Chapter 2
Classic Beginnings

2.1 Introduction

The central theme of this book is the ability to identify the best location of one or more facilities or objects in order to provide some type or level of coverage. For example, let us suppose that we wish to place guards in an art gallery in such a manner that all areas of the gallery are within sight of one or more guards. In essence, we want the set of guard positions to “cover” or view the entire public area of the gallery. There can, of course, be many different configurations in which guards can view the entire gallery; however, it is of practical necessity to seek a pattern that deploys the fewest number of guards. As a second example, consider the case where we desire to provide fire protection to all neighborhoods of a city. In order to respond to a fire in a timely manner, we may set a standard that each neighborhood of the city should be no more than a mile and a half away from their nearest fire station (where fire trucks and crews can be housed to quickly respond when called). The fire service deployment problem can then be defined as finding the fewest fire stations (and their locations) so that each neighborhood is served or covered within a mile and a half of a station. Both the gallery guard positioning problem and the fire station location problem are examples of the Location Set Covering Problem, one of many covering problems that will be addressed in this book.

The definition of “cover” is usually defined within the context of a standard. Such a standard can be defined in terms of maximal distance (e.g., a mile and a half for a fire station), line-of sight (e.g., viewable by a security camera or guard), maximal service time (e.g., an 8 min response time by an available ambulance), the detection limit of a sensor (e.g., radar), audibility (e.g., loud enough to be heard), closer than a competitor (e.g., capture a percentage of customers by providing a closer bank branch), the amount of fabric uniquely covered by a cutout pattern, or by some other contextually meaningful metric (e.g., the presence of a target species at a biological reserve site).

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There are three principal forms of location covering models: those that seek complete coverage, those that maximize coverage subject to budgetary or resource constraints, and those that avoid coverage to the largest extent possible. For example, in terms of fire service deployment we may wish to identify the smallest number of stations and their locations while ensuring that each neighborhood is covered by service. This problem has been defined as the Location Set Covering Problem (LSCP) and seeks to cover all demand while using the fewest number of facilities. This facility location problem was first defined by Toregas et al. (1971) in order to locate emergency facilities. A second major development was based upon recognizing that resources needed to provide complete coverage may not be available. In such a case, the obvious question would be: "how much can be covered while expending no more than an allotted budget?" This, in essence, is the basis for the Maximal Covering Location Problem (Church and ReVelle 1974). A third approach represents an opposite tack, that of avoiding coverage, impact or detection. For example, we may wish to locate several facilities that need to be at least 50 miles from major population centers (e.g., a nuclear power plant) and perhaps 50 miles apart from each other. This last problem has been called the Minimum Impact Location Problem (Murray et al. 1998).

Since the early developments of the Location Set Covering Problem and the Maximal Covering Location Problem, and the latter developments of Minimizing Impact, models based upon some form of coverage have been extended and applied in a number of areas. Examples include remote sensing, watch tower location, surveillance camera placement, trauma care system design, biological reserve design, configuring FDA inspection teams, transit systems design, emergency shelter location, cell tower deployment, equipment storage location, cutting pattern layout of fabric, as well as many others. Since the ground breaking work of Berge (1957), Edmonds (1962), Toregas et al. (1971), and Church and ReVelle (1974), the science of "covering" has become a significant topic in many fields, including regional science, computer science, geography, operations research, industrial engineering, location science, ecology, and systems security. The remainder of this chapter presents two core problems that are central to the field of location covering.

2.2 Theory and Innovation

The field of location science spans a number of disciplines, including regional science, geography, engineering, business, computer science, and operations research. The beginning of this field is rooted in such classic works as Von Thünen (1826), Launhardt (1872), Weber (1909), Christaller (1933), Hotelling (1929), and Isard (1956) involving problems such as land use arrangements, industrial location, competition, and settlement patterns. This fundamental science was based upon mathematical constructs involving a blend of economics, behavior, and physics. For example, Launhardt (1872) attempted to identify the optimal location for a factory by finding the location which minimizes the costs of transport investment (e.g., a rail line) along with the costs of transporting needed raw materials to the factory and the cost of transporting the final product to the market. Such constructs, although easy to conceive, can often be hard to solve optimally due to issues of computational complexity. This fact alone hindered the development of this field until the invention of the modern computer. One might easily characterize the field of location science as having two distinct eras, one before digital computers were available and one since the development of the computer.

In the 1960s five different problem definitions emerged in the location literature: the p-median problem (Hakimi 1964, 1965), the p-center problem (Hakimi 1965), the simple plant location problem (Manne 1964), the plant layout problem (Armour and Buffa 1963), and the vertex cover problem (Fulkerson and Ryser 1961). Although these problems are rooted in a literature that preceded this modern era, there emerged an attempt to both structure and solve realistically sized problems in the 1960s using the resources of IBM's first solid state computer, the 7090 developed in 1959 and the subsequent 360 line of computers which became available in 1964. These key resources, along with computers of Control Data Corporation, Burroughs and Univac, played an instrumental role in the flourish of activity in this field. Although each of these problems is of interest in their own right, our discussion herein will be limited primarily to covering problems.

Berge (1957) was the first to define a cover problem involving a graph of nodes and edges.[1] Berge (1957) defined the cover problem as one of finding the smallest subset of nodes, B, such that every arc in the graph was adjacent to at least one node of B. Fulkerson and Ryser (1961) extended this concept by defining the minimal set representation problem. This problem involved sets and elements. Each set contains a list of elements that are members of that set, where specific elements can be members of more than one set. The minimal set representation problem can therefore be defined as follows:

Find the smallest number of elements in which each set contains at least one of the selected elements.

Define the following notation:

j = index of elements
i = index of sets

$$\delta_{ij} = \begin{cases} 1, & \text{if element } j \text{ is a member of set } i \\ 0, & \text{if otherwise} \end{cases}$$

[1]We have spent considerable time tracking references to various forms of "covering" in order to identify the original academic source. The trail leads to Berge (1957), Quine (1955) and Hall (1935). Hall (1935) defines a form of covering where a distinct representative of sets must be a group of members of which each set must contain one of these members. Quine (1955) was the first to discover how a set of truth functions could be reduced. Although Hall (1935) and Quine (1955) define the problems based upon representative sets and logic functions, Berge (1957) appears to be the first to define the cover problem within the context of a geometric shape such as a network structure.

$$x_j = \begin{cases} 1, \text{if element } j \text{ is selected} \\ 0, \text{ otherwise} \end{cases}$$

Fulkerson and Ryser (1961) formulated the minimal set representation problem as an integer-linear program as follows:

$$Minimize \sum_{j=1}^{n} x_j \tag{2.1}$$

Subject to:

$$\sum_j \delta_{ij} x_j \geq 1 \quad \forall i \tag{2.2}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.3}$$

The objective (2.1) minimizes the number of times that elements are selected. Constraints (2.2) ensure that each set, i, is represented by at least one of the chosen elements, j. Constraints (2.3) impose binary integer requirements.

By definition, the goal is to select a minimum number of elements, so it is not necessary to choose an element more than once. In fact, selecting an element twice would be superfluous. Consequently, when requiring that each set is represented at least once, the decision variables, x_j, can be restricted to be binary, zero–one. For a given set i, constraints (2.2) ensure that at least one of the selected elements, j, will be a member of that set. Note that the coefficient of each variable in the constraint is either zero or one. Because of this, the left hand side of constraint (2.2) boils down to the sum of the element selection variables that are members of set i. For example, suppose that a set contained elements 1, 3, and 7. Then the constraint for this set would be equivalent to:

$$x_1 + x_3 + x_7 \geq 1 \tag{2.4}$$

Essentially, for constraint (2.4) to be satisfied, at least one of the three variables, x_1, x_3 and x_7, must be equal to one. Doing so ensures that this specific set is represented by at least one of its members, but the question is which ones should be selected if the number of selected elements is a minimum.

This formulation can be easily transformed to represent the edge adjacent covering problem of Berge (1957) by using i to represent edges, j to represent nodes, and by defining the δ_{ij} values as follows:

$$\delta_{ij} = \begin{cases} 1, \text{ if node } j \text{ is adjacent to edge } i \\ 0, \text{ if otherwise} \end{cases}$$

The model, (2.1)–(2.3), then captures the essence of the set covering problem as it involves minimizing the number of nodes selected such that each edge is incident to at least one selected node (see also Balinski 1965). Karp (1972) demonstrated that the problem of finding a solution which covers all edges with fewer than K vertices was NP-complete. In fact, the vertex cover problem was one of six fundamental problems initially used to define the NP-complete set (Garey and Johnson 1979). This initial set of problems included among others the clique problem, the Hamiltonian circuit problem, and the set partitioning problem. From a broader perspective, this means that this optimization problem, (2.1–2.3), is NP-hard.

Although early work in set representation and network edge covering problems focused on properties and formulations, Roth (1969) proposed a reductions algorithm for solving (2.1–2.3). This reductions process, when applied to the form given above, can effectively reduce many practical problems to a size that may be solved by hand or inspection.

One notable issue lacking in much of the initial work was a sense of application and use. Hakimi (1965) was perhaps the first to suggest a practical application involving the location of policemen on a highway network so that everyone was within a specified distance of a policeman. Unfortunately, his discussion was based upon the use of a Boolean function that proved to be less than fruitful as an approach for solving larger problems because it was thought to essentially require enumeration.

2.3 Location Set Covering Problem

In 1970, Constantine Toregas working under the direction of Charles ReVelle at Cornell University wrote a Master's thesis titled "A Covering Formulation for the Location of Public Service Facilities." This thesis was the first to structure a location model within the context of covering using an integer-linear programming formulation. Specifically, Toregas (1970) reasoned that emergency services, such as fire equipment and EMS vehicles, should be ideally located according to a response time standard. He suggested that there should be a maximum allowable response time in service provision in order to handle an event, such as a fire, before it gets out of hand. This would ensure a reasonable chance of saving lives and/or property. Simply put, Toregas (1970) described the following public service system design problem:

Minimize the number of facilities needed and locate them so that every demand area is covered within a predefined maximal service distance or time.

It was reasoned that the cost of building and equipping a facility, such as a fire station, would probably not vary much across a planning area as it involved the amortized costs of construction, site acquisition, equipment purchasing, etc. Since only the site acquisition costs would vary, this assumption would roughly hold in most locales. Thus, he argued that overall costs of providing service would indeed be a function of the number of facilities used. This was called the Location Set Covering Problem (LSCP).

2.3.1 *Formulation*

Toregas (1970) viewed the landscape of a region as a road network connecting a set of n nodes, which were defined as places of demand as well as potential facility locations. Consider the following notation:

i = index referencing nodes of the network as demand
j = index referencing nodes of the network as potential facility sites
S = maximal acceptable service distance or time standard
d_{ij} = shortest distance or travel time between nodes i and j
$N_i = \{j \mid d_{ij} \leq S\}$
$x_j = \begin{cases} 1, & \text{if a facility is located at node } j \\ 0, & \text{otherwise} \end{cases}$

Using this notation, Toregas (1970) defined the following model:

$$\text{LSCP}: \qquad \textit{Minimize} \sum_{j=1}^{n} x_j \tag{2.5}$$

Subject to:

$$\sum_{j \in N_i} x_j \geq 1 \quad \forall i \tag{2.6}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.7}$$

The objective (2.5) involves minimizing the number of facilities being located. Constraints (2.6) ensure that for each demand node at least one facility is selected among the set N_i. This is the set of sites that are close enough to node i to provide service within the maximal service distance or time standard. Binary integer restrictions are imposed in constraints (2.7). Again, Toregas (1970) considered specifically the case where facilities were fire stations.

The LSCP is compact, involving only n constraints and n variables. In his Ph.D. dissertation at Cornell University, also under the supervision of Charles ReVelle, Toregas (1971) proposed two approaches to solve the LSCP, one based upon the use of integer-linear programming software and the other based upon a method called reductions.

What can be observed about the LSCP, (2.5–2.7), is that mathematically it is essentially the same as the minimal set representation problem structured in (2.1–2.3). The difference is context, where the LSCP explicitly involves spatial decisions that correspond to spatial service coverage. Accordingly, there are four associated points of contrast.

First, the variables x_j are spatially defined, where a specific node j represents a precise site on the surface of the earth. Second, although the sum in constraint (2.2) is over all j, as the coefficient δ_{ij} is used, the LSCP sums the x_j terms over the set $j \in N_i$.

Nevertheless, the constraints are equivalent as the only time when site j covers demand i is when $\delta_{ij} = 1$. More formally, this set may be stated as $N_i = \{j | \delta_{ij} = 1\}$, where the matrix $[\delta_{ij}]$ is called the coverage or set representation matrix. Third, the set N_i is derived and defined based on travel distance or time, and whether a demand location i can be served by a facility sited at j. This spatially defined set may be contiguous, non-contiguous, convex, non-convex, regularly or irregularly shaped, etc., depending on the context and associated criteria used in the assessment of attaining the desired standard of service. Fourth, the LSCP was conceived in an applied context for expressly helping to site emergency service facilities, such as fire stations.

From one perspective, mathematically the Toregas (1970, 1971) model is the same as that of Fulkerson and Ryser (1961), but from another it represents a substantial leap forward in location modeling. Whereas the vertex cover problem was more of a mathematical puzzle, the LSCP encapsulates the essence of design in efficiently allocating facilities across geographic space. In particular, the facilities of interest to Toregas (1970) were emergency services, such as fire protection resources or an ambulance/pre-hospital care. It is this significant real world application context which led to the modern foundation of location models using a covering objective. The applied setting involving geographic space often presents many challenges, ranging from representation to solution. It is precisely for these reasons, along with practical utility, that the LSCP has come to be so important and significant.

2.3.2 Planning Facility Deployment

Toregas et al. (1971) argued that the LSCP could be used for the purposes of planning by solving the model over a range of maximal service distance or time standards. By doing this, one can generate a tradeoff curve between the number of needed facilities in providing complete coverage as a function of the coverage standard, S. The tradeoff between the needed number of facilities and the maximal service standard will have a stair-step profile as the value of S is increased. When $S = 0$, presumably the number of needed facilities will equal, or nearly equal, the number of demand points in the problem. But as S is increased in value, the number of needed facilities will decline in number, until at some point, the value of S is large enough that all demand can be covered by a single facility capable of travel to all demand locations within the response standard.

In order to illustrate service response capabilities under different assumed standards, an example of the associated tradeoff is given in Fig. 2.1. In this case, an LSCP application instance was solved for each discrete value of S ranging from 1 to 29 by increments of 0.005 (a total of 5800 problems were therefore solved). At a value of 10, for example, it takes 9 facilities to cover all demand within this response standard. Alternatively, at a value of 12 only 7 facilities are required for coverage of all demand. There are several discrete increments in which the needed number of facilities decreases by more than one facility. Toregas et al. (1971) reasoned that this type of

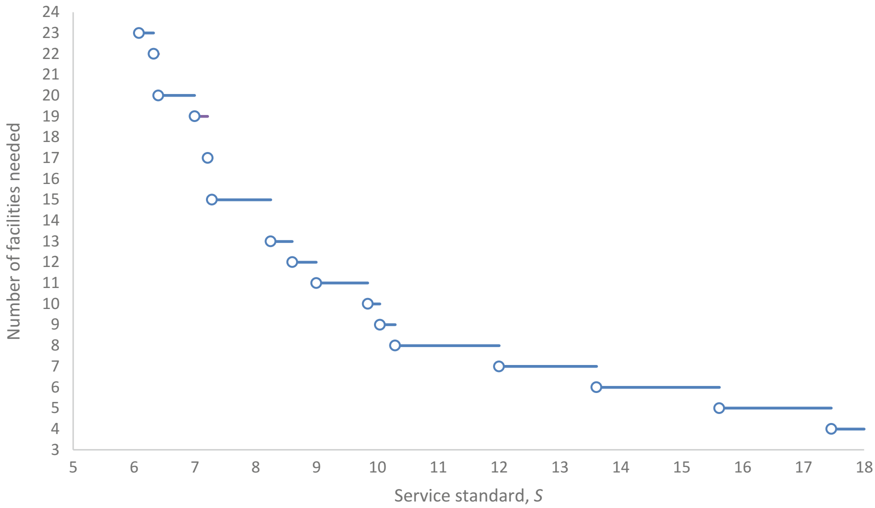

Fig. 2.1 Tradeoff curve between number of needed facilities and service standard, *S*

curve is useful in planning as one could identify the smallest maximal service distance for which the needed number of facilities equaled what was affordable. Worth noting is that one can think of these solutions as optimal for associated *p*-center problems.[2]

2.3.3 Applications

Even though the application focus of the LSCP helped to establish the use of a covering objective in location science, there were four other factors that helped to pique interest in this type of location model. The first factor was that the problem was simple to understand and it represented a major step towards making public safety services more efficient. The second factor was that Toregas (1971) and Toregas and ReVelle (1973) demonstrated that the use of a reductions algorithm in solving realistically sized LSCP instances was possible. Reductions approaches were first developed to reduce the size of logic and switching circuit problems (Quine 1955; Roth 1969). Toregas and ReVelle (1973) demonstrated that the principles used in reduction algorithms could be used to help solve the LSCP. The third factor was the availability of modest computing resources provided by the IBM 360 family of computers. Toregas (1971) solved 66 LSCP application instances and of these he was able to prove optimality for all but nine problems using the reductions process alone. For the remainder of the problems he used the reductions process followed by

[2]The *p*-center problem involves minimizing the furthest service distance that any demand must travel to their closest facility when locating exactly *p*-facilities. More discussion on this problem and its relationship to coverage problems is left for the chapters that follow.

the use of linear programming applied to a reduced form of the model given above. Having a simple solution process that was easy to program and could be supported by computer systems that were available in most large cities meant that the application frontier for this model was fertile. The fourth major factor was that after graduate school Constantine Toregas joined Public Technology Inc., an organization funded by member cities with the objective of solving municipal problems involving communication, public safety, environmental, and other emerging problems. Toregas, through Public Technology Inc., pushed the application of the LSCP and during the 1970s over 100 cities reorganized their fire protection plans based upon the use of the LSCP.

The application of the LSCP to fire station siting was the tip of the iceberg in terms of potential utility. It has been used or suggested for use in the location of ambulances (Berlin and Liebman 1974), the assignment of fire equipment to fire houses (Walker 1974), the selection of reserve sites for biological protection (Underhill 1994), the design of FDA inspection teams (Klimberg et al. 1991), dynamic repositioning of fire equipment (Kolesar and Walker 1974), accessibility of needed services for those with disabilities (Kwan et al. 2003), the location of warning sirens (Current and O'Kelly 1992), bus stop siting (Gleason 1975; Murray 2001), vehicle emissions testing stations (Swersey and Thakur 1995), weather radar (Agnetis et al. 2009), oil spill equipment location (Psaraftis and Ziogas 1985), security camera placement (Murray et al. 2007), art gallery guard placement (Chvatal 1975), and many others. Figure 2.2 shows LSCP results for the siting of emergency warning sirens in the coastal city highlighted in Chap. 1. The existing configuration of eight sirens is only capable of serving some 70% of the demand region. Use of the LSCP indicates that it would take 15 sirens to suitably serve the entire region in this case.

2.4 Maximal Covering Location Problem

The LSCP requires that each demand be covered or served within a maximal service distance or time standard. Recognizing that there may be situations where covering all demand within a strict standard, Toregas (1971) suggested that the maximal standard could be defined differently for each demand, e.g., S_i. For example, in the County of Santa Barbara it is recommended that ambulances be positioned in the region so that all urban areas are served within 10 min response time and that all rural areas be served within a maximal response time of 30 min. But, even when one defines such a standard, there are many circumstances when the resources required to provide complete coverage for urban and rural service exceeds what an agency may spend. This means that a service system configuration identified using the LSCP may be too expensive to implement. In such a circumstance it is reasonable to ask how much coverage can be provided with less investment. In particular, what can be achieved by deploying a fixed number of facilities? In a design sense, we can state this problem as:

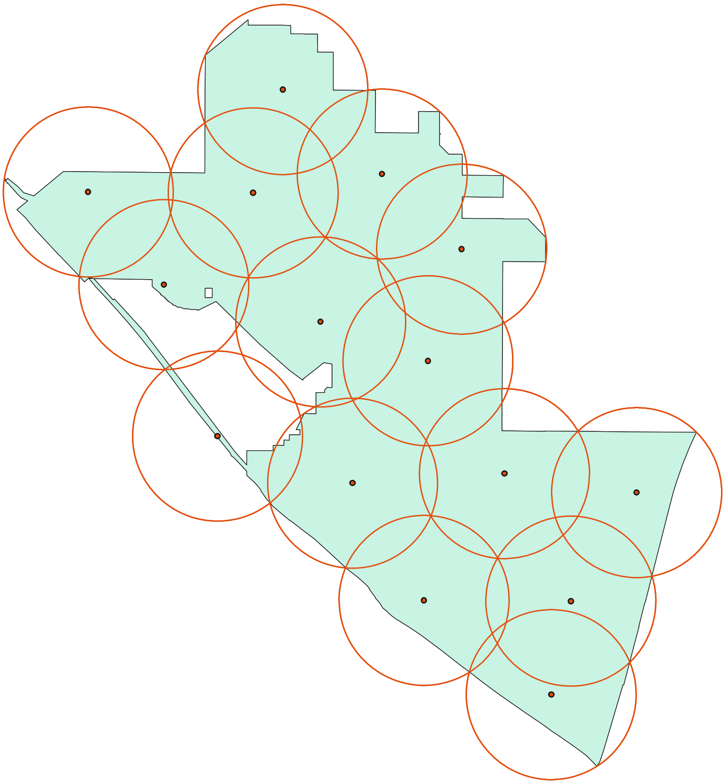

Fig. 2.2 LSCP solution for warning siren coverage in Huntington Beach, CA. This LSCP solution requires 15 siren locations to cover all of Huntington Beach

Maximize the amount of demand covered within a maximal service distance or time standard by locating a fixed number of facilities

This problem is called the Maximal Covering Location Problem (MCLP) and was defined originally in the Ph.D. dissertation of Church (1974) at The Johns Hopkins University as well as in Church and ReVelle (1974).[3]

Interestingly, at the same time White and Case (1974) also argued for more flexibility in determining a cover solution, stating that "... in a number of practical situations, it is not possible to provide the number of facilities required to cover *totally* all customers; rather, the number of facilities available for location is only sufficient to cover *partially* the set of customers." With this argument, White and Case (1974) defined the partial cover problem where the total number of customers (one customer per node) covered is maximized subject to the use of a fixed number

[3]Church and ReVelle in 1973 presented this at the North American Regional Science Council meetings in Atlanta, Georgia that was subsequently published as Church and ReVelle (1974).

of facilities. Both problem definitions are arguably important, as they expand the potential problem domain of location covering models. However, the issue of addressing an available budget makes them even more appealing.

2.4.1 Formulation

The following additional notation is introduced:

p = number of facilities to be located
a_i = service load or population at demand i
$y_i = \begin{cases} 1, & \text{if demand } i \text{ is covered within the service standard} \\ 0, & \text{otherwise} \end{cases}$

The formulation detailed in Church (1974) and Church and ReVelle (1974) can be stated as:

$$\text{MCLP}: \qquad \textit{Maximize} \sum_{i=1}^{n} a_i y_i \tag{2.8}$$

Subject to:

$$\sum_{j \in N_i} x_j \geq y_i \quad \forall i \tag{2.9}$$

$$\sum_{j} x_j = p \tag{2.10}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.11}$$

$$y_i \in \{0, 1\} \quad \forall i \tag{2.12}$$

The objective, (2.8), seeks to maximize the total amount of demand covered by the placement of p facilities. Constraint (2.9) defines whether coverage to a given demand is provided or not. Essentially, if one or more facilities within the coverage set N_i is (are) selected, then the sum, $\sum_{j \in N_i} x_j$, is greater than equal to one in value. This allows the coverage decision variable, y_i, to be one, indicating that coverage has been provided to demand i. If no facilities have been selected among the coverage set, N_i, then the sum $\sum_{j \in N_i} x_j = 0$ and y_i is forced to be zero, indicating that demand i is not covered. The number of facilities to be located is restricted to equal p in constraint (2.10). Constraints (2.11) and (2.12) restrict decision variables to be binary, zero or one.

The MCLP contains $2n$ binary decision variables and $n + 1$ linear constraints. Formally, Church (1974) showed that this problem can be solved to optimality when relaxing the integer restrictions on the y_i variables, imposing only simple upper bounds, $y_i \leq 1$. This means that the MCLP has the same number of needed binary variables (e.g., x_j) as the LSCP for a given application.

Finally, if $a_i = 1$ for each demand i, then the MCLP would represent the partial covering problem of White and Case (1974). This implies then that the partial covering model is a special case of the MCLP.

From the outset, Church and ReVelle (1974) recognized that a heuristic approach would be necessary for solving large maximal covering applications. They suggested a procedure that was based upon "greedy" and "substitution" strategies. Since then, other strategies have been applied for solving large MCLP instances, including simulated annealing (Murray and Church 1996), genetic algorithms (Tong et al. 2009), heuristic concentration (ReVelle et al. 2008), vertex substitution (Gerrard et al. 1996), Lagrangean relaxation (Galvão and ReVelle 1996), and dual-based heuristics with branch and bound (Downs and Camm 1996), among others.

An important and interesting feature of maximizing coverage is that it may be approached as a minimization problem, detailed in Church (1974) and Church and ReVelle (1974). This alternate form is of value theoretically and computationally. Consider the following variables:

$$\bar{y}_i = 1 - y_i \tag{2.13}$$

As $\bar{y}_i \in \{0, 1\}$, then if $\bar{y}_i$ equals to one, the interpretation is that demand i is not covered. Alternatively, if then if $\bar{y}_i$ is equal to zero, the interpretation is that demand i is covered. If we solve the above (2.13) for y_i, then: $y_i = 1 - \bar{y}_i$. Substituting $1 - \bar{y}_i$ for y_i in the MCLP objective (2.8) yields:

$$\sum_{j=1}^{n} a_i\left(1 - \bar{y}_i\right) = \sum_{j=1}^{n} a_i - \sum_{j=1}^{n} a_i\bar{y}_i = A - \sum_{i=1}^{n} a_i y_i \tag{2.14}$$

where $A = \sum_{i=1}^{n} a_i$. Since A is a sum of constants (no decision variables), then it is not optimizable. Therefore, the MCLP objective function is equivalent to:

$$\mathit{Minimize} \quad \sum_{i}^{n} a_i\bar{y}_i \tag{2.15}$$

Similarly, we can also substitute $1 - \bar{y}_i$ for y_i in constraints (2.9) as follows:

$$\sum_{j \in N_i} x_j \geq 1 - \bar{y}_i \tag{2.16}$$

This is equivalent to:

$$\sum_{j \in N_i} x_j + \bar{y}_i \geq 1 \tag{2.17}$$

This condition forces the variable $\bar{y}_i$ to equal one when no sites have been chosen in the set N_i. In essence, when a demand i is not covered, $\bar{y}_i = 1$. Further, when demand i is covered, then $\bar{y}_i = 0$ because the constraint is satisfied without making $\bar{y}_i$ positive as well as combined with the orientation of objective (2.14). Collectively, this will force $\bar{y}_i$ to be zero whenever possible.

This means that the alternative formulation of the maximal covering location model can be summarized as:

$$\text{MCLP2}: \qquad \mathit{Minimize} \sum_{i=1}^{n} a_i \bar{y}_i \tag{2.18}$$

Subject to:

$$\sum_{j \in N_i} x_j + \bar{y}_i \geq 1 \quad \forall i \tag{2.19}$$

$$\sum_{j} x_j = p \tag{2.20}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.21}$$

$$\bar{y}_i \in \{0, 1\} \quad \forall i \tag{2.22}$$

Objective (2.18) involves minimizing the amount of demand not covered. Since the total demand is constant, maximizing coverage is equivalent to minimizing the demand not covered. This model takes the form of a constrained set covering problem. Constraints (2.19) specify coverage, extending constraints (2.6) in the LSCP to include the variable $\bar{y}_i$. Constraints (2.20) specify the number of facilities to be sited, p. Decision variables requirements are indicated in constraints (2.21) and (2.22). However, when solving this problem, it is only necessary to constrain the values of x_j to be binary integer. The integer conditions for $\bar{y}_i$ can be relaxed to simple non-negativity constraints, $\bar{y}_i \geq 0$, as these variables will never be greater than one in value in order to satisfy the covering constraint (2.19) combined with the orientation of the objective (2.18) and the fact that the x_j variables are integer in value.

Computationally, the MCLP2 is somewhat simpler to solve than the maximizing form, MCLP, as an upper bounded variable simplex algorithm is not needed when solving MCLP2 as a minimization problem. The solution results reported in Church and ReVelle (1974) were in fact based upon the MCLP2 formulation.

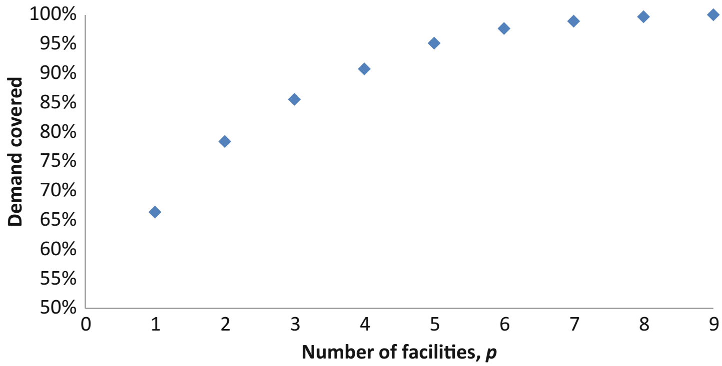

Fig. 2.3 Tradeoff curve between coverage and number of facilities sited

2.4.2 *Planning Facility Deployment*

Church and ReVelle (1974) suggested that the MCLP be solved for a given planning application by varying the number of facilities. Solving over a range of values for p, the number of facilities to site, makes it possible to generate a tradeoff between coverage provided and the investment in facilities. To illustrate this, Fig. 2.2 presents an associated tradeoff curve for a particular service standard. As is often the case, complete coverage requires a significant investment as compared to what is needed to provide 90% coverage, or even 95% coverage. For example, it requires nine facilities to cover all demand in this case within the standard. However, for five facilities coverage of 95% of the demand is possible. The strict elbow-shaped form of the coverage curve shown in Fig. 2.3 is quite common, indicating that in many cases, a high level of coverage can be provided at a substantial reduction to what it may take to provide complete coverage.

2.4.3 *Applications*

Models based upon the maximal covering concept have been applied in a number of fields. These include locating health clinics (Bennett et al. 1982), positioning ambulances (Saydam and McKnew 1985), locating fire stations (Murray 2013), selecting advertising venues from subscriber lists (Dwyer and Evans 1981), cluster analysis (Chung 1986), discriminate analysis (Chung 1986), selecting sites for nature reserves (Church et al. 1996), designing police patrol areas (Curtin et al. 2010), selecting color tabs for matching teeth to prosthetic teeth (Cocking et al. 2009), cutting pattern layout on fabric (Grinde and Daniels 1999), and locating humanitarian relief supplies (Balcik and Beamon 2008), just to name a few. Because the problem is relatively compact in size, many applications can be solved by a straight forward application of commercial software, like that of XPRESS, CPLEX,

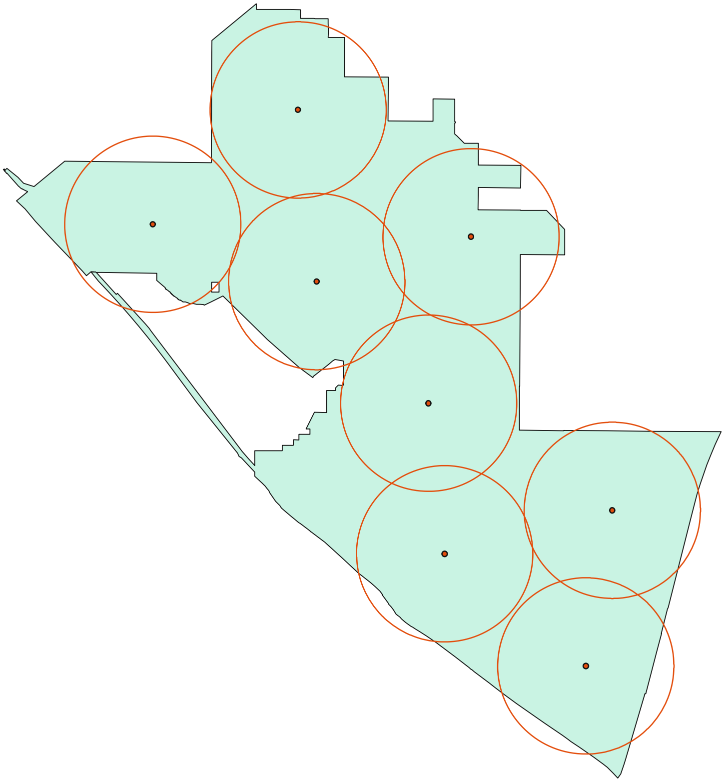

Fig. 2.4 Optimal configuration of eight warning sirens using the MCLP

and GUROBI. Further, some GIS (geographical information systems) software, like ArcGIS by Esri, contain heuristics that can solve the MCLP. Figure 2.4 shows MCLP results for the siting of emergency warning sirens discussed previously (Figs. 1.2 and 2.2). While the existing configuration of eight sirens serves approximately 70% of the demand region, application of the MCLP finds that an optimal configuration of eight sirens could actually serve over 85% of the regional demand in this case.

2.5 Theoretical Linkages

As was evident for the MCLP, alternative ways to mathematically structure a given problem often exist. Further, one formulation may prove more beneficial or desirable than alternatives, under particular conditions. Beyond this, there has been a more general interest in deriving and/or identifying a mathematical model that represents a unifying framework in facility location, and in particular be capable of structuring a range of important location modeling approaches.

Hakimi (1965) defined the following location problem:

Locate a fixed number of facilities such that the resulting sum of travel distances is minimized.

Four problem features were assumed: (1) each facility had the capability of handling all of the demand assigned to it; (2) each user is assigned to their closest located facility; (3) demand was concentrated at nodes of the network; and, (4) facilities could be located anywhere along the network (i.e., nodes and arcs). The motivating application for Hakimi (1965) was that of locating telephone switching centers. It was assumed that each customer would be connected by wire to their closest switching center. The objective was to minimize the total length of wire in serving all customers while locating p telephone switching centers, although it could apply to vehicular travel or any other appropriate form of connection as well. This has been called the p-median problem. Total service distance can be easily computed for any configuration of facilities along a network by first assigning each demand node to their closest facility, then multiplying the assignment distance by the demand to be served, and finally summing all demand weighted distance values. Hakimi (1964, 1965) proved that at least one optimal solution to a given p-median problem would utilize only nodal locations of the network. Since this groundbreaking work, virtually all models and solution techniques for the p-median problem have been based upon the assumption that facilities will be located only at nodes as such a restriction does not preclude an optimal solution from being identified. For example, ReVelle and Swain (1970) developed an integer-linear programming model for the p-median problem based upon this assumption. Accordingly, each node of the network was a place of demand as well as a potential facility site. Thus, a network of n nodes would have n demands and n sites. Consider the following notation:

$$z_{ij} = \begin{cases} 1, \text{if demand } i \text{ is assigned to facility } j \\ 0, \text{otherwise} \end{cases}$$

$$z_{jj} = \begin{cases} 1, \text{if node } j \text{ has been selected for a facility and assigns to itself} \\ 0, \text{otherwise} \end{cases}$$

ReVelle and Swain (1970) structured the following model:

$$p\text{-median}: \qquad Minimize \sum_i \sum_j a_i d_{ij} z_{ij} \tag{2.23}$$

Subject to:

$$\sum_j z_{ij} = 1 \quad \forall i \tag{2.24}$$

$$\sum_j z_{jj} = p \tag{2.25}$$

$$z_{ij} \le z_{jj} \quad \forall i, j \& i \ne j \tag{2.26}$$

$$z_{ij} \in \{0, 1\} \quad \forall i, j \tag{2.27}$$

Objective (2.23) minimizes the total weighted distance of demand assignment. Each demand i is forced to assign to a facility in constraints (2.24). Constraint (2.25) establishes that exactly p facilities are to be sited. Constraints (2.26) ensure that demand i cannot assign to facility j (i.e., $z_{ij} = 1$) unless a facility at j has been sited (i.e., $z_{jj} = 1$). Integer restrictions are imposed in constraints (2.27).

A few observations regarding the p-median problem are in order. First, it has been widely applied and the focus of much academic research. Second, it is has been extended in various ways to address a range of nuances encountered in application. Third, whereas the LSCP has n variables and n constraints and the MCLP has $2n$ variables and $n + 1$ constraints (assuming for the moment that the number of demand locations is the same as the number of potential facility sites), the p-median problem formulation is comparatively quite large as it has n^2 variables and $n^2 + 1$ constraints. For example, a problem of 1000 nodes would entail one million variables and a million and one constraints. As a consequence, many researchers have devoted considerable effort to designing heuristic approaches to solving the p-median problem (Teitz and Bart 1968; Densham and Rushton 1992; Rolland et al. 1997; Rosing and ReVelle 1997; Mladenović and Hansen 1997).

There is, however, a practical and theoretical reason as to why this problem and model formulation is of importance. First, the LSCP and MCLP can be structured as a p-median problem, and it is possible to convert these covering models using a polynomial bounded algorithm. Second, all solution procedures for the p-median problem, even heuristics, can be directly applied to solve an equivalent LSCP or MCLP, after making an appropriate problem transformation. Church (1974) was the first to recognize how a covering problem could be converted into an equivalent p-median problem. For the case where each node is a demand location and a potential facility site, Church (1974) proposed the following distance (or travel time) transformation:

$$d'_{ij} = \begin{cases} 0, \text{if } d_{ij} \le S \\ 1, \text{otherwise} \end{cases} \tag{2.28}$$

If the above p-median problem formulation, (2.23)–(2.27), is solved using transformed distances, d'_{ij}, instead of the actual distances, d_{ij}, the objective function of minimizing total weighted assignment distance (i.e., $\sum_i \sum_j a_i d'_{ij} z_{ij}$ substituted in objective (2.23)) would count each demand i that is served within their standard, S, as zero times a_i, and each demand that is served beyond the standard as one times a_i. This means that the objective seeks to minimize the amount of demand that is assigned outside the service standard. As this is equivalent to maximizing the total demand served within the standard, the above distance transformation when used with a p-median problem represents a covering problem, precisely along the lines of the MCLP2 structured above. Of course, the MCLP can be used to solve the LSCP simply by starting at any value of p, the number of facilities to be sited, such that coverage of all demand is not possible, then increasing p and re-solving until all demand is covered. The case where all demand is covered and p is the smallest value possible represents an optimal solution for the LSCP.

Hillsman (1984) presented a transformation to formulate an LSCP as a uniform linear model, which is a general location construct that he proposed to solve a variety of discrete location problems. The transformation of Hillsman (1984) can be used in part to formulate an LSCP as a p-median problem. Such a transformation is a bit more nuanced as the p-median problem involves locating a fixed number of facilities whereas the LSCP deals with finding the fewest number of needed facilities. We can make such a transformation by first setting the value of p to be equal to the value of n, the original number of potential facility sites. For the sake of generality, we again assume that each node is a place of demand as well as a potential facility site. In addition, we need to add $n - 1$ pseudo nodes. Essentially, these pseudo nodes serve as places of pseudo demand and potential facility sites. These sites do not cover any real demand and cost nothing to use. We will refer to the total list of nodes in terms of $i, j = 1, 2, \ldots, n$ as the original nodes in the problem and $i, j = n + 1, \quad n + 2, \ldots, 2n - 1$ as the pseudo nodes. We can further define the demand at each node (pseudo or real) as one. The added "pseudo nodes" are included so that the total number of selected sites will always add to n, a property which will be explained below. Consider the following distance transformation:

$$\tilde{d}_{ij} = \begin{cases} 0, & \forall i,j \text{ when } d_{ij} \leq S, i \neq j, i \leq n \& j \leq n \\ 0, & \forall i,j \text{ when } i \geq n+1 \\ 1, & \forall i,j \text{ when } i = j, \quad i \leq n \& j \leq n \\ M, & \forall i,j \text{ when } d_{ij} > S_i, i \leq n \& j \leq n \\ M, & \forall i,j \text{ when } \ i \leq n \& j \geq n+1 \end{cases} \tag{2.29}$$

where M is any number greater than n. The last condition of this transformation function creates a cost of serving any original demand by a facility placed at a pseudo node as the value of M. This is a high enough value that no original demand point will ever assign to a facility placed at a pseudo node (unless no feasible solution exists to the LSCP being solved). The cost of assignment for any pseudo demand is zero, whether it is to an original site or a pseudo site. This is established in the second condition of the transformation. Therefore, the added pseudo demands do not affect the objective function. If a pseudo site self assigns (i.e., $x_{jj} = 1$ when $j \geq n + 1$), then it means that it has been selected for one of the p facilities. The cost of selection (i.e., self-assignment) for a pseudo site is zero as well. Thus, such a selection does not affect the overall objective value. The rest of the transformation is explained by reviewing the effective distances for the original demand and facility nodes. If a real node (as opposed to a pseudo node) is selected for a facility, the self assignment distance is 1 in value (condition 3 of the transformation), which will be multiplied by the demand weight of 1, creating a total cost of assignment of 1. When an original demand assigns to a site that can cover that demand, the cost of assignment is zero (the first condition). Finally, if an original demand is forced to assign to a facility that is further than the coverage distance to a real site, the cost of assignment is M (condition 4). Assigning a real demand to a pseudo site will incur this same, prohibitive cost (condition 5). Since the cost of assigning any original demand beyond a coverage distance is so high compared to the cost of locating another real facility, a real facility will be located to cover each demand, reducing all assignment costs to zero except self-assignment. Thus, the assignment costs will ensure that all real demand points are covered by selected real sites. The only cost incurred in doing that is 1 for each selected real site (pseudo sites cost nothing). Thus, the objective will force the number of real sites selected to be as small as possible. Since pseudo sites cost nothing, enough of them will be selected to meet the p facility condition (that has been set arbitrarily large enough to have no impact on the problem). Consequently, when solving a p-median problem using the transformation (2.29), the resulting model [i.e., $\sum_i \sum_j a_i \tilde{d}_{ij} z_{ij}$ substituted in objective (2.23)] will minimize the number of real sites needed, while ensuring that each real demand node is covered. As a final note, for completeness transformation (2.29) is designed for the ReVelle and Swain (1970) formulation of the p-median problem, (2.23–2.27), with a complete $(2n - 1)$ by $(2n - 1)$ distance matrix, but by modifying the formulation it is possible to eliminate the representation of pseudo demands and all assignment variables to pseudo sites.

The significance of the two transformations, (2.28) and (2.29), is that any LSCP or MCLP can be solved as a p-median problem. Of course, this means that the alternative formulations now have an order of n^2 complexity. Since a wide variety of heuristics have been developed to solve the p-median problem, they can be employed without modification for solving these two covering problems, using the above transformations. This is precisely how the location-allocation routine found in ArcGIS allows one to solve covering and median problems. It performs all of the necessary transformations, upon the selection of a model option. However, it is important to note that the performance capabilities of this approach for some heuristics has not been tested or evaluated extensively, if at all. Thus, further testing is needed to fully vet the quality of obtained results using a specific heuristic in solving a covering problem through a transformation as a p-median problem. A final note is that Toregas et al. (1971) suggested that after deciding which level of deployment to pursue, p and S, for a given LSCP application, then it made sense to solve a distance constrained p-median problem. This involves modification of the p-median problem, and in particular constraints (2.24). The objective is to find the solution which minimizes total weighted distance, while locating p facilities and ensuring that each demand is within the S standard of their closest facility. This is commonly referred to as the p-median problem with maximum distance constraints.

2.6 Fixed Charges

One of the basic assumptions of Toregas et al. (1971) was that the cost of individual sites did not vary much, especially when compared to the cost of facility construction, the cost of equipment being housed, and the cost of crews needed. They argued that the number of facilities needed was a surrogate measure for cost, and by keeping the number of located facilities as low as possible meant that overall costs would be minimized as well. Such an assumption would be especially true if, for example, each fire station housed the same type and number of equipment, was no more expensive to build in one location as another, and site acquisition costs were approximately the same. But, there are applications where such an assumption would not necessarily be true. For example, consider the case where an early warning siren system is being located. The objective is to cover a political jurisdiction with siren towers such that all inhabitants can hear a siren when in use. Although the land costs for siren placement may be minimal, the site preparation costs for one site may be quite different from another. Sirens need to be connected to a protected communication network, provided a source of power, and have a solid geotechnical foundation. Utility connections could be considerable for one site and miniscule for another. The same is true for preparing a structural base. Consequently, site costs may be an important consideration in siren location. Another issue is almost always present: some facilities already exist. The cost of new facilities should be accounted for differently than existing facilities that have already been amortized. To address this issue, we can consider a cost sensitive form of the LSCP and MCLP.

In the case of the LSCP, this means that the objective should consider variable costs. This cost may be defined as follows:

$c_j =$ cost of siting facility j

This means that the fixed cost LSCP (FC-LSCP) can be structured as:

$$\text{FC-LSCP}: \qquad Minimize \ \sum_{j=1}^{n} c_j x_j \tag{2.30}$$

Subject to:

$$\sum_{j \in N_i} x_j \geq 1 \quad \forall i \tag{2.31}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.32}$$

Objective (2.30) involves minimizing the total cost of siting facilities. Constraints (2.31) ensure that for each demand node at least one facility is selected among the set N_i. Binary integer restrictions are imposed in constraints (2.32). Worth noting is that FC-LSCP is mathematically equivalent to the set covering problem, extending the (2.1–2.3). Of course, the above mentioned points regarding the spatial significance of the FC-LSCP would apply in terms of its uniqueness and distinction from the set covering problem.

The MCLP also can be extended to account for fixed costs. Here a formulation is detailed based on MCLP2. The fixed cost MCLP2 (FC-MCLP2) is as follows:

$$\text{FC-MCLP2}: \qquad Minimize \sum_{i=1}^{n} a_i \bar{y}_i \tag{2.33}$$

$$Minimize \sum_{j=1}^{n} c_j x_j \tag{2.34}$$

Subject to:

$$\sum_{j \in N_i} x_j + \bar{y}_i \geq 1 \quad \forall i \tag{2.35}$$

$$\sum_j x_j = p \tag{2.36}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.37}$$

$$\bar{y}_i \in \{0, 1\} \quad \forall i \tag{2.38}$$

FC-MCLP2 is a multi-objective model. Objective (2.33) minimizes the amount of demand not covered. Objective (2.34) minimizes the total cost of siting facilities. Constraints (2.35) specify coverage of demand. Constraints (2.36) indicate the number of facilities to be sited. Technically speaking, this constraint is not necessary in the sense that objective (2.34) accounts for costs, so ultimately dictates the number of facilities that would be sited as a tradeoff of the importance to cover demand. Whether to include it partly a preference as this constraint represents another model parameter. Decision variable requirements are indicated in constraints (2.37) and (2.38).

Solution of problems involving multiple objectives can be challenging. Chapter 3 discusses this issue in more detail. Here we simply note that methods exist for dealing with multiple objectives in a manner that associated trade off solutions can be identified.

The FC-MCLP2 was first proposed by Church and Davis (1992). They show that site costs can make a difference in which solutions are non-inferior with respect to the tradeoff between site cost and coverage. To solve this problem, they used DUALOC, a dual ascent solution procedure (Erlenkotter 1978) designed to solve the classical fixed charge simple plant location problem. In casting the fixed charge maximal covering problem (they called this MCFix) as a simple plant location problem, Church and Davis (1992) used a transformation function like that given in (2.28).

Highlighted in this chapter is that there are often alternative ways to mathematically structure any given problem of interest. This was observed for the MCLP, as an example, but also that the MCLP and LSCP could be structured as a transformed p-median problem. Accordingly, fixed costs could readily be incorporated in the MCLP or even the p-median problem.

Both the FC-MCLP2 and FC-LSCP are simple, extended forms of MCLP and LSCP, respectively. They add a needed feature when specific site costs are particularly important. The majority of research involving covering problems has been based upon the assumption of uniform site costs. As stated earlier, site costs often represent only a small portion of the total system cost, and using the number of facilities (as in LSCP and MCLP) as a surrogate measure of overall cost is a reasonable assumption in many cases.[4]

2.7 Summary and Concluding Comments

The early roots in optimization models involving the notion of cover began in logic and set representation problems. Hakimi (1965) was one of the first to propose that the use of a coverage standard in allocating policemen across a network so that all parts of the network are within a prescribed distance of a policeman. Toregas et al. (1971) were the first to propose a mathematical formulation for a location model that involved covering a region with fire stations. This formulation, the LSCP, (2.5–2.7), along with that of the MCLP, (2.8–2.12), formed the basis for an important field of location science. Even though there are a large number of applications of these two models alone, there are also a number of elements that are not captured exactly by these two constructs. In addition, large LSCP and MCLP application instances might not be easy to solve. This has led to two major thrusts in covering model research: the development of different solution strategies, and the development of extended model forms.

[4]In Chap. 3 we present an interesting form of site cost proposed by Plane and Hendrick (1977).

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