



Regional Synchronization of a Probabilistic Cellular Automaton

Franco Bagnoli^{1,2}(✉) and Raúl Rechtman³

¹ Dipartimento di Fisica e Astronomia and CSDC, Università di Firenze,
Via G. Sansone 1, 50019 Sesto Fiorentino, Italy

`franco.bagnoli@unifi.it`

² INFN, sez. Firenze, Sesto Fiorentino, Italy

³ Instituto de Energías Renovables, Universidad Nacional Autónoma de México,
Apdo. Postal 34, 62580 Temixco, Morelos, Mexico

`rrs@ier.unam.mx`

Abstract. We study the regional master-slave synchronization of a one dimensional probabilistic cellular automaton with two absorbing states. The master acts on the boundary of an interval, the region, of a fixed size. For some values of the parameters, this is enough to achieve synchronization in the region. For other values, we extend the regional synchronization to include a fraction of sites inside the region of interest. We present four different ways of doing this and show which is the most effective one, in terms of the fraction of sites inside the region and the time needed for synchronization.

1 Introduction

Cellular Automata (CA) are spatially extended systems that are widely used for modelling various problems ranging from physics to biology, engineering, medicine, ecology and economics [1–8].

Cellular automata are discrete systems in time and space. The state at each node, here 0 or 1, changes in time according to the transition probabilities of assuming a certain state knowing the state of neighbouring nodes. When the transition probabilities are either zero or one, the automata is deterministic, otherwise it is probabilistic. Despite their simplicity, cellular automata may exhibit a large number of different features.

In particular, deterministic cellular automata may exhibit “chaotic” trajectories, in which a initial small disturbance (a “defect”) amplifies or spreads, in average, over time. This is also called the “damage spreading” feature.

Deterministic cellular automata may be considered discrete dynamical systems, and one is interested in the problem of controlling the resulting trajectories. The control problem can be divided in two sub-problems: how to drive a system into a desired state (reachability problem) and how to make it follow a desired trajectory, which can be also a fixed point (drivability problem). We are

interested here in the regional version of this problem, i.e., how to control just a given region of a system. Clearly, one has at least to act on the boundaries of such a region in order to promote this control, but this can be insufficient, especially for chaotic CA.

As shown in Refs. [9, 10], while it is possible to make a system reach a desired state acting on the boundaries, it is in general not possible to impose a trajectory which is not “natural”, i.e., a trajectory different from one that the system would follow if starting from a proper initial configuration and with proper boundary conditions. Except for simple states like fixed points or cycles, the identification of a “natural” trajectory is best done using a replica of the system, that evolves freely. The drivability problem is related to master-slave synchronization. The problem of regional control, that is, where the control is applied in on the boundary of a region with a fixed number of sites is discussed in Ref. [11].

One of the problems in studying discrete cellular automata is that it is not possible to continuously vary their dynamical properties, so that it is difficult to observe bifurcations and changes of behaviour. On the contrary, this is possible with probabilistic cellular automata which however are intrinsically stochastic, and therefore in principle impossible to synchronize. A review of phase transitions for probabilistic cellular automata may be found in Ref. [12].

However, it is possible to “convert” probabilistic cellular automata into deterministic ones, considering that the actual computation of a trajectory of such systems makes use of random numbers, used to choose, for each site and each time step, among the possible alternatives. One may assume that the set of all needed random numbers is extracted at the beginning of the simulation for all sites and all time steps, thus constituting a quenched random field. The evolution of the automata over such a random field becomes deterministic, and therefore it is possible to consider the problem of the divergence of initially similar trajectories (damage spreading) also for Probabilistic Cellular Automata (PCA). The advantage of such an approach is that the behaviour of PCA can be fine-tuned by means of their control parameters, and therefore it is possible to investigate in details the elements that contribute to chaoticity, control and synchronization, a task that is much more difficult with Deterministic Cellular Automata (DCA).

In principle the synchronization characteristics depend on the quenched random field, but in practice these systems are always self-averaging so that a large enough simulation already gives the same value of observables as if one performs an averaging over many realizations of the random field.

In Ref. [13], the problem of synchronization of DCA was addressed, showing that it is possible achieve this goals by randomly choose at each time step a large enough fraction of sites in which the state of sites in the slave system is imposed to be that of the corresponding sites in the master one and it was shown that the synchronization threshold is related to the chaotic properties of automata. In Ref. [9], a similar technique, called pinching synchronization, was applied to control problems, looking for the most efficient way of achieving the synchronization goal. In Ref. [10] this procedure was applied to the regional

control problem of DCA. We want here to apply, and extend, this technique to PCA (see also Ref. [14]).

In what follows, we investigate different strategies of regional master-slave pinching synchronization of a one-dimensional three-state probabilistic cellular automaton with two absorbing states [15]. The state at any site in the lattice at time $t + 1$ depends probabilistically on the states of the site itself and its two next-nearest neighbours at time t , and two probabilities. The master and the slave are two realizations of the same PCA starting from different initial states and the slave is forced to follow the master at the boundary of a given region of width L . Since this is in general insufficient to synchronize the two systems, the slave is additionally forced to take the state of the master at certain sites inside the target region, at every time step.

In Sect. 2 we present this cellular automaton. In our first attempt, boundary regional synchronization, or simply L -synchronization, the master imposes his state on the border of a region of size L on the slave and we find that for some values of the probabilities, there is synchronization in the sense that the slave follows the master in the region of length L . This is discussed in Sect. 3. When there is no L -synchronization, we discuss in Sect. 4 four different pinching synchronization schemes at a fraction π of sites inside the region of size L and show which one is the most successful one in the sense that synchronization occurs with the smallest value of π and the shortest time. We finish with some conclusions in Sect. 5.

2 The Probabilistic Cellular Automaton

We recall the definition of the probabilistic cellular automaton with two absorbing states presented in Ref. [15]. The state at site i at time t , $x_i^{(t)}$, with $i = 0, \dots, N - 1$ and $t = 0, 1, \dots$, can take two values, $x_i^{(t)} = 0$, dry, or $x_i^{(t)} = 1$, wet. The state of the cellular automaton at time t is $\mathbf{x}^{(t)} = (x_0^{(t)}, \dots, x_{N-1}^{(t)})$ and $x_i^{(t+1)}$ depends on the number of wet sites in its neighbourhood and four parameters or probabilities p_0, \dots, p_3 . With

$$\sigma_i^{(t)}(\mathbf{x}) = \sum_{j=-1}^1 x_{i+j}^{(t)},$$

$\sigma_i^{(t)}(\mathbf{x}) = 0, \dots, 3$, and the sum on the sub-indices taken modulo N to account for periodic boundary conditions,

$$x_i^{(t+1)} = \sum_{s=0}^3 \left[r_i^{(t)} \leq p_s \right] \left[\sigma_i^{(t)}(\mathbf{x}) = s \right]. \quad (1)$$

In this expression $r_i^{(t)}$ is a random number uniformly distributed between 0 and 1 and $[\cdot] = 1$ if \cdot is true and zero otherwise. In what follows $p_0 = 0$, and $p_3 = 1$, which means that if the neighbours are all dry (wet), the central site

will be dry (wet) at the next time step. Then, the states $\mathbf{x} = \mathbf{0} = (0, \dots, 0)$ and $\mathbf{x} = \mathbf{1} = (1, \dots, 1)$ are absorbing. The activity $a(t)$ is defined by

$$a(t) = a(\mathbf{x}^{(t)}) = \frac{1}{N} \sum_{i=0}^{N-1} x_i^{(t)}. \tag{2}$$

We indicate with a the asymptotic value of $a(t)$.

In Fig. 1(a) we show the phase diagram of the average activity \bar{a} over M samples with random initial conditions with $a(0) \simeq 1/2$. In the bottom left part (in white), any random initial configuration will end in the absorbing state $\mathbf{x} = \mathbf{0}$, and in the upper right part, (in black), any random initial configuration will end in the absorbing state $\mathbf{x} = \mathbf{1}$. In the lower right part there is a region where $0 < \bar{a} < 1$.

We can also define the damage spreading problem for such a model. Two replicas, \mathbf{x} and \mathbf{y} , starting from different random initial conditions, evolve in time with the same random numbers $r_i^{(t)}$,

$$\begin{aligned} x_i^{(t+1)} &= \sum_{s=0}^3 \left[r_i^{(t)} \leq p_s \right] \left[\sigma_i^{(t)}(\mathbf{x}) = s \right], \\ y_i^{(t+1)} &= \sum_{s=0}^3 \left[r_i^{(t)} \leq p_s \right] \left[\sigma_i^{(t)}(\mathbf{y}) = s \right]. \end{aligned} \tag{3}$$

The Hamming distance between the two replicas, in a region of width L , is defined as

$$h_L = \frac{1}{L} \sum_{i=1}^L x_i \oplus y_i \tag{4}$$

where \oplus is the logical exclusive disjunction (sum modulo two).

In Fig. 1(b) we show the phase diagram of the average normalized regional Hamming distance $\overline{h_L}$, that takes values different from zero at the phase boundaries of the activity a , since in these cases it is possible that a replica goes into a state and the other into another state, and in the “chaotic” region for high values of p_1 and low values of p_2 .

3 L -synchronization

Let us now consider the problem where the two replicas, \mathbf{x} and \mathbf{y} , evolve in time starting from different initial conditions chosen at random with the same random numbers $r_i^{(t)}$ but where at the fixed sites $i = 0$ and $i = L + 1$, a distance L apart, $y_0^{(t)}$ and $y_{L+1}^{(t)}$ take the values of $x_0^{(t)}$ and $x_{L+1}^{(t)}$, respectively, before updating as in Eq. (1). In other words, the master, \mathbf{x} imposes his state at two fixed sites on the slave \mathbf{y} or \mathbf{x} and \mathbf{y} are pinched together at $i = 0$ and $i = L + 1$. The normalized regional damage h_L is still defined as in Eq. (4).

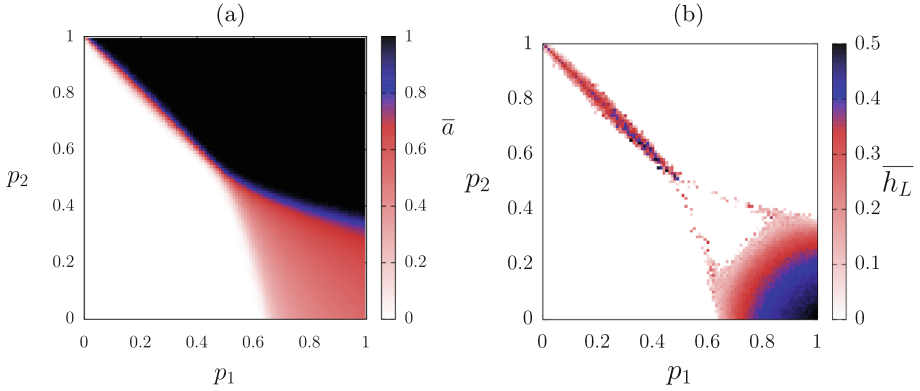


Fig. 1. (a) Phase diagram of the average activity \bar{a} with $N = 1,000$ sites, after $T = 500$ time steps, and $M = 100$ samples with different random initial conditions $\mathbf{x}^{(0)}$ with $a(\mathbf{x}^{(0)}) \simeq 1/2$. (b) Phase diagram of the average normalized regional Hamming distance \bar{h}_L as a function of p_1 and p_2 for the same values of N, T , and M as in (a) and $L = 100$ sites. (Color online)

If $h_L = 0$ at some time t we say there is L -synchronization. In Fig. 1(b) we show the phase diagram of the average \bar{h}_L over M samples as functions of p_1 and p_2 . The area where $h_L > 0$ with $L = N$ is known as the chaotic phase [16] but we prefer to call it the L -damage spreading phase for any value of L .

In Fig. 2(a) we show \bar{h}_L , as a function of p_1 on the diagonal $p_2 = 1 - p_1$. There are three different behaviors of \bar{h}_L , separated by $\xi_1 \simeq 0.5$ and $\xi_2 \simeq 0.75$. For $0 < p_1 \leq \xi_1$, \bar{h}_L grows with L , with fixed T . For $\xi_1 < p_1 \leq \xi_2$, $\bar{h}_L = 0$ and for $\xi_2 < p_1 \leq 1$, \bar{h}_L seems to become independent of L for large L . However, in the first interval of p_1 , \bar{h}_L goes to zero as T grows.

In Fig. 2(c) we show the average time for synchronization \bar{T}_s as a function of p_1 , $p_2 = 1 - p_1$, with $0 < p_1 < \xi_1$ for different values of L . This average time grows with L as expected.

In the third interval $\xi_2 < p_1 \leq 1$ the quantity \bar{h}_L is practically independent of T . Clearly, since the automata is probabilistic and ergodic, and the synchronized state is absorbing, the asymptotic state is always the synchronized one, but the time required for achieving this result is so large, for large enough L , that it is practically unachievable. Indeed, the synchronization task is essentially the same of a percolation problem for defects [17].

In summary L -synchronization is successful for $0 < p_1 < \xi_1$ although it may take a long time T_s that grows with L . For $\xi_1 < p_1 \leq \xi_2$, L -synchronization is present and for $\xi_2 < p_1 < 1$ there is no L -synchronization. In the next section, we present four strategies that achieve regional synchronization when $\xi_2 < p_1 \leq 1$ by adding a fraction π of sites where \mathbf{y} follows \mathbf{x} .

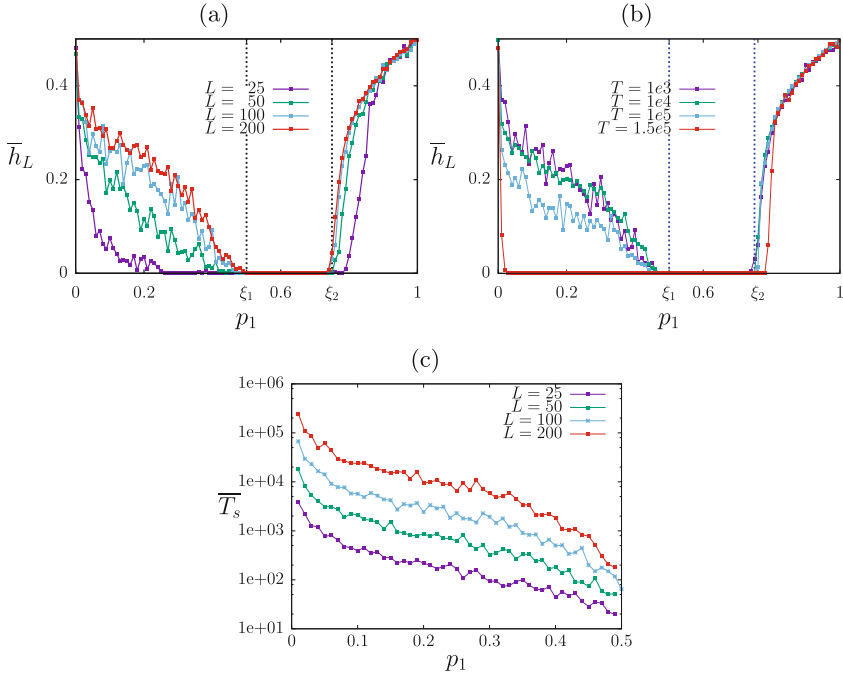


Fig. 2. (a) The average damage \bar{h}_L for $L = 25, 50, 100, 200$ as a function of p_1 on the diagonal $p_2 = 1 - p_1$. We estimate that $\xi_1 \simeq 0.5$ and $\xi_2 \simeq 0.75$. The number of sites is $N = 1,000$ and the average is taken over $M = 100$ samples after a time $T = 1,000$. (b) The average damage \bar{h}_L for $L = 100$ and different total times, $T = 1e3, 1e4, 1e5, 1.5e5$ as a function of p_1 with $p_2 = 1 - p_1$, $N = 1,000$ and $M = 100$. (c) The average synchronization time \bar{T}_s for $L = 25, 50, 100, 200$ as a function of p_1 , $0 \leq p_1 \leq \xi_1$, with $N = 1,000$ and $M = 100$. (Color online)

4 $L\pi$ -synchronization

By $L\pi$ -synchronization we mean that in the region of size L , besides the sites a distance L apart, a fraction π of sites, denoted by j , are chosen and at every time step the slave takes the values of the master, that is $y_j^{(t)} = x_j^{(t)}$.

In other words, \mathbf{x} and \mathbf{y} are pinched together at those sites. We propose four strategies of $L\pi$ -synchronization, L -divide pinching synchronization, L -quenched pinching synchronization, L -annealed pinching synchronization and L -random walk pinching synchronization. In what follows we refer to them as LDP, LQP, LAP and LRWP synchronization respectively.

In the four strategies, a fraction πL , $0 < \pi \leq 1/2$, of sites in the region of length L are chosen. In the first strategy, LDP, the sites j divide the region of length L into equally spaced intervals. In the second one LQP, the sites j are chosen at random in the region L while in the third one, LAP, the sites j are chosen at random at every time step. In the fourth strategy, LRWP, the fraction

πL of sites are the starting point of random walkers that at every time step can move one site to the right or left with the same probability.

Walkers do not know the others' position, cannot coordinate with them, and may cross each other. When they reach the border of the region at $i = 0$ or $i = L + 1$, they bounce back. In Fig. 3 we show examples of the four strategies.

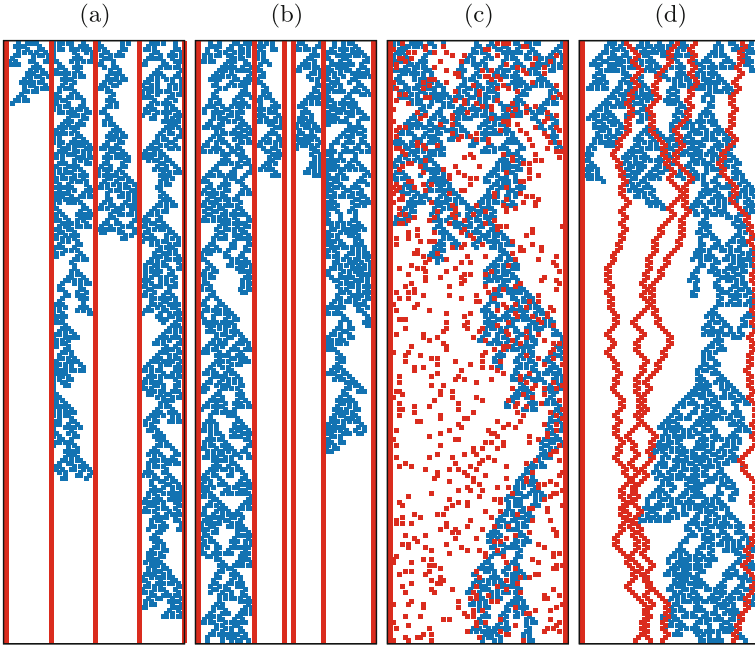


Fig. 3. Space-time diagrams of the four bulk synchronization schemes. (a) L -divide pinching synchronization. (b) L -quenched pinching synchronization. (c) L -annealed pinching quenched synchronization. (d) L -random walk pinching synchronization. In all cases, $L = 60$, $p_1 = 0.8$, $p_2 = 0.2$, $\pi = 0.1$ and the region of size $L = 60$ is shown during $T = 200$ time steps. (Color online)

In Fig. 4(a) and (b) we show $\overline{h_L}$ and $\overline{T_s}$ as functions of π with $p_1 = 0.85$ and $p_2 = 0.15$ respectively. The best strategy, in the sense of achieving synchronization for the smallest value of π in the shortest time, is LDP synchronization. This is valid for other values of p_1 . To simplify our results, if $T_s > T$, that occurs for small p_1 , we write $T_s = T$.

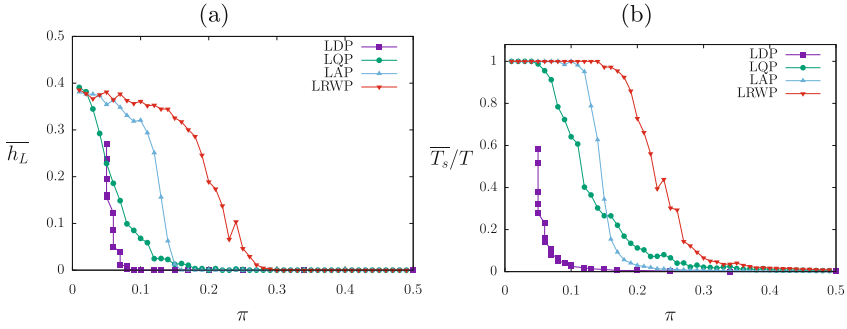


Fig. 4. The average normalized Hamming distance \overline{h}_L in (a), and the average normalized synchronization time fraction \overline{T}_s/T in (b) as functions of π with πL the fraction of sites in the region of size L where synchronization is imposed for $p_1 = 0.85$ and $p_2 = 1 - p_1 = 0.15$ in the four strategies. In (a) and (b) the data correspond, from left to right, to L divide pinching synchronization, LDP (in magenta), L quenched pinching synchronization, LQP (in green), L annealed pinching synchronization, LAP (in blue), and L random walk pinching synchronization, LRWP (in red). The parameters are $N = 1,000$, $L = 100$, $M = 100$ and $T = 10,000$. (Color online).

5 Conclusions

We presented regional synchronization, the synchronization of two extended systems to a sub-domain, the region. As an example, we discussed some properties the three state probabilistic cellular automaton and showed that regional synchronization has three different behaviours on the diagonal $p_2 = 1 - p_1$ of Fig. 1(b). In the first one, $0 < p_1 < \xi_1 \simeq 0.5$, L -synchronization occurs for long times. In the second one, $\xi_1 < p_1 < \xi_2 \simeq 0.75$, L -synchronization is always present, and in the third one, $\xi_2 < p_1 < 1$, L -synchronization is not possible. If we insist on trying to synchronize \mathbf{y} with \mathbf{x} in this third case, we have to split the region in subregions and we presented four different strategies and show which one is the most effective. It might prove interesting to extend the analysis of $L\pi$ -synchronization to the whole phase diagram of Fig. 1 and to other cellular automata.

Acknowledgments. We thank S. El Yacoubi for useful comments. R.S. acknowledges partial financial support from PPA-DGAPA-UNAM.

References

1. Bandini, S., Chopard, B., Tomassini, M. (eds.): ACRI 2002. LNCS, vol. 2493. Springer, Heidelberg (2002). <https://doi.org/10.1007/3-540-45830-1>
2. Sloot, P.M.A., Chopard, B., Hoekstra, A.G. (eds.): ACRI 2004. LNCS, vol. 3305. Springer, Heidelberg (2004). <https://doi.org/10.1007/b102055>

3. El Yacoubi, S., Chopard, B., Bandini, S. (eds.): ACRI 2006. LNCS, vol. 4173. Springer, Heidelberg (2006). <https://doi.org/10.1007/11861201>
4. Umeo, H., Morishita, S., Nishinari, K., Komatsuzaki, T., Bandini, S. (eds.): ACRI 2008. LNCS, vol. 5191. Springer, Heidelberg (2008). <https://doi.org/10.1007/978-3-540-79992-4>
5. Bandini, S., Manzoni, S., Umeo, H., Vizzari, G. (eds.): ACRI 2010. LNCS, vol. 6350. Springer, Heidelberg (2010). <https://doi.org/10.1007/978-3-642-15979-4>
6. Sirakoulis, G.C., Bandini, S. (eds.): ACRI 2012. LNCS, vol. 7495. Springer, Heidelberg (2012). <https://doi.org/10.1007/978-3-642-33350-7>
7. Waş, J., Sirakoulis, G.C., Bandini, S. (eds.): ACRI 2014. LNCS, vol. 8751. Springer, Cham (2014). <https://doi.org/10.1007/978-3-319-11520-7>
8. El Yacoubi, S., Waş, J., Bandini, S. (eds.): ACRI 2016. LNCS, vol. 9863. Springer, Cham (2016). <https://doi.org/10.1007/978-3-319-44365-2>
9. Bagnoli, F., El Yacoubi, S.: Control of cellular automata. *Phys. Rev. E* **86**, 066201 (2012). <https://doi.org/10.1103/PhysRevE.86.066201>
10. Bagnoli, F., El Yacoubi, S., Rechtman, R.: Toward a boundary regional control problem for Boolean cellular automata. *Nat. Comput.* (2017). <https://doi.org/10.1007/s11047-017-9626-1>
11. Bagnoli, F., Dridi, S., El Yacoubi, S., Rechtman, R.: Regional control of probabilistic cellular automata. In: Mauri, G., et al. (eds.) ACRI 2018. LNCS, vol. 11115, pp. 243–254. Springer, Heidelberg (2018)
12. Bagnoli, F., Rechtman, R.: Phase transitions of cellular automata. In: Louis, P.-Y., Nardi, F.R. (eds.) Probabilistic Cellular Automata. ECC, vol. 27, pp. 215–236. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-65558-1_15
13. Bagnoli, F., Rechtman, R.: Synchronization and maximum Lyapunov exponent in cellular automata. *Phys. Rev. E* **59**, R1307 (1999). <https://doi.org/10.1103/PhysRevE.59.R1307>
14. Bagnoli, F., El Yacoubi, S., Rechtman, R.: Control of cellular automata. In: Robert, A.M. (ed.) Encyclopedia of Complexity and Systems Science. Springer, Heidelberg (2018). https://doi.org/10.1007/978-3-642-27737-5_710-1
15. Bagnoli, F., Boccara, N., Rechtman, R.: Nature of phase transitions in a probabilistic cellular automaton with two absorbing states. *Phys. Rev. E* **63**, 046116 (2001). <https://doi.org/10.1103/PhysRevE.63.046116>
16. Martins, M.L., Verona de Resende, H.F., Tsallis, C., Magalhães, A.C.N.: Evidence for a new phase in the Domany-Kinzel cellular automaton. *Phys. Rev. Lett.* **66**, 20145 (1991). <https://doi.org/10.1103/PhysRevLett.66.2045>
17. Grassberger, P.: Are damage spreading transitions generically in the universality class of directed percolation? *J. Stat. Phys.* **79**, 13 (1995). <https://doi.org/10.1007/BF02179381>