



# Finite-State Classical Mechanics

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**Abstract.** Reversible lattice dynamics embody basic features of physics that govern the time evolution of classical information. They have finite resolution in space and time, don't allow information to be erased, and easily accommodate other structural properties of microscopic physics, such as finite distinct state and locality of interaction. In an ideal quantum realization of a reversible lattice dynamics, finite classical rates of state-change at lattice sites determine average energies and momenta. This is very different than traditional continuous models of classical dynamics, where the number of distinct states is infinite, the rate of change between distinct states is infinite, and energies and momenta are not tied to rates of distinct state change. Here we discuss a family of classical mechanical models that have the informational and energetic realism of reversible lattice dynamics, while retaining the continuity and mathematical framework of classical mechanics. These models may help to clarify the informational foundations of mechanics.

## 1 Introduction

The physics of continuous classical materials and fields is pathological. For example, in classical statistical mechanics, each degree of freedom of a system at thermal equilibrium has about the same finite average energy, proportional to the temperature. This implies that a continuous material—which has an infinite number of degrees of freedom—will be able to absorb energy at a finite rate forever without heating up. Exactly this pathology, evident in the radiation field inside hot cavities (black bodies), led to the overturn of classical mechanics as a fundamental theory and the advent of quantum mechanics [1].

A similar pathology of the continuum exists in the classical mechanics of particles. Quantum mechanics provides a fundamental definition of energy in terms of rate of change in time: *frequency*. What we call energy in the classical realm is (in fundamental units) just the average frequency of a large quantum system—and this is precisely the maximum rate at which the system can transition between perfectly distinct states [2–4]. Since all physical systems have finite energy, they all have a finite rate of distinct state change. But in continuous classical mechanics, each infinitesimal interval of time brings a perfectly distinct new state, and so the rate of state change is infinite. Similarly, finite momentum only allows a finite rate of distinct state change due to motion, not the infinite rate required by infinite resolution in space.

It would be nice to have a version of classical dynamics that avoids the pathologies of infinite state and infinite resolution in space and time, while still being a subset of ordinary classical mechanics. Fredkin’s billiard ball model of computation [5] illustrates that this is possible: a carefully designed classical mechanical system with discrete constraints on initial conditions can be equivalent, at discrete times, to a reversible finite-state dynamics. In this paper, we discuss Fredkin’s model as well as others where the equivalence is even more direct. In these examples, local rates of state change in the finite-state dynamics play the roles of mechanical energies and momenta. This property is physically realistic, and arises from the fact that ordinary reversible computations, such as these, can be interpreted as special cases of quantum computations [6], and hence inherit quantum definitions of energy and momentum based on rates of state change.

## 2 Energy and Momentum of Reversible Lattice Gases

Some of the simplest models of physical systems are lattice models with classical finite state. Dynamical models of this sort with local interactions are often referred to as *cellular automata*, but here we will favor the more physical term *classical lattice gas*, which encompasses both deterministic and stochastic physical models [7–13]. We discuss reversible classical lattice gases as foundational models of both classical and quantum mechanics.

By foundational models we mean here the simplest examples of systems that exactly incorporate basic physical properties and principles (*cf.* [14]). That a world with finite entropy has classical lattice gas foundations is not surprising, and is well accepted in statistical mechanics [7, Sect. 2.4], where finite-state lattice models have provided great insight into the foundations of the field. The idea that classical lattice gases are foundational is, however, much less accepted in ordinary mechanics, where the close relationship of reversible finite-state lattice gases to continuous-time and continuous-space mechanics—and the fundamental link that energy and momentum provide—are not widely appreciated.

### 2.1 Continuous Space and Time

The fundamental models of mechanics are continuous in space and time, and hence seem very non-discrete. In fact, though, all realistic physical systems have finite resolution in space and time, usually described using continuous mathematics, and similar descriptions can be applied to discrete models.

The finite resolution in space and time of quantum systems can be expressed in terms of uncertainty relations [15], but is better thought of as akin to the effective discreteness of a finite-bandwidth classical signal [3, 4]. Interestingly, this kind of discreteness was recognized around the same time that the founders of quantum mechanics discovered uncertainty [16]. The discoverer, Harry Nyquist, was thinking about how many dots and dashes could be put into a telegraph signal, and he realized that bandwidth was the key quantity that set the bound.

He gave a simple argument, first considering a signal periodic in time, and then generalizing to the average rate for an infinitely long period. Consider a complex valued periodic wave. This is composed of a discrete set of Fourier components that fit the period: for period  $T$  the possible frequencies are  $1/T$ ,  $2/T$ ,  $3/T$ , etc. With a limited range of frequencies (a limited *bandwidth*), the Fourier sum describing the wave has only a finite number of terms with a finite number (say  $N$ ) coefficients. With  $N$  coefficients we can only freely choose the value of the sum at  $N$  times. Thus the minimum range of frequencies  $\nu_{\max} - \nu_{\min}$  needed to have  $N$  distinct values of the sum is given by the minimum separation between frequencies  $1/T$ , times the minimum number of separations  $N - 1$ :

$$\nu_{\max} - \nu_{\min} \geq \frac{N - 1}{T}. \quad (1)$$

For a long wave, the bandwidth  $\nu_{\max} - \nu_{\min}$  is the maximum average density of distinct values,  $N/T$ . Turning the argument around, if we know the values of a periodic signal with finite bandwidth at enough discrete points, we can determine all coefficients in the finite Fourier sum: the rest of the continuous wave is then determined and carries no additional information [17]. Thus waves with finite bandwidth are effectively discrete.

## 2.2 Finite Resolution in Quantum Mechanics

The argument above goes through essentially unchanged for the wavelike evolution of quantum systems. The wavefunction for an isolated system is expressed in the energy basis as a superposition of frequency components:  $\nu_n = E_n/h$ , where  $E_n$  is the  $n$ -th energy eigenvalue, and  $h$  is Planck's constant. To have  $N$  distinct (mutually orthogonal) states in a periodic time evolution, there must be a superposition of at least  $N$  distinct energy eigenfunctions, with distinct frequencies. Again the minimum frequency separation is  $1/T$  if the period is  $T$ , so the minimum range of energy eigenfrequencies is again given by (1).

For systems that exactly achieve this bound on orthogonal evolution, it is easy to show that the  $N$  equally spaced frequency components must be equally weighted, and that the same minimum-bandwidth distribution minimizes all reasonable measures of frequency width [3,4]. For example, for the minimizing distribution, average frequency  $\bar{\nu}$  minus the lowest  $\nu_{\min}$  is half the bandwidth, so

$$2(\bar{\nu} - \nu_{\min}) \geq \frac{N - 1}{T}. \quad (2)$$

In quantum mechanics, the average energy of an isolated system is  $E = h\bar{\nu}$ . The lowest (ground state) energy  $E_0 = h\nu_{\min}$  is like the lowest frequency used in a classical signal: it is the start of the frequency range available for the dynamics of the isolated system. If the ground state energy  $E_0$  is taken to be zero and  $N \gg 1$ , then letting  $\nu_{\perp} = N/T$  be the average density of distinct states in time, and choosing units with  $h = 2$  (so  $E = 2\bar{\nu}$ ), (2) becomes

$$E \geq \nu_{\perp}. \quad (3)$$

Thus energy is the maximum average rate of distinct state change. This can also be regarded as an uncertainty relation between average orthogonalization time  $\tau = 1/\nu_{\perp}$  and average energy width:  $(E - E_0)\tau \geq 1$ . All uncertainty relations between  $\tau$  and any other width  $\Delta\nu$  of the energy-frequency distribution of the wavefunction are similar [3, 4], with the choice of width and number of distinct states changing the bound by only a factor of order one—a periodic oscillation between just two distinct states is the fastest [2, 18]. The same kind of Fourier analysis also applies to waves in space, rather than in time. All such bounds are attained simultaneously for minimum bandwidth, in which case quantum evolution becomes equivalent to a discrete evolution on a spacetime lattice: only the values of the wavefunction at lattice points in space and time are distinct [3, 4, 19, 21]. The rest of the continuous state is redundant.

### 2.3 Motion Defines Momentum

A moving particle has both extra distinct states due to its distinct positions, and extra energy due to its motion. In the particle's rest frame, it has neither. For a large ( $\approx$ classical) system moving between two events, evolving at the maximum rate of distinct state change allowed by its energy, we can use (3) in the two frames to count the extra distinct states due to the motion. If the system has velocity  $v$  and magnitude of momentum  $p$ , and the events are separated by time  $\Delta t$  and distance  $\Delta x$  in the laboratory frame, and  $\Delta t_r$  in the rest frame, with  $E$  and  $E_r$  the corresponding energies, the invariant time-energy interval is

$$E\Delta t - p\Delta x = E_r\Delta t_r. \quad (4)$$

But from (3),  $E\Delta t$  is simply the number of distinct states seen in the laboratory frame, and  $E_r\Delta t_r$  the number seen in the rest frame. The difference is the number of distinct states due to the motion, which from (4) is simply  $p\Delta x$ . Thus if  $\mu_{\perp}$  is the average density in space of states distinct due to the motion,

$$p \geq \mu_{\perp}. \quad (5)$$

If we multiply (5) by  $v = \Delta x/\Delta t$ , we get that  $vp \geq \Delta x \mu_{\perp}/\Delta t \equiv \nu_{\text{motion}}$ , the number of distinct states per unit time due to motion. So we also have

$$p \geq \nu_{\text{motion}}/v. \quad (6)$$

Thus motion and speed define a minimum  $p$ . In a quantum realization of a reversible lattice gas, a hop of an isolated particle from one lattice site to another is a distinct state change, and speed is just distance over time. In our discussion of energy and momentum conserving lattice gases, we will use the minimum possible momentum for any quantum system, from (6), as our estimate of the momentum of a freely moving isolated particle in an ideal realization [19, 20].

### 2.4 Minimum Energy

Choosing units with the speed of light  $c = 1$ , it is always true relativistically for a freely moving particle that  $E = p/v$ , so we can compute the energy of a free

particle once we know its momentum and velocity. From (6),  $E \geq \nu_{\text{motion}}/v^2$ , so energy is smallest when  $v$  is as large as possible—we can treat  $\nu_{\text{motion}}$  as constant here, since it doesn't depend on the distance between lattice sites, and we can make our particles travel faster by increasing only the distance between lattice sites without changing the time. Now, given a lattice gas dynamics with a set of particle velocities related by the lattice geometry, there is a family of equivalent evolutions that only differ in the choice of the fastest particle speed. Of these, the evolution with the least possible energy has its fastest-moving particles traveling at the speed of light. This makes sense physically, since a system with a non-zero rest-frame energy has a non-trivial internal dynamics—time passes in the rest frame. If we want to just model a logical evolution and nothing extra, the fastest-moving particles should have no internal dynamics.

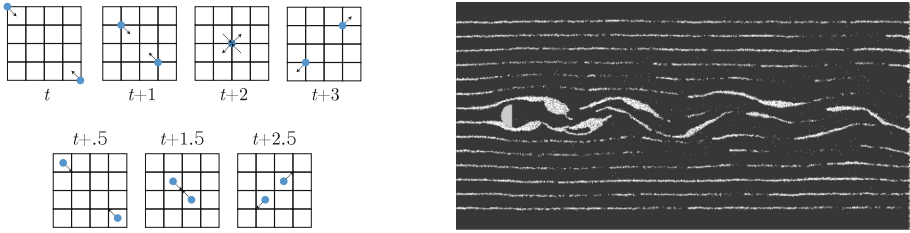
### 3 Finite-State Classical Mechanics

We discuss three reversible lattice gases, each with a finite-state dynamics that reproduces discrete samples of a classical particle evolution, and one model that samples a classical field. In its discrete form, the field example also turns into a reversible lattice gas. Although these models could represent macroscopic systems with any given energy, we're interested here in looking at intrinsic minimum energy and momentum defined by state change on the lattice. We analyze this for just the last two models—the first two are introductory. The field example is particularly interesting because energy and momentum are bound together on the lattice, moving as a relativistic particle with a discrete set of possible speeds. This behavior is intimately related to a biased random walk.

#### 3.1 Sampled Particles

**Lattice Gas Fluids.** The first lattice gases that reproduced samples of a classical mechanical evolution were models of fluids [22]. Lattice gas fluids are stylized molecular dynamics models, with particles started at points of a lattice, moving at one of a discrete set of velocities, and colliding at points of the lattice in a manner that conserves momentum and guarantees particles will again be on the lattice at the next integer time. We illustrate this in Fig. 1 (Left). We show two particles of a four-velocity 2D lattice gas. In the top row we show the particles at integer times, in the bottom row halfway between integer times. The dynamics shown is invertible and is momentum and particle conserving, but is too stylized to be a realistic fluid. Four directions aren't enough to recover fully symmetric Navier Stokes fluid flow in the large scale limit—but six are [11–13, 23]! Figure 1 (Right) is a snapshot of a simulation of a six-velocity lattice gas, with obstacles and a second tracer fluid (smoke) added, showing flow past an obstacle [24].

The reason we can talk here about momentum conserving collisions is because the discrete lattice gas is, conceptually, embedded in a continuous dynamics where we know what momentum is. In classical mechanics, a dynamics with continuous symmetry under translations in space defines a conserved linear

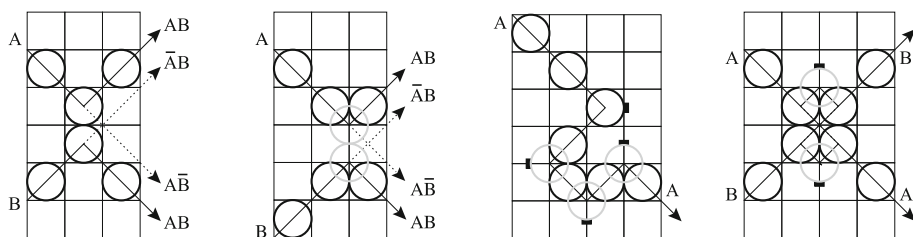


**Fig. 1. Lattice gas molecular dynamics.** *Left:* Particle and momentum conserving collisions in a single-speed four-velocity lattice gas. Particles are at lattice points at integer times. At half integer times, they are midway between. *Right:* A single-speed six-velocity lattice gas run on two million sites of a triangular lattice, with obstacles and a visualizing “smoke” gas added to the model. It exhibits realistic fluid behavior.

momentum, and a continuous rotational symmetry defines a conserved angular momentum [25]. The embedding allows us to define discrete conserved quantities that derive from continuous symmetries which cannot exist on a discrete lattice. The full continuous symmetries associated with the conservations [25] can only emerge in a lattice model in the macroscopic limit. This makes conservation more fundamental than continuous symmetry in lattice models [26].

In order to emulate the reversibility of microscopic physics, a local lattice dynamics must have a structure where data on the lattice are partitioned into separate groups for updating [27–29]; then if the transformation of each group is invertible, this property is inherited by the overall dynamics. At least two different partitions, used at different times, are required—with only one, each group would be forever isolated. In a continuous dynamics that has been initialized to act discretely, the alternation of partitions does not involve any explicit time dependence. For example, at the integer-times of Fig. 1 (Top Left), we see that all collisions happen at lattice locations, and the data at each lattice site are transformed independently of all other sites—this constitutes one partition. Not only invertibility, but particle and momentum conservation are guaranteed by the collision rule. In between collisions, particles travel straight to adjacent lattice locations without interacting. This constitutes the second partition.

We get a different view of partitioning for the same continuous dynamics if we define a lattice gas from the half-integer-time states of Fig. 1 (Bottom Left). In this case, we catch all particles when they are going straight, in between lattice sites. Particles are spread out in space, rather than piled up at lattice sites, and we can tell which way they are going from where they are, when. Groups of four locations that can contain particles converging on a lattice site define a partition—for example, the middle  $2 \times 2$  block of the middle time step. These are replaced by their values after the collision, independently for each block. The outgoing particles then converge on a new set of lattice sites, defining a second partition—the update rule for the two partitions is the same. From the point of view of the continuous dynamics, the  $2 \times 2$  blocks are just imaginary boxes we’ve drawn around regions where particles are converging at a given moment.

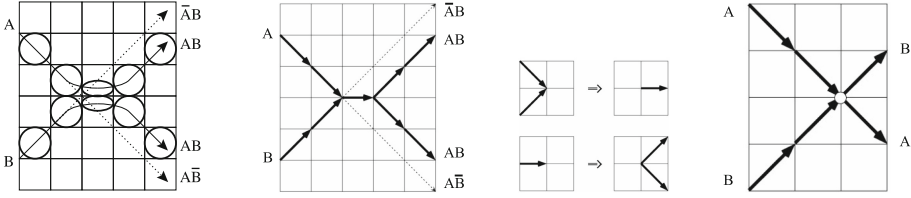


**Fig. 2. Fredkin's billiard ball model of computation.** Perfect billiard balls moving on a lattice can perform reversible computation. Ball presence is a 1, absence a 0. Left to right, we show (a) a right-angle collision where different logical combinations come out at different places, (b) a logical collision that happens in between integer times, (c) infinitely massive mirrors used to route signals, and (d) mirrors used to allow signal paths to cross, regardless of the signal values.

**The Billiard Ball Model.** In Fig. 2 we illustrate the lattice gas that Ed Fredkin invented [5] to try to silence skeptics who claimed reversible computing was physically impossible. This is a perfectly good reversible, particle and energy conserving classical mechanical model that uses the collisions of hard spheres to perform computation: ball or no-ball is a one or a zero, and collisions separate different logical cases into separate paths. The model uses infinitely massive mirrors, which are allowed in non-relativistic classical mechanics, to route signals. The equivalence to a discrete lattice gas is incomplete, though, since certain collisions (such as head on ones) will take the balls off the lattice—it is not enough to merely start all balls at lattice locations. The model can be completed by simply mandating that all problematic cases cause balls to pass through each other without interacting. This is a general feature of sampled classical dynamics: it is typically necessary to add a form of *classical tunneling* to the continuous dynamics, in order to maintain its digital character. From the point of view of the mathematical machinery of classical mechanics, there is really nothing wrong with doing this: it doesn't impact invertibility, conservations, relativity, etc.

There is still a problem, though, with turning the billiard ball model into a lattice gas that acts as a faithful sampling of a classical mechanical dynamics. Because of the hard collisions, the number of locations that need to be updated as a group in order to ensure invertibility and particle conservation is rather large: in Fig. 2(b), when the ball coming in at **B** is in the second column and about to interact, the next value of the location marked **AB** at the top depends on the presence or absence of the ball coming from **B**. To implement this as a lattice gas requires lattice sites that hold many particles, or the use of rather large partitions. This structure also implies extra constraints, not present in the continuous classical model, on the positions where collisions can occur, so that particles only converge on places where all particles can be updated as a group.

**Soft Sphere Model.** We can avoid all of these issues with a simple modification of the classical billiard ball dynamics, illustrated in Fig. 3(a). If we make the



**Fig. 3. Soft sphere model of computation.** Compressible balls collide. Left to right we show (a) collisions displace the colliding balls inwards, putting the AB cases on different paths, (b) we recast this as a lattice gas, with particles located at the tails of the arrows, (c) there is interaction in only two cases (and their rotations); otherwise all particles move straight, (d) adding a rest particle to the model allows particle paths to cross; this still follows the rule “otherwise all particles move straight.”

collisions very elastic, rather than hard, colliding particles spend a finite amount of time colliding, and are deflected onto inward paths, rather than outward as in the billiard ball model. This soft sphere model [30] is equivalent to a lattice gas where interactions can happen at a point, as in Fig. 1 (left-top). The discrete and continuous models can exactly coincide everywhere at integer times.

Figure 3(b) shows a direct translation of (a) into a lattice gas. As in (a), we have two streams of particles coming in at **A** and **B** and depict the state at an integer time, so we see particles at each stage of the collision at different points in space. In the lattice gas diagram, the particles are located at the tails of the arrows, and the arrows indicate direction information carried by the particles. The rule (c) is very simple: diagonal particles colliding at right angles turn into horizontal particles and vice versa—plus  $90^\circ$  rotations of these cases. In all other cases, the particles pass through each other unchanged. In (d) we add an unmoving *rest particle* to the model, so we can place it at any signal crossing to prevent interaction: the rule is unchanged, since moving straight is already the behavior in “all other cases.” This allows the model to perform computation without the addition of separate infinite-mass mirrors [30]. Similar computing lattice gases can be defined on other lattices, in 2D or 3D.

We can analyze the minimum energy and momentum for a unitary quantum implementation of this reversible classical dynamics. Looking at Fig. 3(b), we count state change and direction of motion in the middle of each arrow—during the time when the particle is moving freely between lattice sites. In this way we always see a single isolated particle moving with a definite velocity, and can apply (6) directly to get the minimum momentum. Taking the time between lattice sites as our unit of time, each particle motion constitutes one change per unit time. The particles moving diagonally are the fastest moving particles, so we take their speed  $v = 1$  to get a minimum energy model—they each have ideal (minimum) momentum  $p = 1$ . From the geometry of the model, we see that the horizontal particle must then be moving at speed  $1/\sqrt{2}$ , and there is again one change in a unit of time as it moves, so its ideal momentum from (6) is  $\sqrt{2}$ . This agrees with conservation of momentum, since each of the two incoming particles



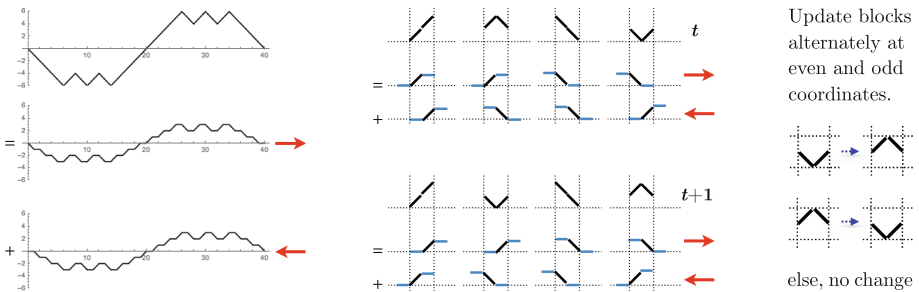
has a horizontal component of momentum of  $1/\sqrt{2}$ . The horizontally moving particle is moving slower than light, and so has a mass. By energy conservation, since the sum of the incoming energies is 2, that must be the energy of the horizontal particle. Then  $m = \sqrt{E^2 - p^2} = \sqrt{2}$  is its mass. This is a classical mechanical system with intrinsic energy, momentum and mass (*cf.* [20]).

### 3.2 Sampled Field

We discuss here a simple classical field dynamics in which constraints on the continuous initial state make it equivalent to a reversible lattice gas at integer times and positions (*cf.* [20, 31–33]). This example illustrates the mechanism behind a phenomenon that was discovered experimentally in reversible cellular automata models: the spontaneous appearance of realistic waves [9, 33, 34].

In Fig. 4 (left) we show, at the top, a continuous wave that is the superposition of continuously shifting rightgoing and leftgoing waves—we assume periodic boundaries so what shifts off one edge reappears at the other. It is a general property of the one dimensional wave equation that any solution is a superposition of an unchanging rightgoing waveform and an unchanging leftgoing waveform. In this case, each of the two component waves contain segments that have slope 0, alternating with segments that have slope  $+1$  or  $-1$ . As the component waves shift continuously in space, at certain moments the slope 0's of each wave align with the  $\pm 1$ 's of the other and we get a discrete sum, composed only of segments with slope  $\pm 1$ .

Because of the discrete constraints, at integer moments of time the space can be partitioned into pairs of adjacent columns where the non-zero slopes are converging towards the center of each partition. This is illustrated in Fig. 4 (Middle). At these times, the next integer-time configuration for each pair of columns is completely determined by the current configuration, and doesn't depend on



**Fig. 4. Continuous wave dynamics equivalent to a finite-state dynamics.** *Left:* Any one-dimensional wave obeying the continuous wave equation is a superposition of a rightgoing and a leftgoing wave. We constrain the two components to always give a discrete sum at integer times, so we can sample then. *Middle:* Each component wave alternates flat intervals with intervals that have a slope of  $\pm 1$ . There are only four cases possible at integer times. *Right:* Block rule for directly evolving the sum wave.

any information outside the pair. We see, from Fig. 4 (Middle), that there are only four distinct cases, and only two of them change the sum: slopes  $\searrow$  turn into  $\swarrow$  and vice versa. We flip hills and troughs; nothing else changes.

Figure 4 (Right) is thus the evolution rule for a *discrete string dynamics* that exactly follows the continuous wave equation at integer times and positions. The rightgoing and leftgoing waves can be reconstructed from the sum—only the sum is evolved as a lattice gas. This *flip* rule must be applied alternately to the two possible partitions into pairs of columns. The rule works just as well, though, if we also partition pairs of rows, so the rule is separately applied to  $2 \times 2$  blocks.

**Transverse Motion.** It is interesting to analyze the energy and momentum for a discrete string, evolving under the *flip* rule, that has a net motion up or down. If we decompose such a string into a superposition of a rightgoing and a leftgoing wave, we find that each has a net slope across the space. As long as these two net slopes add to zero, the string itself will have no net slope and so meets itself correctly at the edges. The rightgoing and leftgoing waves should be thought of as infinite repeating patterns of slopes, rather than as meeting themselves at the edges of one period of the string.

Let  $2N$  be the width of one period of the string, in units of the width of a slope segment. The repeating pattern of the string can be decomposed into the sum of a repeating pattern of  $N$  non-zero rightgoing slopes and  $N$  non-zero leftgoing slopes. Let  $R_+$  be the number of the  $N$  rightgoing slopes that are positive,  $R_-$  the number that are negative, and similarly with  $L_+$  and  $L_-$  for the leftgoing wave. The net slope of the string will be zero as long as  $R_+ + L_+ = R_- + L_-$ .

Our unit of distance is the width (=height) of a slope segment; our unit of time the time needed for a slope segment to move one width. Thus the rightgoing wave will be displaced upward a distance  $R_- - R_+$  in time  $2N$ , and the leftgoing wave up by a distance  $L_+ - L_-$  in the same time, so the net upward displacement of the string will be  $D = R_- - R_+ + L_+ - L_-$  in time  $2N$ .

We can calculate this another way. Consider a partition of the rightgoing and leftgoing waves into pairs of columns at an integer time. A pair containing slopes  $\searrow\searrow$  or  $\swarrow\swarrow$  contributes a net of zero to  $D$ , since  $R_+$  and  $L_+$  enter with different signs, as do  $R_-$  and  $L_-$ . Thus all of the net motion can be attributed to the  $\searrow\swarrow$  and  $\swarrow\searrow$  cases, which respectively contribute  $+2$  and  $-2$ . If there are  $N_{\searrow\swarrow}$  partitions ready to flip up, and  $N_{\swarrow\searrow}$  partitions ready to flip down, the net upward velocity of the string is

$$v = \frac{D}{2N} = \frac{N_{\searrow\swarrow} - N_{\swarrow\searrow}}{N}. \quad (7)$$

**String Momentum and Energy.** Using (6) we can assign minimum momenta. We take the maximum of  $v$  to be the speed of light (assuming this string speed is physically possible), and get one unit of momentum upward for each isolated motion  $\searrow\swarrow \rightarrow \swarrow\searrow$ , one downward for each  $\swarrow\searrow \rightarrow \searrow\swarrow$ , so net momentum upward is

$$p = N_{\searrow\swarrow} - N_{\swarrow\searrow}. \quad (8)$$

Treating the overall motion of the string as that of a relativistic particle, the total energy  $E = p/v$ , so from (7) and (8), the total relativistic energy

$$E = N. \quad (9)$$

We can compare this to the energy of the string, treated as a continuous classical mechanical system. Using the notation  $\Psi_t$  to denote partial derivative with respect to  $t$ , and with  $c = 1$ , the continuous 1D wave equation is  $\Psi_{tt} = \Psi_{xx}$ , and the classical energy density of the string is proportional to  $\Psi_t^2 + \Psi_x^2$ . Since all of the slopes are  $\pm 1$  the integral of this sum of squares at integer times is just  $4N$ , twice the length of the string. The integral is also the same at all intermediate times. This follows from  $\Psi_{tt} = \Psi_{xx}$  but we can also see this directly. Let  $l(x, t)$  be the amplitude of the leftgoing wave, and  $r(x, t)$  the rightgoing. Then  $\Psi = l + r$ ,  $\Psi_x = l_x + r_x$  and  $\Psi_t = l_x - r_x$ , so  $\Psi_t^2 + \Psi_x^2 = 2(l_x^2 + r_x^2)$ , and the integral is  $4N$ .

Similarly, the classical Lagrangian density  $\mathcal{L} \propto \Psi_t^2 - \Psi_x^2$  can be compared. In this case, again using  $\Psi = l + r$  etc.,  $\mathcal{L} \propto l_x r_x$ . At integer times the integral over the width of the space is zero, since every non-zero slope is aligned with a zero slope. Halfway between integer times, half the columns contain pairs of non-zero slopes that are passing each other, the other half contain pairs of zero slopes, which don't contribute to the integral. Each non-zero pair of equal slopes contributes  $+1$  to the integral; each pair of unequal slopes contributes  $-1$ . This is a count of potential minus kinetic energy: halfway through a transition  $\setminus / \rightarrow / \wedge$  or  $/ \wedge \rightarrow \setminus /$  of a continuous string, all of the energy would be kinetic, and for an unchanging length of stretched string  $//$  or  $\setminus \setminus$  all energy would be potential.

**Rest Frame Energy.** For a string moving according to this discrete wave dynamics, there is a maximum amount of internal evolution of string configurations when the net vertical velocity is zero, and no internal evolution when the string is moving vertically as fast as possible, since then there is a unique configuration

$$\dots \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \setminus / \dots \quad (10)$$

that moves upward at the speed of light. The rest frame energy of the string characterizes how much dynamics can happen internally. This is given directly by  $\sqrt{E^2 - p^2}$  as

$$E_r = \sqrt{N^2 - (N_{/\wedge} - N_{\setminus /})^2}. \quad (11)$$

The more the string has a net motion, the less rest energy it has. This makes the string a strange relativistic particle, since its total energy is independent of its speed! In the relation  $E_r = E/\gamma$ , we have  $E_r \rightarrow 0$  as  $1/\gamma \rightarrow 0$ .

This is not surprising, though, given the limited scope of the model: it includes no interactions that could cause the string's travel to be sped up or slowed down. A realistic model of particle motion would have to include interactions with other systems, that can add or remove particle energy while conserving energy and momentum overall. We might still use the simple string model, though, as a proxy for a more realistic finite-state model of inertia, by simply

specifying a statistical interaction that can change the length of the string to change  $E$ .

**Statistical Inertia.** To contemplate a statistical coupling to another system, it is helpful to recast the analysis of the string model in a population-statistics format. Recall that, for a given population of rightgoing and leftgoing slopes, the number going each way is the same and, for the string to meet itself at the edges, the sum of all the slopes is zero. Together, these two constraints imply that  $L_- = R_+$  and  $L_+ = R_-$ : the population statistics for rightgoing and leftgoing waves are mirror images. Therefore we can analyze the motion looking at the statistics for just one of the component waves. From (7), using  $N_{\setminus} - N_{/} = D/2 = L_+ - L_-$  and letting  $p_+ = L_+/N$  and  $p_- = L_-/N$ ,

$$v = p_+ - p_-, \quad (12)$$

with  $p_+ + p_- = 1$ . Here  $p_+$  is the average frequency of upward steps per unit time, and  $p_-$  the frequency of downward. To see this for  $p_+ = L_+/N$ , notice that each leftgoing  $/$  contributes one unit of upward motion in the course of  $2N$  steps of evolution, and so does the mirror image  $\setminus$  moving rightward, so together they take us one position up in  $N$  steps;  $p_-$  is similar. Thus the transverse motion of the string is like a one-dimensional random walk—which is known to exhibit similarities to relativistic particle motion [32, 35, 36].

These frequencies could become true probabilities if the populations were statistically coupled to some environment. For example, imagine the string acting as a mass coupled to a spring, to form an oscillating system. As the spring stretches it slows down the mass, removing energy and changing the bias  $p_+ - p_- = v$ . Eventually it turns the mass around and speeds it up, etc. If the populations are stochastic the velocity and energy determine the entropy of the string, which would change cyclically with time in an oscillator. This introduces a rather thermodynamic flavor into a discussion of inertia in classical mechanics.

## 4 Conclusions

Reversible lattice gas dynamics derived by sampling classical mechanical evolution are foundational models for all of mechanics, in the same way that classical lattice gases have long been foundational for statistical mechanics. They are paradoxically both continuous and discrete, both classical and quantum. They have an intrinsic energy and momentum based on counting classical state change given by the minimum allowed by the general properties of energy and momentum in quantum mechanics. They are non-trivial models and can in fact be computation universal.

These models are foundational rather than fundamental. We are not at all suggesting that nature is, at base, a classical cellular automaton [10, 26, 37], but rather that these reversible classical systems that are also special cases of unitary quantum systems provide a simplified context in which to study the foundations

of mechanics. This is exactly the role that classical lattice gases play in statistical mechanics: classical special cases of quantum systems [7, Sect. 2.4]. These dynamical counterparts should be studied not just for academic or pedagogical interest—though that is reason enough—but also because there are fundamental informational issues that are not understood in physics, and these models introduce a realistic counting of distinct quantum states into classical mechanics and classical field theory. It might be, for example, that some of the (very macroscopic) informational paradoxes of general relativity depend only on the reversibility of the dynamics, and not on the full unitarity of quantum evolution.

The present analysis only scratches the surface of what seems to be a rich field. Even discovering what kinds of macroscopic phenomena *cannot* be modeled in this manner may tell us something about the essential role that quantum mechanics plays, that couldn't be played by a classical informational substratum. These finite-state classical mechanical models also turn some foundational questions on their head, since they can be regarded as special cases of unitary quantum evolution, rather than macroscopic decoherent limits.

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