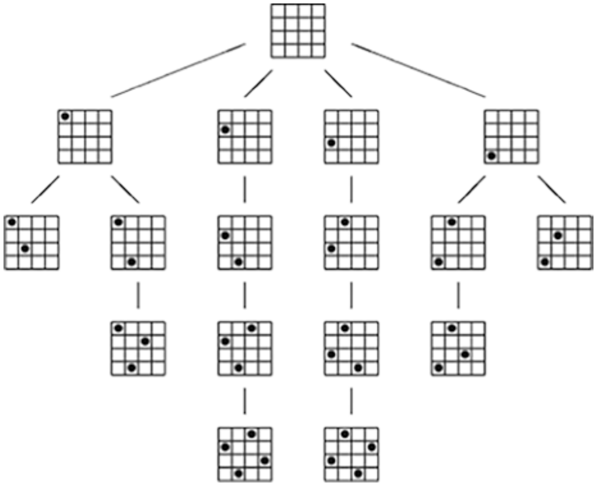


Fig. 25.9 Knuth (1968): an example of a tree for searching solutions for a 4 × 4 chessboard

This principle works for all *n*, but finding solutions for a bigger *n*, such as *n* = 8 (as already described using a determinant-based method), it becomes impossible to handle manually. Hence, it is meaningful to turn to computer programming as an aid for a complete search.

In fact, Wirth (1985), who took the Knight Tours Puzzle and N-Queens puzzle as examples to illustrate the power of computer programming to help in a search for solutions, noticed, among other things, that the absence of an analytic solution is a common characteristic of these problems (p. 97):

> they require large amounts of exacting labor, patience, and accuracy. Such algorithms have therefore gained relevance almost exclusively through the automatic computer, which possesses these properties to a much higher degree than people, and even geniuses, do.

At the same time, construction of (an efficient) algorithm is not an easy task and requires some very particular and creative strategies which, for the *Eight Queens* puzzle, were described in a very general way by Dijkstra (1971).

First of all, Dijkstra points to the following two conditions:

1. There are eight queens on the board
2. No two of the queens can take each other.

If a 'first-glance' approach would be to think about looking for 'all positions with eight queens on board' and for 'all configurations where, among N queens, no two could attack each other', this approach would lead to a huge amount of possible configurations and eventually to an inefficient algorithm.

According to the author, adding other conditions that add more restrictions to the search could help:

3. No row may contain more than one queen; and each row is to contain precisely one queen. The same for the columns.

4. There are 15 'upward' diagonals, each of them containing at most one queen; i.e. eight upward diagonals contain precisely one queen and seven upward diagonals are empty. The same for downward diagonals.
5. Given any non-empty configuration of queens such that no two of them can take each other, removal of any of these queens results in a configuration sharing that property.

As result of continuous process of refining the procedure, and finally reaching the programming stage which would lead to the 92 well-known solutions, Dijkstra concludes his description with several remarks that emphasize not exactly knowing how to program the puzzle but rather how to extract from the process of construction the algorithm's important qualities which identify the process and a learning process, when learning means making discoveries.

The first type of discovery is related to the method (not learning the method, in this case backtracking, but learning to discover the method which would reinforce someone's capabilities, when facing with a different problem to be solved by a different method, to be able to see how the latter may be discovered by a very similar method). The second type of discovery is related to the use of recursion which is a very common element of construction of this kind of algorithms. Again, according to Dijkstra, the matter is less about learning the technique of recursion, and of how it works in a concrete program, but rather a 'collection of considerations leading to the discovery that in this case recursion was an appropriate tool' (idem). Finally, Dijkstra argues for applying an approach which prioritizes thinking about algorithms in much detail prior to putting it into a concrete representation. This puts a lot of emphasis on an analysis of a problem.

When going back to the history of the puzzle, and to the approaches used to finding solutions, many aspects can be seen of this kind of thinking. Today these forms of thinking are called 'algorithmic thinking' or even in more modern manner 'computational thinking'. Whatever the term, these forms of thinking become a crucial part of learning that combines affective components (such as pleasure of taking on challenges, or perseverance in attending the goals), creativity (here capacity to make important discoveries out of the in-depth analysis of the problem), and technology (affordances which are available at a certain moment in time). This observation leads to the conclusive notes on the educational value of recreational mathematics in general and chess puzzles, in particular.

25.8 Conclusion: On the Educational Value of the *Eight Queens* Puzzles and Their Multiple Connections to Mathematics, Computer Science, and More Largely Lifetime Skills and Attitudes

While reflecting on the educational value of recreational mathematics, it is plausible to affirm that it does have potential to increase students' motivation to learn. Unsurprisingly, according to Kulkarni (2013), who makes reference to the work of

Novak and Reiter, students who enjoy solving number puzzles, similar to Sudoku, (Gomez & Novak, 2014; Murawska, 2018; Reiter, et al, 2013) "can develop positive attitudes toward other forms of math in non-puzzle contexts as well".[4] This is particularly true when recreational problems are presented in a form which is called 'low floor and high ceiling' which means using, as an entry point, a context which does not require high-level skills; rather, it helps to engage all students, even weaker ones, into the process of problem-solving, yet it should have several levels of complexity, to challenge 'high-flying' students (https://www.edutopia.org/article/encouraging-persistence-math), and overall, making math "an enjoyable experience" (Kulkarni, 2013).

Yet enlarging learners' opportunities to more challenging and rich mathematics using puzzles is not only a matter of capabilities. It is also a way of introducing learners of all ages into the beautiful world of mathematics and showing them that navigating in this world can be real pleasure. Our research on introducing Bachet's game with elementary school students (Applebaum & Freiman, 2014) shows that playing with 15 counters, and using simple rules of taking a number of them and of determining a winner, can lead to important mathematical discoveries and conjectures, along with opportunities to reason and to attempt to prove them.

Moreover, it can introduce them to a culture of investigation where the rules of the game could be modified, new questions can be asked, and solving a problem could be a continuous process which, eventually, would shape students' minds in terms of curiosity, self-determination, and appreciation of the beauty of mathematical thinking, other qualities which Kruteskii (1976) includes into a concept of mathematical habit of mind and which needs to be developed from a very early age. For instance, Nickerson (2011, p. 284) cites the example from the biography of Wiles who said to have been introduced to the Last Theorem of Fermat when he was 7–8 years old and got interested in finding the proof. Another example cited is one of Alain Connes who is said to have discovered an 'intense pleasure' and 'joy of mathematics' when he was a young boy.

A particular value of recreational puzzles, such as the *Eight Queens* problem, is also remarkable in a context of a possibility to bring more advanced mathematical content into the classroom, such as combinatorics, graph theory, optimization theory, and number theory (Petkovic, 2009). Wildenberg (2002) reports how modern aspects of chess puzzles (such as Knight Tours), introducing backtracking and heuristics, can be brought into a classroom in a form of a project.

Besides the learning about algorithms and methods of programming, modern technology can provide students with an access to rich resources of challenging mathematics (NRICH-project, Recreomath, CAMI), along with the affordances of social media and dynamic software which enhances modelling and investigations; all this can contribute to five aspects of mathematical creativity: enabling, enriching, engaging, encouraging, and empowering (Freiman & Tassell, 2018, in press). As recent example of such opportunities, Tabesh, in a lecture given during the 13th ICME Congress (2016), discusses a case study of a digital learning platform that

[4] https://www.edutopia.org/blog/recreational-educational-value-math-puzzles-deepak-kulkarni.

enables creative engagement, develops mathematical skills, supports a growth mathematical mindset, and helps to become collaborative and social.

Nevertheless, there is a need to adopt a very selective approach to the choice of examples if these are to support the very complex venture of looking for connections between problem-solving, affect, technology, and creativity in mathematics. By choosing recreational problems, and, more specifically, chess puzzles to illustrate this complexity, my intention is to highlight the role of the history in providing not only explanations about the origins of the puzzles and details of their analysis by renowned mathematicians but also in featuring multiple representations, observations, rigorous investigations, and observations that go beyond finding a solution of particular problem towards elaboration of new methods and theories while sharing feelings, emotions, trials, and errors, along with endless pleasures of continued engagement in an intellectual endeavour, thus contributing to opening up the beauty of mathematical challenges and discoveries and finally constructing bridges to modern times. In brief, an ongoing collective human enterprise that needs to be investigated more deeply.

References

Applebaum, M., & Freiman, V. (2014). It all starts with the Bachet's game. *Mathematics Teacher, 241*, 22–26.

Bachet, Sr., & de Méziriac, C. G. (1884). Problèmes plaisans et délectables, qui se font par les nombres. (5e édition). Paris

Ball, W. W. R., & Coxeter, H. S. M. (1940). *Mathematical recreations and essays* (11th ed.). London, UK: Macmillan & Co.

Bellavitis, G. (1861). Q. 251 Disposizione sullo scacchiere di otto regine. *Atti dell' Istituto Veneto, 3*(6), 434–435.

Campbell, P. (1977). Gauss and the eight queens problem: A study in miniature of the propagation of historical error. *Historia Mathematica, 4*, 397–404.

Cretaine, A. (1865). *Etudes sur le Problème de la Marche du Cavalier au Jeu des Echecs et Solution du Problème des Huit Dames Paris* (Libraries Rue des Bons Enfants 28, Ancienne Maison Silvestre).

Crook, J. F. (2009). A pencil-and-paper algorithm for solving Sudoku puzzles. *Notices of the AMS, 56*(4), 460–468.

Danesi, M. (2009). The Appeal of Sudoku. Why is Sudoku an obsession? Psychology Today, https://www.psychologytoday.com/us/blog/brain-workout/200906/the-appeal-sudoku.

Dijkstra, E. W. (1971). EWD316: A short introduction to the art of programming. https://www.cs.utexas.edu/users/EWD/transcriptions/EWD03xx/EWD316.8.html

Euler, L. (1759). Solution d'une question curieuse que ne paroit soumise à aucune analyse. *Mémoires de l'académie des sciences de Berlin, 15*, 310–337.

Euler L., (1766). *Memoire de Berlin for 1759*. Berlin, pp. 310–337.

Freiman, V. (2006). Problems to discover and to boost mathematical talent in early grades: A challenging situations approach. *The Mathematics Enthusiast, 3*(1), 51–75.

Freiman, V. (2018). Complex and open-ended tasks to enrich mathematical experiences of kindergarten students. In M. Singer (Ed.), *Mathematical creativity and mathematical giftedness: Enhancing creative capacities in mathematically promising students (373–404). ICME-13 monographs*. Cham, Switzerland: Springer International Publishing.

Freiman, V., & Tassell, J. (Eds.). (2018, in press). *Creativity and technology in mathematics education*. Series Mathematics Education in the Digital Era. Cham, Switzerland: Springer.
Glaisher, J. W. L. (1874). On the problem of the eight queens. *Philosophical Magazine, 48*(4), 457–467.
Gomez, C., & Novak, D. (2014). Oki-Doku: Number puzzles. *Teaching Children Mathematics, 20*(7), 436–442.
Gunther, S. (1874). Zur mathematischen Theorie des Schachbretts. *Archiv der Mathematik und Physik, 56*, 281–292.
Knuth, D. E. (1968). *Art of computer programming, volume 4B, fascicle 5: The: Mathematical preliminaries redux; backtracking; dancing links*. Boston: Addison-Wesley.
Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: The University of Chicago Press.
Kulkarni, D. (2013). Recreational and educational value of math puzzles. Edutopia: Game-based learning. https://www.edutopia.org/blog/recreational-educational-value-math-puzzles-deepak-kulkarni
Lehman, W. (1862). "Doscendo Discimus": Solution to Eight Queen's Problem. In: Loewenthal, J. (Ed.). The Chess Congress of 1862: A Collection of Games and a selection of the problems sent in the competition. London: Britsh Chess Association.
Lionnet, F. J. E. (1852). Question 251. *Nouvelles Annales de Mathématiques, 11*, 114.
Lionnet, F. J. E. (1869). Question 963. *Nouvelles Annales de Mathématiques, 28*, 560.
Lucas (1891). Récréations mathématiques: Les traversees. Les ponts. Les labyrinthes. Les reines. Le solitaire. la numération. Le baguenaudier. Le taquin. 2e édition. Paris: Gauthier-Villars et fils.
Murawska, J. M. (2018). KenKen Puzzles for Developing Number Sense AndPositive Mathematics Identify in Elementary School. *Success in High-Need Schools Journal, 14*(1), 21–25.
Nauck, F. (1850). Schach: Eine in *das Gebiet der Mathematik fallende Aufgabe von Herrn Dr. Nauck in Schleusingen*. *Illustrirte Zeitung, 14*(361), 352.
Nickerson, R. (2011). *Mathematical reasoning: Patterns, problems, conjectures, and proofs. Mathematical reasoning: Patterns, problems, conjectures, and proofs*. New York: Taylor & Francis Group.
Parmentier (Le Général). (1884). Problème de huit reines. In *Compte rendu de l'association française pour l'avancement des sciences* (Vol. 12, pp. 197–213). Paris: Association française pour l'avancement des sciences.
Petkovic, M. (2009). *Famous puzzles of great mathematicians*. Providence, RI: American Mathematical Society.
Reiter, H. B., Thornton, J., & Vennebush, G. P. (2013). Using KenKen to build reasoning skills. *Mathematics Teacher, 107*(5), 341–347.
Stewart, I. (1997). Mathematical recreations: Knight's tours. *Scientific American, 276*, 102–104.
Tabesh, Y. (2016). Digital pedagogy in mathematical learning. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited lectures from the 13th international congress on mathematical education. ICME-13 monographs*. Cham, Switzerland: Springer International Publishing.
von Jaenisch, C. F. (1862). *Traité des Applications de 1'Analyse Mathematique au Jeu des Echecs* (Vol. I). St. Petersburg (Dufour): Lange.
Wildenberg, G. (2002). Heuristics as an aid to backtracking, a classroom project. *Journal of Computing Sciences in Colleges, 17*(6), 39–44.
Wirth, N. (1985). *Algorithms and data structures*. Englewood Cliffs, NJ: Prentice-Hall.
Yeager, D. S., & Dweck, C. S. (2012). Mindsets that promote resilience: When students believe that personal characteristics can be developed. *Educational Psychologist, 47*, 302–314. https://doi.org/10.1080/00461520.2012.722805

Index

N. Amado et al. (eds.), *Broadening the Scope of Research on Mathematical Problem Solving*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-319-99861-9

D

E

N

O

P

R