# Symbolic-Numeric Simulation of Satellite Dynamics with Aerodynamic Attitude Control System

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**Abstract.** The dynamics of the rotational motion of a satellite, subjected to the action of gravitational, aerodynamic and damping torques in a circular orbit is investigated. Our approach combines methods of symbolic study of the nonlinear algebraic system that determines equilibrium orientations of a satellite under the action of the external torques and numerical integration of the system of linear ordinary differential equations describing the dynamics of the satellite. An algorithm for the construction of a Gröbner basis was implemented for determining the equilibria of the satellite for specified values of the aerodynamic torque, damping coefficients, and principal central moments of inertia. Both the conditions of the satellite's equilibria existence and the conditions of asymptotic stability of these equilibria were obtained. The transition decay processes of the spatial oscillations of the satellite for various system parameters have also been studied.

## 1 Introduction

The study of the satellite dynamics under the influence of gravitational and aerodynamic torques is an important topic for practical implementation of attitude control systems of the artificial satellites. The gravity orientation systems are based on the result that a satellite with unequal moments of inertia in the central Newtonian force field in a circular orbit has stable equilibrium orientations [1–3]. An important property of the gravity orientation systems is that these systems can operate for a long time without fuel consumption. However, at altitudes from 250 up to 500 km, the rotational motion of a satellite is subjected to aerodynamic torque too. Therefore, it is necessary to study the joint action of gravitational and aerodynamic torques and, in particular, to analyze the possible satellite equilibria and conditions of stability of these equilibria in a circular orbit. The dynamics of a satellite subjected to gravitational and aerodynamic torques was considered in many papers indicated in [2]. The problem of determining the classes of equilibrium orientations for general values of aerodynamic


V. P. Gerdt et al. (Eds.): CASC 2018, LNCS 11077, pp. 214–229, 2018.
https://doi.org/10.1007/978-3-319-99639-4_15

torque was considered in [4–6]. In [7,8], some equilibrium orientations were found in special cases, when the center of pressure is located on a satellite's principal central axis of inertia and on a satellite's principal central plane of inertia. In [9], all equilibrium orientations were found in the case of axisymmetric satellite. In [10], all cases when the center of pressure is located in the satellite's principal central plane of inertia were considered using Computer Algebra methods. The basic problems of the satellite dynamics with an aerodynamic attitude control system have been presented in [2,6,11]. In [11], necessary and sufficient conditions for the stability of the aligned equilibrium position of the satellite with the aerodynamic orientation system using the damping moments of the gyroscopes were obtained.

In this paper, we consider a new problem, when the satellite is subjected to aerodynamic, gravitational, and active damping torques. The dynamics of the gravitationally-oriented satellite under the action of the damping torque, without taking into account the influence of the atmosphere on the motion of the satellite, was studied in detail in [12]. The main extension here, in comparison with [12], is the consideration of the additional influence of the atmosphere on the dynamics of the satellite under the action of the damping torque. Adding the action of the aerodynamic moment to the satellite leads to the appearance of new parameters in the equations of motion, which complicates their solution, but at the same time, it allows us to obtain new equilibrium solutions. In particular, the appearance of an additional aerodynamic parameter in the algebraic equations determining the stationary motions of the satellite seriously affects the runtime and memory requirements of symbolic computations for solving these equations.

We assume that the center of pressure of aerodynamic forces is located on one of the principal central axes of inertia of the satellite and the damping torque depends on the projections of the angular velocity of the satellite. This damping torque may be provided by using the angular velocity sensors. The action of damping torques can ensure the asymptotic stability of the equilibria of the satellite with aerodynamic attitude control system. The investigation of satellite equilibria was performed by using the Computer Algebra Gröbner basis methods. The regions with an equal number of equilibria were specified by using the Meiman theorem [19] for the construction of discriminant hypersurfaces. The conditions of equilibria stability are determined as a result of an analysis of the linearized equations of motion using the Routh–Hurwitz criterion [20]. The types of transition decay processes of spatial oscillations of the satellite at different aerodynamic and damping parameters have been investigated numerically.

The question of finding regions of parameter space with certain equilibria properties also occurred in relevance to a biology problem and was presented at the CASC 2017 Workshop [21].

## 2 Equations of Motion

Consider the attitude motion of the satellite subjected to gravitational, aerodynamic, and damping torques in a circular orbit. We assume that the satellite is a triaxial rigid body, and active damping torques depend on the projections of the angular velocity of the satellite.

To write the equations of motion we introduce two right-handed Cartesian coordinate systems with origin at the satellite's center of mass $O$. The orbital coordinate system is $OXYZ$, where the $OZ$ axis is directed along the radius vector from the Earth center of mass to the satellite center of mass; the $OX$ axis is in the direction of the satellite orbital motion. Then, the $OY$ axis is directed along the normal to the orbital plane. The satellite body coordinate system is $Oxyz$, where $Ox$, $Oy$, and $Oz$ are the principal central axes of inertia of the satellite. The orientation of the satellite body coordinate system $Oxyz$ with respect to the orbital coordinate system is determined by means of the aircraft angles of pitch ($\alpha$), yaw ($\beta$), and roll ($\gamma$) (Fig. 1), and the direction cosines in the transformation matrix between the orbital coordinate system $OXYZ$ and $Oxyz$ are expressed in terms of aircraft angles using the relations [2]:

$$\begin{aligned}
a_{11} &= \cos(x, X) = \cos\alpha\cos\beta, \\
a_{12} &= \cos(y, X) = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma, \\
a_{13} &= \cos(z, X) = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma, \\
a_{21} &= \cos(x, Y) = \sin\beta, \\
a_{22} &= \cos(y, Y) = \cos\beta\cos\gamma, \\
a_{23} &= \cos(z, Y) = -\cos\beta\sin\gamma, \\
a_{31} &= \cos(x, Z) = -\sin\alpha\cos\beta, \\
a_{32} &= \cos(y, Z) = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma, \\
a_{33} &= \cos(z, Z) = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma.
\end{aligned} \tag{1}$$

For small oscillations of the satellite, the angles of pitch, yaw, and roll correspond to the rotations around the $OY$, $OZ$, and $OX$ axes, respectively.

In the derivation of the equations of motion, we will make the following assumptions [2]:

(1) the atmospheric effect on the satellite is reduced to the drag force applied at the center of pressure and directed against the velocity of the satellite center of mass relative to the air; the pressure center is located on the axis $Ox$ of the satellite. This assumption is fulfilled accurately enough for the shape of the satellite close to the spherical;
(2) the atmospheric effect on the translational motion of the satellite is negligible;
(1) the atmospheric drag by the rotating Earth is neglected.

These assumptions make it possible to simplify the mathematical model of the effect of the atmosphere on the rotational motion of the satellite and neglect its influence on the parameters of the circular orbit.

Let the damping torque, in addition to the aerodynamic torque, act on the satellite. Their integral vector projections on the axis $Ox$, $Oy$, and $Oz$ are equal to the following values: $M_x = \bar{k}_1 p_1$, $M_y = \bar{k}_2 q_1$, and $M_z = \bar{k}_3 r_1$. Here $\bar{k}_1$, $\bar{k}_2$, and

$\bar{k}_3$ are the damping coefficients; $p_1$, $q_1$, and $r_1$ are the projections of the satellite angular velocity vector onto the axes $Ox$, $Oy$, and $Oz$; $\omega_0$ is the angular velocity of the orbital motion of the satellite's center of mass. Then the equations of satellite attitude motion can be written in the Euler form:

$$\begin{aligned}
&Ap_1' + (C-B)q_1r_1 - 3\omega_0^2(C-B)a_{32}a_{33} + \bar{k}_1p_1 = 0,\\
&Bq_1' + (A-C)r_1p_1 - 3\omega_0^2(A-C)a_{31}a_{33} + \omega_0^2H_1a_{13} + \bar{k}_2q_1 = 0,\\
&Cr_1' + (B-A)p_1q_1 - 3\omega_0^2(B-A)a_{31}a_{32} - \omega_0^2H_1a_{12} + \bar{k}_3r_1 = 0,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
p_1 &= (\alpha' + \omega_0)a_{21} + \gamma',\\
q_1 &= (\alpha' + \omega_0)a_{22} + \beta'\sin\gamma,\\
r_1 &= (\alpha' + \omega_0)a_{23} + \beta'\cos\gamma.
\end{aligned} \tag{3}$$

Moreover, here $A$, $B$, and $C$ are the principal central moments of inertia of the satellite. And $H_1 = -Qa/\omega_0^2$, $Q$ is the drag force acting on the satellite, and $(a, 0, 0)$ are the coordinates of the satellite center of pressure in the reference frame $Oxyz$. For the aerodynamically stable construction of the satellite, the center of pressure lies behind its center of gravity and, therefore, $a < 0$. The prime denotes the differentiation with respect to time $t$.

Over the systems (2) and (3) applying the change of variables $(p, q, r) = (p_1/\omega_0, q_1/\omega_0, r_1/\omega_0)$ and after this introducing dimensionless parameters $\theta_A = A/B$, $\theta_C = C/B$, $\tilde{k}_1 = \bar{k}_1/B\omega_0$, $\tilde{k}_2 = \bar{k}_2/B\omega_0$, $\tilde{k}_3 = \bar{k}_3/B\omega_0$, $h_1 = H_1/B$, and $\tau = \omega_0 t$, we can rewrite (2) and (3), and finally put respectively (because it is transforming (2) and (3))

$$\begin{aligned}
\theta_A\dot{p} + (\theta_C - 1)qr - 3(\theta_C - 1)a_{32}a_{33} + \tilde{k}_1p = 0,\\
\dot{q} + (\theta_A - \theta_C)rp - 3(\theta_A - \theta_C)a_{31}a_{33} + h_1a_{13} + \tilde{k}_2q = 0,\\
\theta_C\dot{r} + (1 - \theta_A)pq - 3(1 - \theta_A)a_{31}a_{32} - h_1a_{12} + \tilde{k}_3r = 0,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
p &= (\dot{\alpha} + 1)a_{21} + \dot{\gamma},\\
q &= (\dot{\alpha} + 1)a_{22} + \dot{\beta}\sin\gamma,\\
r &= (\dot{\alpha} + 1)a_{23} + \dot{\beta}\cos\gamma.
\end{aligned} \tag{5}$$

The dot denotes the differentiation with respect to $\tau$.

## 3 Equilibrium Orientations of Satellite

Assuming the initial condition $(\alpha, \beta, \gamma) = (\alpha_0 = \text{const}, \beta_0 = \text{const}, \gamma_0 = \text{const})$ and also $A \neq B \neq C$ $(\theta_A \neq \theta_C \neq 1)$, we obtain from (4) and (5) the equations

$$\begin{aligned} a_{22}a_{23} - 3a_{32}a_{33} + ka_{21} &= 0, \\ (1-\nu)(a_{21}a_{23} - 3a_{31}a_{33}) + h(a_{21}a_{32} - a_{22}a_{31}) + ka_{22} &= 0, \\ \nu(a_{21}a_{22} - 3a_{31}a_{32}) - h(a_{23}a_{31} - a_{21}a_{33}) + ka_{23} &= 0, \end{aligned} \quad (6)$$

which allow us to determine the satellite equilibria in the orbital coordinate system. Here we consider the special case when $\tilde{k}_1/(\theta_C - 1) = \tilde{k}_2/(1 - \theta_C) = \tilde{k}_3/(1 - \theta_C) = k$. This reduction in the number of parameters makes it possible to simplify the system of equations and solve the problem. In (6), $h = h_1/(1-\theta_C)$ and $\nu = (1 - \theta_A)/(1 - \theta_C)$.

Substituting the expressions for the direction cosines from (1) in terms of the aircraft angles into Eq. (6), we obtain three equations with three unknowns $\alpha$, $\beta$, and $\gamma$. Another way of closing Eq. (6) is to add the following three conditions for the orthogonality of direction cosines:

$$\begin{aligned} a_{21}^2 + a_{22}^2 + a_{23}^2 - 1 &= 0, \\ a_{31}^2 + a_{32}^2 + a_{33}^2 - 1 &= 0, \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0. \end{aligned} \quad (7)$$

Equations (6) and (7) form a closed system of equations with respect to the six direction cosines identifying the satellite equilibrium orientations. For this system of equations, we formulate the following problem: for given values of $h$, $k$, and $\nu$, it is required to determine all the nine directional cosines, i.e., all satellite equilibrium orientations in the orbital coordinate system. After $a_{21}$, $a_{22}$, $a_{23}$, $a_{31}$, $a_{32}$, and $a_{33}$ are found, the direction cosines $a_{11}$, $a_{12}$, and $a_{13}$ can be determined from the conditions of orthogonality.

To find solutions of the algebraic system (6), (7) we used the algorithm for constructing the Gröbner bases [13]. The method for constructing a Gröbner basis is an algorithmic procedure for complete reduction of the problem involving systems of polynomials in many variables to consideration of a polynomial in one variable.

In our study, for Gröbner bases construction, we applied the command `Groebner[Basis]` from the package `Groebner` implemented in the computer algebra system Maple 15 [14]. We constructed the Gröbner basis of the system of six second-order polynomials (6), (7) with six variables $a_{ij}$ $(i = 2, 3; j = 1, 2, 3)$, with respect to the lexicographic ordering of variables by using option `plex`. In the list of polynomials F:=$[f_i, i = 1, 2, 3, 4, 5, 6]$, $f_i$ are the left–hand sides of the algebraic equations (6), (7). Thus, the Maple command used was as follows:

```
G:=map(factor,Groebner[Basis](F,plex(a31,a32,a33,a21,a22,a23)));
```

Here, calculating the Gröbner basis over the field of rational functions in $h$, $k$, and $\nu$ we compute the generic solutions of our problem only. In our task from the area of the satellite dynamics with aerodynamic attitude control system, the main goal of the study is to estimate a range of system parameters for which the satellite's equilibria exist.

It should be taking into account that in practice, it is difficult to ensure a constant value of the aerodynamic moment on the orbit and there are errors of the angular velocity sensors and the errors of the signals, which generate damping torques, the exact bifurcation values of the coefficients are very difficult to obtain. We are interested in estimating the size of regions in the space of system parameters where equilibria exist. In the case of parametric dynamical system solving, when the parameters reach non-generic solutions, the symbolic application based on comprehensive Gröbner bases [15], discriminant varieties [16], and comprehensive triangular decomposition [17,18] methods are used. In our task, we did not use these methods because we did not consider the cases of bifurcation values of the parameters, and for our problem, these methods are rather complicated.

Here we write down the polynomial in the Gröbner basis that depends only on one variable $x = a_{23}$. This polynomial has the form

$$P(x) = P_1(x)P_2(x) = 0, \tag{8}$$

where

$$\begin{aligned}
P_1(x) &= x(x^2 - 1),\\
P_2(x) &= p_0x^4 + p_1x^2 + p_2 = 0,\\
p_0 &= \big(16(k^2 + (1-\nu)^2)(k^2+\nu^2)h^4\\
&\quad - 24(k^2 + \nu(1-\nu))\big(k^2 - 2\nu(1-\nu)\big)^2 h^2\\
&\quad + 9(k^2 - 2\nu(1-\nu))^4\big)^2,\\
p_1 &= -h^2\big(64(k^2+4\nu^2)(k^2+(1-\nu)^2)^2h^8\\
&\quad + 16\big((2+8\nu)k^8 + (72\nu^3 - 50\nu^2 + 8\nu + 7)k^6\\
&\quad - 4(1-\nu)(48\nu^4 - 58\nu^3 + 20\nu^2 - 8\nu + 1)k^4\\
&\quad + 4\nu(1-\nu)^2(32\nu^4 - 104\nu^3 + 100\nu^2 - 25\nu + 6)k^2\\
&\quad + 192\nu^3((1-\nu)^5\big)h^6 + 12(k^2 - 2\nu(1-\nu))^2\big((40\nu - 21)k^6\\
&\quad + 4(32\nu^3 - 28\nu^2 + 5\nu + 6)k^4\\
&\quad + 4(1-\nu)(56\nu^4 - 78\nu^3 + 24\nu^2 + 13\nu + 3)k^2\\
&\quad + 288\nu^2(1-\nu)^4\big)h^4\\
&\quad - 36(k^2 - 2\nu(1-\nu)^4\big(2(8\nu - 5)k^4 + (16\nu^3 - 24\nu + 17)\\
&\quad + 48\nu(1-\nu)^3\big)h^2 + 27\big(k^2 - 2\nu(1-\nu)\big)^6((8\nu-5)k^2\\
&\quad + 12(1-\nu)^2)\big),\\
p_2 &= p_{21}p_{22},\\
p_{21} &= -h^4k^2(k^2 + 4\nu^2 - 6\nu)^2\\
p_{22} &= 4(k^2+4\nu^2)h^6 - 4(4k^4 + (14\nu^2 - 2\nu + 1)k^2\\
&\quad + 4\nu^2(1 + 4\nu - 5\nu^2)h^4\\
&\quad + 3\big(k^2 - 2\nu(1-\nu)\big)^2(7k^2 + 8\nu + 4\nu^2)h^2 - 9\big(k^2 - 2\nu(1-\nu)\big)^4.
\end{aligned}$$

The left-hand side of (8) becomes zero under the conditions $P_1(x) = 0$, $P_2(x) = 0$. Whence follows that in order to determine the equilibria it is required to consider separately the three cases: the first $a_{23}^2 = 1$, the second $a_{23} = 0$, and the third $P_2(a_{23}) = 0$. It should also be taken into account that equilibrium solutions are determined only by such real roots (8) whose absolute values should be less than or equal to 1.

In the first case, when $a_{23} = \pm 1$, $(a_{21} = a_{22} = 0)$, system (6), (7) takes the form

$$\begin{aligned} -3\nu a_{31}a_{32} - ha_{31}a_{23} + ka_{23} &= 0,\\ a_{31}^2 + a_{32}^2 &= 1,\\ a_{23}^2 &= 1,\\ a_{33} = a_{21} = a_{22} &= 0. \end{aligned} \tag{9}$$

The first two equations of system (9) can be written in a simpler form

$$P_3(a_{32}) = 9\nu^2 a_{32}^4 \pm 6\nu h a_{32}^3 + (h^2 - 9\nu^2)a_{32}^2 \mp 6\nu h a_{32} + k^2 - h^2 = 0, \quad (10)$$
$$a_{31} = \pm\frac{k}{(3\nu a_{32} \pm h)}.$$

Having solved system (10), one can determine all six direction cosines of system (9). The number of real roots of equations (10) does not exceed 8. It is possible to show that each real root $a_{32}$ of equations (10) corresponds to one equilibrium solution of the original system (6), (7).

In studying the satellite equilibrium orientations in the first case, we determine the conditions for the existence of real roots of equations (10). To identify these conditions, we use the Meiman theorem [19], which yields that the decomposition of the space of parameters into domains with equal number of real roots is determined by the discriminant hypersurface.

In our case, the discriminant hypersurface is given by the discriminant of polynomial $P_3(a_{32})$. This hypersurface contains a component of codimension 1, which is the boundary of domains with equal number of real roots. The set of singular points of the discriminant hypersurface in the space of parameters $k$, $h$, and $\nu$ is given by the following system of algebraic equations:

$$P_3(y) = 0, \quad P_3'(y) = 0. \tag{11}$$

Here $y = a_{32}$, and the prime denotes the differentiation with respect to $y$.

We eliminate the variable $y$ from system (11) by calculating the determinant of the resultant matrix of Eq. (11) and obtain an algebraic equation of the discriminant hypersurface as

$$P_4(k, h, \nu) = h^6 - (k^2 + 27\nu^2)h^4 + 9\nu^2(20k^2 + 27\nu^2)h^2 - 9\nu^2(4k^2 - 9\nu^2)^2 = 0. \tag{12}$$

Now we should check the change in the number of equilibria when the surface (12) is intersected. This can be done numerically by determining the number of equilibria at a point of each domain $P_4(k, h, \nu) = 0$ in the space of parameters $k$, $h$, and $\nu$.

Figure 2 presents an example of the properties and form of the discriminant hypersurface $P_4(k, h, \nu) = 0$, which are two-dimensional cross sections of the surface in the plane $(k, h)$ at the fixed value of parameter $\nu = 0.5$. Figure 2 shows the distributions of domains with equal number of real roots of Eq. (10) and indicates the domains where four and two real solutions exist as well as the domains where no real solutions exist (marked by 0).

In the second case, when $a_{23} = 0$, system (6), (7) takes the form

$$\begin{aligned}
ka_{21} - 3a_{32}a_{33} &= 0,\\
ka_{22} - 3(1-\nu)a_{31}a_{33} + h(a_{21}a_{32} - a_{22}a_{31}) &= 0,\\
\nu(a_{21}a_{22} - 3a_{31}a_{32}) + ha_{21}a_{33} &= 0,\\
a_{21}^2 + a_{22}^2 = 1, \quad a_{21}a_{31} + a_{22}a_{32} &= 0,\\
a_{31}^2 + a_{32}^2 + a_{33}^2 - 1 &= 0.
\end{aligned} \tag{13}$$

From (13) we can obtain the following solutions:

$$\begin{aligned}
a_{21} = a_{23} = a_{32} = 0, \quad a_{22}^2 &= 1,\\
P_5(a_{33}) = 9(1-\nu)^2 a_{33}^4 &\pm 6(1-\nu)ha_{33}^3\\
+(h^2 - 9(1-\nu)^2)a_{33}^2 \mp 6(1-\nu)ha_{33} + k^2 - h^2 &= 0,\\
a_{31} = \pm\frac{k}{3(1-\nu)a_{33} \pm h}&.
\end{aligned} \tag{14}$$

Note that if in the expressions for the coefficients $P_5$ from (14) the term $(1-\nu)$ is replaced by $\nu$, we obtain the form of the coefficients of the polynomial $P_3$ from (10). Therefore, the conditions for the existence of real roots of Eq. (14) will be determined by the discriminant (12), in which the term $\nu$ is replaced by $(1-\nu)$. For example, for the value $\nu = 0.5$, the conditions for the existence of real roots of Eqs. (10) and (14) will be the same (see Fig. 2).

Now let us consider the third case for which the satellite equilibria are determined by the real roots of the biquadratic equation $P_2(a_{23}) = 0$ from (8). The number of real roots of the biquadratic equation $P_2(a_{23}) = 0$ is even and not greater than 4. For each solution, one can find from the second polynomial from the constructed Gröbner basis two values of $a_{22}$ and, then, their respective values $a_{21}$. For each set of values $a_{21}$, $a_{22}$, and $a_{23}$, one can unambiguously define from original system (6), (7) the respective values of the direction cosines $a_{31}$, $a_{32}$, and $a_{33}$. Thus, each real root of the biquadratic Eq. (6) is matched with two sets of values $a_{ij}$ (two equilibrium orientations). Since the number of real roots of biquadratic Eq. (6) does not exceed 4, the satellite at the third case can have no more than 8 equilibrium orientations.

Real solutions of the biquadratic equation from (8) exist in the case when the discriminant

$$D(k, h, \nu) = p_1^2 - 4p_0p_2 \tag{15}$$

is non-negative. Using symbolic computations, it is possible to factorize the discriminant (15) in rather simple form

$$D(k, h, \nu) = h^4 D_1(k, h, \nu)\big(D_2(k, h, \nu)\big)^2, \tag{16}$$

where

$$\begin{aligned}D_1(k,h,\nu) &= 4h^4 + 4\big(k^4 - (1+4\nu(1-\nu))k^2 - 6\nu(1-\nu)\big]h^2 \\ &\quad - (4k^2-9)[k^2 - 2\nu(1-\nu)\big)^2, \\ D_2(k,h,\nu) &= 27\big(k^2 - 4(1-\nu)^2\big)\big(k^2 - 2\nu(1-\nu)\big)^5 \\ &\quad - 32\big((k^2+(1-\nu)^2\big)^2(k^2+4\nu^2)h^6 \\ &\quad + 24\big(4k^8 + (22\nu^2 - 12\nu - 1)k^6 \\ &\quad + 2(1-\nu)^2(1+4\nu+\nu^2)k^4 \\ &\quad - 4\nu(1-\nu)^2(6\nu - 21\nu^2 + 19\nu^3 - 1)k^2 \\ &\quad + 48\nu^3(1-\nu)^5\big)h^4 \\ &\quad - 18\big(k^2 - 2\nu(1-\nu)\big)^3\big(5k^4 + 2(\nu^2+7\nu-5)k^2 \\ &\quad - 24\nu(1-\nu)^3\big)h^2.\end{aligned}$$

For the existence of real roots of biquadratic equation from (8), it is necessary to satisfy the inequality $D(k,h,\nu) \geq 0$ ($D_1(k,h,\nu) \geq 0$). In case of the $D_1(k,h,\nu) > 0$ ($D_2(k,h,\nu) \neq 0$) and $0 \leq a_{23}^2 \leq 1$ inequalities fulfillment, biquadratic Eq. (8) has four real roots $a_{23}$. The boundary of the regions of the necessary conditions for the existence of these solutions is the curve $D_1(k,h,\nu) = 0$.

The regions of the necessary conditions for the existence of the real solutions of biquadratic Eq. (8) on the plane $(k,h)$ are presented in Figs. 3 and 4 for $\nu = 0.2$ and $\nu = 0.5$. For the values $\nu$ and $(1-\nu)$ these regions coincide.

Thus, from Eq. (8), we can obtain all possible values of the direction cosine $a_{23}$ and corresponding values $a_{21}$, $a_{22}$, $a_{31}$, $a_{32}$, and $a_{33}$ satisfying the initial system (6), (7). Once the set of six values $a_{21}$, $a_{22}$, $a_{23}$, $a_{31}$, $a_{32}$, and $a_{33}$ is found, the remaining three values $a_{11}$, $a_{12}$, and $a_{13}$ can be uniquely determined from the conditions of the orthogonality of the directional cosines. So we can determine all the equilibrium orientations of the satellite under the influence of aerodynamic, gravitational, and damping torques.

## 4 Necessary and Sufficient Conditions of Asymptotic Stability Of the Equilibrium Orientations of Satellite

In order to study the necessary and sufficient conditions of asymptotic stability of the equilibrium orientations of System (6) and (7), let us linearize the system of differential Eqs. (4) and (5) in the vicinity of the specific equilibrium solution, from the case 2 ($a_{22}^2 = 1, a_{21} = a_{23} = 0$):

$$\alpha = \alpha_0, \quad \beta_0 = \gamma_0 = 0. \tag{17}$$

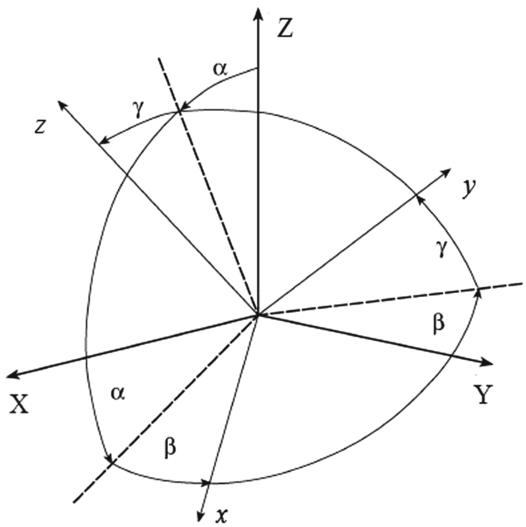


**Fig. 1.** Orientation of body–fixed axes with respect to the orbital coordinate system

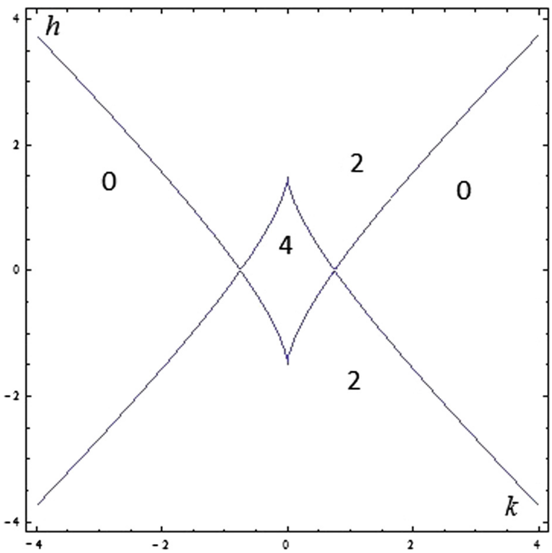


**Fig. 2.** The regions with the fixed number of equilibria for $\nu = 0.5$ for the cases 1, 2

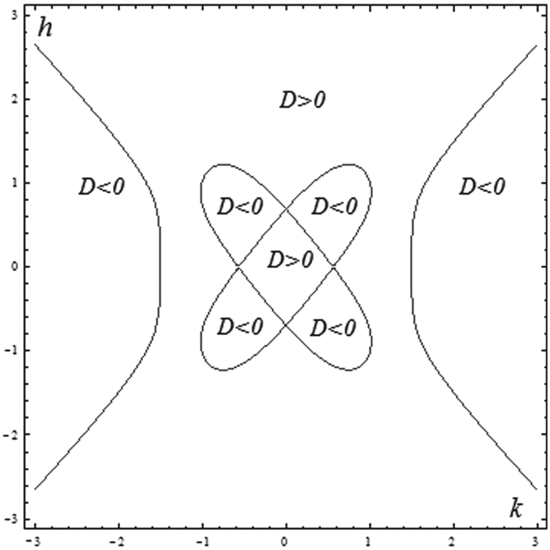


**Fig. 3.** The regions where the necessary conditions for the existence of equilibria are satisfied for $\nu = 0.2$ in case 3

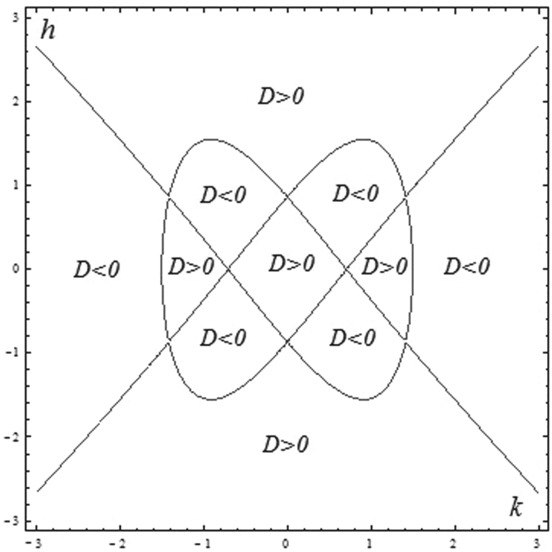


**Fig. 4.** The regions where the necessary conditions for the existence of equilibria are satisfied for $\nu = 0.5$ in case 3

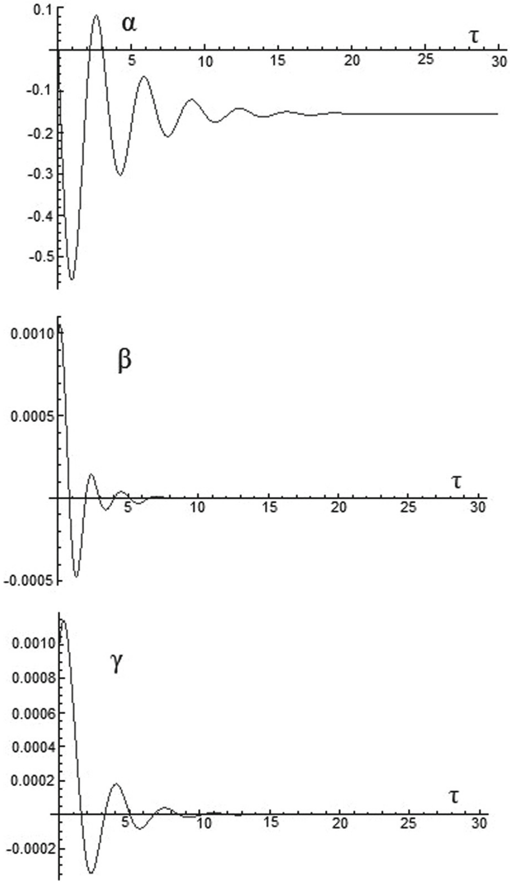


**Fig. 5.** The transitional process of damping oscillations for $k = 1.0; h = 5.0$

Here $\alpha_0 = \arccos(a_{33})$, where $a_{33}$ is a real root of algebraic Eq. (14). We represent $\alpha$, $\beta$, and $\gamma$ in the form $\alpha = \alpha_0 + \bar{\alpha}$, $\beta = \beta_0 + \bar{\beta}$, $\gamma = \gamma_0 + \bar{\gamma}$, where $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are small deviations from the equilibrium orientation (17) of the satellite. The linearized system of equations of motion takes the following form:

$$\begin{aligned}
&\ddot{\bar{\alpha}} + (1-\theta_C)k\dot{\bar{\alpha}} + \big((1-\theta_C)h\cos\alpha_0 + 3(\theta_A-\theta_C)\cos 2\alpha_0\big)\bar{\alpha} = 0,\\
&\theta_C\ddot{\bar{\beta}} + (1-\theta_C)k\dot{\bar{\beta}} - (\theta_A+\theta_C-1)\dot{\bar{\gamma}} + \big((1-\theta_C)h\cos\alpha_0\\
&\quad + (1-\theta_A)(1+3\sin^2\alpha_0)\big)\bar{\beta} + \big(1.5(1-\theta_A)\sin 2\alpha_0\\
&\quad - (1-\theta_C)((1-\theta_A)k + h\sin\alpha_0)\big)\bar{\gamma} = 0, \qquad (18)\\
&\theta_A\ddot{\bar{\gamma}} - (1-\theta_C)k\dot{\bar{\gamma}} + (\theta_A+\theta_C-1)\dot{\bar{\beta}} + (1-\theta_C)(1+3\cos^2\alpha_0)\bar{\gamma}\\
&\quad + (1-\theta_C)(1.5\sin 2\alpha_0 - k)\bar{\beta} = 0.
\end{aligned}$$

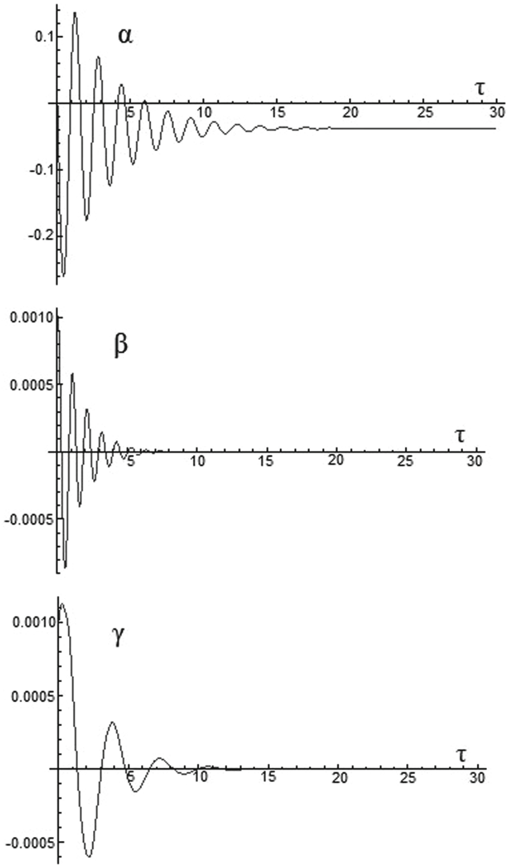


**Fig. 6.** The transitional process of damping oscillations for $k = 1.0; h = 25.0$

The characteristic equation of system (18)

$$(\lambda^2 + A_{01}\lambda + A_{02})(A_0\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4) = 0 \qquad (19)$$

decomposes into quadratic and 4th degree equations. Here the following notations are introduced:

$$\begin{aligned}
A_{01} &= (1-\theta_C)k, \quad A_{02} = (1-\theta_C)h\cos\alpha_0 + 3(\theta_A - \theta_C)\cos 2\alpha_0,\\
A_0 &= \theta_A\theta_C, \quad A_1 = (1-\theta_C)(\theta_A - \theta_C)k,\\
A_2 &= (\theta_A + \theta_C - 1)^2 - (1-\theta_C)^2k^2 + (1-\theta_C)(\theta_A h + \theta_C(1+3\cos^2\alpha_0))\\
&\quad + \theta_A(1 - \theta_A(1+3\sin^2\alpha_0),\\
A_3 &= k(1-\theta_C)\big((1-\theta_C)(1+3\cos^2\alpha_0 - h\cos\alpha_0) - (1-\theta_A)(1+3\sin^2\alpha_0)\big)
\end{aligned}$$

$$+ (\theta_A + \theta_C - 1)\big((1-\theta_C)(h\mathrm{sin}\alpha_0 + 1.5\mathrm{sin}2\alpha_0) - 1.5(1-\theta_A)\mathrm{sin}2\alpha_0],$$
$$A_4 = (1-\theta_C)(1+3\mathrm{cos}^2\alpha_0)\big((1-\theta_C)h\mathrm{cos}\alpha_0 + (1-\theta_A)(1+3\mathrm{sin}^2\alpha_0)\big)$$
$$+ (\theta_A + \theta_C - 1)\big((1-\theta_C)(k + h\mathrm{sin}\alpha_0 - 1.5(1-\theta_A)\mathrm{sin}2\alpha_0\big).$$

The necessary and sufficient conditions for asymptotic stability (Routh–Hurwitz criterion) of the equilibrium solution (17) take the following form:

$$\begin{aligned} &(1-\theta_C)k > 0, \quad (1-\theta_C)h\cos\alpha_0 + 3(\theta_A - \theta_C)\cos 2\alpha_0 > 0,\\ &\varDelta_1 = A_1 > 0,\\ &\varDelta_2 = A_1A_2 - A_0A_3 > 0,\\ &\varDelta_3 = A_1A_2A_3 - A_0A_3^2 - A_1^2A_4 > 0,\\ &\varDelta_4 = \varDelta_3A_4 > 0. \end{aligned} \tag{20}$$

The detailed analysis of the fulfillment of inequalities (20), under which necessary and sufficient conditions for stability are satisfied was performed numerically for fixed values of the parameters $\theta_A$, $\theta_C$, $k$, and $h$. One should take into account also the following triangle inequalities for the real bodies, which parameters ($\theta_A$ and $\theta_C$) should fulfill: $\theta_A + \theta_C > 1$, $\theta_C + 1 > \theta_A$, $\theta_A + 1 > \theta_C$. The triangle conditions isolate the infinite half-band in the $(\theta_A, \theta_C)$ plane.

The numerical integration of system (4) and (5) was carried out for the fixed values of the parameters $\theta_A$, $\theta_C$, $k$, and $h$ where the conditions of asymptotic stability (20) and the triangle inequalities hold. The different types of transition decay processes of spatial oscillations of the satellite at different inertial, aerodynamic, and damping parameters are presented in Figs. 5 and 6. The initial values of variables in the calculations were taken to be equal to 0.001.

Figure 5 shows that for rather small values of the damping coefficient and for small values of the aerodynamic torque ($k = 1, h = 5$; $\theta_A = 0.7, \theta_C = 0.4$), the system reaches the equilibrium solution (18) for $\alpha$ angle, when the $\tau$ value exceeds 15, and for $\beta$ and $\gamma$ angles, when the $\tau$ values are equal to about 10. Here equilibrium value $\alpha_0 = \arccos(a_{33}) = -0.155$ and $a_{33} = 0.988$ is the real root of algebraic Eq. (14).

When the value of the aerodynamic torque $h$ increases the satellite oscillation frequency increases in angles $\alpha$ and $\beta$ and the time of the transient process for $h = 25$, $k = 1.0$, ($\theta_A = 0.7, \theta_C = 0.4$) (Fig. 6) is close to 15 for $\alpha$ angle and less than 10 for $\beta$ and $\gamma$ angles. In Fig. 6, $\alpha_0 = -0.0377$. The value of the $\alpha$ angle approaches zero when the aerodynamic moment significantly increases.

In the case of the satellite with an aerogyroscopic stabilization system, when studying the dynamics of this system in [11] it was also shown that the satellite oscillation frequency increased in angles $\alpha$ and $\beta$ when the magnitude of aerodynamic moment increased.

When the value of the damping coefficient increases, the time of the transient process of the system to the equilibrium solution decreases, for example when $k = 1.5$, $h = 25$ ($\theta_A = 0.7, \theta_C = 0.4$), the time of the transient process is less than 10 for all three angles. For $k = 2.0$, $h = 25$ ($\theta_A = 0.7, \theta_C = 0.4$), the transition time becomes less than 7 also for all three angles, which corresponds to one satellite turnover in the orbit.

## 5 Conclusion

In this paper, we present the study of the dynamics of the rotational motion of the satellite subject to the gravitational, aerodynamic, and active damping torques, which depend on the projections of satellite angular velocity.

The computer algebra method (based on the construction of Gröbner basis) of determining all equilibrium orientations of the satellite in the orbital coordinate system with given values of aerodynamic torque, damping coefficients and principal central moments of inertia was presented. The conditions for existence of these equilibria were obtained. We have made a detailed analysis of the evolution of domains of existence of equilibrium orientations in the plane of system parameters $h$ and $k$ for the fixed values of parameter $\nu$.

For the special equilibrium orientation, when two axes of the satellite-centered coordinate system coincide with two axes of the orbital coordinate system, the necessary and sufficient conditions for asymptotic stability are obtained.

The numerical study of the character of transient processes of system, entering the special equilibrium orientation, has been carried out for various values of aerodynamic and damping parameters. It has been shown that there is a wide range of values of aerodynamic and damping parameters from which, choosing the required values of parameters, one can provide the asymptotic stability of the equilibrium orientation. The obtained results can be used to design aerodynamic attitude control systems for the artificial Earth satellites.

**Acknowledgments.** The authors thank the reviewers for very useful remarks and suggestions.

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