Qualitative Analysis of a Dynamical System with Irrational First Integrals

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Abstract. We conduct qualitative analysis for a completely integrable system of differential equations with irrational first integrals. These equations originate from gas dynamics and describe adiabatical motions of a compressible gas cloud with homogeneous deformation. We study the mechanical analog of this gas dynamical system – the rotational motion of a spheroidal rigid body around a fixed point in a potential force field described by an irrational function. Within our study, equilibria, pendulum oscillations and invariant manifolds, which these solutions belong to, have been found. The sufficient conditions of their stability in Lyapunov's sense have been derived and compared with the necessary ones. The analysis has been performed with the aid of computer algebra tools which proved to be essential. The computer algebra system "Mathematica" was employed.

1 Introduction

Many different natural phenomena and processes can be described mathematically by the same equations. Such a mathematical analogy allows one to apply the methods developed for studying and an interpretation of phenomena and processes of one type to phenomena and processes of other type. Let us consider, e.g., the equations of adiabatical motions of an ideal gas in the form [9]:

$$\begin{aligned} &\operatorname{div} \boldsymbol{v} = -\frac{1}{(\gamma-1)}\frac{d}{dt}\ln T \\ &\partial_t \boldsymbol{v} = \boldsymbol{v} \wedge \operatorname{rot} \boldsymbol{v} + T\boldsymbol{\nabla} S - \boldsymbol{\nabla}\Big(\frac{\boldsymbol{v}^2}{2} + \frac{\gamma T}{\gamma-1}\Big) \\ &\partial_t S + \boldsymbol{v}\cdot\boldsymbol{\nabla} S = 0, \end{aligned} \tag{1}$$

where $\boldsymbol{v}$ is the vector of velocity of the gas, T is the gas temperature, γ is the adiabatical index, and S is the entropy.

As was shown [6,16], in a Lagrangian formalism, when S is a quadratic function of Lagragian coordinates, and $\boldsymbol{v}$ depends on these linearly, partial differential equations (1) are reduced to ordinary ones and describe the motions of an ellipsoidal cloud of a compressible gas expanding freely in vacuum. The mechanical

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interpretation of these equations was given [6]. They are identical with the equations of motion of a point mass in nine-dimensional Euclidean space. This gas dynamical model was studied in a series of works, e.g., [1,15]. The present paper is based on the results [7,9].

In [7], under some assumptions, such as the gas is monatomic with the adiabatic index $\gamma = 5/3$, and there is neither rotation nor vorticity of the gas cloud, the above gas dynamical model was reduced to three second order ordinary differential equations. It was shown that they are equivalent to the equations of motion of a point mass on the unit 2-sphere, and an additional integral of 3rd degree in momenta has been derived for them. More general case was considered in [9] when the gas ellipsoid rotates around one of its principal axes. Then the equations of motion possess an additional first integral of 6th degree in momenta.

The study of the gas dynamical model proposed [16], [6] is ongoing to the present time towards generalizations of the found integrable cases [8]. A topological analysis of the integrable cases with the additional first integrals of 3rd and 6th degree has been done in [4]. In this work, the mechanical analog for the gas cloud – the motion of a point mass on the 2-sphere – was investigated. According to [3], the dynamics of a point mass on the 2-sphere is equivalent to the motion of a spheroidal rigid body in a potential force field at zeroth level of area integral. Thus, one can use this mechanical analog to study the gas dynamical model and to apply the methods developed for the analysis of dynamical systems of such type.

In the present work, the latter mechanical model is used for the qualitative analysis of the gas dynamical system. We analyze the differential equations of the spheroidal body in the above-mentioned integrable cases and obtain new results for both the gas system and the mechanical one. As is well-known, the problem of the qualitative analysis of differential equations is to find special solutions (equilibria, periodic motions, etc.) of these equations and to study their stability and bifurcations. Based on computer algebra methods, the computer analysis of the above problems can be performed in analytical form. The latter enables us to investigate the properties of the solutions under continuous (smooth) variation of their parameters. The research technique based on computer algebra methods as applied to the qualitative analysis of differential equations with first integrals is presented in the paper. The symbolic analysis is performed using built-in procedures of the computer algebra system "Mathematica" (CAS) and the "Mathematica" software package [2]. The procedures are used to solve computational problems arising in the study and to manipulate mathematical expressions. The package is employed to investigate the stability of the special solutions.

For finding the special solutions, the Routh–Lyapunov method [13] and its generalizations [12] are applied. By these methods, the qualitative analysis of differential equations with polynomial first integrals is reduced to algebraic problems solved efficiently by CAS. The first integrals in the problem under consideration are irrational; that is a special feature of the given problem. To avoid the use of fractional exponents and fractions (that is usually difficult in CAS), we transform irrational expressions to polynomial ones by introducing new variables.

In addition to the above methods for finding the special solutions, the chains of differential consequences [10] are applied. This technique mainly uses symbolic differentiation of expressions and is well suited for both algebraic expressions and irrational ones.

The paper is organized as follows. In Sect. 2, we analyze the equations of motion of the body when these possess the additional cubic integral in momenta. The special solutions of the equations are found and their stability is investigated. In Sect. 3, the same problems are solved for the equations of motion of the body when these have the additional integral of 6th degree in momenta. Finally, we discuss the obtained results and give a conclusion in Sect. 4.

2 The Integrable Case with the Additional Cubic Integral

2.1 Formulation of the Problem

Euler–Poisson's differential equations describing the motion of a spheroidal rigid body around a fixed point in a force field with the potential $2V = 3a\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)(\gamma_1\gamma_2\gamma_3)^{-2/3}$ can be written as [5]

$$\begin{aligned}
\dot{M}_1 &= -[a\,(\gamma_2^2 - \gamma_3^2)\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)]\,(\gamma_2\gamma_3)^{-5/3}\,\gamma_1^{-2/3}, \dot{\gamma}_1 = \gamma_2 M_3 - \gamma_3 M_2,\\
\dot{M}_2 &= \ [a\,(\gamma_1^2 - \gamma_3^2)\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)]\,(\gamma_1\gamma_3)^{-5/3}\,\gamma_2^{-2/3}, \dot{\gamma}_2 = \gamma_3 M_1 - \gamma_1 M_3,\\
\dot{M}_3 &= -[a\,(\gamma_1^2 - \gamma_2^2)\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)]\,(\gamma_1\gamma_2)^{-5/3}\,\gamma_3^{-2/3}, \dot{\gamma}_3 = \gamma_1 M_2 - \gamma_2 M_1,
\end{aligned} \tag{2}$$

where M_i are the components of the kinetic momentum vector, γ_i are the direction cosines of "the vertical", a is some constant.

The above equations under the corresponding interpretation of the variables describe an expansion of the gas ellipsoid (without rotation) in vacuum. In this case, M_i are the impulses, $\gamma_i = A_i/\sqrt{\sum A_i^2}$, where A_i are the lengths of principal axes of the ellipsoid.

Equation (2) admit the following first integrals:

$$\begin{aligned}
&2H = M_1^2 + M_2^2 + M_3^2 + 3a\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\,(\gamma_1\gamma_2\gamma_3)^{-2/3} = 2h,\\
&V_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,\ V_2 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = 0,\\
&V_3 = M_1M_2M_3 - 3a\,(\gamma_1\gamma_2\gamma_3)^{1/3}(M_1\gamma_1^{-1} + M_2\gamma_2^{-1} + M_3\gamma_3^{-1}) = c_1.
\end{aligned} \tag{3}$$

Here V_3 is the additional integral derived in [7]. It is cubic with respect to M_1, M_2, M_3. This integral exists when the constant of the integral V_2 is equal to zero.

We can use the integrals having fixed constants for eliminating a part of the variables from differential equations (2) and the rest of the integrals to reduce the dimension of the problem. Let us eliminate the variable M_1 from Eq. (2) with the aid of $V_2 = 0$. They become:

$$\begin{aligned}
&\dot{M}_2 = a\,(\gamma_1^2 - \gamma_3^2)\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\,(\gamma_1\gamma_3)^{-5/3}\,\gamma_2^{-2/3},\\
&\dot{M}_3 = -a\,(\gamma_1^2 - \gamma_2^2)\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\,(\gamma_1\gamma_2)^{-5/3}\,\gamma_3^{-2/3},\\
&\dot{\gamma}_1 = \gamma_2 M_3 - \gamma_3 M_2, \dot{\gamma}_2 = -[M_2\gamma_2 + M_3(\gamma_1^2 + \gamma_3)]\,\gamma_3\gamma_1^{-1},\\
&\dot{\gamma}_3 = [M_2(\gamma_1^2 + \gamma_2) + M_3\gamma_3]\,\gamma_2\gamma_1^{-1}.
\end{aligned} \tag{4}$$

The first integrals of the above equations are:

$$2\tilde{H} = (M_2\gamma_2 + M_3\gamma_3)^2\,\gamma_1^{-2} + M_2^2 + M_3^2 + 3a\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\,(\gamma_1\gamma_2\gamma_3)^{-2/3} = 2\tilde{h},$$
$$V_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \tilde{V}_3 = -M_2M_3\,(M_2\gamma_2 + M_3\gamma_3)\,\gamma_1^{-1}$$
$$-3a\,(\gamma_1\gamma_2\gamma_3)^{1/3}[M_2\gamma_2^{-1} + M_3\gamma_3^{-1} - (M_2\gamma_2 + M_3\gamma_3)\,\gamma_1^{-2}] = \tilde{c}_1. \tag{5}$$

Further, we conduct the qualitative analysis of Eq. (4). In the general case, this problem is to find special solutions (equilibria, periodic motions) and to investigate their qualitative properties. In the case of conservative systems, the variety of the special solutions is expanded through stationary sets. By these sets, we mean sets of any finite dimension on which the problem's first integrals (or their combinations) assume a stationary value. Zero-dimensional sets having this property are known as stationary solutions, while we shall call positive dimensional sets the stationary invariant manifolds (IMs).

Our goal is to find the stationary solutions and the IMs of Eq. (4) and to investigate their stability.

2.2 Finding Invariant Manifolds

According to the Routh–Lyapunov method, the stationary solutions and the IMs of the differential equations under consideration can be obtained by solving the conditional extremum problem for the first integrals of these equations. For this purpose, a linear or nonlinear combination of the first integrals (a family of the first integrals) is constructed and the necessary extremum conditions for this family with respect to the phase variables are written. Thus, in the case of algebraic first integrals, the problem of finding stationary solutions and IMs is reduced to solving a system of algebraic equations.

Following the technique chosen, we take the complete linear combination of the first integrals of the problem:

$$2K = 2\lambda_0\tilde{H} - \lambda_1 V_1 - 2\lambda_3\tilde{V}_3, \tag{6}$$

where $\lambda_0, \lambda_1, \lambda_3$ are the parameters of the family of the integrals K, and write the necessary conditions for the integral K to have an extremum with respect to the phase variables:

$$\partial K/\partial M_2 = 0,\ \partial K/\partial M_3 = 0,\ \partial K/\partial \gamma_i = 0\ (i = 1, 2, 3). \tag{7}$$

The solutions of system (7), when it is degenerate (its Jacobian is identically equal to zero), allow one to define the IMs (or their families) for differential equations (4) which correspond to the family of the first integrals K.

System (7) is that of five irrational equations with the parameters $a, \lambda_0, \lambda_1, \lambda_3$. We should transform these equations to polynomial ones to use computer algebra methods, e.g., Gröbner basis method, for finding their solutions. For this purpose, we introduce the new variables:

$$M_2 = M_2,\ M_3 = M_3,\ x_1 = \gamma_1,\ x_2 = \gamma_2^{1/3}\gamma_1^{-1/3},\ x_3 = \gamma_3^{1/3}\gamma_1^{-1/3}. \tag{8}$$

In the above variables, the equations of motion (4) and first integrals (5) take the form

$$\begin{array}{l}\dot{M}_2 = -a(x_3^6 - 1)(x_2^6 + x_3^6 + 1)\, x_2^{-2} x_3^{-5}, \;\; 3\dot{x}_2 = -M_3\, (x_2^6 + x_3^6 + 1)\, x_2^{-2}, \\ \dot{M}_3 = \;\; a(x_2^6 - 1)(x_2^6 + x_3^6 + 1)\, x_2^{-5} x_3^{-2}, \;\; 3\dot{x}_3 = M_2\, (x_2^6 + x_3^6 + 1)\, x_3^{-2}, \\ \dot{x}_1 = x_1\, (M_3 x_2^3 - M_2 x_3^3), \end{array} \tag{9}$$

$$\begin{array}{l} 2\hat{H} = M_2^2 + M_3^2 + (M_2 x_2^3 + M_3 x_3^3)^2 + 3a\, (x_2^6 + x_3^6 + 1)\, x_2^{-2} x_3^{-2} = 2\hat{h}, \\ \hat{V}_1 = x_1^2\, (x_2^6 + x_3^6 + 1) = 1, \\ \hat{V}_3 = -M_2 M_3\, (M_2 x_2^3 + M_3 x_3^3) - 3a x_2 x_3\, [M_2\, (x_2^{-3} - x_2^3) + M_3\, (x_3^{-3} - x_3^3)] = \hat{c}_1, \end{array}$$

and the conditions for stationarity of the integral K can be written as:

$$\begin{array}{l} [\lambda_0 x_2^2\, (M_2 + x_2^3\, (M_2 x_2^3 + M_3 x_3^3)) + \lambda_3\, (M_3 x_2^2\, (2M_2 x_2^3 + M_3 x_3^3) \\ -3a x_3\, (x_2^6 - 1))\,]\, x_2^{-2} = 0, \\ [\lambda_0 x_3^2\, (M_3 + x_3^3\, (M_2 x_2^3 + M_3 x_3^3)) + \lambda_3\, (M_2 x_3^2\, (M_2 x_2^3 + 2M_3 x_3^3) \\ -3a x_2\, (x_3^6 - 1))\,]\, x_3^{-2} = 0, \;\; \lambda_1 x_1\, (x_2^6 + x_3^6 + 1) = 0, \\ [\lambda_0\, (M_2\, x_2^5 x_3^2\, (M_2 x_2^3 + M_3 x_3^3) + a\, (2x_2^6 - x_3^6 - 1)) - \lambda_2 x_1^2 x_2^8 x_3^2 \\ +\lambda_3\, [M_2^2 M_3 x_2^5 x_3^2 - a\, (2M_2 x_3^3\, (2x_2^6 + 1) + M_3 x_2^3\, (x_3^6 - 1))\,]]\, x_2^{-3} x_3^{-2} = 0, \\ [\lambda_0\, (a\, (x_2^6 - 2x_3^6 + 1) - M_3 x_2^2 x_3^5\, (M_2 x_2^3 + M_3 x_3^3)) + \lambda_2 x_1^2 x_2^2 x_3^8 \\ -\lambda_3\, [M_2 M_3^2 x_2^2 x_3^5 - a\, (M_2 x_3^3\, (x_2^6 - 1) + 2M_3 x_2^3\, (2x_3^6 + 1))\,]]\, x_2^{-2} x_3^{-3} = 0. \end{array} \tag{10}$$

First, we find the IMs of maximal codimension for Eq. (9). As the first integrals of the problem define IMs and families of IMs of codimension 1, we start with the IMs of codimension 2. As said before, the IMs can be derived as the solutions of system (10) when it is degenerate. To this end, we compute a lexicographical basis for the polynomials in square brackets (10) with respect to a part of the phase variables and the parameters, e.g., $\lambda_0, \lambda_1, M_2, M_3$ (the polynomials have least degrees with respect to these variables). Here the number of the phase variables determines the codimension of the desired IM. This technique enables us to obtain both the IMs and the conditions under which the stationary equations become degenerate (see., e.g., [11]).

The "Mathematica" program *GroebnerBasis* is applied to compute the basis:

```
GroebnerBasis[ polys, {lambda0, lambda2, M2, M3},
 CoefficientDomain -> RationalFunctions]
```

Here *polys* is the list of the polynomials in square brackets (10). All computations are performed on a computer with processor Intel Core 7i (3.6 GHz) and 32 GB RAM. The program has returned the basis in 21 s. So, we have the following system:

$$\begin{array}{l} \sigma_0 M_3^8 + \sigma_2 M_3^6 + \sigma_4 M_3^4 + \sigma_6 M_3^2 + \sigma_8 = 0, \\ \sigma M_2 + \sigma_1 M_3^7 + \sigma_3 M_3^5 + \sigma_5 M_3^3 + \sigma_7 M_3 = 0, \end{array} \tag{11}$$

$$\lambda_1 = 0, \;\; f(M_3, x_2, x_3, \lambda_0, \lambda_3, a) = 0, \tag{12}$$

where σ_j $(j = 0, \ldots, 8)$, σ are the polynomials of a, x_2, x_3 (their full form is given in the Appendix), f is a linear function of λ_0.

It is easy to verify by IM definition that Eq. (11) determine the IM of codimension 2 of differential equations (9): the derivative of (11) calculated by virtue of Eq. (9) vanishes on the given expressions.

The first of expressions (11) ($\lambda_1 = 0$) is the condition of degeneration of system (10). The latter expression ($f = 0$) allows one to derive the first integral for the equations of vector field on IM (11).

By this technique, one can also find an IM of codimension 3. First, under the condition $\lambda_1 = 0$, we compute a Gröbner basis with respect to elimination monomial order for the polynomials in square brackets (10):

```
gb = GroebnerBasis[ polys, {x3}, {M2, M3}, CoefficientDomain ->
RationalFunctions, MonomialOrder -> EliminationOrder]
```

Then, we construct a lexicographical basis:

```
 GroebnerBasis[ gb, { M2, M3, x3},
    CoefficientDomain -> RationalFunctions]
```

As a result, we have:

$$\begin{aligned}
&\lambda_0^8 x_3^{12} - 2\lambda_0^2 \rho_1 u x_3^6 - 12 a_1 \lambda_3^2 \rho_2 x_2^4 x_3^4 + \lambda_0^2\,(16 a_1^3 \lambda_3^6 x_2^6 + \lambda_0^6 v^2) = 0,\\
&2\lambda_0^3 \lambda_3 x_2 [(\lambda_0^{12} + 64 a_1^6 \lambda_3^{12})\, x_2^6 + 8 a_1^3 \lambda_0^6 \lambda_3^6\,(x_2^{12} + 1)]\, M_3 + \lambda_0^4\,(16 a_1^3 \lambda_3^6\, \rho_2 + \lambda_0^{12})\, u x_2^4\\
&-4 a_1^2 \lambda_3^4\,[16 a_1^3 \lambda_3^6\,(\lambda_0^6 v^2 + 12 a_1^3 \lambda_3^6 x_2^6) - \lambda_0^{12}\,(v^2 - x_2^6)]\, x_3^2 - 2 a_1 \lambda_0^2 \lambda_3^2\,[\lambda_0^{12} - 32 a_1^3\\
&\times \lambda_3^6\, \rho_2]\, u x_2^2 x_3^4 - \lambda_0^{10}\, \rho_1\, x_2^4 x_3^6 - 2 a_1 \lambda_0^8 \lambda_3^2 x_3^8\,[2 a_1 \lambda_0^4 \lambda_3^2 u - \rho_1\, x_2^2 x_3^4] = 0,\\
&[2\lambda_0 \lambda_3 v\,(\lambda_0^{12} v^2 + 8 a_1^3 \lambda_3^6 x_2^6\,(16 a_1^6 \lambda_3^6 x_2^6 + \lambda_0^6\,(x_2^{12} - 2v + 1)))]\, M_2 - 2 a_1 \lambda_3^2\,[8 a_1^3 \lambda_3^6 v\\
&-\lambda_0^6 (v-2)]\, \rho_3 x_2^4 x_3 - \lambda_0^2\,[\lambda_0^{12} (u+2) v^2 - 64 a_1^6 \lambda_3^{12}\,(2v + 3v^2)\, x_2^6\\
&+8 a_1^3 \lambda_0^6 \lambda_3^6 (5 v^2 x_2^6 - 2u)]\, x_3^3 + 4 a_1^2 \lambda_0^4 \lambda_3^4 x_2^2\,[16 a_1^3 \lambda_3^6 ((u+1) v^2 - 3)\\
&+\lambda_0^6 (4 - 3u^2 - v^3 + 16 x_2^6)]\, x_3^5 - \lambda_0^6 x_3^7\,[2 a_1 \lambda_3^2 x_2^4 \rho_3 - \lambda_0^2 \rho_3 x_3^2\\
&-4 a_1^2 \lambda_0^4 \lambda_3^4 (v^2 - 2)\, x_2^2 x_3^4] = 0,
\end{aligned} \tag{13}$$

where $u = x_2^6 + 1$, $v = x_2^6 - 1$, $a_1 = a/3$, $\rho_1 = \lambda_0^6 - 8 a_1^3 \lambda_3^6$, $\rho_2 = \lambda_0^6 - 4 a_1^3 \lambda_3^6$, $\rho_3 = \lambda_0^6 v^2 + 16 a_1^3 \lambda_3^6 x_2^6$. The total time to compute the basis is 8 s.

Likewise as above, it is easy to verify by IM definition that Eq. (13) define the family of IMs of codimension 3 for differential equations (9). Here λ_0, λ_3 are the parameters of the family. In the terms of the paper, it is the family of stationary IMs, since the integral $\hat{K} = \lambda_0 \hat{H} - \lambda_3 \hat{V}_3$ assumes a stationary value on the elements of this family.

One can show that the elements of IMs family (13) are the submanifolds of IM (11). Let us find their intersection. To this end, we compute a lexicographical basis with respect to the variables M_2, M_3, x_3 for the polynomials of the system composed of Eqs. (11), (13). The resulting equations are the family of IMs (13). So, the original assumption is true.

With (8), we can return to the initial variables $M_2, M_3, \gamma_1, \gamma_2, \gamma_3$ in Eqs. (11), (13). In the initial variables, these equations define, respectively, the IM of codimension 2 and the family of IMs of codimension 3 for differential equations (4) that can be verified by IM definition.

Other IMs of codimension 2 for the equations of motion (4) have been obtained by the chains of differential consequences of the kind [10]:

$$W_0' = \varphi_1(x)\, W_1(x),\ W_1' = \varphi_2(x)\, W_2(x), \ldots,\ W_{k-1}' = \varphi_k(x)\, W_k(x), \ldots \quad (14)$$

Here $x = (M_2, M_3, \gamma_1, \gamma_2, \gamma_3)$, and $W_j(x)$ $(j = 0, \ldots)$, $\varphi_m(x)$ $(m = 1, \ldots)$ are some smooth functions of x, W_j' $(j = 1, \ldots)$ are their derivatives by virtue of differential equations (4).

We call the chain of differential consequences (14) cyclical one if for some k:

$$W_k' = \sum_{i=0}^{k} \bar{\varphi}_i(x)\, W_i(x), \quad (15)$$

where $\bar{\varphi}_i(x)$ are the smooth functions.

Statement 1. If system (4) admits cyclical chain (15) then it has the IM defined by the equations $W_0(x) = W_1(x) = \ldots = W_k(x) = 0$. The proof is obvious.

In the given approach, computer algebra tools play an auxiliary role. They give us a possibility to make computational experiments, e.g., for finding the functions W_i that would be most "suitable" to generate the chain. The "Mathematica" program *PolynomialReduce* is used to test criterion (15).

Let be $W_0 = M_2 + M_3$. On differentiating this expression by virtue of Eq. (4) we obtain $W_1 = \gamma_2 - \gamma_3$. The subsequent differentiation of W_1 shows that differential equations (4) admit the following cyclical chain:

$$W_0' = [a\,(\gamma_1^2 + \gamma_2\gamma_3)\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\,(\gamma_1\gamma_2\gamma_3)^{-5/3}]\, W_1,$$
$$W_1' = -[(\gamma_1^2 + \gamma_2^2 + \gamma_2\gamma_3)\,\gamma_1^{-1}]\, W_0 + [M_3\,(\gamma_2 + \gamma_3)\,\gamma_1^{-1}]\, W_1.$$

According to Statement 1, the expressions

$$M_2 + M_3 = 0,\ \gamma_2 - \gamma_3 = 0 \quad (16)$$

determine the IM of codimension 2 of differential equations (4).

The vector field on IM (16) is given by

$$\dot{M}_3 = -a\,(\gamma_1^2 - \gamma_3^2)\,(\gamma_1^2 + 2\gamma_3^2)\,\gamma_1^{-5/3}\gamma_2^{-7/3},\ \dot{\gamma}_1 = 2M_3\gamma_3,\ \dot{\gamma}_3 = -M_3\gamma_1. \quad (17)$$

In the same way, the IM defined by the equations

$$M_2 - M_3 = 0,\ \gamma_2 + \gamma_3 = 0 \quad (18)$$

has been derived.

The vector field on this IM is described by

$$\dot{M}_3 = a\,(-\gamma_3)^{1/3}(\gamma_3^2 - \gamma_1^2)\,(\gamma_1^2 + 2\gamma_3^2)\,\gamma_1^{-5/3}\gamma_3^{-8/3},$$
$$\dot{\gamma}_1 = -2M_3\gamma_3,\ \dot{\gamma}_3 = M_3\gamma_1. \quad (19)$$

Note that IMs (16), (18) are stationary. The integral $\Omega = \tilde{V}_3^2$ takes a stationary value on them.

All found IMs for differential equations (4) can be "lifted up" into the phase space of system (2). For this purpose, it is sufficient to add expression $V_2 = 0$ (3) to the equations of these IMs. In particular, equations IMs (16), (18) take the form

$$M_2 + M_3 = 0,\ \gamma_2 - \gamma_3 = 0,\ M_1\gamma_1 = 0 \tag{20}$$

and $M_2 - M_3 = 0,\ \gamma_2 + \gamma_3 = 0,\ M_1\gamma_1 = 0$, respectively,

From the physical viewpoint, in the case of the spheroidal body, the above equations together with (17), (19) define pendulum-like oscillations of the body. From the formulation of the problem it follows that IM (20) is related to the problem of the expanding gas cloud only. Equation (20) together with (17) describe the periodical changes of the cloud sizes.

2.3 Finding Stationary Solutions

As mentioned before, stationary solutions are usually found by the Routh–Lyapunov method from the conditions for stationarity of a family of problem's first integrals. In the case of polynomial first integrals, this approach leads to solving a system of polynomial equations. When the first integrals are not polynomial or the polynomials have high degrees, the technique applied in [11] is more suitable. The given technique is used in the present work.

Equate the right-hand sides of differential equations (4) to zero and add relation $V_1 = 1$ (5) to them:

$$\begin{array}{l} a\,(\gamma_1^2-\gamma_3^2)\,(\gamma_1^2+\gamma_2^2+\gamma_3^2)\,(\gamma_1\gamma_3)^{-5/3}\,\gamma_2^{-2/3} = 0, \gamma_2 M_3 - \gamma_3 M_2 = 0,\\ -a\,(\gamma_1^2-\gamma_2^2)\,(\gamma_1^2+\gamma_2^2+\gamma_3^2)\,(\gamma_1\gamma_2)^{-5/3}\,\gamma_3^{-2/3} = 0, \gamma_1^2+\gamma_2^2+\gamma_3^2-1 = 0,\\ -\gamma_3\gamma_1^{-1}\,[M_2\gamma_2 + M_3(\gamma_1^2+\gamma_3)] = 0,\\ \gamma_2\gamma_1^{-1}\,[M_2(\gamma_1^2+\gamma_2) + M_3\gamma_3] = 0. \end{array} \tag{21}$$

Next, construct a lexicographical Gröbner basis with respect to M_2, M_3, γ_1, γ_2, γ_3 for the polynomials of the subsystem

$$\begin{array}{l} \gamma_1^2-\gamma_3^2 = 0,\ M_2\gamma_2 + M_3(\gamma_1^2+\gamma_3) = 0,\ M_2(\gamma_1^2+\gamma_2) + M_3\gamma_3 = 0,\\ \gamma_1^2-\gamma_2^2 = 0,\ \gamma_2 M_3 - \gamma_3 M_2 = 0,\ \gamma_1^2+\gamma_2^2+\gamma_3^2-1 = 0 \end{array}$$

of system (21). As a result, we have:

$$3\gamma_3^2 - 1 = 0,\ 1 - 3\gamma_2^2 = 0,\ 1 - 3\gamma_1^2 = 0, M_2 = 0,\ M_3 = 0.$$

The latter system has the following solutions:

$$\begin{array}{l} M_2 = 0,\ M_3 = 0,\ \gamma_1 = \pm 3^{-1/2},\ \gamma_2 = \gamma_3 = 3^{-1/2},\\ M_2 = 0,\ M_3 = 0,\ \gamma_1 = \pm 3^{-1/2},\ \gamma_2 = \gamma_3 = -3^{-1/2}. \end{array} \tag{22}$$

$$M_2 = 0,\ M_3 = 0,\ \gamma_1 = \pm 3^{-1/2},\ \gamma_2 = -3^{-1/2},\ \gamma_3 = 3^{-1/2},$$
$$M_2 = 0,\ M_3 = 0,\ \gamma_1 = \pm 3^{-1/2},\ \gamma_2 = 3^{-1/2},\ \gamma_3 = -3^{-1/2}. \tag{23}$$

On substituting these solutions into Eq. (4) they are satisfied.

Now, let us derive the family of the integrals which takes a stationary value on solutions (22), (23). When these solutions are substituted into Eq. (7), we find that the equations are satisfied under $\lambda_1 = 0$.

On substituting $\lambda_1 = 0$ into (6), we have:

$$\tilde{K} = \lambda_0 \tilde{H} - \lambda_3 \tilde{V}_3. \tag{24}$$

Thus, the family of the integrals $\tilde{K}$ assumes a stationary value on solutions (22), (23). Each integral belonging to this family also takes a stationary value on the above solutions. It is verified by direct calculation. In particular, the integral $\tilde{V}_3$ is identically equal to zero on all solutions (22), (23).

In the same way as the IMs in Subsect. 2.2, the stationary solutions can be "lifted up" into the phase space of system (2). From the physical viewpoint, in the original phase space, these solutions correspond to the equilibria of the spheroidal body, and only one of these solutions is related to the problem of the expanding gas cloud: $M_1 = M_2 = M_3 = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 3^{-1/2}$. It was also found in [4]. This solution corresponds to the cloud of the spherical shape without changing sizes.

One can show that stationary solutions (22), (23) belong to IM (11). To this end, we substitute these solutions into the equations of the IM (they must be written in the initial variables $M_2, M_3, \gamma_1, \gamma_2, \gamma_3$). The equations turn into identities. Thus, solutions (22), (23) belong to IM (11).

In the same way, we reveal that solutions (22) and (23) belong to IM (16) and IM (18), respectively. Hence, IM (11) and IM (16) have the common points (i.e., the points of intersection of these IMs) defined by relations (22). Analogously, relations (23) define the points of intersection of IM (11) and IM (18).

2.4 On Stability of Stationary Solutions

The integrals and their families, which take a stationary value on solutions (22), (23), are used to investigate the stability of these solutions by the Routh–Lyapunov method. The problem is to verify the sign-definiteness conditions for the 2nd variation of the family of integrals which is obtained in the neighborhood of the solution under study. These conditions are analyzed on the linear manifold defined by the variations of the "conditional" integrals.

Let us investigate the stability of one of solutions (22), e.g.,

$$M_2 = M_3 = 0,\ \gamma_1 = \gamma_2 = \gamma_3 = 3^{-1/2}, \tag{25}$$

which is related to the problem of the expanding gas cloud.

We use the family of integrals $\tilde{K}$ (24). In the deviations $y_1 = \gamma_1 - 3^{-1/2}$, $y_2 = \gamma_2 - 3^{-1/2}$, $y_3 = \gamma_3 - 3^{-1/2}$, $y_4 = M_2$, $y_5 = M_3$ on the linear manifold $\delta V_1 = 2(y_1 + y_2 + y_3)/\sqrt{3} = 0$, the 2nd variation of $\tilde{K}$ in the neighborhood of solution (25) can be written as:

$$\delta^2 \tilde{K} = \lambda_0 \left[18a\left(y_1^2 + y_1 y_2 + y_2^2\right) + y_4^2 + y_4 y_5 + y_5^2\right] + 6\sqrt{3} a \lambda_3 \left[y_1\left(y_4 + 2y_5\right) \right.$$
$$\left. + y_2\left(y_5 - y_4\right)\right]. \tag{26}$$

The conditions for the quadratic form $\delta^2 \tilde{K}$ to be positive definite in the form of Sylvester's inequalities are given by $a\lambda_0 > 0$, $a^2\lambda_0^2 > 0$, $a^2\lambda_0\left(\lambda_0^2 - 6a\lambda_3^2\right) > 0$, $a^2(\lambda_0^2 - 6a\lambda_3^2)^2 > 0$.

These inequalities are consistent under the following constraints on a, λ_0, λ_3:

$$a > 0,\ \lambda_3 > 0,\ \lambda_0 > \sqrt{6}\sqrt{a}\,\lambda_3. \tag{27}$$

Inequalities (27) are split up into 2 groups. The first ($a > 0$) is the sufficient condition for the stability of solution (25), and the rest of the inequalities separates some subfamily from the family of integrals $\tilde{K}$ (24), the elements of which give us a possibility to derive this condition.

Let us show that the sufficient condition of stability is also necessary. To this end, we use Lyapunov's linear stability theorem [14].

In the case studied, the equations of first approximation, in the deviations $y_i\,(i = 1, \ldots, 5)$, are:

$$\sqrt{3}\,\dot{y}_1 = y_5 - y_4, \sqrt{3}\,\dot{y}_2 = -(y_4 + 2y_5),\ \sqrt{3}\,\dot{y}_3 = 2y_4 + y_5,$$
$$\dot{y}_4 = 6\sqrt{3}a\,(y_1 - y_3),\ \dot{y}_5 = 6\sqrt{3}a\,(y_2 - y_1).$$

The characteristic equation $\lambda\,(\lambda^2 + 18a)^2 = 0$ of the above system has only zero and pure imaginary roots when $a > 0$. On comparing the latter inequality with (27), we conclude that the condition $a > 0$ is necessary and sufficient for the stability of solution (25). For the rest of the stationary solutions, we have obtained similar results.

Now, we investigate the stability of IM (16), which solution (25) belongs to.

For the equations of perturbed motion, in the deviations $y_1 = M_2 + M_3$, $y_2 = \gamma_2 - \gamma_3$, on the linear manifold $\delta V_1 = 2\gamma_3\,y_2 = 0$, the 2nd variation of the integral $\Omega = \tilde{V}_3^2$ is:

$$\delta^2 \Omega = \left[3a\left(\gamma_3^2 - \gamma_1^2\right) + \gamma_1^{2/3}\gamma_3^{4/3}M_3^2\right]^2 \gamma_1^{-10/3}\gamma_3^{-2/3}\,y_1^2. \tag{28}$$

On IM (16), the integral $\tilde{H}$ assumes the form:

$$\bar{H} = \left[M_3^2 + 3a\left(\gamma_1^2 + 2\gamma_3^2\right)\right]\left(2\gamma_1^{-2/3}\gamma_3^{-4/3}\right) = h_1. \tag{29}$$

Eliminate M_3 from (28) with (29):

$$4\delta^2 \Omega = \left(9a\gamma_1^{4/3} - 2\,h_1\gamma_3^{4/3}\right)^2 \gamma_1^{-2}\gamma_3^{-2/3}\,y_1^2.$$

Equate the numerator of the latter expression to zero and eliminate γ_1 from the resulting equation with the integral $V_1 = 1$. As a result, we obtain the following boundary value for γ_2:

$$\gamma_2 = \Big(\frac{2h_1}{9a} + 1\Big)^{3/4},$$

under which there exist the stable oscillations of the spheroidal body. As to the gas ellipsoid, the latter relation allows one to determine the limit values for the lengths of its principal axes under which the periodical changes of the cloud sizes are stable.

When the stability of stationary solutions and IMs is studied on the base of Lyapunov's linear stability theorems and the 2nd Lyapunov method, we need often to derive the sign-definiteness conditions for a quadratic form as well as the characteristic equation for a system of linear differential equations with constant coefficients. The computer program codes of these procedures are included in the "Mathematica" software package [2]. This package has been developed to do the qualitative analysis of conservative systems on the base of the approach described in the this paper. It is applied as an auxiliary tool at different stages of analysis of the systems. In the above calculations, for the given solution and the given combination of the first integrals, the package has constructed the sign-definiteness conditions for the quadratic form $\delta^2\tilde{K}$ (26) in the form of Sylvester's inequalities. The subsequent analysis of these inequalities was made by computer algebra tools. In a similar manner, the package is used to investigate the stability on the base of Lyapunov's linear stability theorems.

3 The Integrable Case with the Additional 6th Degree Integral

3.1 Formulation of the Problem

The equations of motion of the spheroidal body in a force field with the potential

$$2V = G\,[3a\,(\gamma_1\gamma_2\gamma_3)^{-2/3} + 4c^2\,(\gamma_1^2+\gamma_2^2)\,(\gamma_1^2-\gamma_2^2)^{-2}]$$

can be written as:

$$\begin{array}{l}
\dot{M}_1 = -G\,[a\,(\gamma_2^2-\gamma_3^2)\,(\gamma_2\gamma_3)^{-5/3}\,\gamma_1^{-2/3} + 4c^2\gamma_2\gamma_3\,(3\gamma_1^2+\gamma_2^2)\,(\gamma_1^2-\gamma_2^2)^{-3}],\\
\dot{M}_2 = \ \ G\,[a\,(\gamma_1^2-\gamma_3^2)\,(\gamma_1\gamma_3)^{-5/3}\,\gamma_2^{-2/3} - 4c^2\gamma_1\gamma_3\,(\gamma_1^2+3\gamma_2^2)\,(\gamma_1^2-\gamma_2^2)^{-3}],\\
\dot{M}_3 = -G\,[a\,(\gamma_1^2-\gamma_2^2)\,(\gamma_1\gamma_2)^{-5/3}\,\gamma_3^{-2/3} - 16c^2\gamma_1\gamma_2\,(\gamma_1^2+\gamma_2^2)\,(\gamma_1^2-\gamma_2^2)^{-3}],\\
\dot{\gamma}_1 = \gamma_2 M_3 - \gamma_3 M_2,\ \dot{\gamma}_2 = \gamma_3 M_1 - \gamma_1 M_3,\ \dot{\gamma}_3 = \gamma_1 M_2 - \gamma_2 M_1.
\end{array} \tag{30}$$

Here the variables $M_i, \gamma_i\,(i = 1,2,3)$ are interpreted as in Sect. 2, $G = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$.

The first integrals of Eq. (30) are given by

$$\begin{array}{l}
2H = M_1^2 + M_2^2 + M_3^2 + G\,[3a\,(\gamma_1\gamma_2\gamma_3)^{-2/3} + 4c^2(\gamma_1^2+\gamma_2^2)\,(\gamma_1^2-\gamma_2^2)^{-2}] = 2h,\\
V_1 = \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1,\ V_2 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = 0,\\
V_3 = (F_3+F_c)^2 + 4\varPhi\,[\bar{\varPhi}\,\gamma_1^2\,\gamma_3^{-2} + 3a]\,[\bar{\varPhi}\,\gamma_2^2\,\gamma_3^{-2} + 3a\Big] = c_1,
\end{array} \tag{31}$$

where

$$F_3 = M_1 M_2 M_3 - 3a\,(\gamma_1\gamma_2\gamma_3)^{1/3}\,(M_1\gamma_1^{-1} + M_2\gamma_2^{-1} + M_3\gamma_3^{-1}),$$
$$F_c = 4c^2 M_3\gamma_1\gamma_2\gamma_3^2\,(\gamma_1^2 - \gamma_2^2)^{-2},$$
$$\Phi = 4c^2\gamma_3^2(\gamma_1\gamma_2\gamma_3)^{2/3}\,(\gamma_1^2 - \gamma_2^2)^{-2},\ \bar{\Phi} = M_1 M_2\,(\gamma_1\gamma_2\gamma_3)^{2/3}\,\gamma_1^{-1}\gamma_2^{-1} + \Phi - 3a.$$

Here V_3 is the additional 6th degree integral with respect to M_1, M_2, M_3. It has been derived in [9]. This integral exists when the constant of the integral V_2 is equal to zero. Note that the potential energy V in this problem has a singularity when $\gamma_1 = \gamma_2$.

Likewise as in Sect. 2, we shall consider the equations of motion of the body on the manifold $V_2 = 0$. On this manifold, differential equations (30) and first integrals (31) take the form:

$$\begin{aligned}
&\dot{M}_1 = -G\,[a\,(\gamma_2^2 - \gamma_3^2)\,(\gamma_2\gamma_3)^{-5/3}\,\gamma_1^{-2/3} + 4c^2\gamma_2\gamma_3\,(3\gamma_1^2 + \gamma_2^2)\,(\gamma_1^2 - \gamma_2^2)^{-3}],\\
&\dot{M}_2 = G\,[a\,(\gamma_1^2 - \gamma_3^2)\,(\gamma_1\gamma_3)^{-5/3}\,\gamma_2^{-2/3} - 4c^2\gamma_1\gamma_3\,(\gamma_1^2 + 3\gamma_2^2)\,(\gamma_1^2 - \gamma_2^2)^{-3}],\\
&\dot{\gamma}_1 = -[M_1\gamma_1\gamma_2 + M_2\,(\gamma_2^2 + \gamma_3^2)]\,\gamma_3^{-1},\ \dot{\gamma}_2 = [M_1\,(\gamma_1^2 + \gamma_3^2) + M_2\gamma_1\gamma_2]\,\gamma_3^{-1},\\
&\dot{\gamma}_3 = \gamma_1 M_2 - \gamma_2 M_1.
\end{aligned} \tag{32}$$

$$\begin{aligned}
&2\tilde{H} = M_1^2 + M_2^2 + (M_1\gamma_1 + M_2\gamma_2)^2\,\gamma_3^{-2} + G\,[3a\,(\gamma_1\gamma_2\gamma_3)^{-2/3}\\
&+4c^2(\gamma_1^2 + \gamma_2^2)\,(\gamma_1^2 - \gamma_2^2)^{-2}] = 2\tilde{h},\ V_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,\\
&\tilde{V}_3 = (\tilde{F}_3 + \tilde{F}_c)^2 + 4\Phi\left[\bar{\Phi}\,\gamma_1^2\,\gamma_3^{-2} + 3a\right][\bar{\Phi}\,\gamma_2^2\,\gamma_3^{-2} + 3a] = \tilde{c}_1,\ \text{where}\\
&\tilde{F}_3 = -M_1 M_2\,(M_1\gamma_1 + M_2\gamma_2)\,\gamma_3^{-1} - 3a\,(\gamma_1\gamma_2\gamma_3)^{1/3}\,(M_1\gamma_1^{-1} + M_2\gamma_2^{-1}\\
&-(M_1\gamma_1 + M_2\gamma_2)\,\gamma_3^{-2}),\ \tilde{F}_c = -4c^2(M_1\gamma_1 + M_2\gamma_2)\gamma_1\gamma_2\,(\gamma_1^2 - \gamma_2^2)^{-2}.
\end{aligned} \tag{33}$$

They have been derived from (30), (31) by eliminating the variable M_3 from them with the aid of $V_2 = 0$.

In the present work, we restrict our consideration to the problem of finding the stationary solutions for Eq. (32) and the investigation of their stability.

3.2 Finding Stationary Solutions

We apply the same technique as in Subsect. 2.3 to obtain the stationary solutions of differential equations (32). For this purpose, these equations are written in the variables $M_1 = M_1$, $M_2 = M_2$, $x_1 = \gamma_1$, $x_2 = \gamma_2^{1/3}\gamma_1^{-1/3}$, $x_3 = \gamma_3^{1/3}\gamma_1^{-1/3}$:

$$\begin{aligned}
&\dot{M}_1 = -\bar{G}\,[a(x_2^6 - 1)^3(x_2^6 - x_3^6) - 4c^2x_2^8x_3^8\,(x_2^6 + 3)]\,x_2^{-5}x_3^{-5}(x_2^6 - 1)^{-3},\\
&\dot{M}_2 = \bar{G}\,[(4c^2x_2^2x_3^8\,(3x_2^6 + 1) - a(x_2^6 - 1)^3(x_3^6 - 1)]\,x_2^{-2}x_3^{-5}\,(x_2^6 - 1)^{-3},\\
&\dot{x}_1 = -[M_1x_2^3 + M_2(x_2^6 + x_3^6)]\,x_1x_3^{-3},\ 3\dot{x}_2 = \bar{G}\,[M_1 + M_2x_2^3]\,x_2^{-2}x_3^{-3},\\
&3\dot{x}_3 = (x_2^6 + x_3^6 + 1)\,M_2x_3^{-2},
\end{aligned} \tag{34}$$

where $\bar{G} = x_2^6 + x_3^6 + 1$.

Next, we equate the right-hand sides of Eq. (34) to zero and consider the following subsystem

$$
\begin{aligned}
&a(x_2^6-1)^3(x_2^6-x_3^6)-4c^2x_2^8x_3^8\,(x_2^6+3)=0,\\
&(4c^2x_2^2x_3^8\,(3x_2^6+1)-a(x_2^6-1)^3(x_3^6-1)=0,\\
&M_1x_2^3+M_2(x_2^6+x_3^6)=0,\ M_1+M_2x_2^3=0,\ M_2=0
\end{aligned}
\tag{35}
$$

of the resulting system.

From the latter three equations (35), it follows that $M_1=M_2=0$. For the polynomials of the rest of the equations, we compute a Gröbner basis with respect to the ordering $x_3>x_2$. Taking into account the above values for M_1, M_2, we have:

$$
\begin{aligned}
&a^3(x_2^6-1)^{12}\,(x_2^{12}+6x_2^6+1)-16384\,c^6x_2^{30}(x_2^6+1)^4=0,\\
&16384a^2c^2x_3^2-16384c^6x_2^{22}(x_2^6+1)^3(31x_2^6+32)(33x_2^6+32)+a^3x_2^4(x_2^6-1)^4\\
&\times(1023x_2^{54}-1021x_2^{48}-21488x_2^{42}+86920x_2^{36}-136858x_2^{30}+71014x_2^{24}\\
&+72584x_2^{18}-138224x_2^{12}+88067x_2^6-22529)=0,\\
&M_1=0,\ M_2=0.
\end{aligned}
\tag{36}
$$

It is easy to verify by IM definition that Eq. (36) define the one-dimensional IM of differential equations (34). The vector field on this IM is described by the equation $\dot{x}_1=0$. It has the following solution:

$$x_1=x_1^0=\text{const.} \tag{37}$$

Equation (36) together with (37) and the condition

$$x_1^2\,(x_2^6+x_3^6+1)=1, \tag{38}$$

which is the integral V_1 in the variables x_1, x_2, x_3, determine the set of fixed points for system (34).

In the initial variables $M_1, M_2, \gamma_1, \gamma_2, \gamma_3$, Eqs. (36) and (36)–(38) determine the one-dimensional IM and the set of fixed points for system (32), respectively. In the same way as in Sect. 2, these solutions can be "lifted up" into the phase space of system (30).

From the physical viewpoint, in the original phase space, the solutions defined by (36)–(38) correspond to the equilibria of the spheroidal body (the gas ellipsoid). From equations (36)–(38) it follows that the number of the equilibria is no more than 336 $\forall\ a\neq 0,\ c\neq 0$. One can also see from these equations that they can have one real positive solution only. Thus, in the problem of the expanding gas cloud, there exists no more than one equilibrium position for each fixed pair of values of the parameters $a\neq 0,\ c\neq 0$. The latter agrees with the result [4]. Further, we find the equilibria under some conditions imposed on the parameters a and c.

System (34) is defined in the domain: $x_1^2\,(x_2^6+x_3^6+1)=1,\ x_i\neq 0\ (i=1,2,3),\ x_2\neq 1$. We choose a value of x_2 from this domain, e.g. $x_2=1/2^{1/6}$, and then substitute it into the 1st equation of system (36). Whence, one can obtain $a = 192\,(6/17)^{1/3}c^2$. Under the above values of x_2, a, from the rest of Eqs. (36)–(38), we find x_1, x_3. So, for the given values of x_2, a, system (36)–(38) has the following solutions:

$$M_1 = M_2 = 0,\ x_1 = (34/3)^{1/2}\,5^{-1},\ x_2 = 2^{-1/6},\ x_3 = \pm 2^{1/3}\,(3/17)^{1/6},$$
$$M_1 = M_2 = 0,\ x_1 = -(34/3)^{1/2}\,5^{-1},\ x_2 = 2^{-1/6},\ x_3 = \pm 2^{1/3}\,(3/17)^{1/6}.$$

In the initial variables, the above solutions are:

$$M_1 = M_2 = 0,\ \gamma_1 = (34/3)^{1/2}\,5^{-1},\ \gamma_2 = (17/3)^{1/2}\,5^{-1},\ \gamma_3 = \pm 2\sqrt{2}\,5^{-1},$$
$$M_1 = M_2 = 0,\ \gamma_1 = -(34/3)^{1/2}\,5^{-1},\ \gamma_2 = -(17/3)^{1/2}\,5^{-1},$$
$$\gamma_3 = \pm 2\sqrt{2}\,5^{-1}. \qquad (39)$$

On substituting these solutions into differential equations (32) they are satisfied.

From the physical viewpoint, in the original phase space, solutions (39) correspond to the equilibria of the spheroidal body. Only one of these solutions is related to the problem of the expanding gas cloud:

$$M_1 = M_2 = M_3 = 0,\ \gamma_1 = (34/3)^{1/2}\,5^{-1},\ \gamma_2 = (17/3)^{1/2}\,5^{-1},\ \gamma_3 = 2\sqrt{2}\,5^{-1}.$$

It corresponds to the gas cloud of ellipsoidal shape. This ellipsoid is prolate along its principal axis Ox.

As in Sect. 2, one can show that the family of integrals

$$\tilde{K} = \lambda_0 \tilde{H} - \lambda_3 \tilde{V}_3 \qquad (40)$$

(and each integral of this family) assumes a stationary value on solutions (39). The family of integrals $\tilde{K}$ (40) is used for the investigation of stability of the given solutions.

3.3 On Stability of Stationary Solutions

In order to study the stability of stationary solutions (39), we apply the same approach, methods and computing tools as in Sect. 2.

First, let us investigate the stability of one of solutions (39) which is related to the problem of the expanding gas cloud:

$$M_1 = M_2 = 0,\ \gamma_1 = (34/3)^{1/2}\,5^{-1},\ \gamma_2 = (17/3)^{1/2}\,5^{-1},\ \gamma_3 = 2\sqrt{2}\,5^{-1}. \quad (41)$$

In the deviations $y_1 = M_1$, $y_2 = M_2$, $y_3 = \gamma_1 - (34/3)^{1/2}\,5^{-1}$, $y_4 = \gamma_2 - (17/3)^{1/2}\,5^{-1}$, $y_5 = \gamma_3 - 2\sqrt{2}\,5^{-1}$, on the linear manifold $\delta V_1 = 2\,[\sqrt{51}\,(\sqrt{2}y_3 + y_4) + 6\sqrt{2}y_5]/15 = 0$, the 2nd variation of the family of integrals $\tilde{K}$ in the neighborhood of the solution under study is written as: $\delta^2\tilde{K} = Q_1 + Q_2$, where

$$83521\,Q_1 = 15000\,c^2\,[204\,(221\lambda_0 - 161792\,c^4\lambda_3)\,y_4^2 + \sqrt{102}\,(3961\lambda_0 + 7651328\,c^4\lambda_3)\,y_4 y_5 + (19822\lambda_0 - 795295744\,c^4\lambda_3)\,y_5^2],$$
$$816\,Q_2 = (986\lambda_0 - 14450688\,c^4\lambda_3)\,y_1^2 + 2\sqrt{2}\,(289\lambda_0 + 10764288\,c^4\lambda_3)\,y_1 y_2 + (697\lambda_0 - 17842176\,c^4\lambda_3)\,y_2^2.$$

The conditions for the family of the quadratic forms Q_1, Q_2 to be positive definite are sufficient for the stability of solution (41). In the form of Sylvester's inequalities, they are:

$$221\lambda_0 - 161792\,c^4\lambda_3 > 0,\ 289\lambda_0^2 - 22282240\,c^4\lambda_0\lambda_3 + 14495514624\,c^8\lambda_3^2 > 0,$$
$$986\lambda_0 - 14450688\,c^4\lambda_3 > 0. \qquad (42)$$

Inequalities (42) are compatible under the following constraints on the parameters λ_0, λ_3, c: $17\lambda_0 > 16384\,(40+\sqrt{1546})\,c^4\lambda_3$. The latter condition separates the subfamily from the family of integrals $\tilde{K}$ (40), the elements of which enable us to derive the sufficient conditions for the stability of solution (41). Comparison of the above sufficient condition with the relation $a = 192\,(6/17)^{1/3}c^2$ gives us the following sufficient condition for the stability of solution (41): $a > 0$.

For solution (41), we have also derived the conditions of its stability on the base of Lyapunov's linear stability theorem. The resulting necessary stability conditions coincide with the sufficient ones.

Similar results have been obtained for the rest of solutions (39).

4 Conclusion

In the given work, ordinary differential equations with irrational first integrals were studied. These equations describe a series of dynamical systems, such as an expansion of the gas ellipsoidal cloud in vacuum, the rotation of the spheroidal body in a potential force field, the motion of a point mass on the spherical surface. We analyzed the equations in the cases when they possess the additional first integrals of 3rd and 6th degree in momenta. The purpose of the study was to find the stationary solutions and IMs of the equations and to investigate their stability. To solve these problems, computer algebra methods and tools were applied. The first integrals in the problem are rather complicated irrational functions. Computer algebra methods were used for transforming irrational equations to polynomial ones and for finding their solutions.

In the problem of the expanding gas cloud, in addition to previously known solutions, new IMs of codimension 2, 3 as well one-dimensional IM have been obtained, and the physical interpretation for some of them has been done. It was established that the previously known solutions belong to these IMs. It was also shown that these solutions are stationary. For the stationary solutions and IMs, the sufficient conditions of their stability on the base of the 2nd Lyapunov method have been derived. The "Mathematica" software package developed by the authors together with their colleagues was used to investigate the stability of the found solutions. It should be noted that in the problem of the rotational motion of the spheroidal body, there exists a greater number of stationary solutions and IMs than in the above problem. Some of them have been found and represented in the paper. The analysis of their stability has also been done.

The obtained results, their consistency with those known before, show that the approach used as well the computing tools are rather efficient for the study of the dynamical systems of the considered type.

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A Appendix

The coefficients of equation (11):

$$\sigma_0 = x_2^8 x_3^8 \left(4x_3^6 - (x_2^6 - x_3^6 - 1)^2\right),\ \sigma_2 = -2ax_2^6 \Big[5\left(x_2^6+1\right)x_3^{18} - 2\left(8x_2^6 + 5\left(x_2^6+1\right)^2\right)x_3^{12} + \left(5\left(x_2^{18}+1\right) - 9x_2^6\left(x_2^6+1\right)\right)x_3^6 + 2x_2^6\left(x_2^6-1\right)^2\Big],$$

$$\sigma_4 = -36\,a^2 x_2^4 x_3^4 \Big[\left(\left(x_2^6+1\right)^2 + x_2^6\right)x_3^{12} - 2\left(x_2^{18} + 6x_2^6\left(x_2^6+1\right)+1\right)x_3^6 + \left(x_2^{24} - 2x_2^6\left(x_2^{12}+x_2^6+1\right)+1\right)\Big],\ \sigma_6 = -54\,a^3 x_2^2 x_3^2 \Big[\left(x_2^{18} + 7x_2^6\left(x_2^6+1\right) + 1\right)x_3^{12} - 2\left(\left(x_2^{12}+1\right)^2 + 14\,x_2^{12} + 7x_2^6\left(x_2^{12}+1\right)\right)x_3^6 + \left(x_2^6-1\right)^2\left(x_2^6+1\right)^3\Big],$$

$$\sigma_8 = -27a^4\Big[\left(\left(x_2^6+1\right)^2 + 4x_2^6\right)x_3^6 - \left(x_2^6+1\right)^3\Big]^2,$$

$$\sigma = -18a^3 x_2^3 x_3 \Big[\left(x_2^6-1\right)\left[\left(x_2^6+1\right)^2+4x_2^6\right]^2 x_3^{30} - \left[3(x_2^{36}-1) - 8x_2^6(x_2^{24}-1) - 153x_2^{12}(x_2^{12}-1)\right]x_3^{24} + 2\left[x_2^{42} - 15x_2^6(x_2^{30}-1) - 3x_2^{12}(x_2^{18}-1) + 269x_2^{18}(x_2^6-1) - 1\right]x_3^{18} + 2\left[x_2^{48} - 2x_2^6(x_2^{36}-1) - 82x_2^{12}(x_2^{24}-1) + 102x_2^{18}(x_2^{12}-1) - 1\right]x_3^{12} - \left(x_2^6+1\right)^4\left[3(x_2^{30}-1) - 23x_2^6(x_2^{18}-1) + 86x_2^{12}(x_2^6-1)\right]x_3^6 + \left(x_2^6-1\right)^3\left(x_2^6+1\right)^7\Big],$$

$$\sigma_1 = x_2^6 x_3^{10}\Big[\left[5\left(x_2^{12}+1\right)+6x_2^6\right]x_3^{30} - 8\left[2(x_2^{18}+1) + x_2^6(x_2^6+1)\right]x_3^{34} + 2\left[7(x_2^{24}+1) - 16x_2^6(x_2^{12}+1) - 30x_2^{12}\right]x_3^{18} + 4\left[(x_2^{30}+1) + 12x_2^6(x_2^{18}+1) + 3x_2^{12}(x_2^6+1)\right]x_3^{12} - \left(x_2^6-1\right)^2\left[11(x_2^{24}+1) + 20x_2^6(x_2^{12}+1) + 2x_2^{12}\right]x_3^6 + 4(x_2^6-1)^4\left(x_2^{18}+1\right)\Big],$$

$$\sigma_3 = ax_2^4 x_3^2\Big[4\left[17x_2^6\left(x_2^6+1\right) + 11(x_2^{18}+1)\right]x_3^{36} - \left[322x_2^{12} + 292x_2^6\left(x_2^{12}+1\right) + 139(x_2^{24}+1)\right]x_3^{30} - 4\left[\left(277x_2^{12}\left(x_2^6+1\right) + 16x_2^6\left(x_2^{18}+1\right) - 29(x_2^{30}+1)\right)\right]x_3^{24} + 2\left[72x_2^{18} + 281x_2^{12}\left(x_2^{12}+1\right) + 300x_2^6(x_2^{24}+1) + 23(x_2^{36}+1)\right]x_3^{18} - 4\left[60x_2^{18}\left(x_2^6+1\right) - 191x_2^{12}(x_2^{18}+1) + 41x_2^6\left(x_2^{30}+1\right) + 26\left(x_2^{42}+1\right)\right]x_3^{12} + (x_2^6-1)^2\left[84x_2^{18} - 117x_2^{12}(x_2^{12}+1) - 90x_2^6\left(x_2^{24}+1\right) + 37\left(x_2^{36}+1\right)\right]x_3^6 + 6x_2^6\left(x_2^6-1\right)^4(x_2^{18}+1)\Big],$$

$$\sigma_5 = 3a^2 x_2^2\Big[\left(x_2^6+1\right)^2\left[43(x_2^{12}+1) + 42x_2^6\right]x_3^{36} - 2\left[188x_2^{12}\left(x_2^6+1\right) + 257x_2^6\left(x_2^{18}+1\right) + 67(x_2^{30}+1)\right]x_3^{30} - 2\left[632x_2^{18} + 701x_2^{12}\left(x_2^{12}+1\right)\right.$$

$$
\begin{aligned}
&-4x_2^6\,(x_2^{24}+1)-53(x_2^{36}+1)]\,x_3^{24}+4\,[272x_2^{18}\,(x_2^6+1)+119x_2^{12}\,(x_2^{18}+1)\\
&+203x_2^6\,(x_2^{30}+1)+14(x_2^{42}+1)]\,x_3^{18}-[510x_2^{24}+20x_2^{18}\,(x_2^{12}+1)\\
&-876x_2^{12}\,(x_2^{24}+1)+236x_2^6(x_2^{36}+1)+109(x_2^{48}+1)]\,x_3^{12}+2\,(x_2^6-1)^2\\
&\times[81x_2^{18}\,(x_2^6+1)-9x_2^{12}(x_2^{18}+1)-59x_2^6\,(x_2^{30}+1)+19(x_2^{42}+1)]\,x_3^6\\
&-4x_2^6\,(x_2^{12}-1)^4\Big],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_7 = 9a^3x_3^4\Big[&2\,((x_2^6+1)^2+4x_2^6)\,[7\,(x_2^{12}+1)+6x_2^6(x_2^{12}+2)]\,x_3^{30}-[37\,(x_2^{36}+1)\\
&+x_2^{24}(208x_2^6+161)+x_2^{12}(16x_2^6-145)+6(32x_2^6+1)]\,x_3^{24}+4\,[7\,(x_2^{42}+1)\\
&+x_2^6(x_2^{30}-14)-2x_2^{24}(22x_2^6+141)-x_2^{12}(13x_2^6+47)+1]\,x_3^{18}+2\,[9(x_2^{48}+1)\\
&+2x_2^{36}(62x_2^6+83)+4x_2^{24}(128x_2^6-95)+2x_2^{12}(358x_2^6+1)+2(60x_2^6+1)]\,x_3^{12}\\
&-2\,(x_2^6+1)\,[16\,(x_2^{48}+1)+9x_2^{36}(4x_2^6+1)+x_2^{24}(x_2^{18}+11)-499x_2^{24}(x_2^6-2)\\
&-x_2^6(8x_2^{12}-23)-35x_2^{12}(11x_2^6-1)+3]\,x_3^6+(x_2^6-1)^2\,(x_2^6+1)^4\,[11\,(x_2^{24}+1)\\
&-2x_2^6\,(23x_2^{12}+21)+2\,(40x_2^{12}+1)]\Big].
\end{aligned}
$$

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