

Robust Reliable H_∞ Control and Input-to-State Stabilization for Uncertain Hybrid Systems



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Abstract The main objective of this paper to design a robust reliable H_∞ control and a switching law for a class of uncertain switched systems under an average dwell time switching signal that guarantees ISS not only when all the actuators are operational, but also when some of them experience failure. The faulty actuator output is assumed to be nonzero, which is treated as a disturbance signal that is augmented with the system disturbance input. The input-to-state stability (ISS) property is analyzed by the multiple Lyapunov functions and comparison principle approach. A numerical example is introduced to illustrate the validity of the theoretical results.

Keywords Switched systems · ISS · Average dwell time · Reliable control H_∞ control

1 Introduction

There has been a growing interest in studying switched systems in the last three decades due to their widespread applications in different fields such as aircraft, automotive industry, robotics, control systems, biological, epidemic disease models; see [7, 8, 10] and the references therein. By a switched system we mean a special class of hybrid systems that consist of a family of continuous- or discrete-time dynamical subsystems (or modes), and a switching rule (or signal). The role of the switching signal is to govern the jump among the subsystems. The stability of switched systems has received much attention and has been studied using either the common Lyapunov function method [11], or the multiple Lyapunov function method [2]. It has been realized that it is more convenient to use multiple Lyapunov functions than

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the common Lyapunov function since having only one Lyapunov function for all the modes under study is not practical and is difficult to construct.

The reliable control is the controller that tolerates actuator and/or sensor failures. In reality, the failure of control components is frequently encountered, yet the immediate repair may not be feasible, such as in the case of aerospace or submarine system, etc. Therefore, designing a reliable controller to guarantee an acceptable level of performance becomes crucial. The trend to design reliable controllers has been increased; see for instance [4, 12, 15, 16, 18]. In most of the available results about reliable control, the faulty actuators are modelled as outages i.e., the output is assumed to be zero. In [1, 9, 12, 16], the output signal is considered as a disturbance signal with bounded magnitude that is augmented with the system disturbance signal.

The H_∞ control has received a great deal of attention in control theory [3, 17]. It is a useful measure used to guarantee the performance of the plant when dealing with control problems that involve robust design. However, in the event of control component failures, the stability or performance of the plant may not be achieved by such designs. Therefore, it would be advantageous if it is associated with a reliable control design to handle such failures when they occur. One may refer to [1, 9, 12, 18].

In practice, most of the real control systems are subject to some disturbance inputs. ISS notion, introduced in [13] which addresses the system response to a bounded disturbance when the unforced system is asymptotically stable, is an efficient tool to deal with these disturbances. As a result, it becomes important in the modern nonlinear control theory and design [1, 13, 14].

This paper is organized as follows. Section 2 involves the problem description, definitions, and a useful lemma. The main results and proofs are stated in Sect. 3. A numerical example with simulations is presented in Sect. 4. The conclusion is given in Sect. 5.

2 Problem Formulation and Preliminaries

Throughout this paper, \mathbb{R}^n denotes the n -dimensional Euclidean space; \mathbb{R}_+ refers to the nonnegative real numbers; $\mathbb{R}^{n \times m}$ is the class of all $n \times m$ real matrices. A symmetric matrix P is said to be positive definite if all its eigenvalues are positive. Moreover, If $P \in \mathbb{R}^{n \times n}$, denote by $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) the maximum (minimum) eigenvalue of P . If $V(x) = x^T P x$, the following inequalities are always true

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2, \quad (1)$$

where $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$. If $x \in \mathbb{R}^n$, then $\|x\|$ refers to the Euclidean vector norm of x . $L_2[t_0, \infty)$ is the space of square integrable vector-valued functions on $[t_0, \infty)$ and $\|\cdot\|_2$ denotes $L_2[t_0, \infty)$ -norm (i.e., $w \in L_2[t_0, \infty)$ means $\|w\|_2^2 = \int_{t_0}^{\infty} \|w(t)\|^2 dt < \infty$). Consider a class of uncertain switched systems given by

$$\begin{cases} \dot{x} = (A_{\rho(t)} + \Delta A_{\rho(t)})x + B_{\rho(t)}u + G_{\rho(t)}w + f_{\rho(t)}(x), \\ z = C_{\rho(t)}x + F_{\rho(t)}u, \quad x(t_0) = x_0, \end{cases} \quad (2)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^q$ is the control input, $w \in \mathbb{R}^p$ is an input disturbance, which is assumed to be bounded, and $z \in \mathbb{R}^r$ is the controlled output. ρ is the switching law which is a piecewise constant function defined by $\rho : [t_0, \infty) \rightarrow \mathcal{S} = \{1, 2, \dots, N\}$. The role of ρ is to switch among the system modes. For each $i \in \mathcal{S}$, A_i is a non Hurwitz matrix, $K_i \in \mathbb{R}^{q \times n}$ is the control gain matrix such that $u = K_i x$, where (A_i, B_i) is assumed to be stabilizable, $f_i(x) \in \mathbb{R}^n$ is some nonlinearity, A_i, B_i, G_i, C_i and F_i are known real constant matrices, and ΔA_i is a deterministic piecewise continuous function of time t which represents parameter uncertainty with bounded norm and it also gives the structure of the system uncertainty. For any $i \in \mathcal{S}$, the closed-loop system is

$$\begin{cases} \dot{x} = (A_i + \Delta A_i + B_i K_i)x + G_i w + f_i(x), \\ z = C_i x, \quad x(t_0) = x_0, \quad \text{where } C_{ic} = C_i + F_i K_i. \end{cases} \quad (3)$$

To analyze the reliable stabilization with respect to actuator failures, the q control actuators are divided into two sets. $\Sigma \subseteq \{1, 2, \dots, q\}$ the set of actuators that are susceptible to failure, i.e., they may occasionally fail, and $\bar{\Sigma} \subseteq \{1, 2, \dots, q\} - \Sigma$ the other set of actuators which are robust to failures and essential to stabilize the given system. The elements of Σ are redundant in terms of the stabilization but necessary to improve the system performance, while the elements of $\bar{\Sigma}$ are required to stabilize the system and assumed that they never fail, i.e., the pair $(A_i, B_{i\bar{\Sigma}})$ is assumed to be stabilizable.

For $i \in \mathcal{S}$, consider the decomposition of the control matrix $B_i = B_{i\Sigma} + B_{i\bar{\Sigma}}$, where $B_{i\Sigma}, B_{i\bar{\Sigma}}$ are the control matrices associated with $\Sigma, \bar{\Sigma}$ respectively, and $B_{i\Sigma}, B_{i\bar{\Sigma}}$ are generated by zeroing out the columns corresponding to $\bar{\Sigma}$ and Σ , respectively. For a fixed $i \in \mathcal{S}$, let $\sigma \subseteq \Sigma$ corresponds to some of the actuators that experience failure, and assume that the output of faulty actuators is any arbitrary energy-bounded signal (or disturbance input) which belongs to $L_2[t_0, \infty)$. Then, the decomposition becomes $B_i = B_{i\sigma} + B_{i\bar{\sigma}}$, where $B_{i\sigma}$ and $B_{i\bar{\sigma}}$ have the same definition of $B_{i\Sigma}$ and $B_{i\bar{\Sigma}}$, respectively. Furthermore, the augmented disturbance input to the system becomes

$$w_\sigma^F = (w^T \quad (u_\sigma^F)^T)^T,$$

where $u_\sigma^F \in \mathbb{R}^q$ is the failure vector whose elements corresponding to the set of faulty actuators σ , and F here stands for ‘‘failure’’. Since the control input u is applied to the system through the normal actuators, and the outputs of the faulty actuators are assumed to be arbitrary signals, the closed-loop system becomes

$$\begin{cases} \dot{x} = (A_i + \Delta A_i + B_{i\bar{\sigma}} K_i)x + G_{ic} w_\sigma^F + f_i(x), \quad i \in \mathcal{S} = \{1, 2, \dots, N\}, \\ z = C_{ic} x, \quad x(t_0) = x_0, \quad \text{where } G_{ic} = (G_i \quad B_{i\sigma}). \end{cases} \quad (4)$$

Definition 2.1 [6] (*Class-K function*) A function $\rho : [0, r) \rightarrow [0, \infty)$ is said to belong to class \mathcal{K} (i.e., $\rho \in \mathcal{K}$) if it is continuous, strictly increasing, and $\rho(0) = 0$.

Definition 2.2 (*Input-to-State Stability*) System (3) is said to be robustly globally exponentially ISS if there exist positive constants $\lambda, \bar{\lambda}$ and a function $\rho \in \mathcal{K}$ such that, for any solution $x(t) = x(t, t_0, x_0)$,

$$\|x\| \leq \bar{\lambda} \|x_0\| e^{-\lambda(t-t_0)} + \rho \left(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\| \right), \quad \forall t \geq t_0.$$

Definition 2.3 (*input-to-state stability with an H_∞ -norm (ISS- H_∞)*) Given a constant $\gamma > 0$, system (3) is said to be ISS- H_∞ if there exists a state feedback law $u(t) = K_i x(t)$, such that, for any admissible parameter uncertainties ΔA_i , the closed loop system (3) is globally exponentially ISS, and the controlled output z satisfies

$$\|z\|_2^2 = \int_{t_0}^{\infty} \|z\|^2 dt \leq \gamma^2 \|w\|_2^2 + m_0,$$

for some positive constant m_0 .

Assumption A For $i \in \mathcal{S}$, the admissible parameter uncertainties are defined by

$$\Delta A_i(t) = D_i \mathcal{U}_i(t) H_i, \quad \forall t \in \mathbb{R}_+,$$

with D_i, H_i being known real matrices with appropriate dimensions that give the structure of the uncertainty, and $\mathcal{U}_i(t)$ being unknown real time-varying matrix representing the uncertain parameter and satisfying $\|\mathcal{U}_i(t)\| \leq 1$.

Lemma 2.4 For any arbitrary positive constants ξ_1, ξ_2 and ξ_3 , and a positive definite matrix P , we have

- (i) $2x^T P (\Delta A) x \leq x^T \left(\xi_1 P D D^T P + \frac{1}{\xi_1} H^T H \right) x.$
- (ii) $2x^T P G w \leq x^T \left(\xi_2 P G G^T P \right) x + \frac{1}{\xi_2} w^T w.$
- (iii) $2x^T P f(x) \leq x^T \left(\xi_3 P^2 + \frac{1}{\xi_3} \delta I \right) x$ such that $\|f(x)\|^2 \leq \delta \|x\|^2$ with $\delta > 0$.

Proof For (i), we have

$$\begin{aligned} 0 &\leq (\sqrt{\xi_1} x^T (P D \mathcal{U}) - \frac{1}{\sqrt{\xi_1}} x^T H^T) (\sqrt{\xi_1} x^T (P D \mathcal{U}) - \frac{1}{\sqrt{\xi_1}} x^T H^T)^T \\ &= \xi_1 x^T P D D^T P x + \frac{1}{\xi_1} x^T H^T H x - 2x^T (P D \mathcal{U} H) x, \end{aligned}$$

which leads to $2x^T (P D \mathcal{U} H) x \leq x^T \left(\xi_1 P D D^T P + \frac{1}{\xi_1} H^T H \right) x$, which yields the desired result. The inequalities in (ii) and (iii) can be proved similarly. \square

Average Dwell Time Condition (ADTC) [5]. The number of switches $N(t_0, t)$ in the interval (t_0, t) for a finite t satisfies $N(t_0, t) \leq N_0 + \frac{t-t_0}{\tau_a}$, where $N_0 > 0$ is the chatter bound, and τ_a is the average dwell time.

Here, by *dwell time* we mean the time between two consecutive switches, while the *chatter bound* is an upper bound for the number of switches in an interval of length smaller than τ_a .

3 Main Results

Theorem 3.1 *Let the controller gain K_i and the constant $\gamma_i > 0$ be given, and assume that Assumption A holds. Then, the switched control system (3) is robustly globally exponentially ISS with an H_∞ -norm bound γ if the ADTC holds, and there exist positive constants ξ_{1i} , ξ_{2i} , ξ_{3i} , and a positive definite matrix P_i satisfying the Riccati-like equation*

$$\begin{aligned} (A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + C_{ic}^T C_{ic} \\ + \xi_{2i} P_i G_i G_i^T P_i + \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I + \alpha_i P_i = 0, \end{aligned} \quad (5)$$

where $\delta_i > 0$ such that $\|f_i(x)\|^2 \leq \delta_i \|x\|^2$, and $\alpha_i > 0$ is the decay rate of mode i .

Proof Let $x(t) = x(t, t_0, x_0)$ be the solution of system (3). For any $i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate for the i th mode. Then,

$$\begin{aligned} \dot{V}_i(x) &= x^T [(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i)]x + 2x^T P_i (\Delta A_i)x \\ &\quad + 2x^T P_i G_i w + 2x^T P_i f_i(x) \\ &\leq x^T [(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i \\ &\quad + \xi_{2i} P_i G_i G_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i \\ &\quad + \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I]x + \frac{1}{\xi_{2i}} w^T w = -\alpha_i V_i(x) + \frac{1}{\xi_{2i}} w^T w, \end{aligned}$$

where we used $\|f_i(x)\|^2 \leq \delta_i \|x\|^2$ and Lemma 2.4 in the second bottom line, and condition (5) in the last line. Hence, for each subinterval $[t_{k-1}, t_k)$ we have, after adding-subtracting the term $\theta_i V_i(x)$,

$$\begin{aligned} \dot{V}_i(x) &\leq -(\alpha_i - \theta_i) V_i(x) - \theta_i V_i(x) + w^T w / \xi_{2i} \\ &\leq -\bar{\alpha}_i V_i(x) - \theta_i V_i(x) + w^T w / \xi_{2i}, \end{aligned}$$

where $\bar{\alpha}_i = \alpha_i - \theta_i$ for some $0 < \theta_i < \alpha_i$. The foregoing inequality implies that

$$\dot{V}_i(x) \leq -\bar{\alpha}_i V_i(x), \quad \text{for all } t \in [t_{k-1}, t_k),$$

provided that the sum $-\theta_i V_i(x) + w^T w / \xi_{2i} < 0$ or $V_i(x) > \frac{1}{\theta_i \xi_{2i}} \|w\|^2$. By (1), $\|x\| > \frac{\|w\|}{\sqrt{\theta_i c_2 \xi_{2i}}} =: \rho_i(\|w\|)$. Then, for all $t \in [t_{k-1}, t_k)$, $V_i(x(t)) \leq V_i(x(t_{k-1})) e^{-\bar{\alpha}_i(t-t_{k-1})}$ provided that $\|x\| > \rho(\|w\|)$, where $\rho(\|w\|) = \max_{i \in \mathcal{S}} \{\rho_i(\|w\|)\}$. From (1), we have for any $i, j \in \mathcal{S}$

$$V_j(x(t)) \leq \mu V_i(x(t)), \quad \mu = c_2/c_1, \quad (6)$$

where $c_1 = \min_{i \in \mathcal{S}} \{\lambda_{\min}(P_i)\}$ and $c_2 = \max_{i \in \mathcal{S}} \{\lambda_{\max}(P_i)\}$. Then, for $i \in \mathcal{S}$ and $t \in [t_{k-1}, t_k)$, we have $V_i(x(t)) \leq \mu^{k-1} e^{-\bar{\alpha}_i(t-t_{k-1})} e^{-\bar{\alpha}_{i-1}(t_{k-1}-t_{k-2})} \dots e^{-\bar{\alpha}_1(t_1-t_0)} V_1(x_0)$ provided that $\|x\| > \rho(\|w\|)$. Letting $\alpha^* = \min\{\bar{\alpha}_i; i \in \mathcal{S}\}$, one may get

$$V_i(x(t)) \leq \mu^{k-1} e^{-\alpha^*(t-t_0)} V_1(x_0) = e^{(k-1) \ln \mu - \alpha^*(t-t_0)} V_1(x_0)$$

provided that $\|x\| > \rho(\|w\|)$. Using the ADTC with $N_0 = \frac{\eta}{\ln \mu}$, $\tau_a = \frac{\ln \mu}{\alpha^* - \nu}$, ($\nu < \alpha^*$), for some arbitrary positive constant η , we get

$$V_i(x(t)) \leq e^{\eta - \nu(t-t_0)} V_1(x_0) \quad \text{provided that } \|x\| > \rho(\|w\|).$$

This implies that [6] $\|x\| \leq b \|x_0\| e^{-\nu(t-t_0)/2} + \gamma(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\|)$, $t \geq t_0$, where $b = \sqrt{e^\eta c_2 / c_1}$, and $\gamma(s) = \sqrt{\frac{c_2}{c_1}} \rho(s)$, which completes the proof of exponential ISS.

To prove the upper bound on the output magnitude $\|z\|$, for any $i \in \mathcal{S}$, we introduce the performance function $J_i = \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w) dt$. Then,

$$\begin{aligned} J_i &\leq \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w) dt + \int_{t_0}^{\infty} \dot{V}_i dt + V_i(x_0) \\ &\leq \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w) dt + V_i(x_0) + \int_{t_0}^{\infty} \{x^T [(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) \\ &\quad + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I - \gamma_i^{-2} P_i G_i G_i^T P_i \\ &\quad + \gamma_i^{-2} P_i G_i G_i^T P_i] x + 2x^T P_i G_i w\} dt \\ &= V_i(x_0) + \int_{t_0}^{\infty} \{x^T [(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i \\ &\quad + \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I + \gamma_i^{-2} P_i G_i G_i^T P_i + C_{ic}^T C_{ic}] x \\ &\quad - \int_{t_0}^{\infty} \gamma_i^2 (w - \gamma_i^{-2} G_i^T P_i x)^T (w - \gamma_i^{-2} G_i^T P_i x) dt. \end{aligned}$$

The last term is strictly negative, so, using condition (5) with $\gamma_i^{-2} = \xi_{2i}$, we get $J_i \leq V_i(x_0)$. Recalling the definition of J_i , we see that $\|z\|_2^2 \leq \gamma^2 \|w\|_2^2 + m_0$, where $m_0 = \max_{i \in \mathcal{S}} \{V_i(x_0)\}$, and $\gamma = \max_{i \in \mathcal{S}} \{\gamma_i\}$. This completes the proof. \square

Theorem 3.2 (Reliability) *Let the constant $\gamma_i > 0$ be given. Assume that Assumption A holds, the switched control system (4) is robustly globally exponentially ISS- H_∞ if the ADTC holds, the controller gain $K_i = -\frac{1}{2}\varepsilon_i B_{i\bar{\sigma}}^T P_i$, for some constants $\varepsilon_i > 0$, and positive definite matrix P_i , and there exist positive constants ξ_{1i} , ξ_{2i} , ξ_{3i} , ε_i , and a positive definite matrix P_i satisfying the Riccati-like equation*

$$\begin{aligned} & A_i^T P_i + P_i A_i + P_i (\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T - \varepsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T + \xi_{3i} I) P_i \\ & + \frac{1}{\xi_{1i}} H_i^T H_i + C_{ic}^T C_{ic} + \frac{1}{\xi_{3i}} \delta_i I + \alpha_i P_i = 0, \end{aligned} \quad (7)$$

where δ_i is a positive constant such that $\|f_i(x)\|^2 \leq \delta_i \|x\|^2$ holds.

Proof Let $x(t) = x(t, t_0, x_0)$ be the solution of system (4). For any $i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate for the i th mode. Then,

$$\begin{aligned} & \dot{V}_i(x) \\ & = x^T [A_i^T P_i + P_i A_i + 2P_i (\Delta A_i) + (B_{i\bar{\sigma}} K_i)^T P_i + P_i B_{i\bar{\sigma}} K_i] x \\ & \quad + 2x^T P_i G_{ic} w_\sigma^F + 2x^T P_i f_i(x) \\ & = x^T [A_i^T P_i + P_i A_i + 2P_i (\Delta A_i) - \varepsilon_i P_i (B_{i\bar{\sigma}}) (B_{i\bar{\sigma}})^T P_i] x \\ & \quad + 2x^T P_i G_{ic} w_\sigma^F + 2x^T P_i f_i(x) \\ & \leq x^T [A_i^T P_i + P_i A_i + \xi_{1i} P_i D_i D_i^T P_i + \xi_{2i} P_i G_{ic} G_{ic}^T P_i \\ & \quad + \frac{1}{\xi_{1i}} H_i^T H_i + \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I - \varepsilon_i P_i (B_{i\bar{\sigma}}) (B_{i\bar{\sigma}})^T P_i] x + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F \\ & \leq x^T [A_i^T P_i + P_i A_i + P_i (\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T - \varepsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T \\ & \quad + \xi_{3i} I) P_i + \frac{1}{\xi_{1i}} H_i^T H_i \\ & \quad + \frac{1}{\xi_{3i}} \delta_i I] x + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F = -\alpha_i V_i(x) + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F, \end{aligned}$$

where we used $\|f_i(x)\|^2 \leq \delta_i \|x\|^2$ and Lemma 2.4 in the third bottom line, the fact that [12] $B_{i\bar{\Sigma}} (B_{i\bar{\Sigma}})^T \leq B_{i\bar{\sigma}} (B_{i\bar{\sigma}})^T$, and condition (5) in the last line. Then, for all $t \in [t_{k-1}, t_k)$, we have

$$\dot{V}_i(x) \leq -\bar{\alpha}_i V_i(x) - \theta_i V_i(x) + (w_\sigma^F)^T w_\sigma^F / \xi_{2i},$$

where $\bar{\alpha}_i = \alpha_i - \theta_i$ and $0 < \theta_i < \alpha_i$. This implies that $\dot{V}_i(x) \leq -\bar{\alpha}_i V_i(x)$, for all $t \in [t_{k-1}, t_k)$ provided that $\|x\| > \frac{\|w_\sigma^F\|}{\sqrt{\theta_i c_2 \xi_{2i}}} =: \rho_i(\|w_\sigma^F\|)$. As done in Theorem 3.1, one may get $V_i(x(t)) \leq e^{\eta - \nu(t-t_0)} V_1(x_0)$ provided that $\|x\| > \rho(\|w\|)$, where $\rho(\|w\|) = \max_{i \in \mathcal{S}} \{\rho_i(\|w\|)\}$. This also implies that [6]

$$\|x\| \leq b\|x_0\|e^{-\nu(t-t_0)} + \gamma \left(\sup_{t_0 \leq \tau \leq t} \|w_\sigma^F(\tau)\| \right), \quad t \geq t_0,$$

where $b = \sqrt{e^\eta c_2 / c_1}$, $\gamma(s) = \sqrt{\frac{c_2}{c_1}} \rho(s)$. As for the upper bound on $\|z\|$, we follow the same steps in Theorem 3.1, where $J_i = \int_{t_0}^{\infty} (z^T z - \gamma_i^2 (w_\sigma^F)^T w_\sigma^F) dt$, to obtain $\|z\|_2^2 \leq \gamma^2 \|w_\sigma^F\|_2^2 + m_0$, where $m_0 = \max_{i \in \mathcal{S}} \{V_i(x_0)\}$, and $\gamma = \max_{i \in \mathcal{S}} \{\gamma_i\}$. \square

4 Numerical Example

Example 1 Consider system (3) where $\mathcal{S} = \{1, 2\}$,

$$A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -6 \end{bmatrix}, B_1 = \begin{bmatrix} -7 & 1 \\ 0.1 & 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 2 & 0.1 \\ 0 & 2 \end{bmatrix}, F_1 = \begin{bmatrix} 0.1 & -2 \\ 0.1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H_1 = [0 \ 1], G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f_1 = 0.01 \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \end{bmatrix}, \mathcal{U}_1 = \sin(t),$$

$\varepsilon_1 = 2$, $\xi_{11} = 0.2$, $\gamma_1 = 0.1$, $\alpha_1 = 2$, $\xi_{21} = \gamma_1^{-2}$, $\xi_{31} = 1$, and $\theta_1 = 1$ with $t_0 = 0$. From $\|f_i(x)\|^2 \leq \delta_i \|x\|^2$, one may get $\delta_1 = 0.01$. As for the second mode, we take

$$A_2 = \begin{bmatrix} -9 & 0.2 \\ 0 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & 0.5 \\ 0.1 & -8 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, F_2 = \begin{bmatrix} 0.1 & 0 \\ -3 & 0.1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H_2 = [1 \ 0], G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f_2 = 0.01 \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \end{bmatrix}, \mathcal{U}_2 = \sin(t),$$

$\varepsilon_2 = 0.5$, $\xi_{12} = 0.3$, $\gamma_2 = 0.15$, $\alpha_2 = 2.5$, $\xi_{22} = \gamma_2^{-2}$, $\xi_{32} = 1$, and $\theta_2 = 1.5$. From $\|f_i(x)\|^2 \leq \delta_i \|x\|^2$, one may get that $\delta_2 = 0.01$. Let the system input disturbance be defined by $w(t) = [\sin(t) \ \sin(t)]^T$.

Case 1 (*All the actuators are operational*) When all the control actuators are operational, from Riccati-like equation,

$$P_1 = \begin{bmatrix} 1.6437 & 0.0149 \\ 0.0149 & 0.2499 \end{bmatrix}, P_2 = \begin{bmatrix} 0.1633 & 0.0859 \\ 0.0859 & 0.2724 \end{bmatrix},$$

with $c_{11} = \lambda_{\min}(P_1) = 0.2498$, $c_{12} = \lambda_{\max}(P_1) = 1.6439$, $c_{21} = \lambda_{\min}(P_2) = 0.1161$, $c_{22} = \lambda_{\max}(P_2) = 0.3197$, so, $c_1 = 0.1161$, $c_2 = 1.6439$, and

$$K_1 = \begin{bmatrix} 11.5047 & 0.0796 \\ -1.6467 & -0.0649 \end{bmatrix}, K_2 = \begin{bmatrix} -0.0062 & -0.0090 \\ 0.1514 & 0.5342 \end{bmatrix}.$$

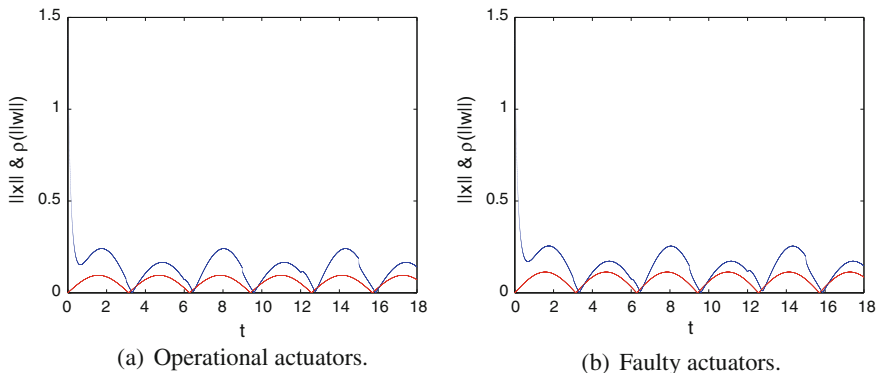


Fig. 1 Input-to-state stabilization

Thus, the matrices $A_1 + B_1K_1$ and $A_2 + B_2K_2$ are Hurwitz. The average dwell time is $\tau_a = \frac{\ln \mu}{\alpha^* - \nu} = 2.7898$, with $\nu = 0.05$.

Figure 1a shows the simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom), where $\rho(s) = \max\{\rho_1(s), \rho_2(s)\}$ and $\rho_i(s) = s/\sqrt{c_2\theta_i\xi_{2i}}$, and $\tau_a = 3$.

Case 2 (*Failure in the second actuator in the first mode and first actuator in the second mode*) When there is a failure in the second actuator, i.e., $B_{1\Sigma} = \{2\}$ and $B_{1\bar{\Sigma}} = \begin{bmatrix} -7 & 0 \\ 0.1 & 0 \end{bmatrix}$, and $B_{2\Sigma} = \{1\}$ and $B_{2\bar{\Sigma}} = \begin{bmatrix} 0 & 0.5 \\ 0 & -8 \end{bmatrix}$, we have from Riccati-like equation,

$$P_1 = \begin{bmatrix} 1.1265 & -0.1913 \\ -0.1913 & 0.3110 \end{bmatrix}, P_2 = \begin{bmatrix} 0.1676 & 0.0980 \\ 0.0980 & 0.2436 \end{bmatrix},$$

with $c_{11} = \lambda_{\min}(P_1) = 0.2683$, $c_{12} = \lambda_{\max}(P_1) = 1.1691$, $c_{21} = 0.1005$, $c_{22} = 0.3107$, so $c_1 = 0.1005$, $c_2 = 1.1691$, and the control gain matrices

$$K_1 = \begin{bmatrix} 7.9046 & -1.3703 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ 0.1751 & 0.4750 \end{bmatrix}.$$

Thus, the matrices $A_1 + B_1K_1$ and $A_2 + B_2K_2$ are Hurwitz, and $\tau_a = 2.5834$. Figure 1b shows the simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom), where $\rho(s) = \max\{\rho_1(s), \rho_2(s)\}$ and $\rho_i(s) = s/\sqrt{c_2\theta_i\xi_{2i}}$, $\tau_a = 3$.

5 Conclusion

We have considered a time-varying parameter uncertainty in the system state, an L_2 norm-bounded input disturbance, and a linearly bounded nonlinear term. The output of the faulty actuators has been treated as a disturbing signal that has been augmented

with the system disturbance. We have shown that, using the average dwell time with multiple Lyapunov functions, the switched system is exponentially input-to-state stabilizable, when every individual mode is exponentially input-to-state stabilized by a reliable feedback controller.

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