Algebras from Semiconcepts in Rough Set Theory

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Abstract. In this article, we propose a notion of a semiconcept in the framework of Yao's object oriented concepts. A study of the algebra of such 'object oriented semiconcepts' is carried out, in the line of the study by Wille for the algebra of semiconcepts in formal concept analysis. Two further unary operators, 'semi-topological' in nature, are introduced on these structures. On abstraction, the properties of these operators lead to the definition of a 'semi-topological double Boolean algebra', of which the algebra of object oriented semiconcepts becomes an instance.

Keywords: Formal concept analysis · Rough sets
Object oriented concepts

1 Introduction

Rough set theory [10] and formal concept analysis (FCA) [4] provide two related methodologies for data analysis. Both investigate the notion of concepts, albeit from different perspectives. Classical rough set theory is developed based on an equivalence relation on a domain of objects. Generalized formulations have been proposed by using a binary relation on two domains, one a set of objects and the other a set of properties – such a binary relation on two domains is called a *formal context* in FCA. Many efforts have been made to compare and combine the two theories [1,3,5,6,9,19].

The central notion in FCA is that of a *concept lattice* on a context $\mathbb{K}$, denoted $\underline{\mathfrak{B}}(\mathbb{K})$. Düntsch and Gediga, and Yao introduced two kinds of 'rough concept lattices' in rough set theory, based on operators defined in [2]. The former defined *property oriented concept lattices* [1], and Yao proposed *object oriented concept lattices* [17]. Yao also studied the relationship between these two kinds of rough concept lattices and concept lattices of FCA in [17]. It is shown that object oriented concept lattices are dually isomorphic to concept lattices, while property oriented concept lattices are isomorphic to concept lattices. Further algebraic properties of rough concept lattices were investigated in [16].

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There is also a study of logic in the direction of FCA [13,14]. To formulate what is called *contextual logic*, 'negation of a concept' has to be formalized and Boole's correspondence between negation and set-complement is taken as a basis for the purpose. However, there turns out to be a problem of closure if set-complement is used to define negation of a formal concept. So the latter notion is generalized successively to that of *semiconcept*, *protoconcept* and *preconcept* [13,15]. Our interest also lies in defining a negation, in the context of rough concepts. This article does so in the framework of Yao's object oriented concepts. We define *object oriented semiconcepts* in Sect. 3, and follow the line of study in [13]. An algebraic structure is developed on the set $\mathfrak{S}(\mathbb{K})$ of all object oriented semiconcepts. We show that it forms a dual of *double Boolean algebra* [13], and contains two special Boolean subalgebras. In Sect. 4, two further unary operators are defined on $\mathfrak{S}(\mathbb{K})$, which turn out to be 'semi-topological' [11] in nature. The properties of these operators lead us to define a *semi-topological double Boolean algebra*, of which $\mathfrak{S}(\mathbb{K})$ becomes an instance.

Considering Boole's correspondence mentioned above, Wille defined another (weak) negation in [13], which can be generated by the negations defined on semiconcepts. This operator gives rise to a 'concept algebra', the abstraction of which is a 'dicomplemented lattice'. In [8], weakly dicomplemented lattices are defined which constitute a superclass of the class of dicomplemented lattices. In Sect. 4.1, we show that weakly dicomplemented lattices are different from the double Boolean algebras considered in this work. Section 5 concludes the article.

In the next section, we give the preliminaries required for the work presented in the rest of the paper.

2 Preliminaries

Definition 1 [4]. *A* formal context *is a triple* $\mathbb{K} := (G, M, R)$*, where* G, M *are sets of* objects *and* properties *respectively, and* $R \subseteq G \times M$. gRm *is interpreted as* object g has property m. *For* $A \subseteq G$ *and* $B \subseteq M$*,*

$$A^{'} := \{m \in M \mid gRm \, for\, all\, g \in A\},$$
$$B^{'} := \{g \in G \mid gRm \, for\, all\, m \in B\}.$$

A concept *of* $\mathbb{K}$ *is defined to be a pair* (A, B) *where* $A \subseteq G$, $B \subseteq M$, $A^{'} = B$ *and* $B^{'} = A$. A *is called the* extent *and* B *the* intent *of the concept* (A, B). *The set of all concepts of* $\mathbb{K}$ *is denoted by* $\mathcal{B}(\mathbb{K})$.
For concepts (A_1, B_1) *and* (A_2, B_2) *in* $\mathbb{K}$ *an order is defined as:*

$$(A_1, B_1) \leq (A_2, B_2) \, if\, and\, only\, if\, A_1 \leq A_2.$$

$(\mathfrak{B}(\mathbb{K}), \leq)$ *forms a complete lattice, and is called the* concept lattice *of* $\mathbb{K}$.

Definition 2 [1]. *For a formal context* $\mathbb{K} := (G, M, R)$, $\mathbb{K}^c := (G, M, -R)$*, is called a* complement *of* $\mathbb{K}$*, where* $-R = \{(x, y) \in G \times M : (x, y) \notin R\}$.

Example 1 [15]. The following table gives an example of a formal context. Objects are family members, properties are genders and age variables (Table 1).

Table 1. A formal context

	Male (Ma)	Female (Fe)	Old	Young
Father (Fa)	*		*	
Mother (Mo)		*	*	
Son (So)	*			*
Daughter (Da)		*		*

2.1 Semiconcept Algebra

As mentioned in Sect. 1, there is a problem of closure if set-complement is used to define negation of a formal concept. More explicitly, if (A, B) is a formal concept in a context (G, M, R), the complement $G \setminus A$ $(M \setminus B)$ of the extent (intent) A (B) may not be an extent (intent). The notion of formal concept was then generalized by defining a *semiconcept*.

Definition 3 [12]. *A* semiconcept *of a formal context* $\mathbb{K} := (G, M, R)$ *is defined as a pair* (A, B) *with* $A \subseteq G$ *and* $B \subseteq M$ *such that* $A = B^{'}$ *or* $B = A^{'}$*.*

The set of all semiconcepts of $\mathbb{K}$ is denoted by $\mathfrak{H}(\mathbb{K})$. The following algebraic operations $\sqcap, \sqcup, \neg, \lrcorner, \bot$ and $\top$ are introduced on $\mathfrak{H}(\mathbb{K})$:

$$
\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)^{'}) \\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)^{'}, B_1 \cap B_2) \\
\neg(A, B) &:= (G \setminus A, (G \setminus A)^{'}) \\
\lrcorner(A, B) &:= ((M \setminus B)^{'}, M \setminus B) \\
\top &:= (G, \phi) \\
\bot &:= (\phi, M)
\end{aligned}
$$

$\mathfrak{H}(\mathbb{K})$ with the operations $\sqcap, \sqcup, \neg, \lrcorner, \bot$ and $\top$ is called the *algebra of semiconcepts* of $\mathbb{K}$, and denoted by $\underline{\mathfrak{H}}(\mathbb{K})$. The following sets of idempotent elements are considered, and shown to form Boolean algebras in [12,13]:

$$\mathfrak{H}_{\sqcap} := \{(A, A^{'}) \in \mathfrak{H}(\mathbb{K}) : A \subseteq G\} \text{ and } \mathfrak{H}_{\sqcup} := \{(B^{'}, B) \in \mathfrak{H}(\mathbb{K}) : B \subseteq M\}.$$

2.2 Object Oriented Concept Lattice

Let G and M be two non-empty sets, and R $\subseteq$ G $\times$ M be a relation. For each $x \in G$, the R-range of x is $R(x) := \{y \in M : xRy\}$. The converse R^0 of R is $R^0 := \{(y, x) \in M \times G : xRy\}$.

For a given formal context $\mathbb{K} := (G, M, R)$, ${}^{\Box}, {}^{\Diamond} : 2^G \to 2^M$ constitute a pair of dual approximation operators defined as:

$$X^{\Diamond} := \{y \in M : X \bigcap R^0(y) \neq \emptyset\}, \qquad X^{\Box} := \{y \in M : R^0(y) \subseteq X\}.$$

On the other hand, $^{\Box},^{\Diamond} : 2^M \to 2^G$ constitute another pair of dual approximation operators defined as:

$$Y^{\Diamond} := \{x \in G : Y \bigcap R(x) \neq \emptyset\}, \qquad Y^{\Box} := \{x \in G : R(x) \subseteq Y\}.$$

$^{\Diamond}$ is called the possibility operator and $^{\Box}$ the necessity operator. Note that if we take $G = M$ and R to be an equivalence relation on G then the $^{\Diamond},^{\Box}$ operators coincide respectively with the upper and lower approximation operators (on the approximation space (G, R)) of rough set theory.

Now we list some properties of $^{\Diamond},^{\Box}$. For proof we refer to [1,17,18].

Proposition 1. *Let $\mathbb{K} := (G, M, R)$ be a context. For any $X, X_1, X_2 \subseteq G$ and $Y, Y_1, Y_2 \subseteq M$, the following hold.*

1. $G^{\Box} = M$ *and* $\phi^{\Diamond} = \phi$.
2. $M^{\Diamond} = G$ *if and only if* $R(x) \neq \phi$ *for all* $x \in G$.
3. $\phi^{\Box} = \phi$ *if and only if* $R^0(y) \neq \phi$ *for all* $y \in M$.
4. *if* $X_1 \subseteq X_2$ *then* $X_1^{\Box} \subseteq X_2^{\Box}$ *and* $X_1^{\Diamond} \subseteq X_2^{\Diamond}$.
5. *if* $Y_1 \subseteq Y_2$ *then* $Y_1^{\Box} \subseteq Y_2^{\Box}$ *and* $Y_1^{\Diamond} \subseteq Y_2^{\Diamond}$.
6. $X^{\Box\Diamond} \subseteq X \subseteq X^{\Diamond\Box}$ *and* $Y^{\Box\Diamond} \subseteq Y \subseteq Y^{\Diamond\Box}$.
7. $(X)_R^{\Box} = (X^c)'_{-R}$ *and* $(Y)_R^{\Box} = (Y^c)'_{-R}$.
8. $X^{c\Box} = X^{\Diamond c}$ *and* $Y^{c\Box} = Y^{\Diamond c}$.
9. $X^{\Box c} = X^{c\Diamond}$ *and* $Y^{\Box c} = Y^{c\Diamond}$.
10. $(X \cap Y)^{\Box} = X^{\Box} \cap Y^{\Box}$.
11. $(X \cup Y)^{\Diamond} = X^{\Diamond} \cup Y^{\Diamond}$.
12. $(X \cap Y)^{\Box\Diamond} \subseteq X^{\Box\Diamond} \cap Y^{\Box\Diamond}$ *and* $X^{\Diamond\Box} \cup Y^{\Diamond\Box} \subseteq (X \cup Y)^{\Diamond\Box}$.
13. $X^{\Box\Diamond\Box} = X^{\Box}$ *and* $Y^{\Box\Diamond\Box} = Y^{\Box}$.
14. $X^{\Diamond\Box\Diamond} = X^{\Diamond}$ *and* $Y^{\Diamond\Box\Diamond} = Y^{\Diamond}$.

Proposition 2.

1. $^{\Box\Diamond}$ *mapping X to $X^{\Diamond\Box}$, is a closure operator.*
2. $^{\Diamond\Box}$ *mapping X to $X^{\Box\Diamond}$, is an interior operator.*

For a given set of objects $A \subseteq G$, the map $^{\Box} : 2^G \to 2^M$ assigns to it a set of properties $A^{\Box}$, while the map $^{\Diamond} : 2^M \to 2^G$ assigns to a set of properties $B \subseteq M$, an object set $B^{\Diamond}$. For special pairs (A, B), we have the following.

Definition 4 [17,18]. *An* object oriented concept *of the context $\mathbb{K}$ is defined as a pair (A, B) with $A \subseteq G$, $B \subseteq M$ such that $A^{\Box} = B$ and $B^{\Diamond} = A$. A is the* extent *and B the* intent *of the object oriented concept (A, B). The set of all object oriented concepts of $\mathbb{K}$ is denoted by $RO{-}L(\mathbb{K})$.*

With this definition, it is shown in [18] that object oriented concepts are described by disjunctions of properties, whereas formal concepts are described by conjunctions of properties. The two theories together can thus give a more complete picture of data.

An order is defined on the set $RO - L(\mathbb{K})$ of object oriented concepts:

$(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (which is equivalent to $B_1 \subseteq B_2$). $(RO - L(\mathbb{K}), \leq)$ forms a complete lattice. Moreover, we have

Theorem 1 [16]. *$RO{-}L(\mathbb{K})$ is dually isomorphic to $\underline{\mathfrak{B}}(\mathbb{K}^c)$.*

3 Object Oriented Semiconcept

We are interested to study the notion of negation in the context of rough concepts. In [13], Wille studied a negation in FCA, by separately negating the extent and intent of a concept, using set-complement. In this work, we consider object oriented concepts and introduce negation using Wille's approach. For a given object oriented concept we also have two negations, one by taking the complement of its extent and the other by taking the complement of its intent.

Example 2. Let us continue with the context given in Example 1. In Tables 2 and 3 below, we list for all $A \subseteq G, B \subseteq M$ respectively, $A^{\Box}, A^{\Box\Diamond}$ and $B^{\Diamond}, B^{\Diamond\Box}$.

Table 2. Subsets A of G giving object oriented semiconcepts $(A, A^{\Box})$

$\boldsymbol{A \subseteq G}$	$\boldsymbol{A^{\Box}}$	$\boldsymbol{A^{\Box\Diamond}}$
ϕ	ϕ	ϕ
$\{Fa\}$	ϕ	ϕ
$\{Mo\}$	ϕ	ϕ
$\{so\}$	ϕ	ϕ
$\{Da\}$	ϕ	ϕ
$\{Fa, Mo\}$	$\{old\}$	$\{Fa, Mo\}$
$\{Fa, So\}$	$\{Ma\}$	$\{Fa, So\}$
$\{Fa, Da\}$	ϕ	ϕ
$\{Mo, So\}$	ϕ	ϕ
$\{Mo, Da\}$	$\{Fe\}$	$\{Mo, Da\}$
$\{So, Da\}$	$\{Young\}$	$\{So, Da\}$
$\{Fa, Mo, So\}$	$\{Old, Ma\}$	$\{Fa, Mo, So\}$
$\{Fa, Mo, Da\}$	$\{Fe, Old\}$	$\{Fa, Mo, Da\}$
$\{Mo, So, Da\}$	$\{Young, Fe\}$	$\{Mo, So, Da\}$
$\{Fa, So, Da\}$	$\{Ma, Young\}$	$\{Fa, So, Da\}$
G	M	G

Consider the pair $(\{Mo, So, Da\}, \{Young, Fe\})$. It is clear from Table 2 that it is an object oriented concept. The complement of the extent $A = \{Mo, So, Da\}$ is $\{Fa\}$, which is not the extent of any object oriented concept of $\mathbb{K}$ (cf. Table 3). Now consider the pair $(\{So, Da\}, \{Young\})$, which is also an object oriented concept. The complement of the intent $B = \{Young\}$ is $C = \{Ma, Fe, Old\}$ and C is not the intent of any object oriented concept of $\mathbb{K}$. Analogous to the situation in FCA, simply taking the set-complement of the extent or intent of an object oriented concept, may not result in an object oriented concept. One then relaxes the requirement to consider pairs of the form $(A^c, A^{c\Box})$ and $(B^{c\Diamond}, B^c)$

Table 3. Subsets B of M giving Object oriented Semiconcepts $(B^{\Diamond}, B)$

$\boldsymbol{B \subseteq M}$	$\boldsymbol{B^{\Diamond}}$	$\boldsymbol{B^{\Diamond\Box}}$
ϕ	ϕ	ϕ
$\{old\}$	$\{Fa, Mo\}$	$\{old\}$
$\{Ma\}$	$\{Fa, So\}$	$\{Ma\}$
$\{Fe\}$	$\{Mo, Da\}$	$\{Fe\}$
$\{Young\}$	$\{So, Da\}$	$\{Young\}$
$\{Old, Ma\}$	$\{Fa, Mo, So\}$	$\{Old, Ma\}$
$\{Fe, Old\}$	$\{Fa, Mo, Da\}$	$\{Fe, Old\}$
$\{Young, Fe\}$	$\{Mo, So, Da\}$	$\{Young, Fe\}$
$\{Ma, Young\}$	$\{Fa, So, Da\}$	$\{Ma, Young\}$
$\{Ma, Fe\}$	G	M
$\{Old, Young\}$	G	M
$\{Ma, Fe, Old\}$	G	M
$\{Ma, Old, Young\}$	G	M
$\{Ma, Fe, Young\}$	G	M
$\{Fe, Old, Young\}$	G	M
M	G	M

to define negation, as $A^{c\Box}$ collects properties of the objects of A^c only, while $B^{c\Diamond}$ contains all objects that have properties belonging to B^c. (Note that these pairs still need not be concepts, as we shall see in an example below). This idea is generalized to give the definition of an *object oriented semiconcept*.

Definition 5. *Let* $\mathbb{K} := (G, M, R)$ *be a formal context. An* object oriented semiconcept *of* $\mathbb{K}$ *is defined as a pair* (A, B) *with* $A \subseteq G, B \subseteq M$ *such that* $A^{\Box} = B$ *or* $B^{\Diamond} = A$*. The set of all object oriented semiconcepts of* $\mathbb{K}$ *is denoted by* $\mathfrak{S}(\mathbb{K})$*.*

Thus object oriented semiconcepts of $\mathbb{K}$ are pairs of the form $(A, A^{\Box})$ or $(B^{\Diamond}, B)$. Tables 2 and 3 in Example 2 give us all the object oriented semiconcepts of the context (G, M, R). It may be then observed that an object oriented semiconcept may not always be an object oriented concept: $(\{Fa\}, \phi)$ is an object oriented semiconcept but not an object oriented concept.

Now is there any relation between semiconcepts of FCA and object oriented semiconcepts defined above? The answer is given by

Proposition 3. *For a context* $\mathbb{K}$, $(A, B) \in \mathfrak{S}(\mathbb{K})$ *if and only if* $(A^c, B) \in \mathfrak{H}(\mathbb{K}^c)$*, the set of all semiconcepts of the complement of the context* $\mathbb{K}$*.*

3.1 Algebra of Object Oriented Semiconcepts

An order $\leq$ and algebraic operations $\sqcap, \sqcup, \neg, \lrcorner, \top$ and $\bot$ are considered on $\mathfrak{S}(\mathbb{K})$:

Definition 6. *For* $(A_1, B_1),\ (A_2, B_2) \in \mathfrak{S}(\mathbb{K})$,

(a) $(A_1, B_1) \leq (A_2, B_2)$ *if and only if* $A_1 \subseteq A_2$ *and* $B_1 \subseteq B_2$,
(b) $(A_1, B_1) \sqcap (A_2, B_2) := ((B_1 \cap B_2)^{\Diamond}, B_1 \cap B_2)$,
(c) $(A_1, B_1) \sqcup (A_2, B_2) := (A_1 \cup A_2, (A_1 \cup A_2)^{\Box})$,
(d) $\neg(A, B) := (G \setminus A, (G \setminus A)^{\Box})$,
(e) $\lrcorner(A, B) := ((M \setminus B)^{\Diamond}, M \setminus B)$,
(f) $\top := (G, M)$,
(g) $\bot := (\phi, \phi)$.

The meet ($\sqcap$) and join ($\sqcup$) operations taken in $RO{-}L(\mathbb{K})$ are extended to $\mathfrak{S}(\mathbb{K})$. It is clear from Definition 5 and Proposition 1(1) that $\mathfrak{S}(\mathbb{K})$ is closed with respect to all the operations defined above. The tuple $(\mathfrak{S}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$ is called the *algebra of object oriented semiconcepts* of $\mathbb{K}$ and is denoted by $\underline{\mathfrak{S}}(\mathbb{K})$.

Proposition 4. $(A_1, B_1) \sqcap (A_2, B_2)$ *is a lower bound of* (A_1, B_1) *and* (A_2, B_2), *and* $(A_1, B_1) \sqcup (A_2, B_2)$ *is an upper bound of* (A_1, B_1) *and* (A_2, B_2) *in* $(\mathfrak{S}(\mathbb{K}), \leq)$.

Proof. $(A_1, B_1) \sqcap (A_2, B_2) := ((B_1 \cap B_2)^{\Diamond}, B_1 \cap B_2)$ and $(A_1, B_1) \sqcup (A_2, B_2) := (A_1 \cup A_2, (A_1 \cup A_2)^{\Box})$. We have the following cases.

Case I: Suppose $A_1 = B_1^{\Diamond}$ and $A_2 = B_2^{\Diamond}$. Then $(B_1 \cap B_2)^{\Diamond} \subseteq B_1^{\Diamond} = A_1$ and $(B_1 \cap B_2)^{\Diamond} \subseteq B_2^{\Diamond} = A_2$ by (5) of Proposition 1. Now $(A_1 \cup A_2)^{\Box} = (B_1^{\Diamond} \cup B_2^{\Diamond})^{\Box}$. Using Proposition 1(11) on the rhs, we have $(A_1 \cup A_2)^{\Box} = (B_1 \cup B_2)^{\Diamond\Box}$ and using Proposition 1(6), we have $B_1, B_2 \subseteq (B_1 \cup B_2)^{\Diamond\Box} = (A_1 \cup A_2)^{\Box}$.
Case II: $A_1^{\Box} = B_1$ and $A_2^{\Box} = B_2$. This case is dealt similarly by replacing $^{\Box}$ with $^{\Diamond}$ as Case I.
Case III: Now let $A_1^{\Box} = B_1$ and $A_2 = B_2^{\Diamond}$. We have $(B_1 \cap B_2)^{\Diamond} \subseteq A_1^{\Box\Diamond} \subseteq A_1$ and $(B_1 \cap B_2)^{\Diamond} \subseteq B_2^{\Diamond} = A_2$, using Proposition 1(5) and (6). From Proposition 1(4) and (6), we have $B_1 = A_1^{\Box} \subseteq (A_1 \cup A_2)^{\Box}$ and $B_2 \subseteq B_2^{\Diamond\Box} \subseteq (A_1 \cup A_2)^{\Box}$. □

Are these the greatest and least upper bounds? Not necessarily so. In Example 2, consider the two elements $(\{Mo, Da\}, \{Fe\})$ and $(\{So, Da\}, \{Young\})$ in $\mathfrak{S}(\mathbb{K})$. $(\{Mo, Da\}, \{Fe\}) \sqcap (\{So, Da\}, \{Young\}) = (\phi, \phi)$ is a lower bound but is not the greatest lower bound as $(\{Da\}, \phi)$ is also a lower bound of the two object oriented semiconcepts. On the other hand, we can consider $(\{Fa\}, \phi)$ and $(\{Mo, Fa\}, \phi)$, for which $(\{Fa\}, \phi) \sqcup (\{Mo, Fa\}, \phi) = (\{Fa, Mo\}, \{old\})$, which is an upper bound but not least as $(\{Mo, Fa\}, \phi)$ is an upper bound of the two object oriented semiconcepts.

Following the approach of Wille, we now consider the set of idempotent elements in $\underline{\mathfrak{S}}(\mathbb{K})$ with respect to the operations $\sqcup$ and $\sqcap$.

$$\mathfrak{S}(\mathbb{K})_{\sqcup} := \{(A, B) \in \mathfrak{S}(\mathbb{K}) : (A, B) \sqcup (A, B) = (A, B)\}, \text{ and}$$
$$\mathfrak{S}(\mathbb{K})_{\sqcap} := \{(A, B) \in \mathfrak{S}(\mathbb{K}) : (A, B) \sqcap (A, B) = (A, B)\}.$$

It can be easily observed that

$$\mathfrak{S}(\mathbb{K})_{\sqcup} = \{(A, B) \in \mathfrak{S}(\mathbb{K}) : (A, A^{\Box}) = (A, B)\} = \{(A, A^{\Box}) : A \subseteq G\}, \text{ and}$$
$$\mathfrak{S}(\mathbb{K})_{\sqcap} = \{(A, B) \in \mathfrak{S}(\mathbb{K}) : (B^{\Diamond}, B) = (A, B)\} = \{(B^{\Diamond}, B) : B \subseteq M\}.$$

Note: For two object oriented semiconcepts $(A_1, B_1), (A_2, B_2)$, if the pair with componentwise set-theoretic intersection, viz. $(A_1 \cap A_2, B_1 \cap B_2)$, belongs to $\mathfrak{S}(\mathbb{K})$ then it must be the greatest lower bound of $(A_1, B_1), (A_2, B_2)$. A similar observation holds for $(A_1 \cup A_2, B_1 \cup B_2)$ and least upper bound of $(A_1, B_1), (A_2, B_2)$.

We obtain in a straightforward manner, the following results for any context $\mathbb{K}$.

Proposition 5.

1. $\mathfrak{S}(\mathbb{K})_\sqcap \cap \mathfrak{S}(\mathbb{K})_\sqcup = RO - L(\mathbb{K})$.
2. $\mathfrak{S}(\mathbb{K})_\sqcap \cup \mathfrak{S}(\mathbb{K})_\sqcup = \mathfrak{S}(\mathbb{K})$.
3. $(A_1, B_1) \sqcap (A_2, B_2) = (A_1, B_1) \sqcap (A_1, B_1)$ *and* $(A_1, B_1) \sqcup (A_2, B_2) = (A_2, B_2) \sqcup (A_2, B_2)$ *if and only if* $(A_1, B_1) \leq (A_2, B_2)$.

As done for semiconcepts, we define two operations on $\underline{\mathfrak{S}}(\mathbb{K})$:

$$x \vee y := \lrcorner(\lrcorner x \sqcap \lrcorner y), \text{ and } x \wedge y := \neg(\neg x \sqcup \neg y), \text{ for all } x, y \in \underline{\mathfrak{S}}(\mathbb{K}).$$

Theorem 2. *The following equations are valid in* $\underline{\mathfrak{S}}(\mathbb{K})$*:*

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$ *(1b)* $(x \sqcup x) \sqcup y = x \sqcup y$
(2a) $x \sqcap y = y \sqcap x$ *(2b)* $x \sqcup y = y \sqcup x$
(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ *(3b)* $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
(4a) $\lrcorner(x \sqcap x) = \lrcorner x$ *(4b)* $\neg(x \sqcup x) = \neg x$
(5a) $x \sqcap (x \sqcup y) = x \sqcap x$ *(5b)* $x \sqcup (x \sqcap y) = x \sqcup x$
(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$ *(6b)* $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
(7a) $x \sqcap (x \vee y) = x \sqcap x$ *(7b)* $x \sqcup (x \wedge y) = x \sqcup x$
(8a) $\lrcorner\lrcorner(x \sqcap y) = x \sqcap y$ *(8b)* $\neg\neg(x \sqcup y) = x \sqcup y$
(9a) $x \sqcap \lrcorner x = \bot$ *(9b)* $x \sqcup \neg x = \top$
(10a) $\lrcorner\bot = \top \sqcap \top$ *(10b)* $\neg\top = \bot \sqcup \bot$
(11a) $\neg\bot = \top$ *(11b)* $\lrcorner\top = \bot$
(12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$.

Observe that the equations stated in Theorem 2 are dual with respect to $\sqcup$ and $\sqcap$ in the equations defining a double Boolean algebra [13].

In our next result, we prove that $\underline{\mathfrak{S}}(\mathbb{K})$ is dually isomorphic to $\underline{\mathfrak{H}}(\mathbb{K}^c)$. In other words, we show the following for the algebraic structure $\underline{\mathfrak{H}}^\partial(\mathbb{K}^c)$ that is obtained from $\underline{\mathfrak{H}}(\mathbb{K}^c)$ by replacing $\sqcap$ with $\sqcup$ and $\sqcup$ with $\sqcap$.

Theorem 3. *For a context* $\mathbb{K}$*,* $\underline{\mathfrak{S}}(\mathbb{K})$ *is isomorphic to* $\underline{\mathfrak{H}}^\partial(\mathbb{K}^c)$.

Proof. We define a map $h : \mathfrak{S}(\mathbb{K}) \rightarrow \mathfrak{H}(\mathbb{K}^c)$ such that $h((A, B)) := (A^c, B)$, where $(A, B) \in \mathfrak{S}(\mathbb{K})$. This map is well-defined and onto by Proposition 3. It is trivially one-one. To show h is a homomorphism, we check the case for $\sqcap$.

$$
\begin{aligned}
h((A,B)\sqcap(A_1,B_1)) =& h((B\cap B_1)^{\Diamond}, B\cap B_1)\\
=& ((B\cap B_1)^{\Diamond c}, B\cap B_1)\\
=& ((B\cap B_1)'_{-R}, B\cap B_1) \quad \text{(by Proposition 1(4) and (5))}\\
=& (A^c,B)\sqcup(A_1^c,B_1) \text{ in } \underline{\mathfrak{H}}(\mathbb{K}^c)\\
=& (A^c,B)\sqcap(A_1^c,B_1) \text{ in } \underline{\mathfrak{H}^{\partial}}(\mathbb{K}^c)\\
=& h((A,B))\sqcap h((A_1,B_1))
\end{aligned}
$$

$h((G,M)) = (\phi, M) = \bot$, which is the top element of $\underline{\mathfrak{H}^{\partial}}(\mathbb{K}^c)$ and $h((\phi,\phi)) = (G,\phi) = \top$, the bottom element of $\underline{\mathfrak{H}^{\partial}}(\mathbb{K}^c)$. The case for $\sqcup$ is similar. □

Recall the algebras of idempotent elements of semiconcepts defined in Sect. 2.1. We get the following relationships.

Corollary 1.

1. *$\underline{\mathfrak{S}}(\mathbb{K})_{\sqcap}$ is dually isomorphic to $\underline{\mathfrak{H}}(\mathbb{K}^c)_{\sqcup}$.*
2. *$\underline{\mathfrak{S}}(\mathbb{K})_{\sqcup}$ is dually isomorphic to $\underline{\mathfrak{H}}(\mathbb{K}^c)_{\sqcap}$.*

Proof. (1) Let $(A,B) \in \mathfrak{S}(\mathbb{K})$ then $(A^c,B) \in \mathfrak{H}(\mathbb{K}^c)$. Using definitions of $\sqcap, \sqcup$ in algebras of object oriented semiconcepts and semiconcepts respectively, we have

$$(A,B)\sqcap(A,B) = (B^{\Diamond},B) \text{ and } (A^c,B)\sqcup(A^c,B) = (B|'_{-R},B).$$

Therefore $(A,B) = (A,B)\sqcap(A,B)$ if and only if $(A,B) = (B^{\Diamond},B)$, i.e. if and only if $A^c = B^{\Diamond c}$.

On the other hand, $(A^c,B) = (A^c,B)\sqcup(A^c,B)$ if and only if $(A^c,B) = (B|'_{-R},B)$, i.e. if and only if $A^c = B|'_{-R}$. From Proposition 1(7), we have $B^{\Diamond c} = B|'_{-R}$ and hence $(A,B)\sqcap(A,B) = (A,B)$ if and only if $(A^c,B)\sqcup(A^c,B) = (A^c,B)$. Similarly one can show that $(A,B)\sqcup(A,B) = (A,B)$ if and only if $(A^c,B)\sqcap(A^c,B) = (A^c,B)$. Therefore image of $\underline{\mathfrak{S}}(\mathbb{K})_{\sqcap}$ under h defined in Theorem 3 is equal to $\underline{\mathfrak{H}}(\mathbb{K}^c)_{\sqcup}$ and it is also clear that h is an isomorphism from $\underline{\mathfrak{S}}(\mathbb{K})_{\sqcap}$ to $\underline{\mathfrak{H}}^{\partial}(\mathbb{K}^c)_{\sqcup}$.
Proof of (2) is similar. □

4 Semi-topological Operators on $\mathfrak{S}(\mathbb{K})$

Rough concept analysis deals with the necessity and possibility operators $^{\Box}$ and $^{\Diamond}$. As mentioned in Proposition 2, $^{\Box\Diamond}$ is a closure operator and $^{\Diamond\Box}$ is an interior operator. We use this idea and define two unary operators C, I on the set $\mathfrak{S}(\mathbb{K})$ of object oriented semiconcepts. As we shall see, the two operators turn out to have *semi-topological* properties [11].

Definition 7. *For any $(A,B) \in \mathfrak{S}(\mathbb{K})$,*

$$C((A,B)) := (A^{\Diamond\Box}, A^{\Diamond\Box\Box}),$$
$$I((A,B)) := (B^{\Box\Diamond\Diamond}, B^{\Box\Diamond}).$$

Note. Using the algebraic operations on object oriented semiconcepts, we get for any $x \in \mathfrak{S}(\mathbb{K})$, $C(x) = \neg(\neg x \sqcap \neg x)$, and $I(x) = \lrcorner(\lrcorner x \sqcup \lrcorner x)$.

Lemma 1. *Let $x, y \in \mathfrak{S}(\mathbb{K})$. I has the following properties.*

1. *If $x \leq y$ then $I(x) \leq I(y)$.*
2. *$II(x) = I(x)$.*
3. *$I(x) \sqcap x = I(x) = I(x) \sqcap I(x)$ and $x \sqcup I(x) = x \sqcup x$.*
4. *$I(\top) = \lrcorner(\bot \sqcup \bot)$.*
5. *$I(x \sqcap y) \leq I(x) \sqcap I(y)$.*

Proof. (1) Let $x, y \in \mathfrak{S}(\mathbb{K}) = \mathfrak{S}(\mathbb{K})_\sqcup \cup \mathfrak{S}(\mathbb{K})_\sqcap$ such that $x \leq y$.

Case I: Suppose $x, y \in \mathfrak{S}(\mathbb{K})_\sqcap$. Without loss of generality, we assume that $x = (A^\Diamond, A)$ and $y = (B^\Diamond, B)$ where $A, B \subseteq M$. Then

$$I(x) = \lrcorner(\lrcorner(A^\Diamond, A) \sqcup \lrcorner(A^\Diamond, A)) = \lrcorner(A^{c\Diamond}, A^{c\Diamond\Box}) = (A^{c\Diamond\Box c\Diamond}, A^{c\Diamond\Box c}).$$

Similarly we deduce that $I(y) = (B^{c\Diamond\Box c\Diamond}, B^{c\Diamond\Box c})$. Now $x \leq y$ implies that $A \subseteq B$, which implies that $A^{c\Diamond\Box c} \subseteq B^{c\Diamond\Box c}$ and from this we have $A^{c\Diamond\Box c\Diamond} \subseteq B^{c\Diamond\Box c\Diamond}$ and hence $I(x) \leq I(y)$.

Case II: If $x, y \in \mathfrak{S}(\mathbb{K})_\sqcup$ then let $x = (A, A^\Box)$ and $y = (B, B^\Box)$, $A, B \subseteq G$. Then

$$I(x) = (A^{\Box c\Diamond\Box c\Diamond}, A^{\Box c\Diamond\Box c}) \text{ and } I(y) = (B^{\Box c\Diamond\Box c\Diamond}, B^{\Box c\Diamond\Box c}).$$

As $x \leq y$, $A^\Box \subseteq B^\Box$, which implies that $A^{\Box c\Diamond\Box c} \subseteq B^{\Box c\Diamond\Box c}$. So $A^{\Box c\Diamond\Box c\Diamond} \subseteq B^{\Box c\Diamond\Box c\Diamond}$ and we get $I(x) \leq I(y)$.

Case III: If $x \in \mathfrak{S}(\mathbb{K})_\sqcup$ and $y \in \mathfrak{S}(\mathbb{K})_\sqcap$, we assume that $x = (A, A^\Box)$ and $y = (B^\Diamond, B)$. Then $I(x) = (A^{\Box c\Diamond\Box c\Diamond}, A^{\Box c\Diamond\Box c})$ and $I(y) = (B^{c\Diamond\Box c\Diamond}, B^{c\Diamond\Box c})$. $x \leq y$ implies that $A^\Box \subseteq B$, which gives $A^{\Box c\Diamond\Box c} \subseteq B^{c\Diamond\Box c}$. From this we have $A^{\Box c\Diamond\Box c\Diamond} \subseteq B^{c\Diamond\Box c\Diamond}$ and hence $I(x) \leq I(y)$.

(2) Let $x \in \mathfrak{S}(\mathbb{K})$.

$$\begin{aligned} I(I(x)) &= \lrcorner(\lrcorner\lrcorner(\lrcorner x \sqcup \lrcorner x) \sqcup \lrcorner\lrcorner(\lrcorner x \sqcup \lrcorner x)) \\ &= \lrcorner(((\lrcorner x \sqcup \lrcorner x) \sqcap (\lrcorner x \sqcup \lrcorner x)) \sqcup ((\lrcorner x \sqcup \lrcorner x) \sqcap (\lrcorner x \sqcup \lrcorner x))) \\ &= \lrcorner(((\lrcorner x \sqcup \lrcorner x) \sqcup (\lrcorner x \sqcup \lrcorner x)) \sqcap ((\lrcorner x \sqcup \lrcorner x) \sqcup (\lrcorner x \sqcup \lrcorner x))) \\ &= \lrcorner((\lrcorner x \sqcup \lrcorner x) \sqcap (\lrcorner x \sqcup \lrcorner x)) \\ &= \lrcorner(\lrcorner x \sqcup \lrcorner x) = I(x). \end{aligned}$$

(3) Let $x \in \mathfrak{S}(\mathbb{K}) = \mathfrak{S}(\mathbb{K})_\sqcup \cup \mathfrak{S}(\mathbb{K})_\sqcap$.

Case I: Let $x \in \mathfrak{S}(\mathbb{K})_\sqcup$. Without loss of generality we assume that $x = (A, A^\Box)$, for some $A \subseteq G$. $I(x) = (A^{\Box c\Diamond\Box c\Diamond}, A^{\Box c\Diamond\Box c})$ and from this we get $I(x) \sqcap x = (A^{\Box c\Diamond\Box c\Diamond}, A^{\Box c\Diamond\Box c}) \sqcap (A, A^\Box) = ((A^{\Box c\Diamond\Box c} \cap A^\Box)^\Diamond, A^{\Box c\Diamond\Box c} \cap A^\Box)$. Now $A^{\Box c} \subseteq A^{\Box c\Diamond\Box}$ for any subset A of G. So $A^{\Box c\Diamond\Box c} \subseteq A^\Box$ and hence $I(x) \sqcap x = (A^{\Box c\Diamond\Box c\Diamond}, A^{\Box c\Diamond\Box c}) = I(x)$.

Case II: If $x \in \mathfrak{S}(\mathbb{K})_\sqcap$, let $x = (B^\Diamond, B)$ for some $B \subseteq M$. Then $I(x) = (B^{c\Diamond\Box c\Diamond}, B^{c\Diamond\Box c})$ whence $I(x) \sqcap x = (B^{c\Diamond\Box c\Diamond}, B^{c\Diamond\Box c}) \sqcap (B^\Diamond, B) = ((B^{c\Diamond\Box c} \cap B)^\Diamond, (B^{c\Diamond\Box c} \cap B)) = (B^{c\Diamond\Box c\Diamond}, B^{c\Diamond\Box c}) = I(x)$, as $B^{c\Diamond\Box c} \subseteq B$.

Since for any $x \in \mathfrak{S}(\mathbb{K})$ say $x = (A, B)$, $I(x) = (B^{c\Diamond\Box c\Diamond}, B^{c\Diamond\Box c}) = (D^\Diamond, D)$, where $D = B^{c\Diamond\Box c}$, we have $I(x) \in \mathfrak{S}(\mathbb{K})_\sqcap$ for all $x \in \mathfrak{S}(\mathbb{K})$). Thus $I(x) \sqcap I(x) = I(x)$ and so $I(x) \sqcap x = I(x) = I(x) \sqcap I(x)$.

Now we will show that $x \sqcup I(x) = x \sqcup x$. Let $x \in \mathfrak{S}(\mathbb{K}) = \mathfrak{S}(\mathbb{K})_{\sqcup} \cup \mathfrak{S}(\mathbb{K})_{\sqcap}$.

Case I: If $x \in \mathfrak{S}(\mathbb{K})_{\sqcup}$, say $x = (A, A^{\Box})$ for some $A \subseteq G$.

$$\begin{aligned}(A, A^{\Box}) \sqcup I((A, A^{\Box})) &= (A, A^{\Box}) \sqcup (A^{\Box c \Diamond \Box c \Diamond}, A^{\Box c \Diamond \Box c}) \\ &= (A \cup A^{\Box c \Diamond \Box c \Diamond}, (A \cup A^{\Box c \Diamond \Box c \Diamond})^{\Box}) \\ &= (A, A^{\Box}) \text{ because } A^{\Box c \Diamond \Box c \Diamond} \subseteq A^{\Box \Diamond} \subseteq A \\ &= (A, A^{\Box}) \sqcup (A, A^{\Box}) = x \sqcup x.\end{aligned}$$

Case II: If $x \in \mathfrak{S}(\mathbb{K})_{\sqcap}$, let $x = (B^{\Diamond}, B)$ for some $B \subseteq M$.

$$\begin{aligned}(B^{\Diamond}, B) \sqcup I((B^{\Diamond}, B)) &= (B^{\Diamond}, B) \sqcup (B^{c \Diamond \Box c \Diamond}, B^{c \Diamond \Box c}) \\ &= (B^{\Diamond} \cup B^{c \Diamond \Box c \Diamond}, (B^{\Diamond} \sqcup B^{c \Diamond \Box c \Diamond})^{\Box}) \\ &= (B^{\Diamond}, B^{\Diamond \Box}) \text{ because } B^{c \Diamond \Box c} \subseteq B \\ &= (B^{\Diamond}, B) \sqcup (B^{\Diamond}, B) = x \sqcup x.\end{aligned}$$

(4) $I(\top) = \lrcorner(\lrcorner\top \sqcup \lrcorner\top) = \lrcorner(\bot \sqcup \bot)$.
(5) Let $x, y \in \mathfrak{S}(\mathbb{K}) = \mathfrak{S}(\mathbb{K})_{\sqcap} \cup \mathfrak{S}(\mathbb{K})_{\sqcup}$.
Case I: Let $x \in \mathfrak{S}(\mathbb{K})_{\sqcup}$ and $y \in \mathfrak{S}(\mathbb{K})_{\sqcap}$. Without loss of generality we assume that $x = (A, A^{\Box})$ and $y = (B^{\Diamond}, B)$, where $A \subseteq G$ and $B \subseteq M$. Then $I(x) = (A^{\Box c \Diamond \Box c \Diamond}, A^{\Box c \Diamond \Box c})$ and $I(y) = (B^{c \Diamond \Box c \Diamond}, B^{c \Diamond \Box c})$.
$I(x) \sqcap I(y) = ((A^{\Box c \Diamond \Box c} \cap B^{c \Diamond \Box c})^{\Diamond}, A^{\Box c \Diamond \Box c} \cap B^{c \Diamond \Box c})$ and $I(x \sqcap y) = I((A^{\Box} \cap B)^{\Diamond}, A^{\Box} \cap B) = ((A^{\Box} \cap B)^{c \Diamond \Box c \Diamond}, (A^{\Box} \cap B)^{c \Diamond \Box c})$. Now $A^{\Box c} \subseteq (A^{\Box} \cap B)^{c}$ and $B^{c} \subseteq (A^{\Box} \cap B)^{c}$. From this inequality we have, $A^{\Box c \Diamond \Box} \subseteq (A^{\Box} \cap B)^{c \Diamond \Box}$ and $B^{c \Diamond \Box} \subseteq (A^{\Box} \cap B)^{c \Diamond \Box}$. This implies that $(A^{\Box} \cap B)^{c \Diamond \Box c} \subseteq A^{\Box c \Diamond \Box c}$ and $(A^{\Box} \cap B)^{c \Diamond \Box c} \subseteq B^{c \Diamond \Box c}$. So $(A^{\Box} \cap B)^{c \Diamond \Box c} \subseteq A^{\Box c \Diamond \Box c} \cap B^{c \Diamond \Box c}$ and $(A^{\Box} \cap B)^{c \Diamond \Box c \Diamond} \subseteq (A^{\Box c \Diamond \Box c} \cap B^{c \Diamond \Box c})^{\Diamond}$ and hence $I(x \sqcap y) \leq I(x) \sqcap I(y)$.
Case II: If $x, y \in \mathfrak{S}(\mathbb{K})_{\sqcup}$, let us assume that $x = (A, A^{\Box})$ and $y = (B, B^{\Box})$.

Then $I(x) \sqcap I(y) = ((A^{\Box c \Diamond \Box c} \cap B^{\Box c \Diamond \Box c})^{\Diamond}, A^{\Box c \Diamond \Box c} \cap B^{\Box c \Diamond \Box c})$ and $I(x \sqcap y) = I((A \cap B)^{\Box \Diamond}, (A \cap B)^{\Box}) = ((A \cap B)^{\Box c \Diamond \Box c \Diamond}, (A \cap B)^{\Box c \Diamond \Box c})$. Now $(A \cap B)^{\Box} \subseteq A^{\Box}$ and $(A \cap B)^{\Box} \subseteq B^{\Box}$. From this we have,

$$\begin{aligned}A^{\Box c} \subseteq (A \cap B)^{\Box c} &\Rightarrow A^{\Box c \Diamond \Box} \subseteq (A \cap B)^{\Box c \Diamond \Box} \\ &\Rightarrow (A \cap B)^{\Box c \Diamond \Box c} \subseteq A^{\Box c \Diamond \Box c}.\end{aligned}$$

Similarly, one can prove that $(A \cap B)^{\Box c \Diamond \Box c} \subseteq B^{\Box c \Diamond \Box c}$. From this inequality we have $(A \cap B)^{\Box c \Diamond \Box c} \subseteq A^{\Box c \Diamond \Box c} \cap B^{\Box c \Diamond \Box c}$ and $(A \cap B)^{\Box c \Diamond \Box c \Diamond} \subseteq (A^{\Box c \Diamond \Box c} \cap B^{\Box c \Diamond \Box c})^{\Diamond}$. Hence $I(x \sqcap y) \leq I(x) \sqcap I(y)$.
Case III: If $x, y \in \mathfrak{S}(\mathbb{K})_{\sqcap}$, the proof is similar to Case II. □

Dually, one can prove the following for the operator C on $\mathfrak{S}(\mathbb{K})$.

Lemma 2. *For all $x, y \in \mathfrak{S}(\mathbb{K})$,*

1. *If $x \leq y$ then $C(x) \leq C(y)$*
2. *$CC(x) = C(x)$*
3. *$C(x) \sqcup x = C(x) = C(x) \sqcup C(x)$ and $x \sqcap C(x) = x \sqcap x$*
4. *$C(\bot) = \neg(\top \sqcap \top)$*
5. *$C(x) \sqcup C(y) \leq C(x \sqcup y)$.*

4.1 Semi-topological Double Boolean Algebra

Recall our observation after Theorem 2 that $\underline{\mathfrak{S}}(\mathbb{K})$ satisfies the dual of all equations defining a double Boolean algebra [13]. In this section, we deal with such an abstract 'dual double Boolean algebra', and for the sake of simplicity, retain the name double Boolean algebra for the structure. More precisely, we have the following definition.

Definition 8. *A* double Boolean algebra $(A, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ *is an abstract algebra which satisfies the following properties: For any $x, y, z \in A$.*

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$
(2a) $x \sqcap y = y \sqcap x$
(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
(4a) $\lrcorner(x \sqcap x) = \lrcorner x$
(5a) $x \sqcap (x \sqcup y) = x \sqcap x$
(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$
(7a) $x \sqcap (x \vee y) = x \sqcap x$
(8a) $\lrcorner\lrcorner(x \sqcap y) = x \sqcap y$
(9a) $x \sqcap \lrcorner x = \bot$
(10a) $\lrcorner\bot = \top \sqcap \top$
(11a) $\neg\bot = \top$

(1b) $(x \sqcup x) \sqcup y = x \sqcup y$
(2b) $x \sqcup y = y \sqcup x$
(3b) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
(4b) $\neg(x \sqcup x) = \neg x$
(5b) $x \sqcup (x \sqcap y) = x \sqcup x$
(6b) $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
(7b) $x \sqcup (x \wedge y) = x \sqcup x$
(8b) $\neg\neg(x \sqcup y) = x \sqcup y$
(9b) $x \sqcup \neg x = \top$
(10b) $\neg\top = \bot \sqcup \bot$
(11b) $\lrcorner\top = \bot$

(12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$,

where $\vee$ and $\wedge$ are defined as $x \vee y := \lrcorner(\lrcorner x \sqcap \lrcorner y)$, and $x \wedge y := \neg(\neg x \sqcup \neg y)$. $\neg$ is called the negation *and $\lrcorner$ the* opposition.

Corollary 2. $\underline{\mathfrak{S}}(\mathbb{K})$ *is a double Boolean algebra.*

A quasi-order (reflexive and transitive relation) on a double Boolean algebra may be defined [7] for all $x, y \in \mathbf{A}$ as:

$$x \sqsubseteq y \text{ if and only if } x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

Remark. As we mentioned in Sect. 1, the algebraic structure of a weakly dicomplemented lattice [8,15] also emerged in the context of defining negations in FCA. We now compare this structure with the double Boolean algebra of Definition 8. Note that these are algebras of the same type (2,2,1,1,0,0). However, it can be seen that these are different with respect to the defining axioms. Firstly, in a weakly dicomplemented lattice $(L, \vee, \wedge, ^{\triangle}, ^{\triangledown}, 1, 0)$, the reduct $(L, \vee, \wedge, 1, 0)$ is a *lattice*, while a double Boolean algebra $(A, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ need not be a lattice with respect to the $\sqcup, \sqcap$ operations, as shown in Sect. 3.1. Secondly, to force another comparison, suppose the lattice meet and join in a weakly dicomplemented lattice are relaxed to be lower and upper bound operations $\sqcup, \sqcap$ satisfying the axioms 1a-b, 2a-b, 3a-b, 5a-b and 12 in Definition 8. The remaining defining axioms of the negations $^{\triangle}, ^{\triangledown}$ (cf. [8]) in a weakly dicomplemented lattice are retained. Will a double Boolean algebra then be a special case of such a structure? We find that the negations in the two structures behave differently as well. In particular, it can be shown that the axiom $(x \wedge y) \vee (x \wedge y^{\triangle}) = x$ for $^{\triangle}$ need not hold in a double Boolean algebra, irrespective of whether $^{\triangle}$ is taken as the negation ($\neg$) or opposition ($\lrcorner$) of the double Boolean algebra. Indeed, consider Example 1: take two object oriented semiconcepts $x := (\{Fa, Da\}, \phi)$ and $y := (\phi, \phi)$. $(x \sqcap y) \sqcup (x \sqcap \neg y) = (\phi, \phi) \neq x$. If we take $x := (G, \{Ma, Fe\})$ and $y := (\{Fa\}, \phi)$ then $(x \sqcap y) \sqcup (x \sqcap \lrcorner y) = (G, M) \neq x$. On the other hand, if we force $\sqcap, \sqcup$ in a double Boolean algebra to be infimum and supremum operators respectively, we get the equations $\lrcorner\lrcorner x = x$ and $\neg\neg x = x$ from (8a) and (8b) of Definition 8. However, these do not hold in general for the negations $^{\triangle}, ^{\triangledown}$ in a weakly dicomplemented lattice, so that the latter is not an example of such a special case of a double Boolean algebra either.

Now we define a semi-topological double Boolean algebra.

Definition 9. *A* semi-topological double Boolean algebra *is an abstract algebra* $\boldsymbol{A} := (A, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \boldsymbol{I}, \boldsymbol{C})$*, where* $(A, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ *is a double Boolean algebra, and the unary operators* $\boldsymbol{I}$ *and* $\boldsymbol{C}$ *satisfy the following equations for any* $x, y \in A$*.*

$(sa)^1$ $\boldsymbol{I}(x) \sqcap x = \boldsymbol{I}(x) \sqcap \boldsymbol{I}(x)$ *and* $x \sqcup \boldsymbol{I}(x) = x \sqcup x$

$(sa)^2$ $\boldsymbol{I}(x \sqcap y) \sqsubseteq \boldsymbol{I}(x) \sqcap \boldsymbol{I}(y)$

$(sa)^3$ $\boldsymbol{I}(\boldsymbol{I}(x)) = \boldsymbol{I}(x)$

$(sb)^1$ $\boldsymbol{C}(x) \sqcup x = \boldsymbol{C}(x) \sqcup \boldsymbol{C}(x)$ *and* $\boldsymbol{C}(x) \sqcap x = x \sqcap x$

$(sb)^2$ $\boldsymbol{C}(x) \sqcup \boldsymbol{C}(y) \sqsubseteq \boldsymbol{C}(x \sqcup y)$

$(sb)^3$ $\boldsymbol{C}(\boldsymbol{C}(x)) = \boldsymbol{C}(x)$

Theorem 4. $\underline{\mathfrak{S}}(\mathbb{K}) := (\mathfrak{S}(\mathbb{K}, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, I_1, C_1)$ *is a semi-topological double Boolean algebra.*

Proof. Follows from Theorem 2 and Lemmas 1, 2. □

5 Conclusion

This work introduces the notion of negation in the framework of object oriented concepts in rough concept analysis, and object oriented semiconcepts are defined. The algebra that these semiconcepts form is shown to be (a dual of) double Boolean algebra. Moreover, two unary operators are introduced in this algebra, leading to the definition of a semi-topological double Boolean algebra.

The proposal opens up several directions of further work, including possible applications. The definition of a new algebraic structure warrants some immediate algebraic investigations, such as investigation for representation theorems. Definition of a negation can now facilitate studies in the direction of contextual logic for rough sets. Besides, one can follow up the entire study in the framework of property oriented concepts.

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