Chapter 15
A Double Discontinuity and a Triple Approach: Felix Klein's Perspective on Mathematics Teacher Education

Jeremy Kilpatrick

Abstract Felix Klein was the first to identify a central problem in the preparation of mathematics teachers: a double discontinuity encountered in going from school to university and then back to school to teach. In his series of books for prospective teachers, Klein attempted to show how problems in the main branches of mathematics are connected and how they are related to the problems of school mathematics. He took three approaches: The first volume built on the unity of arithmetic, algebra, and analysis; the second volume attempted a comprehensive overview of geometry; and the third volume showed how mathematics arises from observation. Klein's courses for teachers were part of his efforts to improve secondary mathematics by improving teacher preparation. Despite the many setbacks he encountered, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice.

Keywords Klein · Higher standpoint · Mathematical knowledge for teaching
Capstone · Genetic method

As an undergraduate mathematics major at the University of California, Berkeley, in the 1950s, I took a course entitled "Elementary Mathematics for Advanced Students" in which we used two books by Felix Klein with the title *Elementary Mathematics from an Advanced Standpoint*. The subtitles were *Arithmetic, Algebra, Analysis* and *Geometry*. We used Dover reprints (Fig. 15.1) of English translations that had first been published in 1932 and 1939, respectively, by Macmillan.

The course was what would be termed today a "capstone" course, meaning that it came near the end of our program and was designed to demonstrate our mastery of mathematics. Topics in the course included continued fractions and Pythagorean triples represented graphically, quaternions, plane algebraic (normal) curves,

Portions of this chapter first appeared in Kilpatrick (2008).

J. Kilpatrick (✉)
Regents Professor of Mathematics Education Emeritus, University of Georgia,
105 Aderhold Hall, Athens, GA 30602-7124, USA
e-mail: jkilpat@uga.edu

H.-G. Weigand et al. (eds.), *The Legacy of Felix Klein*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-99386-7_15

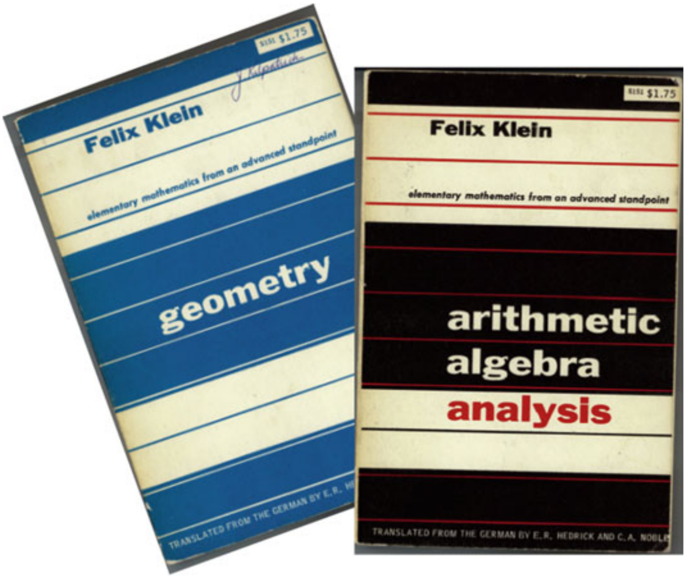

Fig. 15.1 Dover reprints of Klein (1924/1932, 1925/1939)

logarithm defined as area under a hyperbola, and geometric transformations. In the introduction to the volume on geometry, Klein points out that in presenting geometric topics, he first offers a general survey of the field, providing a frame into which students can insert items of mathematical knowledge that they have learned. Then he says, "Only afterward shall I emphasize that *interest in mathematical instruction* which was always my starting point [for the first volume]" (Klein 1925/1939, p. 1). He goes on to say that, as in the volume on arithmetic, algebra, and analysis, he will "draw attention . . . to the *historical development of the science*" (p. 2). He also notes that despite the separation of mathematical topics into two volumes, he definitely advocates

> a tendency which I like best to designate by the phrase "*fusion of arithmetic and geometry*"—meaning by arithmetic, as is usual in the schools, the field which includes not merely the theory of integers, but also the whole of algebra and analysis. (p. 2)

In print for over a century, the volumes of Klein's textbook have been used in countless courses for prospective and practicing teachers. The first two volumes were translated into English in 1924 and 1925, respectively, and into Spanish in 1927. Other translations followed. Not until 2016, however, was the third volume translated into English.

Klein's three volumes provide excellent early examples of what today is termed *mathematical knowledge for teaching* (Ball and Bass 2000; Bass 2005). The organization of the first volume, with pedagogical issues and difficulties facing the teacher taken up after each topic rather than relegated to a final chapter, seems much superior to that of the second. The organization of the first volume allows Klein to make specific suggestions for instruction and references to textbooks and historical treatments of topics, whereas the comments in the second volume tend to be more general. In the third volume, he makes no specific mention of pedagogy except briefly in the preface. Klein's courses for teachers were part of his reform efforts to improve secondary mathematics by improving the preparation of teachers. Despite the many setbacks he encountered, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice.

15.1 A Double Discontinuity

In the introduction to the first volume, Klein (1924/1932) noted the phenomenon of recent interest by university faculty in mathematics and natural sciences in the suitable training of prospective teachers. He noted that until recently, faculty had been exclusively concerned with their science "without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics" (p. 1). Considering the result of this practice, he noted that young university students found themselves, "at the outset, confronted with problems which did not suggest, in any particular, the things with which [they] had been concerned at school. Naturally [they] forgot these things quickly and thoroughly" (p. 1). Then when these students became teachers, they found themselves expected to teach traditional elementary mathematics "in the old pedantic way. . . . [They were] scarcely able, unaided, to discern any connection" (p. 1) between that task and their university mathematics. Therefore, beginning teachers "soon fell in with the time honored way of teaching . . . [Their] university studies remained only a more or less pleasant memory" (p. 1) that had no influence on their teaching.

He went on to say, "There is now a movement to abolish this double discontinuity, helpful neither to the school nor to the university" (Klein 1924/1932, p. 1). In Klein's view, the discontinuity meant that school mathematics and university mathematics typically seemed to have no connection. The courses enshrined in Klein's books assumed that prospective teachers were familiar with the main branches of mathematics, and he attempted to show how problems in those branches are connected and how they are related to the problems of school mathematics.

To eliminate the discontinuity, Klein (1924/1932) had two proposals: (a) update the school mathematics curriculum, and (b) "take into account, in university instruction, the needs of the school teacher" (p. 1). His goal was to show

> the *mutual connection between problems in the various fields,* a thing which is not brought out sufficiently in the usual lecture course, and more especially to emphasize the relations of these problems to those of school mathematics. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge there put before you a living stimulus for your teaching. (pp. 1–2)

In this quotation, one hears echoes of Klein's early views of mathematics education expressed in his inaugural address (*Antrittsrede*) of 1872 when he became professor at Erlangen at the age of 23. The problem of the secondary school curriculum was, for Klein, neither insufficient time nor inadequate content:

> What is required is more interest in mathematics, livelier instruction, and a more spirited treatment of the material! . . .
>
> At stake [for university teachers of mathematics] is the task . . . of raising the standards of mathematical education for later teaching candidates to a level that has not been seen for many years. If we educate better teachers, then mathematics instruction will improve by itself, as the old consigned form will be filled with a new, revitalized content! . . .
>
> [Therefore,] we, as university teachers, require not only that our students, on completion of their studies, know what must be taught in the schools. We want the future [teachers] to stand *above* [their] subject, that [they] have a conception of the present state of knowledge in [their] field, and that [they] generally be capable of following its further development. (Klein, in Rowe 1985, p. 139)

To address the school-to-university discontinuity, Klein proposed (a) taking the function concept as the focus of school instruction, and (b) making calculus the target of the secondary school curriculum. To address the university-to-school discontinuity, he proposed (a) offering university courses that would show connections between problems in various fields of mathematics (e.g., algebra and number theory), and (b) developing university courses in elementary mathematics from a higher standpoint. Finally, to address both discontinuities, Klein argued that instructors should make instruction livelier and more interesting, which meant that school mathematics should be more intuitive, less abstract, and less formal, and university mathematics should include more applied mathematics. Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject.

15.2 A Triple Approach

In each of the three volumes of his books for teachers, Klein took a different approach. In the first volume, to balance existing treatments of topics in school mathematics, Kline attempted to show the prospective teacher specific examples of how three seemingly unrelated branches of mathematics could be integrated. In the second volume, given that there were no unified treatments of geometry in the literature, he offered such a treatment, postponing attention to geometry teaching until the end of the volume. In the third volume, he had yet a different agenda: to show the contrast and emphasize the link between mathematics and its applications.

15.2.1 Arithmetic, Algebra, Analysis

To conclude the introduction to the first volume, Klein cited several recent discussions of mathematics instruction that supplemented the topics he would be treating. He pointed out, however, that some treatments of elementary mathematics build it up "systematically and logically in the mature language of the advanced student, [whereas] the presentation in the schools . . . should be psychological and not systematic. . . . A more abstract presentation will be possible only in the upper classes" (Klein 1924/1932, pp. 3–4). He also pointed out that he was adopting a "progressive" stance:

> We, who are called the reformers, would put the function concept at the very center of instruction, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into instruction as early as possible with constant use of the graphical method, the representation of functional relations in the $x\,y$ system, which is used today as a matter of course in every practical application of mathematics. . . . Strong development of space perception, above all, will always be a prime consideration. In its upper reaches, however, instruction should press far enough into the elements of infinitesimal calculus for the natural scientist or insurance specialist to get at school the tools which will be indispensable [for him or her]. (p. 4)

Klein was anticipating the emphasis that he would put in the subsequent text on applications, geometric illustrations, space perception, and the historical development of the field.

The book is divided into three parts—arithmetic, algebra, analysis—together with supplementary sections on transcendental numbers and set theory. The main topics in the first part are the natural numbers; the extension to negative numbers, fractions, and irrationals; number theory; and complex numbers. An example of Klein's emphasis on practical applications is his extended treatment of the mechanism for calculating machines (see Fig. 15.2, which shows how multiplication is performed). Later in the book, when discussing logarithmic tables, Klein (1924/1932) mentions that such a machine "makes logarithmic tables superfluous. At present, however, this machine is so expensive that only large offices can afford it. When it has become considerably cheaper, a new phase of numerical calculation will be inaugurated" (p. 174)—truly prophetic words.

Klein ends the discussion of arithmetic with a brief survey of the modern development of mathematics. Reviewing the first edition, John Wesley Young (1910) said, "It is a mere sketch, but it is a masterpiece" (p. 258). In the survey, Klein distinguishes two processes by which mathematics has grown, each of which leads to a different plan for instruction. In Plan A, the plan more commonly followed in school and in elementary textbooks, each branch of mathematics is developed separately for its own sake and with its own methods. The major branches—algebraic analysis and geometry—make occasional contact but are not unified. In Plan B, in contrast, "the controlling *thought is that of analytic geometry,* which seeks a *fusion of the perception of number with that of space*" (Klein 1924/1932, p. 77). Mathematics is to be seen as a connected whole, with pure and applied mathematics unified. Not surprisingly, Klein argues that Plan B is more likely than Plan A to engage those

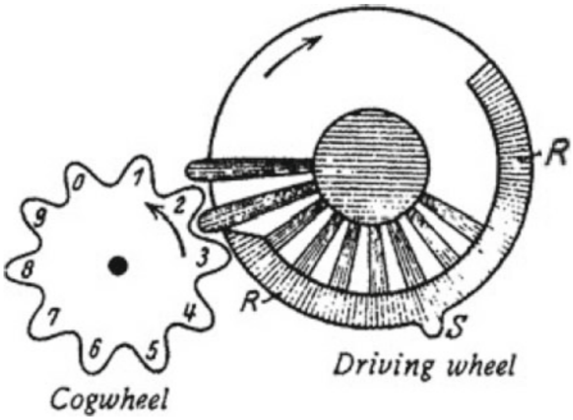

Fig. 15.2 Driving wheel and cogwheel in a calculating machine (Klein, 1933/2016a, p. 22)

pupils "not endowed with a specific abstract mathematical gift" (p. 78). Both plans have their place, and neither should be neglected. But secondary school instruction

> has long been under the one-sided control of the Plan A. Any movement toward reform of mathematical teaching must, therefore, press for more emphasis upon direction B. [Klein is] thinking, above all, of an impregnation with the genetic method of teaching, of a stronger emphasis upon space perception, as such, and, particularly, of giving prominence to the notion of function, under fusion of space perception and number perception!". (p. 85)

Klein then argues that his aim in this volume is to follow Plan B, thereby balancing existing books on elementary mathematics that almost invariably follow Plan A.

The main topics of the second part of the book, on algebra, concern the use of graphical and geometric methods in the theory of equations. Klein begins by citing textbooks on algebra and pointing out that the "one-sided" approach he will take is designed to emphasize material neglected elsewhere that can nevertheless illuminate instruction. His approach to solving real equations uses the duality of point and line coordinates, and he draws on the theory of functions of a complex variable to show how to represent, using conformal mapping, the solution of equations with a complex parameter.

The third part of the book, on analysis, concerns elementary transcendental functions and the calculus. It begins with a discussion of the logarithm, which provides a good illustration of Klein's approach. He first considers how the logarithm is introduced in school—by performing the operation inverse to that of raising to a power—and draws attention to various difficulties and possible confusions that accompany such an approach, including the absence of any justification for using the number e as the base for what are, for the pupil, inexplicably called the "natural" logarithms. After discussing the historical development of the concept, emphasizing the pioneering work of Napier and Bürgi, Klein proposes an introduction that would define the logarithm of a as the area between the hyperbola $xy = 1$, the x-axis, the ordinate $x = 1$, and the ordinate $x = a$, first approximating the area as a sum of rectangles and then taking the integral. The section on the logarithm ends by considering a complex-theoretic view of the function, which Klein argues that teachers should know even though it would not be an appropriate topic in school. In Young's (1910) review of the book, he points at Klein's treatment of the logarithm as the only one of his proposed reforms that would not be practical in the United States (and perhaps not even in Germany) since pupils need to use logarithms before they encounter hyperbolas, not to mention integrals.

The trigonometric functions and hyperbolic functions are also treated from the point of view of the theory of functions of a complex variable, and the part ends with an introduction to the infinitesimal calculus that relies heavily on Taylor's theorem and that includes historical and pedagogical considerations. The supplement at the end of the volume contains a proof of the transcendence of e and π and a brief, lucid introduction to set theory. As noted in Schubring (2016), the two appendices from Klein (1933/2016a) were inexplicably omitted from the first English translation.

15.2.2 Geometry

In the second volume, Klein (1909, 1925, 1925/1939) takes a different approach than in the first. Arguing that there are no unified textbook treatments of geometry, as there are for algebra and analysis, he proposes to give a comprehensive overview of geometry, leaving all discussion of instruction in geometry for a final chapter (unfortunately not included in the first English translation). Two supplements to the third edition that were prepared by Klein's colleague Fritz Seyfarth in consultation with Klein "concern literature of a scientific and pedagogic character which was not considered in the original text" (Klein 1925/1939, p. vi; the supplements were not translated into English either, but they do appear in Klein 1926/2016b).

The second volume, like the first, has three parts. The first concerns the simplest geometric forms; the second, geometric transformations; and the third, a systematic discussion of geometry and its foundations. Not surprisingly, Klein's innovative characterization of geometries as the invariants of their symmetry groups, from his famous Erlangen program (see, e.g., Bass 2005; Schubring n.d.), forms the basis of his discussion of the organization of geometry. In the discussion of foundations, Klein (1925/1939) emphasizes the importance of non-Euclidean geometry "as a very convenient means for making clear visually relations that are arithmetically complicated" (p. 184):

> *Every teacher certainly should know something of non-euclidean geometry*. . . . On the other hand, I should like to advise emphatically against bringing non-euclidean geometry into *regular school instruction* (i.e., beyond occasional suggestions, upon inquiry by interested pupils), as enthusiasts are always recommending. Let us be satisfied if the preceding advice is followed and if the pupils learn really to understand *euclidean* geometry. After all, it is in order for the teacher to know a little more than the average pupil. (p. 185)

The third part ends with a discussion of Euclid's *Elements* in its historical context.

In the final chapter, Klein surveys efforts to reform the teaching of elementary geometry in England, France, Italy, and Germany. The supplement contains some additional observations on questions of elementary geometry and updated material on reform in the four countries, particularly reports prepared for the surveys of teaching practices and curricula that had been initiated in 1908 during Klein's presidency of the *Commission internationale de l'enseignement mathématique* (CIEM, anglicized as the International Commission on the Teaching of Mathematics).

15.2.3 Precision Mathematics and Approximation Mathematics

The final volume in the series, Klein (1928/2016c), takes yet a third approach. As Marta Menghini and Gert Schubring point out in their introduction to the 2016 English translation, Klein maintained the view throughout his career that instruction needs to link mathematics to its applications. Because mathematics arises from observation and then transcends that observation to become abstract, learners need to see how the process works. Klein wanted the process to be intuitive:

> The third volume focuses on those properties that applied mathematicians take for granted when studying certain phenomena from a mathematical point of view. These properties must be seen as supplementary conditions (and constraints) to be required for the ideal objects of pure mathematics. However, in the meantime, these very proper ties prove to be the more intuitive ones. Therefore the comparison moves towards another field: it is a comparison between properties that can be considered only in the theoretical field of abstract mathematics and properties that can be grasped by intuition. Here the problem proves to become pertinent for mathematics teaching. (Menghini and Schubring 2016, pp. vii–viii)

The third volume had been originally published in 1902 but was revised and put at the end of the series because, as Klein (1924/1932) noted in his introduction to the third edition of the first volume, it had been "designed to bridge the gap between the needs of applied mathematics and the more recent investigations of pure mathematics" (p. v.), a somewhat different purpose than that of the first two volumes, which were designed "to bring to the attention of secondary school teachers of mathematics and science the significance for their professional work of their academic studies, especially their studies in pure mathematics" (p. v). As Klein's colleague Seyfarth (1928/2016) pointed out in the introduction to the third edition:

> The lecture notes, in their lithographic form, have for a long time been cited in mathematical literature with the title "applications of differential and integral calculus to geometry (a revision of the principles)". The change of title is due to the personal request of Felix Klein, with whom I had—in the last two months before his death—a series of conversations about the work required for their publication. Klein believed that the new title ["precision mathematics and approximation mathematics"] would better meet the goals of the notes than the former. (p. xiii)

Klein's third volume for prospective teachers, like the previous two, attempts to help them get perspective on their forthcoming practice—to "stand above" its content. Discussing the mathematics a teacher needs to know, Klein (1924/1932) wrote: "The teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them" (p. 162). The metaphor here is that of guide, someone who knows the mathematical terrain well and can conduct his or her pupils through it without them getting lost or injured. Klein went on to discuss how the novice teacher needs to be equipped to counteract common misperceptions of mathematical ideas:

> If you lack orientation, if you are not well informed concerning the intuitive elements of mathematics as well as the vital relations with neighboring fields, if, above all, you do not know the historical development, your footing will be very insecure. You will then either withdraw to the ground of the most modern pure mathematics, and fail to be understood in the school, or you will succumb to the assault, give up what you learned at the university and even in your teaching allow yourself to be buried in the traditional routine. (p. 236)

Klein's goal in the third volume was to help the prospective teacher of mathematics maintain the link between the different scientific fields and understand how mathematics arises from observation.

15.3 Klein and Mathematics Teacher Education

Like many mathematicians, Felix Klein spent much of his time working on issues of mathematics education once he was no longer doing research in mathematics. Unlike most of them, however, he had pursued such issues throughout his career. As noted above, Klein's Erlangen inaugural address of 1872 dealt with mathematics education (Rowe 1983, 1985). In it, he deplored the lack of mathematical knowledge among educated people. He saw that lack as symptomatic of a growing division between humanistic and scientific education, a division in which mathematics is uniquely positioned: "Mathematics and those fields connected with it are hereby relegated to the natural sciences and rightly so considering the indispensability of mathematics for these. On the other hand, its conceptual content belongs to neither of the two categories" (Rowe 1985, p. 135). Observing that like all sciences, mathematics is undertaken for its own sake, Klein goes on to argue that "it also exists in order to serve the other sciences as well as for the formal educational value that its study provides" (p. 137).

In the inaugural address at Erlangen, Klein expressed a neohumanistic view of how mathematics ought to appear in school and university instruction, a view he was later to modify in light of his experience. After teaching at the technical institute in Munich from 1875 to 1880, for example, he adopted a more expansive outlook on the mutual roles of mathematics, science, and technology in modern education. When he became professor of geometry at Leipzig in 1880, he began to promote the teaching of applied mathematics in universities as well as in technical institutes. Klein's ultimate goal was to make mathematics a foundational discipline in tertiary education, and to achieve that goal, he initiated a reform of secondary mathematics education so that it would include the calculus. In Erlangen, however, he had said that livelier teaching rather than new subject matter was what the secondary schools needed: In autobiographical notes he made in 1913 (Rowe 1985, p. 125), he summarized what he had said in that address: "*An den Gymnasien auszubauen: Interesse. Leben und Geist. Kein neuer Stoff* [To develop in the high schools: Interest. Life and spirit. No new material]." He then added a marginal remark reflecting his revised opinion that the secondary curriculum did need new material: "*Da bin ich nun anderen Sinnes geworden* [I have changed my mind about that]." After 40 years of teaching, Klein also reversed his view that prospective teachers should conduct an independent study on any topic whatsoever. In private notes made available to his colleague Wilhelm Lorey (quoted in Rowe 1985), he wrote:

> I would now suggest that teaching candidates of average talent should confine themselves to such studies as will be of fundamental importance in the later exercise of their profession, while everything beyond this should be reserved for those with unusual talent or favorable circumstances. (p. 128)

A final comment in Klein's (quoted in Rowe 1985) autobiographical notes suggests the toll his battles for reform had taken: "When one is young, one works much more hastily and unsteadily, one also believes the ideals will soon be attained" (p. 126).

Nonetheless, Klein was successful in reforming the secondary school curriculum as well as in creating university courses for teachers. His goal had long been to raise the level of mathematics instruction in both the technical institutes and the universities, and he came to realize that the key to achieving that goal would be to raise the level of secondary mathematics instruction to include the calculus, thereby raising the level of tertiary instruction (Schubring 1989). To push for reform in secondary and tertiary curricula, Klein forged an alliance among teachers, scientists, and engineers, and he also helped the international commission (CIEM) become an agent for curricular change. His courses for teachers were part of his reform efforts to improve secondary mathematics by improving the preparation of teachers.

References

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83–104). Westport, CT: Ablex.

Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society, 42,* 417–430.

Kilpatrick, J. (2008). A higher standpoint. In *ICMI proceedings: Regular lectures* (pp. 26–43). http://www.mathunion.org/icmi/publications/icme-proceedings/materials-from-icme-11-mexico/regular-lectures/. Accessed 1 Mar 2017.

Klein, F. (1908). *Elementarmathematik vom höheren Standpunkte aus: Arithmetik, algebra, analysis [Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis]* (Vol. 1). Leipzig, Germany: Teubner.

Klein, F. (1909). *Elementarmathematik vom höheren Standpunkte aus: Geometrie [Elementary mathematics from a higher standpoint: Geometry]* (Vol. 2). Leipzig, Germany: Teubner.

Klein, F. (1924). *Elementarmathematik vom höheren Standpunkte aus: Arithmetik, algebra, analysis [Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis]* (3rd ed., Vol. 1). Berlin, Germany: Springer.

Klein, F. (1925). *Elementarmathematik vom höheren Standpunkte aus: Geometrie [Elementary mathematics from a higher standpoint: Geometry]* (3rd ed., Vol. 2). Berlin, Germany: Springer.

Klein, F. (1926). *Elementarmathematik vom höheren Standpunkte aus: Geometrie [Elementary mathematics from a higher standpoint: Geometry]* (4th ed., Vol. 2). Berlin, Germany: Springer.

Klein, F. (1928). *Elementarmathematik vom höheren Standpunkte aus: Präzisions- und Approximationsmathematik [Elementary mathematics from a higher standpoint: Precision mathematics and approximation mathematics]* (3rd ed., Vol. 3). Berlin, Germany: Springer.

Klein, F. (1932). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis* (3rd ed., Vol. 1, E. R. Hedrick & C. A. Noble, Trans.). New York, NY: Macmillan. (Original work published 1924).

Klein, F. (1933). *Elementarmathematik vom höheren Standpunkte aus: Arithmetik, algebra, analysis [Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis]* (4th ed., Vol. 1). Berlin, Germany: Springer.

Klein, F. (1939). *Elementary mathematics from an advanced standpoint: Geometry* (3rd ed., Vol. 2, E. R. Hedrick & C. A. Noble, Trans.). New York, NY: Macmillan. (Original work published 1925).

Klein, F. (2016a). *Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis* (Vol. 1, G. Schubring, Trans.). Berlin, Germany: Springer. (Original work published 1933).

Klein, F. (2016b). *Elementary mathematics from a higher standpoint: Geometry* (Vol. 2, G. Schubring, Trans.). Berlin, Germany: Springer. (Original work published 1926).

Klein, F. (2016c). *Elementary mathematics from a higher standpoint: Precision mathematics and approximation mathematics* (Vol. 3, M. Menghini, Trans.). Berlin, Germany: Springer. (Original work published 1928).

Menghini, M., & Schubring, G. (2016). Preface to the 2016 edition. In F. Klein (Ed.), *Elementary mathematics from a higher standpoint: Precision mathematics and approximation mathematics* (Vol. 3, pp. v–viii). Berlin, Germany: Springer.

Rowe, D. E. (1983). A forgotten chapter in the history of Felix Klein's *Erlanger Programm*. *Historia Mathematica, 10,* 448–457.

Rowe, D. E. (1985). Felix Klein's "*Erlanger Antrittsrede*": A transcription with English translation and commentary. *Historia Mathematica, 12,* 123–141.

Schubring, G. (1989). Pure and applied mathematics in divergent institutional settings in Germany: The role and impact of Felix Klein. In D. E. Rowe & J. McCleary (Eds.), *The history of modern mathematics: Vol. 2. Institutions and applications* (pp. 171–220). San Diego, CA: Academic Press.

Schubring, G. (2016). Preface to the 2016 edition. In F. Klein (Ed.), *Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis* (Vol. 1, pp. v–xii). Berlin, Germany: Springer.

Schubring, G. (n.d.). *Felix Klein.* http://www.icmihistory.unito.it/portrait/klein.php. Accessed 1 Mar 2017.

Seyfarth, F. (2016). Preface to the third edition. In F. Klein (Ed.), *Elementary mathematics from a higher standpoint: Precision mathematics and approximation mathematics* (Vol. 3, pp. xiii–xiv, M. Menghini, Trans.). Berlin, Germany: Springer. (Original work published 1928).

Young, J. W. (1910). A synoptic course for teachers [Review of the book *Elementarmathematik vom höheren Standpunkte aus,* Vols. 1 & 2]. *Bulletin of the American Mathematical Society, 16,* 254–265.