From Exponential Analysis to Padé Approximation and Tensor Decomposition, in One and More Dimensions

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Abstract. Exponential analysis in signal processing is essentially what is known as sparse interpolation in computer algebra. We show how exponential analysis from regularly spaced samples is reformulated as Padé approximation from approximation theory and tensor decomposition from multilinear algebra.

The univariate situation is briefly recalled and discussed in Sect. 1. The new connections from approximation theory and tensor decomposition to the multivariate generalization are the subject of Sect. 2. These connections immediately allow for some generalization of the sampling scheme, not covered by the current multivariate theory.

An interesting computational illustration of the above in blind source separation is presented in Sect. 3.

Keywords: Exponential analysis · Parametric method · Multivariate Padé approximation · Tensor decomposition

1 The Univariate Connections

Let us first introduce the problem statement of exponential analysis, which is known in the computer algebra community as sparse interpolation [4,10]. Afterward we rewrite it as a rational approximation problem and as a tensor decomposition problem. In this section, we restrict ourselves to the univariate case.

Let the signal $f(t)$ be given by

$$f(t) = \sum_{j=1}^{n} \alpha_j \exp(\phi_j t), \qquad \alpha_j, \phi_j \in \mathbb{C}, \tag{1}$$

where the objective is to recover the values of the coefficients $\alpha_j, j = 1, \ldots, n$ and the mutually distinct exponents $\phi_j, j = 1, \ldots, n$. Already in 1795, de Prony

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[14] proved that the problem can be solved from $2n$ equidistant samples if the sparsity n is known, as we assume in the sequel.

In the following, we choose a real $\Delta \neq 0$ such that $|\Im(\phi_j)| < \pi/|\Delta|$, in order to comply with [12,17], where $\Im(\cdot)$ denotes the imaginary part of a complex number. The value Δ denotes the sampling step in the equidistant sampling scheme

$$f_k := f(k\Delta) = \sum_{j=1}^{n} \alpha_j \exp(\phi_j k\Delta) = \sum_{j=1}^{n} \alpha_j \Phi_j^k, \qquad \Phi_j = \exp(\phi_j \Delta). \tag{2}$$

With the samples $f_k, k = 0, \ldots, 2n-1$, we fill the Hankel matrices

$$H_n^{(m)} := (f_{m+i+j-2})_{i,j=1}^{n} = \begin{bmatrix} f_m & f_{m+1} & \cdots & f_{m+n-1} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n-1} & f_{m+n} & \cdots & f_{m+2n-2} \end{bmatrix}, \qquad m \geq 0.$$

From the expression (2) for the samples f_k, we immediately find that $H_n^{(m)}$ can be factored as

$$H_n^{(m)} = V_n D_\alpha D_\Phi^m V_n^T,$$

where V_n is the Vandermonde matrix

$$V_n = \left(\Phi_j^{i-1}\right)_{i,j=1}^{n}$$

and D_α and D_Φ are diagonal matrices respectively filled with the vectors $(\alpha_1, \ldots, \alpha_n)$ and $(\Phi_1, \ldots, \Phi_n)$ on the diagonal. So the $\Phi_j, j = 1, \ldots, n$ can be found as the generalized eigenvalues $\lambda_j, j = 1, \ldots, n$ of the problem [11]

$$H_n^{(1)} v_j = \lambda_j H_n^{(0)} v_j, \tag{3}$$

where the $v_j, j = 1, \ldots, n$ are the right generalized eigenvectors. From the generalized eigenvalues $\Phi_j = \exp(\phi_j \Delta)$, the complex values ϕ_j can be extracted uniquely because $|\Im(\phi_j)\Delta| < \pi$. After recovering the Φ_j, the α_j can be computed from the Vandermonde structured linear system

$$\sum_{j=1}^{n} \alpha_j \Phi_j^k = f_k, \qquad k = 0, \ldots, 2n-1. \tag{4}$$

In a noisefree mathematical context, n equations of (4) are linearly dependent because of the relationship (3) between the Φ_j. How to proceed in a noisy context is analyzed in great detail and including several variations in a forthcoming paper and is outside the scope of the current presentation, where we focus on the mathematical interrelationship between seemingly disconnected problem statements.

1.1 From Exponential Analysis to Padé Approximation in 1-D

Instead of filling Hankel matrices with the samples f_k, we construct a formal power series expansion

$$F(z) = \sum_k f_k z^k.$$

The Padé approximant $[m/n]_F$ for $F(z)$ of degree m in the numerator and n in the denominator is defined as the irreducible form of the rational function $p_{m,n}(z)/q_{m,n}(z)$, with

$$p_{m,n}(z) := \sum_{j=0}^{m} a_j z^j,$$

$$q_{m,n}(z) := \sum_{j=0}^{n} b_j z^j,$$

that satisfies

$$F(z) q_{m,n}(z) - p_{m,n}(z) = \sum_{k \geq m+n+1} c_k z^k.$$

The computation of Padé approximants is closely connected to the solution of Toeplitz structured linear systems. The $[m/n]_F$ is computed from putting to zero the terms of degree 0 to $m+n$ in $(F q_{m,n} - p_{m,n})(z)$:

$$\sum_{j=0}^{n} f_{k-j} b_j = a_k, \qquad k = 0, \dots, m,$$

where $f_k = 0$ if $k < 0$, and

$$\sum_{j=0}^{n} f_{m+k-j} b_j = 0, \qquad k = 1, \dots, n.$$

Again using expression (2) for the f_k and under the assumption that the Φ_j are mutually distinct, it is not difficult to see that [2]

$$\begin{aligned} F(z) &= \sum_k f_k z^k \\ &= \sum_k \left(\sum_{j=1}^{n} \alpha_j \Phi_j^k \right) z^k \\ &= \sum_{j=1}^{n} \alpha_j \left(\sum_k \Phi_j^k z^k \right) \\ &= \sum_{j=1}^{n} \frac{\alpha_j}{1 - \Phi_j z}. \end{aligned}$$

So the function $F(z)$ is itself a rational function of degree $n-1$ in the numerator and n in the denominator. The consistency property of Padé approximants guarantees that a rational function like $F(z)$ is reconstructed from its formal series expansion by its $[n-1/n]_F$ Padé approximant, thereby needing only the series coefficients $f_0, \ldots, f_{2n-1}$. So we can also obtain the Φ_j from the Padé denominator

$$\prod_{j=1}^{n}(1-\Phi_j z) \tag{5}$$

and the α_j from the partial fraction decomposition of the approximant $[n-1/n]_F$, through

$$P_{n-1,n}(z) = \sum_{j=1}^{n} \alpha_j L_j(z), \qquad L_j(z) = \sum_{\substack{i=1\\ i\neq j}}^{n}(1-\Phi_i z).$$

The poles $1/\Phi_j$ of $F(z)$ can even directly be computed from the f_k, in the order of increasing magnitude, using the qd-algorithm [1].

1.2 From Exponential Analysis to Tensor Decomposition in 1-D

With the samples f_k we can also fill an order N tensor $T \in \mathbb{C}^{n_1\times\cdots\times n_N}$ where

$$2 \leq n_j \leq n, \qquad 1 \leq j \leq N, \qquad 3 \leq N \leq 2n-1,$$
$$\sum_{j=1}^{N} n_j = 2n + N - 1,$$

and

$$T_{k_1,\ldots,k_N} := f_{k_1+\cdots+k_N-N}, \qquad 1 \leq k_j \leq n_j. \tag{6}$$

The tensor of smallest order $N=3$ is, for instance, of size $n\times n\times 2$ [13] and the one of largest order $N=2n-1$ is symmetric and of size $2\times\cdots\times 2$ [6]. For the sequel, we generalize the definition of the square Hankel matrix above to cover rectangular Hankel structured matrices

$$H_{n_1,n_2}^{(m)} = \begin{bmatrix} f_m & f_{m+1} & \cdots & f_{m+n_2-1} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n_2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n_1-1} & f_{m+n_1} & \cdots & f_{m+n_1+n_2-2} \end{bmatrix}.$$

The tensor slices $T_{\cdot,\cdot,k_3,\ldots,k_N}$ equal

$$T_{\cdot,\cdot,k_3,\ldots,k_N} = H_{n_1,n_2}^{(k_3+\cdots+k_N-N+2)}$$

and so are Hankel structured. The tensor T decomposes as

$$T = \sum_{j=1}^{n} \alpha_j \begin{bmatrix} 1 \\ \Phi_j \\ \vdots \\ \Phi_j^{n_1-1} \end{bmatrix} \circ \cdots \circ \begin{bmatrix} 1 \\ \Phi_j \\ \vdots \\ \Phi_j^{n_N-1} \end{bmatrix}, \tag{7}$$

where still the $\Phi_j = \exp(\phi_j \Delta)$ are mutually distinct and $\circ$ denotes the outer product. Decomposition (7) is easily verified by checking the element at position $(k_1, \ldots, k_N)$ in (7):

$$
\begin{aligned}
T_{k_1,\ldots,k_N} &= \sum_{j=1}^{n} \alpha_j \Phi_j^{k_1-1} \cdots \Phi_j^{k_N-1} \\
&= \sum_{j=1}^{n} \alpha_j \Phi_j^{k_1+\cdots+k_N-N} \\
&= f_{k_1+\cdots+k_N-N}.
\end{aligned}
$$

The factor matrices are the rectangular Vandermonde structured matrices

$$V_{n_k,n} = \left(\Phi_j^{i-1}\right)_{i=1,j=1}^{n_k,n}, \qquad 1 \le k \le N.$$

Because of the Vandermonde structure of the factor matrices with $n_k \le n, k = 1, \ldots, N$, their Kruskal rank equals n_k for all k. Since $n_1 + \cdots + n_N = 2n + N - 1$ we find that the sum of the Kruskal ranks of the N factor matrices of the rank n tensor T is bounded below by $2n + N - 1$. Hence the Kruskal condition is satisfied and the unicity of the decomposition is guaranteed.

2 The Multivariate Connections

The result from de Prony that (1) can be solved from only $2n$ samples if the sparsity n is known and that the recovery of the linear coefficients α_j and the nonlinear parameters ϕ_j can be separated, is only recently truly generalized [5] to d-variate functions of the form

$$f(x_1, \ldots, x_d) = \sum_{j=1}^{n} \alpha_j \exp\left(\phi_{j1} x_1 + \cdots + \phi_{jd} x_d\right), \qquad \alpha_j, \phi_{j\ell} \in \mathbb{C}. \tag{8}$$

In [5], is proved that the $\alpha_j, j = 1, \ldots, n$ and $\phi_{j\ell}, j = 1, \ldots, n, \ell = 1, \ldots, d$ can be recovered from $(d+1)n$ samples in the absence of collisions or cancellations of terms when sampling. In the latter case, the problem is still solvable but requires some additional samples to untangle the collisions or cancellations [5]. For the sequel, we also introduce the vectors $x = (x_1, \ldots, x_d)$ and $\phi_j = (\phi_{j1}, \ldots, \phi_{jd})$ where it is clear from the context whether ϕ_j refers to a complex value as in the previous section or a vector of complex values. Using the vector notation, (8) becomes

$$f(x) = \sum_{j=1}^{n} \alpha_j \exp\left(\langle \phi_j, x \rangle\right).$$

The way to achieve the generalization (8) is by falling back on a one-dimensional projected generalized eigenvalue problem requiring $2n$ samples, complemented with $d-1$ structured linear systems each requiring n samples along linearly independent directions to cover the additional dimensions, and one more structured linear system set up from the first $2n$ samples to recover the linear coefficients α_j.

We introduce the real linearly independent d-dimensional vectors $\Delta_\ell, \ell = 1, \ldots, d$ satisfying $|\Im(\langle \phi_j, \Delta_\ell \rangle)| < \pi, j = 1, \ldots, n, \ell = 1, \ldots, d$. We further collect the samples

$$f_k := f(k\Delta_1), \qquad 0 \le k \le 2n-1,$$
$$f_{k\ell} := f(k\Delta_1 + \Delta_\ell), \qquad 0 \le k \le n-1, \qquad 2 \le \ell \le d$$

and denote $\Phi_{j\ell} := \exp(\langle \phi_j, \Delta_\ell \rangle)$.

We assume now that all the values Φ_{j1} are mutually distinct so that the $\Phi_{j1}, j = 1, \ldots, n$ can be obtained as the generalized eigenvalues of a generalized eigenvalue problem of the form (3) where the Hankel matrices are filled with the samples f_k. Subsequently the α_j are the solution of the Vandermonde linear system

$$\sum_{j=1}^{n} \alpha_j \Phi_{j1}^k = f_k, \qquad k = 0, \ldots, 2n-1. \tag{9}$$

Of course, from $\langle \phi_j, \Delta_1 \rangle$ extracted from Φ_{j1}, the individual $\phi_{j\ell}$ cannot yet be identified. For that purpose, we need the additional $(d-1)n$ samples $f_{k\ell}$ which we reinterpret for each $2 \le \ell \le d$ as

$$\sum_{j=1}^{n} (\alpha_j \Phi_{j\ell})\, \Phi_{j1}^k = f_{k\ell}, \qquad k = 0, \ldots, n-1. \tag{10}$$

In other words, with the samples $f_{k\ell}$ as right hand side for ℓ fixed and with the first n rows of the same Vandermonde coefficient matrix as in (9), we obtain the unknown coefficients $\alpha_j \Phi_{j\ell}$ and subsequently the values $\Phi_{j\ell}$ from

$$\frac{\alpha_j \Phi_{j\ell}}{\alpha_j}, \qquad j = 1, \ldots, n, \qquad 2 \le \ell \le d$$

and $\langle \phi_j, \Delta_\ell \rangle$ from $\Phi_{j\ell}$. We remark that $\Phi_{j\ell}$ is easily paired to its associated generalized eigenvalue Φ_{j1} through the structured linear systems (9) and (10), a problem that remained unsolved in various other approaches [9,15].

We now have extracted all the inner products $\langle \phi_j, \Delta_\ell \rangle, j = 1, \ldots, n, \ell = 1, \ldots, d$ for linearly independent vectors Δ_ℓ and so for each $1 \le j \le n$ the individual $\phi_{j\ell}$ can be retrieved as the solution of the following regular linear system:

$$\begin{bmatrix} \Delta_{11} & \ldots & \Delta_{1d} \\ \vdots & & \vdots \\ \Delta_{d1} & \ldots & \Delta_{dd} \end{bmatrix} \begin{bmatrix} \phi_{j1} \\ \vdots \\ \phi_{jd} \end{bmatrix} = \begin{bmatrix} \langle \phi_j, \Delta_1 \rangle \\ \vdots \\ \langle \phi_j, \Delta_d \rangle \end{bmatrix}.$$

In [6], some preliminary work was done leading to a novel technique based on the use of multivariate Padé approximation, but a proper rewrite of the problem statement (8) in terms of Padé approximants was still lacking. We fill this gap here by turning our attention to the concept of simultaneous Padé approximant. We continue along the same lines with a reformulation into a tensor decomposition problem of smaller order than in [6].

2.1 From Exponential Analysis to Padé Approximation in d-D

Instead of one formal power series, we now set up d formal power series, namely

$$F_1(z) = \sum_k f_k z^k,$$
$$F_\ell(z) = \sum_k f_{k\ell} z^k, \qquad 2 \le \ell \le d.$$

Making use of the expressions (9) and (10) for f_k and $f_{k\ell}$, respectively, we again find that the functions

$$F_1(z) = \sum_{j=1}^{n} \frac{\alpha_j}{1-\Phi_{j1}z},$$
$$F_\ell(z) = \sum_{j=1}^{n} \frac{\alpha_j \Phi_{j\ell}}{1-\Phi_{j1}z}, \qquad 2 \le \ell \le d$$

are rational functions, each of degree $n-1$ in the numerator and degree n in the denominator. Note that for all $\ell = 1, \ldots, d$, the denominator of $F_\ell(z)$ is the same and reveals the generalized eigenvalues Φ_{j1} which are assumed to be mutually distinct.

The rational functions $F_\ell(z), 1 \le \ell \le d$ can be recovered from the multivariate samples $f_k, 0 \le k \le 2n-1$ and $f_{k\ell}, 0 \le k \le n-1, 2 \le \ell \le d$ by computing the simultaneous Padé approximant $[(n-1,\ldots,n-1)/n]_{(F_1,\ldots,F_d)}$ for the vector of functions $(F_1(z),\ldots,F_d(z))$ [3, pp. 415–417], defined more precisely as the vector of irreducible forms of the rational functions $p_{n-1,n,\ell}(z)/q_{n-1,n}(z), 1 \le \ell \le d$ satisfying

$$\left(F_\ell q_{n-1,n} - p_{n-1,n,\ell}\right)(z) = \begin{cases} \displaystyle\sum_{k\ge 2n} c_k z^k, & \ell = 1, \\ \displaystyle\sum_{k\ge n} c_{k\ell} z^k, & 2 \le \ell \le d. \end{cases} \tag{11}$$

So the denominator polynomial $q_{n-1,n}(z) = b_0 + \cdots + b_n z^n$ is computed from the Toeplitz structured linear system

$$\sum_{j=0}^{n} f_{n+k-j} b_j = 0, \qquad k = 0, \ldots, n-1,$$

arising from the accuracy-through-order conditions (11) for $F_1(z)$. We remark that again the α_j and $\Phi_{j\ell}, 2 \le \ell \le d$ are naturally paired to the poles $1/\Phi_{j1}$ of each rational function $p_{n-1,n,\ell}(z)/q_{n-1,n}(z)$, which can be computed directly from the samples using the qd-algorithm [1] applied to the formal series $F_1(z)$. This pairing is essential to recover the individual $\phi_{j\ell}$.

It is worthwhile to note that the Padé formulation of (8) allows a slight generalization compared to the generalized eigenvalue formulation of the multivariate problem. The simultaneous Padé approximant $[(n-1,\ldots,n-1)/n]_{(F_1,\ldots,F_d)}$ can also be computed from ν_1 samples f_k and ν_ℓ samples $f_{k\ell}$ for $2 \le \ell \le d$, where the total number of samples equals

$$\sum_{\ell=1}^{d} \nu_\ell = (d+1)n, \qquad \nu_\ell \ge n,$$

instead of $2n$ samples f_k and n samples $f_{k\ell}$ for $2 \le \ell \le d$. In that setting (11) becomes

$$(F_\ell q_{n-1,n} - p_{n-1,n,\ell})(z) = \sum_{k \ge \nu_\ell} c_{k\ell} z^k, \qquad 1 \le \ell \le d,$$

and the common denominator $q_{n-1,n}(z)$ is computed from the linear system

$$\sum_{j=0}^{n} f_{n+k-j} b_j = 0, \qquad k = 0.\ldots,\nu_1 - n - 1,$$
$$\sum_{j=0}^{n} f_{n+k-j,\ell} b_j = 0, \qquad k = 0,\ldots,\nu_\ell - n - 1, \qquad 2 \le \ell \le d.$$

2.2 From Exponential Analysis to Tensor Decomposition in *d*-D

Along the same lines as above, a connection to a so-called coupled tensor decomposition problem can be made. With the samples $f_k, k = 0,\ldots,2n-1$, a first order N tensor $T^{(1)}$ of dimension $n_1 \times \cdots \times n_N$ is defined as in (6), which decomposes as in (7), but with Φ_j replaced by Φ_{j1}:

$$T^{(1)} = \sum_{j=1}^{n} \alpha_j \begin{bmatrix} 1 \\ \Phi_{j1} \\ \vdots \\ \Phi_{j1}^{n_1-1} \end{bmatrix} \circ \cdots \circ \begin{bmatrix} 1 \\ \Phi_{j1} \\ \vdots \\ \Phi_{j1}^{n_N-1} \end{bmatrix}.$$

As explained in Sect. 1.2, this decomposition is unique as long as the Φ_{j1} are mutually distinct. Remains to recover the $\Phi_{j\ell}, 2 \le \ell \le d$.

To this end, we construct another $d-1$ order N tensors $T^{(\ell)}, 2 \le \ell \le d$ of dimension $n_{1\ell} \times \cdots \times n_{N\ell}$, where

$$2 \le n_{j\ell} \le n, \qquad \sum_{j=1}^{N} n_{j\ell} = n + N - 1,$$

with tensor elements

$$T^{(\ell)}_{k_1,\ldots,k_N} := f_{k_1+\cdots+k_N-N,\ell}, \qquad 2 \le \ell \le d,$$

of which the slices $T^{(\ell)}_{\cdot,\cdot,k_3,\ldots,k_N}$ are still Hankel structured. With

$$H^{(m,\ell)}_{n_1,n_2} = \begin{bmatrix} f_{m,\ell} & f_{m+1,\ell} & \cdots & f_{m+n_2-1,\ell} \\ f_{m+1,\ell} & f_{m+2,\ell} & \cdots & f_{m+n_2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n_1-1,\ell} & f_{m+n_1,\ell} & \cdots & f_{m+n_1+n_2-2,\ell} \end{bmatrix},$$

the tensor slices $T^{(\ell)}_{\cdot,\cdot,k_3,\ldots,k_N}$ equal

$$T^{(\ell)}_{\cdot,\cdot,k_3,\ldots,k_N} = H^{(k_3+\cdots+k_N-N+2,\ell)}_{n_1,n_2}.$$

The tensors $T^{(\ell)}$ decompose as

$$T^{(\ell)} = \sum_{j=1}^{n} \alpha_j \Phi_{j\ell} \begin{bmatrix} 1 \\ \Phi_{j1} \\ \vdots \\ \Phi_{j1}^{n_{1\ell}-1} \end{bmatrix} \circ \cdots \circ \begin{bmatrix} 1 \\ \Phi_{j1} \\ \vdots \\ \Phi_{j1}^{n_{N\ell}-1} \end{bmatrix},$$

where the entries in the factor matrices from $T^{(\ell)}$ can all be obtained from the decomposition of $T^{(1)}$, hence the term coupled tensor decomposition. Only the sizes $n_{j\ell} \times n$ of the factor matrices may be smaller as the sum of the $n_{j\ell}$ is bounded by $n+N-1$ instead of $2n+N-1$. The decomposition of the $T^{(\ell)}$ only serves the purpose of recovering the $\alpha_j \Phi_{j\ell}, j = 1, \ldots, n, 2 \le \ell \le d$. Note again the natural pairing of the α_j and $\alpha_j \Phi_{j\ell}, 2 \le \ell \le d$ to the Φ_{j1}, which is required to recover the individual $\phi_{j\ell}$ in (8).

A similar generalization as in Sect. 2.1 where now

$$\sum_{j=1}^{N} n_j + \sum_{\ell=2}^{d} \sum_{j=1}^{N} n_{j\ell} = (d+1)n + d(N-1)$$

is obviously also possible. Then the order N tensor $T^{(1)}$ of dimension $n_1 \times \cdots \times n_N$ is such that

$$2 \le n_j \le n, \qquad \sum_{j=1}^{N} n_j = \nu_1 + N - 1$$

and decomposes in the same way as $T^{(1)}$ above (only the sum of the dimensions is bounded differently). Similarly $T^{(\ell)}, 2 \le \ell \le d$ of dimension $n_{1\ell} \times \cdots \times n_{N\ell}$ obeys

$$2 \le n_{j\ell} \le n, \qquad \sum_{j=1}^{N} n_{j\ell} = \nu_\ell + N - 1$$

and decomposes as $T^{(\ell)}$ above. Note that Kruskal's condition only guarantees a unique decomposition if $\nu_1 \ge 2n$. However, the unicity is guaranteed through the other formulations of the problem statement, be it as a simultaneous Padé approximation problem or a multivariate exponential analysis.

3 Illustration: Blind Source Separation

We now illustrate the connections between exponential analysis or sparse interpolation with on the one hand Padé approximation and on the other hand tensor decomposition. The emphasis is on the mathematical reformulations of the problem statement and not on the numerical aspects of the various algorithms that can be used in either of the three settings.

We analyze a demo signal consisting of some wild bird chirps mixed with the whistle of a passing train (the original signal is available at our website[1]). The signal is graphed in Fig. 1: it lasts somewhat longer than 1.5 seconds and consists of 12850 samples collected at a rate of 8192 samples per second with a high signal-to-noise ratio. In Fig. 2, the signal's spectrogram is given, put together by applying the short-time Fourier transform to 257 non-overlapping frames of each 50 consecutive samples multiplied by a rectangular window function. It exhibits clearly the Fourier transform's typical leakage. Also the resolution is poor as we consider windows of only 50 samples. The horizontal stripes in the spectrogram represent the train whistle while the bird chirps are found in the higher frequency flame-like elements.

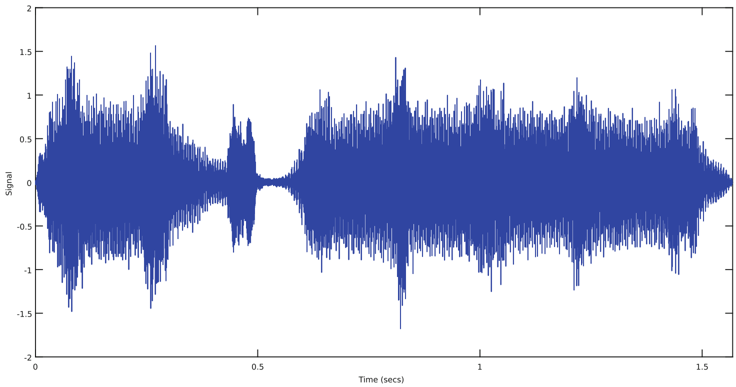

Fig. 1. Real-valued demo signal

The objective now is to identify the bird chirps and the train whistle using a sparse technique instead of the fast Fourier transform, thereby avoiding the leakage and limited resolution. So we recover each contributing α_j and ϕ_j in (1) from the signal samples following the outline of Sect. 1. To this end, we again divide the full data set into 257 non-overlapping windows of 50 samples. In each window, we take the sparsity $n = 20$, meaning that we choose a model consisting of 20 exponential terms, that we fit to 50 samples, in the least squares sense since $50 > 2n$. For the practical computation, use was made of:

[1] http://cma.uantwerpen.be/publications.

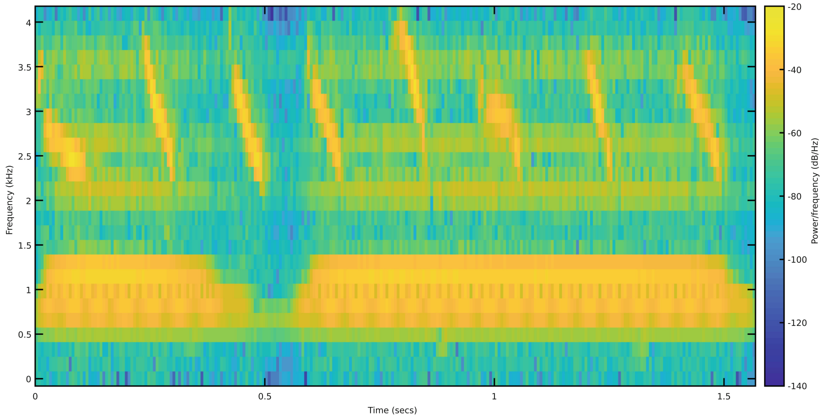

Fig. 2. Spectrogram of the demo signal

– the `ESPRIT` algorithm from [16] for the exponential analysis,
– the qd-algorithm as in [1] for the rational function reformulation,
– `Tensorlab` from [18] for the tensor decomposition.

Complexitywise the Fourier analysis and exponential analysis of each window compare as follows. A Fourier analysis of M samples is $O(M \log M)$ while an exponential analysis using the Hankel structured generalized eigenvalue problem (3) and the Vandermonde structured linear system (4) is $O(n^2 \log n)$. When solving (3)–(4) in a least squares sense from $m > 2n$ samples then the complexity increases to $O((m-n)n^2)$ [7,8]. Note that in practical applications usually $M \gg m$ and hence also $M \gg n$.

In Figs. 3, 4, and 5 at the top, we show the computed $\phi_j, j = 1, \ldots, 20$ from window number 88 (samples number 4351 till 4400), where only the blue coloured ϕ_j are retained, for the exponential analysis, Padé approximation, and tensor decomposition, respectively. The ϕ_j indicated in red are discarded because either their imaginary part was (numerically) zero or their modulus was too large ($|\cdot| > 1.05$). The former does not contribute to a sound signal, while the latter may cause ill-conditioning when setting up the Vandermonde matrices involved.

In the same figures at the bottom, the spectrogram results for each of exponential analysis, Padé approximation, and tensor decomposition is shown. It is clear that the sparse technique of exponential analysis and its reformulations do not suffer from the undesirable leakage and limited resolution, as they identify the frequency content in the signal $f(t)$.

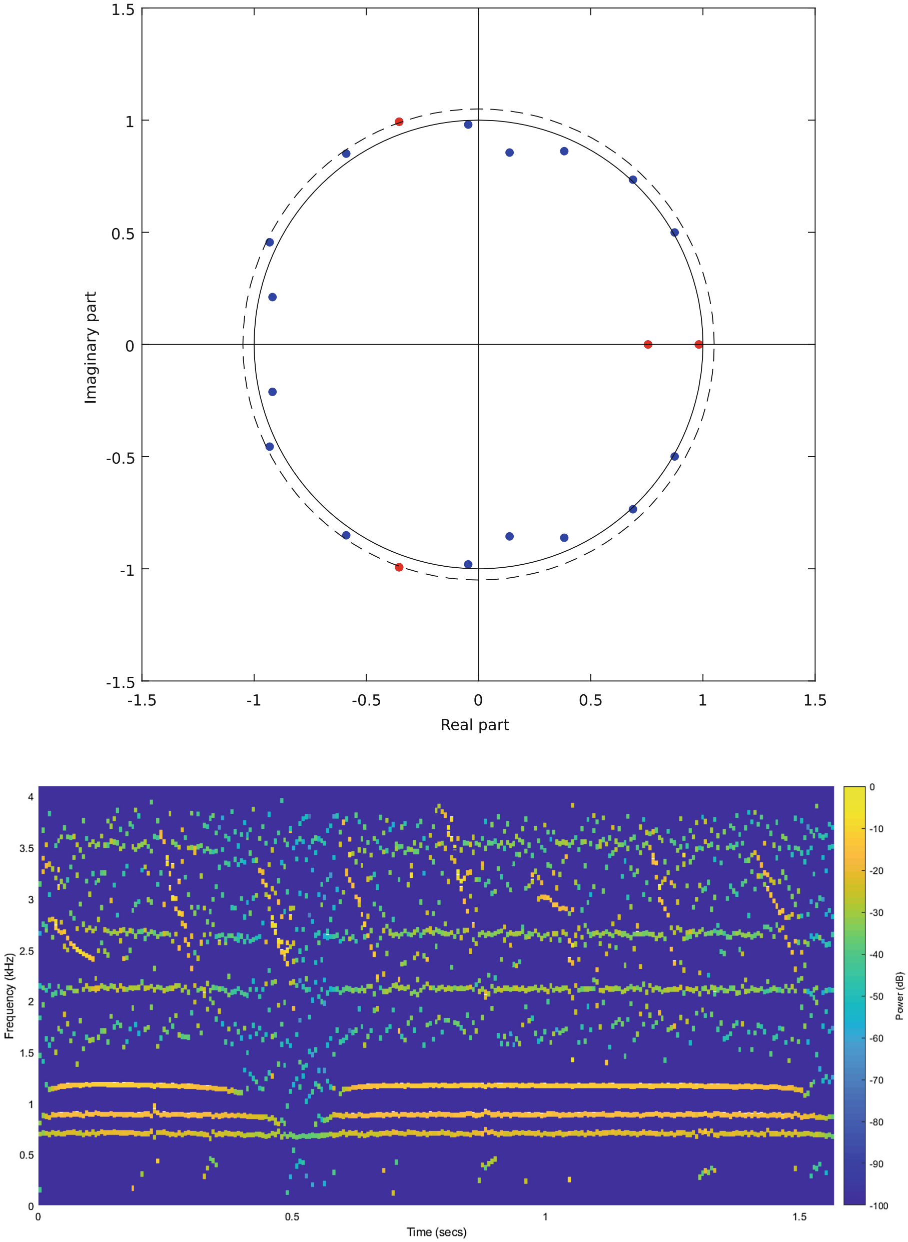

Fig. 3. Extracted $\phi_j, j = 1, \ldots, 20$ using (3) (top) and spectrogram based on retained ϕ_j (bottom) (Color figure online)

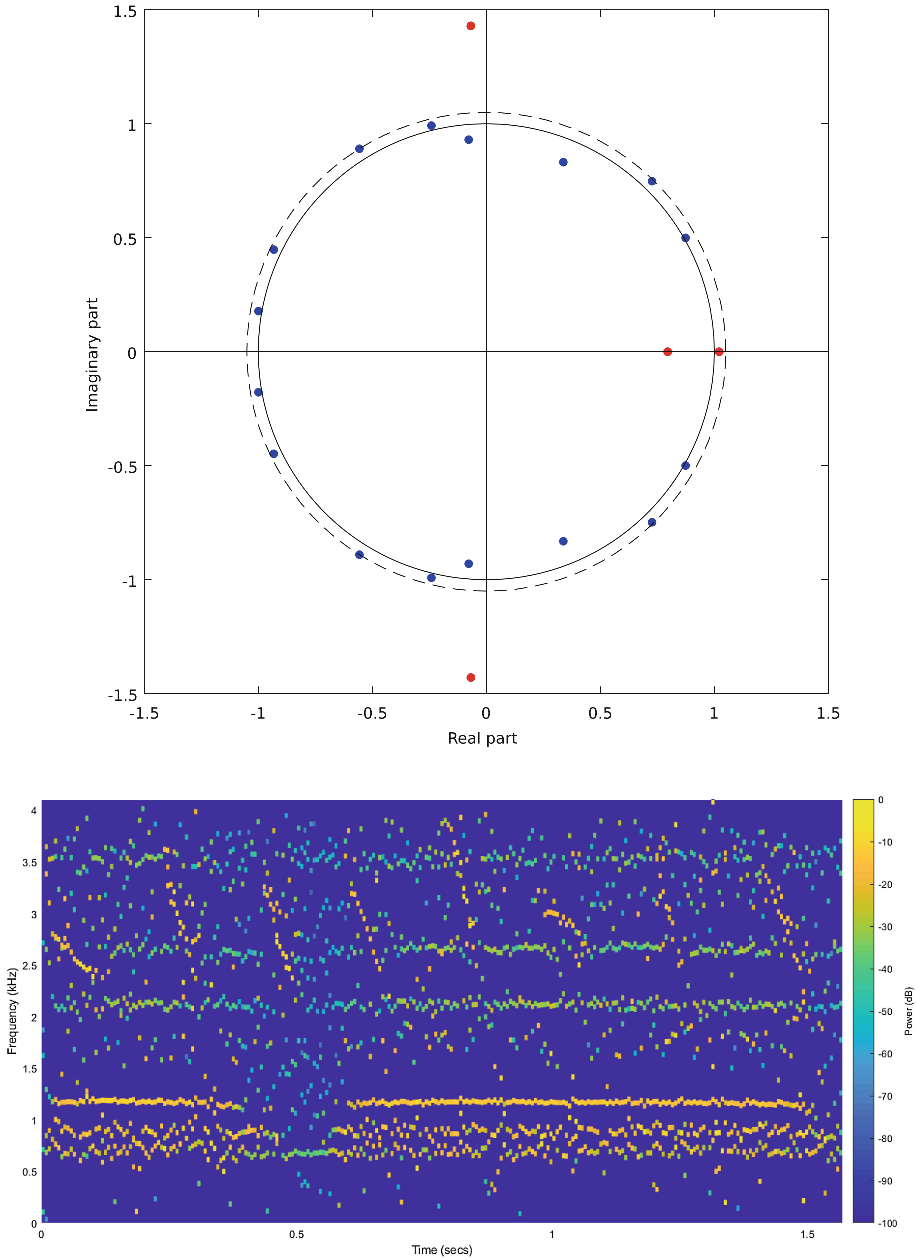

Fig. 4. Extracted $\phi_j, j = 1, \ldots, 20$ using (5) (top) and spectrogram based on retained ϕ_j (bottom) (Color figure online)

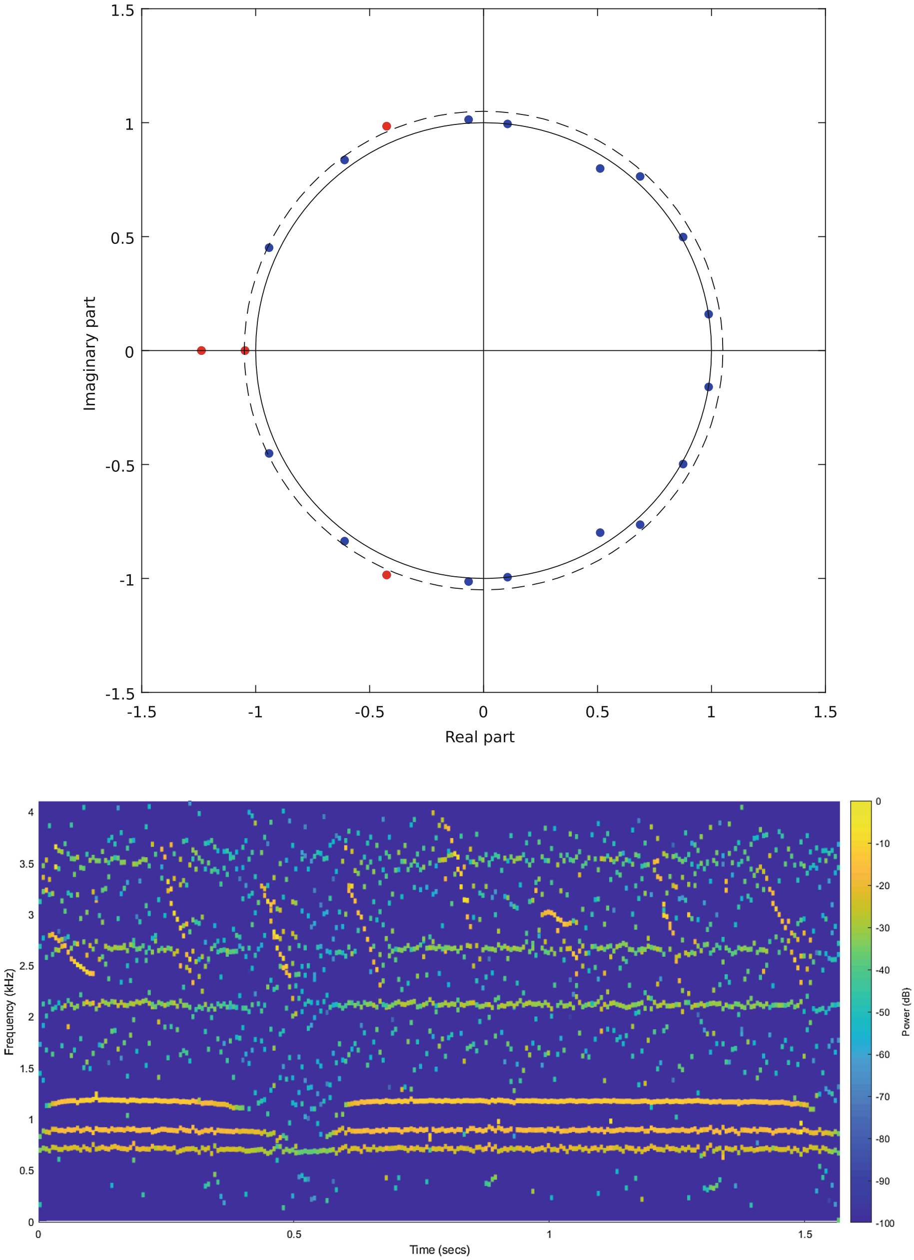

Fig. 5. Extracted $\phi_j, j = 1, \ldots, 20$ using (7) (top) and spectrogram based on retained ϕ_j (bottom) (Color figure online)

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