Rule Induction Based on Indiscernible Classes from Rough Sets in Information Tables with Continuous Values

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Abstract. Rule induction based on indiscernible classes from neighborhood rough sets is described in information tables with continuous values. An indiscernible range that a value has in an attribute is determined by a threshold on that attribute. The indiscernible class of every object is derived from using the indiscernible range. First, lower and upper approximations are described in complete information tables by using indiscernible classes. Rules are obtained from the approximations. A rule that an object supports, which is called a single rule, is short of applicability. To improve the applicability of rules, a series of single rules is put into one rule expressed in an interval value, which is called a combined rule. Second, these are addressed in incomplete information tables. Incomplete information is expressed in a set of values or an interval value. Two types of indiscernible classes; namely, certainly and possibly indiscernible ones, are obtained from in an information table. The actual indiscernibility class is between the certainly and possibly indiscernible classes. The family of indiscernible classes of an object has a lattice structure. The minimal element is the certainly indiscernible class while the maximal one is the possibly indiscernible class. By using certainly and possibly indiscernible classes, we obtain four types of approximations: certain lower, certain upper, possible lower, and possible upper approximations. From these approximations we obtain four types of combined rules: certain and consistent, certain and inconsistent, possible and consistent, and possible and inconsistent ones. These combined rules have greater applicability than single rules that individual objects support.

Keywords: Neighborhood rough sets · Rule induction
Incomplete information · Indiscernible classes
Lower and upper approximations · Continuous values

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1 Introduction

Rough sets, constructed by Pawlak [12], are used as an effective method for data mining. The framework is usually applied to information tables with nominal attributes and creates fruitful results in various fields. However, we are frequently faced with attributes taking continuous values, when we describe properties of an object in our daily life. Therefore, we describe rough sets in information tables with continuous values.

Ways how to deal with attributes taking continuous values are broadly classified into two approaches. One is to discretize a continuous domain by dividing it into a collection of disjunctive intervals. Objects included in an interval are regarded as indistinguishable. From this indistinguishability the family of indiscernible classes is derived [1]. Results strongly depend on how discretization is made. Especially, objects that are located in the proximity of the boundary of intervals are strongly affected by discretization. This leads to that results abruptly change by a little alteration of discretization. The other is a way using neighborhood [7]. In this approach when the distance of an object to another one on an attribute is less than or equal to a given threshold, two objects are regarded as indistinguishable on the attribute. Results gradually change as the threshold changes. So, we use the latter approach.

Rules are induced from lower and upper approximations. Concretely speaking, when objects o and o' are included in the approximations, let single rules $a_i = 3.60 \rightarrow a_j = v$ and $a_i = 3.73 \rightarrow a_j = v$ be induced, where objects o and o' are characterized by values 3.60 and 3.73 of attribute a_i and the set approximated is specified by value v of attribute a_j. For example, value 3.66 of attribute a_i is not indiscernible with 3.60 and 3.73 under the threshold 0.05. Therefore, we cannot say anything from these single rules for a rule supported by an object with value 3.66 of attribute a_i. This means that the single rules are short of applicability. To improve such applicability, we consider a combined rule that is derived from a series of single rules supported by individual objects.

In addition, we are frequently confronted with incomplete information in daily life. We cannot sufficiently utilize information obtained from our daily life unless we deal with incomplete information. We express incomplete information in a partial value or an interval value. A missing value that means unknown in an attribute is expressed in all elements over the domain of the attribute. For example, the domain is given in the interval $[1.23, 4.45]$, the missing value is expressed in $[1.23, 4.45]$.

Most of authors fix the indiscernibility of an object with incomplete information with another object [3,16–18], as was done by Kryszkiewicz [4]. However, object o characterized by a value with incomplete information has two possibilities. One possibility is that the object o may have the same value as another one o'; namely, the two objects may be indiscernible. The other possibility is that o may have a different value from o'; namely, the two objects may be discernible. To fix the indiscernibility is to take into account only one of the two possibilities. Therefore, this treatment creates poor results and induces information loss [9,15]. We do not fix the indiscernibility of objects with incomplete information and simultaneously deal with both possibilities. This can be realized by dealing with objects having incomplete information from viewpoints of certainty and possibility [10], as was done by Lipski in the field of incomplete databases [5,6].

We have an approach based on possible world from the viewpoints of certainty and possibility. This way creates possible tables. Unfortunately, infinite possible tables can be derived from an information table with continuous values. Another way uses possible classes of an object, in which the object is possibly indiscernible with anyone [8]. The number of possible classes grows exponentially, as the number of values with incomplete information increases. However, this difficulty can be avoided by using minimum and maximum possible classes in the case of nominal attributes [10]. In this work, we apply this approach to information tables with continuous values.

The paper is organized as follows. In Sect. 2, an approach using indiscernible classes is addressed in complete information tables. In Sect. 3, we develop the approach in incomplete information tables. This is described from two viewpoints of certainty and possibility. In Sect. 4, conclusions are addressed.

2 Rough Sets by Using Indiscernible Classes in Complete Information Systems with Continuous Values

A data set is represented as a two-dimensional table, called an information table. In the information table, each row and each column represent an object and an attribute, respectively. A mathematical model of an information table with complete information is called a complete information system. The complete information system is a triplet expressed by $(U, AT, \{D(a_i) \mid a_i \in AT\})$. U is a non-empty finite set of objects, which is called the universe. AT is a non-empty finite set of attributes such that $a_i : U \rightarrow D(a_i)$ for every $a_i \in AT$ where $D(a_i)$ is the domain of attribute a_i.

Indiscernible class $[o]_{a_i}$ for object o on a_i is:

$$[o]_{a_i} = \{o' \mid |a_i(o) - a_i(o')| \leq \delta_{a_i}\}, \tag{1}$$

where $a_i(o)$ is the value for attribute a_i of object o and δ_{a_i} is a threshold that denotes a range in which $a_i(o)$ is indiscernible with $a_i(o')$. The indiscernible class is a tolerance class. Using the tolerance class, rough sets are generalized [14]. And recently it is used in decision rule induction [13].

Family $\mathcal{F}_{a_i}$ of indiscernible classes on a_i is:

$$\mathcal{F}_{a_i} = \{[o]_{a_i} \mid o \in U\}, \tag{2}$$

where $\cup_i [o]_{a_i} = U$. Using indiscernible classes, lower approximation $\underline{apr}_{a_i}(\mathcal{O})$ and upper approximation $\overline{apr}_{a_i}(\mathcal{O})$ of set $\mathcal{O}$ of objects for a_i are:

$$\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid [o]_{a_i} \subseteq \mathcal{O}\}, \tag{3}$$

$$\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid [o]_{a_i} \cap \mathcal{O} \neq \emptyset\}. \tag{4}$$

Proposition 1. If $\delta 1 \leq \delta 2$, then $\underline{apr}^{\delta 1}_{a_i}(\mathcal{O}) \supseteq \underline{apr}^{\delta 2}_{a_i}(\mathcal{O})$ and $\overline{apr}^{\delta 1}_{a_i}(\mathcal{O}) \subseteq \overline{apr}^{\delta 2}_{a_i}(\mathcal{O})$, where $\underline{apr}^{\delta 1}_{a_i}(\mathcal{O})$ and $\overline{apr}^{\delta 1}_{a_i}(\mathcal{O})$ are lower and upper approximations under threshold $\delta 1$ of attribute a_i and $\underline{apr}^{\delta 2}_{a_i}(\mathcal{O})$ and $\overline{apr}^{\delta 2}_{a_i}(\mathcal{O})$ are lower and upper approximations under threshold $\delta 2$ of attribute a_i.

For object o in the lower approximation of $\mathcal{O}$, all objects with which o is indiscernible are included in $\mathcal{O}$; namely, $[o]_{a_i} \subseteq \mathcal{O}$. On the other hand, for an object o in the upper approximation of $\mathcal{O}$, some objects with which o is indiscernible are in $\mathcal{O}$; namely, $[o]_{a_i} \cap \mathcal{O} \neq \emptyset$. Thus, $\underline{apr}_{a_i}(\mathcal{O}) \subseteq \overline{apr}_{a_i}(\mathcal{O})$.

Rules are induced from lower and upper approximations. Let $\mathcal{O}$ be specified by restriction $a_j = x$. Object $o \in \underline{apr}_{a_i}(\mathcal{O})$ consistently supports a single rule $a_i = a_i(o) \rightarrow a_j = x$. Object $o \in \overline{apr}_{a_i}(\mathcal{O})$ inconsistently supports a single rule $a_i = a_i(o) \rightarrow a_j = x$. The degree of consistency, called accuracy, is $|[o]_{a_i} \cap \mathcal{O}|/|\mathcal{O}|$.

Since attribute a_i has the continuous domain, the antecedent part of single rules that individual objects support is usually different. We obtain lots of single rules, but they have a drawback for applicability. For example, let two values $a_i(o)$ and $a_i(o')$ be 3.65 and 3.75 for objects o and o' in $\underline{apr}_{a_i}(\mathcal{O})$. When $\mathcal{O}$ is specified by restriction $a_j = x$, o and o' support single rules $a_i = 3.65 \rightarrow a_j = x$ and $a_i = 3.75 \rightarrow a_j = x$, respectively. By using these rules, we can say that a object having value 3.68 of a_i, indiscernible with 3.65 under $\delta_{a_i} = 0.03$, supports $a_i = 3.68 \rightarrow a_j = x$. However, we cannot at all say anything for a rule supported by an object with value 3.70 discernible with 3.65 and 3.75. This shows that a single rule is short of applicability.

To improve the applicability of rules, we combine a series of single rules into one rule, which is called a combined rule. Let objects in U be aligned in ascending order of $a_i(o)$ and be attached the serial superscript with 1 to N_U where $|U| = N_U$. $\underline{apr}_{a_i}(\mathcal{O})$ and $\overline{apr}_{a_i}(\mathcal{O})$ consist of collections of objects with serial superscripts. For example, $\underline{apr}_{a_i}(\mathcal{O}) = \{\cdots, o^h, o^{h+1}, \cdots, o^{k-1}, o^k, \cdots\}$ $(h \leq k)$. Let o^l in $\underline{apr}_{a_i}(\mathcal{O})$ support a single rule $a_i = a_i(o^l) \rightarrow a_j = x$. Then, single rules derived from collection $(o^h, o^{h+1}, \cdots, o^{k-1}, o^k)$ can be put into one combined rule $a_i = [a_i(o^h), a_i(o^k)] \rightarrow a_j = x$.

Next, when a_j is an attribute with the continuous domain, $\mathcal{O}$ is specified by a restriction with an interval value. The interval value has the lower and the upper bounds that are existing values of attribute. Let the objects be aligned in ascending order of values of a_j and be attached the serial superscript with 1 to N_U. For example, using the ordered objects, $\mathcal{O}$ is specified like $\mathcal{O} = \{o \mid a_j(o) \geq a_j(o^m) \wedge a_j(o) \leq a_j(o^n)\}$ with $m \leq n$; in other words, $\mathcal{O}$ is specified by restriction $a_j = [a_i(o^m), a_i(o^n)]$. In the case, the combined rule, derived from collection $(o^h, o^{h+1}, \cdots, o^{k-1}, o^k)$, is expressed with $a_i = [a_i(o^h), a_i(o^k)] \rightarrow a_j = [a_i(o^m), a_i(o^n)]$. The accuracy of the combined rule is $\min_{h \leq s \leq k} |[o^s]_{a_i} \cap \mathcal{O}|/|\mathcal{O}|$.

Proposition 2. Let $\underline{r}$ and $\overline{r}$ be sets of combined rules obtained from $\underline{apr}_{a_i}(\mathcal{O})$ and $\overline{apr}_{a_i}(\mathcal{O})$, respectively. If $(a_i = [l, u] \rightarrow W) \in \underline{r}$, then $\exists l' \leq l, \exists u' \geq u$ $(a_i = [l', u'] \rightarrow W) \in \overline{r}$, where $\mathcal{O}$ is specified by restriction W.

Example 1. Information tables are depicted in Fig. 1. T0 is the original information table. U is $\{o_1, o_2, \cdots, o_{18}, o_{19}\}$. T1, T2, and T3 are derived from T0, where some attributes are projected and objects are aligned in ascending order of values of attributes a_1, a_2, and a_3, respectively.

T0

U	a_1	a_2	a_3	a_4
1	3.11	2.98	3.02	b
2	2.94	3.65	3.44	b
3	2.33	3.69	3.28	f
4	4.78	2.98	3.52	a
5	3.42	2.35	2.67	b
6	3.03	4.52	4.07	c
7	2.81	2.95	2.91	c
8	4.36	3.11	3.49	a
9	3.22	4.63	4.21	b
10	3.07	3.78	3.57	c
11	2.97	3.98	3.68	b
12	2.63	4.81	4.16	c
13	3.91	3.71	3.77	a
14	3.12	2.78	2.88	b
15	3.05	3.29	3.22	c
16	2.95	3.65	3.44	b
17	2.89	3.51	3.32	c
18	2.45	3.96	3.51	f
19	3.86	3.44	3.57	b

T1

U	a_1	a_4
3	2.33	f
18	2.45	f
12	2.63	c
7	2.81	c
17	2.89	c
2	2.94	b
16	2.95	b
11	2.97	b
6	3.03	c
15	3.05	c
10	3.07	c
1	3.11	b
14	3.12	b
9	3.22	b
5	3.42	b
19	3.86	b
13	3.91	a
8	4.36	a
4	4.78	a

T2

U	a_2	a_3
5	2.35	2.67
14	2.78	2.88
7	2.95	2.91
1	2.98	3.02
4	2.98	3.52
8	3.11	3.49
15	3.29	3.22
19	3.44	3.57
17	3.51	3.32
2	3.65	3.44
16	3.65	3.44
3	3.69	3.28
13	3.71	3.77
10	3.78	3.57
18	3.96	3.51
11	3.98	3.68
6	4.52	4.07
9	4.63	4.21
12	4.81	4.16

T3

U	a_3
5	2.67
14	2.88
7	2.91
1	3.02
15	3.22
3	3.28
17	3.32
2	3.44
16	3.44
8	3.49
18	3.51
4	3.52
19	3.57
10	3.57
11	3.68
13	3.77
6	4.07
12	4.16
9	4.21

Fig. 1. T0 is the original information table. T1, T2, and T3 are derived from T0.

Let threshold δ_{a_1} be 0.05. Indiscernible classes of objects are:

$$[o_1]_{a_1} = \{o_1, o_{10}, o_{14}\}, [o_2]_{a_1} = \{o_2, o_{11}, o_{16}, o_{17}\}, [o_3]_{a_1} = \{o_3\}, [o_4]_{a_1} = \{o_4\},$$
$$[o_5]_{a_1} = \{o_5\}, [o_6]_{a_1} = \{o_6, o_{10}, o_{15}\}, [o_7]_{a_1} = \{o_7\}, [o_8]_{a_1} = \{o_8\}, [o_9]_{a_1} = \{o_9\},$$
$$[o_{10}]_{a_1} = \{o_1, o_6, o_{10}, o_{14}, o_{15}\}, [o_{11}]_{a_1} = \{o_2, o_{11}, o_{16}\}, [o_{12}]_{a_1} = \{o_{12}\},$$
$$[o_{13}]_{a_1} = \{o_{13}, o_{19}\}, [o_{14}]_{a_1} = \{o_1, o_{10}, o_{14}\}, [o_{15}]_{a_1} = \{o_6, o_{10}, o_{15}\},$$
$$[o_{16}]_{a_1} = \{o_2, o_{11}, o_{16}\}, [o_{17}]_{a_1} = \{o_2, o_{17}\}, [o_{18}]_{a_1} = \{o_{18}\}, [o_{19}]_{a_1} = \{o_{13}, o_{19}\}.$$

When $\mathcal{O}$ is specified by restriction $a_4 = b$, $\mathcal{O} = \{o_1, o_2, o_5, o_9, o_{11}, o_{14}, o_{16}, o_{19}\}$. Let $\mathcal{O}$ be approximated by objects on attribute a_1 with continuous values.

Using formulas (3) and (4), lower and upper approximations are:

$$\underline{apr}_{a_1}(\mathcal{O}) = \{o_5, o_9, o_{11}, o_{16}\},$$
$$\overline{apr}_{a_1}(\mathcal{O}) = \{o_1, o_2, o_5, o_9, o_{10}, o_{11}, o_{13}, o_{14}, o_{16}, o_{17}, o_{19}\}.$$

Information table T1 is derived from information table T0, where objects are aligned in ascending order of values of attribute a_1 and are attached the serial superscript from 1 to 19. The above approximations are described using the serial superscript as follows:

$$\underline{apr}_{a_1}(\mathcal{O}) = \{o^7, o^8, o^{14}, o^{15}\},$$
$$\overline{apr}_{a_1}(\mathcal{O}) = \{o^5, o^6, o^7, o^8, o^{11}, o^{12}, o^{13}, o^{14}, o^{15}, o^{16}, o^{17}\},$$

where

$$o^5 = o_{17},\ o^6 = o_2,\ o^7 = o_{16},\ o^8 = o_{11},\ o^{11} = o_{10},\ o^{12} = o_1,$$
$$o^{13} = o_{14},\ o^{14} = o_9,\ o^{15} = o_5,\ o^{16} = o_{19},\ o^{17} = o_{13}.$$

From the lower approximation, consistent combined rules are

$$a_1 = [2.95, 2.97] \rightarrow a_4 = b,\ a_1 = [3.22, 3.42] \rightarrow a_4 = b,$$

from collections $\{o^7, o^8\}$ and $\{o^{14}, o^{15}\}$, respectively, where $a_1(o^7) = 2.95$, $a_1(o^8) = 2.97$, $a_1(o^{14}) = 3.22$, and $a_1(o^{15}) = 3.42$. From the upper approximation, inconsistent combined rules are

$$a_1 = [2.89, 2.97] \rightarrow a_4 = b,\ a_1 = [3.07, 3.91] \rightarrow a_4 = b,$$

from collections $\{o^5, o^6, o^7, o^8\}$ and $\{o^{11}, o^{12}, o^{13}, o^{14}, o^{15}, o^{16}, o^{17}\}$, respectively, where $a_1(o^5) = 2.89$, $a_1(o^{11}) = 3.07$, and $a_1(o^{17}) = 3.91$.

Next, we consider the case where $\mathcal{O}$ is specified by a_3 with the continuous domain. Information table T3 is derived from T0, where the objects are aligned in ascending order of values of a_3 and are attached the serial superscript from 1 to 19. Using lower bound $a_3(o^5) = a_3(o_{15}) = 3.22$ and upper bound $a_3(o^{10}) = a_3(o_8) = 3.49$, $\mathcal{O} = \{o^5, o^6, o^7, o^8, o^9, o^{10}\} = \{o_2, o_3, o_8, o_{15}, o_{16}, o_{17}\}$. We approximate $\mathcal{O}$ by attribute a_2. Information table T2 where the objects are aligned in ascending order of values of a_2 is derived from T0. Let δ_{a_2} be 0.05. Indiscernible classes of objects are:

$$[o_1]_{a_2} = \{o_1, o_4, o_7, o_8\}, [o_2]_{a_2} = \{o_2, o_3, o_{16}\}, [o_3]_{a_2} = \{o_2, o_3, o_{13}, o_{16}\},$$
$$[o_4]_{a_2} = \{o_1, o_4, o_7, o_8\}, [o_5]_{a_2} = \{o_5\}, [o_6]_{a_2} = \{o_6\}, [o_7]_{a_2} = \{o_1, o_4, o_7\},$$
$$[o_8]_{a_2} = \{o_8\}, [o_9]_{a_2} = \{o_9\}, [o_{10}]_{a_2} = \{o_{10}\}, [o_{11}]_{a_2} = \{o_{11}, o_{18}\}, [o_{12}]_{a_2} = \{o_{12}\},$$
$$[o_{13}]_{a_2} = \{o_3, o_{13}\}, [o_{14}]_{a_2} = \{o_{14}\}, [o_{15}]_{a_2} = \{o_{15}\}, [o_{16}]_{a_2} = \{o_2, o_3, o_{16}\},$$
$$[o_{17}]_{a_2} = \{o_{17}\}, [o_{18}]_{a_2} = \{o_{11}, o_{18}\}, [o_{19}]_{a_2} = \{o_{19}\}.$$

Using formulas (3) and (4), lower and upper approximations are:

$$\underline{apr}_{a_2}(\mathcal{O}) = \{o_2, o_8, o_{15}, o_{16}, o_{17}\},\ \overline{apr}_{a_2}(\mathcal{O}) = \{o_1, o_2, o_3, o_4, o_8, o_{13}, o_{15}, o_{16}, o_{17}\}.$$

Using information table T2 where objects are aligned in ascending order of values of attribute a_2 and are attached the serial superscript from 1 to 19, the above approximations are described as follows:

$$\underline{apr}_{a_2}(\mathcal{O}) = \{o^6, o^7, o^9, o^{10}, o^{11}\},\ \overline{apr}_{a_2}(\mathcal{O}) = \{o^4, o^5, o^6, o^7, o^9, o^{10}, o^{11}, o^{12}, o^{13}\},$$

From the lower approximation, consistent combined rules are

$$a_2 = [3.11, 3.29] \to a_3 = [3.22, 3.49],\ a_2 = [3.51, 3.65] \to a_3 = [3.22, 3.49],$$

where $a_2(o^6) = 3.11$, $a_2(o^7) = 3.29$, $a_2(o^9) = 3.51$, and $a_2(o^{11}) = 3.65$. From the upper approximation, inconsistent combined rules are

$$a_2 = [2.98, 3.29] \to a_3 = [3, 22, 3.49],\ a_2 = [3.51, 3.71] \to a_3 = [3.22, 3.49],$$

where $a_2(o^4) = 2.98$ and $a_2(o^{13}) = 3.71$.

This example shows that a combined rule is more applicable than single rules. For example, using the above consistent combined rule $a_2 = [3.11, 3.29] \to a_3 = [3.22, 3.49]$, we can say that an object with 3.20 for a value of attribute a_2 supports this rule, because 3.20 is included in interval $[3.11, 3.29]$. On the other hand, using single rules $a_2 = 3.11 \to a_3 = [3.22, 3.49]$ and $a_2 = 3.29 \to a_3 = [3.22, 3.49]$, we cannot say what rule the object supports under a threshold 0.05.

For formulas on sets A and B of attributes,

$$[o]_A = \cap_{a_i \in A}[o]_{a_i}, \tag{5}$$
$$\underline{apr}_A(\mathcal{O}) = \{o \mid [o]_A \subseteq \mathcal{O}\}, \tag{6}$$
$$\overline{apr}_A(\mathcal{O}) = \{o \mid [o]_A \cap \mathcal{O} \neq \emptyset\}. \tag{7}$$

3 Rough Sets by Indiscernible Classes in Incomplete Information Systems with Continuous Domains

An information table with incomplete information is called an incomplete information system. In incomplete information systems, $a_i : U \to s_{a_i}$ for every $a_i \in AT$ where s_{a_i} is a set of values over domain $D(a_i)$ of attribute a_i or an interval on $D(a_i)$. Single value v with $v \in a_i(o)$ or $v \subseteq a_i(o)$ is a possible value that may be the actual one as the value of attribute a_i in object o. The possible value is the actual one if $a_i(o)$ is a single value.

In an incomplete information system[1], an indiscernible class is a possible class that may be the actual indiscernible class. We have lots of indiscernible classes. Family $\mathcal{F}[o]_{a_i}$ of indiscernible class is:

$$\mathcal{F}[o]_{a_i} = \{C[o]_{a_i} \cup e \mid e \in \mathcal{P}(P[o]_{a_i} \backslash C[o]_{a_i})\}, \tag{8}$$

[1] For the sake of simplicity and space limitation, We describe the case of an attribute, although our approach can be easily extended to the case of more than one attribute.

where $\mathcal{P}(P[o]_{a_i} \backslash C[o]_{a_i})$ is the power set of $P[o]_{a_i} \backslash C[o]_{a_i}$, and certainly indiscernible class $C[o]_{a_i}$ and possibly one $P[o]_{a_i}$ on attribute a_i of object o are:

$$C[o]_{a_i} = \{o' \mid o' = o \vee (\forall u \in a_i(o) \forall v \in a_i(o') |u - v| \leq \delta_{a_i})\}, \tag{9}$$

$$P[o]_{a_i} = \{o' \mid o' = o \vee (\exists u \in a_i(o) \exists v \in a_i(o') |u - v| \leq \delta_{a_i})\}. \tag{10}$$

The family of indiscernible classes has a lattice structure. The minimal element is the certainly indiscernible class and the maximal one is the possibly indiscernible class. In other words, $C[o]_{a_i}$ is the minimum indiscernible class and $P[o]_{a_i}$ is the maximum indiscernible class. Objects in the certainly indiscernible class of o are certainly indistinguishable with o. Objects in the possibly indiscernible class of o are possibly indistinguishable with o.

We can derive not the actual, but certain and possible approximations from the viewpoint of certainty and possibility, as Lipski obtained in query processing under incomplete information [5,6]. We cannot definitely obtain whether or not an object belongs to the actual approximations, but we can know whether or not the object certainly and/or possibly belongs to approximations. Therefore, we show certain approximations (resp. possible approximations) whose object certainly (resp. possibly) belongs to the actual approximations.

Let $\mathcal{O}$ be a set of objects. Using certainly and possibly indiscernible classes, certain lower approximation $C\underline{apr}_{a_i}(\mathcal{O})$ and possible one $P\underline{apr}_{a_i}(\mathcal{O})$ for a_i are:

$$C\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid P[o]_{a_i} \subseteq \mathcal{O}\}, \tag{11}$$

$$P\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid C[o]_{a_i} \subseteq \mathcal{O}\}. \tag{12}$$

Similarly, Certain upper approximation $C\overline{apr}_{a_i}(\mathcal{O})$ and possible one $P\overline{apr}_{a_i}(\mathcal{O})$ are:

$$C\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid C[o]_{a_i} \cap \mathcal{O} \neq \emptyset\}, \tag{13}$$

$$P\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid P[o]_{a_i} \cap \mathcal{O} \neq \emptyset\}. \tag{14}$$

As with the case of nominal attributes [10], the following proposition holds.

Proposition 3. $C\underline{apr}_{a_i}(\mathcal{O}) \subseteq P\underline{apr}_{a_i}(\mathcal{O}) \subseteq \mathcal{O} \subseteq C\overline{apr}_{a_i}(\mathcal{O}) \subseteq P\overline{apr}_{a_i}(\mathcal{O})$.

Using four approximations denoted by formulae (11)–(14), lower and upper approximations are expressed in interval sets, as is described in [11][2], as follows:

$$\underline{apr}^{\bullet}_{a_i}(\mathcal{O}) = [C\underline{apr}_{a_i}(\mathcal{O}), P\underline{apr}_{a_i}(\mathcal{O})], \tag{15}$$

$$\overline{apr}^{\bullet}_{a_i}(\mathcal{O}) = [C\overline{apr}_{a_i}(\mathcal{O}), P\overline{apr}_{a_i}(\mathcal{O})]. \tag{16}$$

Certain and possible approximations are the lower and upper bounds of the actual approximation. The two approximations $\underline{apr}^{\bullet}_{a_i}(\mathcal{O})$ and $\underline{apr}^{\bullet}_{a_i}(\mathcal{O})$ depend

[2] Hu and Yao also say that approximations describes by using an interval set in information tables with incomplete information [2].

on each other; namely, the complementarity property $\underline{apr}^{\bullet}_{a_i}(\mathcal{O}) = U - \overline{apr}^{\bullet}_{a_i}(U - \mathcal{O})$ linked with them holds, as is so in complete information systems.

When objects in $\mathcal{O}$ are specified by attribute a_j with incomplete information, $\mathcal{O}$ is specified by using an element in domain $D(a_j)$. In the case where $\mathcal{O}$ is specified by restriction $a_j = x$ with $x \in D(a_j)$, four approximations: certain lower, possible lower, certain upper, and possible upper ones, are:

$$C\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid P[o]_{a_i} \subseteq C\mathcal{O}_{a_j=x}\}, \tag{17}$$
$$P\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid C[o]_{a_i} \subseteq P\mathcal{O}_{a_j=x}\}, \tag{18}$$
$$C\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid C[o]_{a_i} \cap C\mathcal{O}_{a_j=x} \neq \emptyset\}, \tag{19}$$
$$P\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid P[o]_{a_i} \cap P\mathcal{O}_{a_j=x} \neq \emptyset\}, \tag{20}$$

where

$$C\mathcal{O}_{a_j=x} = \{o \in \mathcal{O} \mid a_j(o) = x\}, \tag{21}$$
$$P\mathcal{O}_{a_j=x} = \{o \in \mathcal{O} \mid a_j(o) \supseteq x\}. \tag{22}$$

For rule induction, we can say as follows:

- $o \in C\underline{apr}_{a_i}(\mathcal{O})$ certainly and consistently supports rule $a_i = a_i(o) \rightarrow a_j(o) = x$.
- $o \in C\overline{apr}_{a_i}(\mathcal{O})$ certainly and inconsistently supports rule $a_i = a_i(o) \rightarrow a_j(o) = x$.
- $o \in P\underline{apr}_{a_i}(\mathcal{O})$ possibly and consistently supports $a_i = a_i(o) \rightarrow a_j(o) = x$.
- $o \in P\overline{apr}_{a_i}(\mathcal{O})$ possibly and inconsistently supports $a_i = a_i(o) \rightarrow a_j(o) = x$.

We create combined rules from them.

Let $U^C_{a_i}$ and $U^I_{a_i}$ be sets of objects having complete information and incomplete information for a_i. $o \in U^C_{a_i}$ is aligned in ascending order of $a_i(o)$ and is attached the serial superscript with 1 to N^C_i where $|U^C_{a_i}| = N^C_i$. Objects $o \in (C\underline{apr}_{a_i}(\mathcal{O}) \cap U^C_{a_i})$, $o \in (C\overline{apr}_{a_i}(\mathcal{O}) \cap U^C_{a_i})$, $o \in (P\underline{apr}_{a_i}(\mathcal{O}) \cap U^C_{a_i})$, and $o \in (P\overline{apr}_{a_i}(\mathcal{O}) \cap U^C_{a_i})$ are aligned in ascending order of $a_i(o)$. And then they are expressed by a sequence of collections of objects with a serial superscript like $\{\cdots, o^h, o^{h+1}, \cdots, o^{k-1}, o^k, \cdots\}$ $(h \leq k)$. From collection $(o^h, o^{h+1}, \cdots, o^{k-1}, o^k)$, four types of combined rules expressed with $a_i = [l, u] \rightarrow a_j = x$ are derived. For a certain and consistent combined rule,

$$l = \min(a_i(o^h), \min_Y e) \text{ and } u = \max(a_i(o^k), \max_Y e),$$
$$Y = \begin{cases} e < a_i(o^{k+1}), & \text{for } h = 1 \wedge k \neq N^C_i \\ a_i(o^{h-1}) < e < a_i(o^{k+1}), & \text{for } h \neq 1 \wedge k \neq N^C_i \\ a_i(o^{h-1}) < e, & \text{for } h \neq 1 \wedge k = N^C_i \end{cases}$$
$$\text{with } e \in a_i(o') \wedge o' \in X, \tag{23}$$

where X is $(C\underline{apr}_{a_i}(\mathcal{O}) \cap U^I_{a_i})$.

For certain and inconsistent, possible and consistent, possible and inconsistent combined rules, X is $(C\overline{apr}_{a_i}(\mathcal{O}) \cap U^I_{a_i})$, $(P\underline{apr}_{a_i}(\mathcal{O}) \cap U^I_{a_i})$, and $(P\overline{apr}_{a_i}(\mathcal{O}) \cap U^I_{a_i})$, respectively.

Proposition 4. Let $C\underline{r}$ and $P\underline{r}$ be sets of combined rules obtained from $C\underline{apr}_{a_i}(\mathcal{O})$ and $P\underline{apr}_{a_i}(\mathcal{O})$, respectively. When $\mathcal{O}$ is specified by restriction W, if $(a_i = [l, u] \rightarrow W) \in C\underline{r}$, then $\exists l' \leq l, \exists u' \geq u$ $(a_i = [l', u'] \rightarrow W) \in P\underline{r}$.

Proposition 5. Let $C\overline{r}$ and $P\overline{r}$ be sets of combined rules obtained from $C\overline{apr}_{a_i}(\mathcal{O})$ and $P\overline{apr}_{a_i}(\mathcal{O})$, respectively. When $\mathcal{O}$ is specified by restriction W, if $(a_i = [l, u] \rightarrow W) \in C\overline{r}$, then $\exists l' \leq l, \exists u' \geq u$ $(a_i = [l', u'] \rightarrow W) \in P\overline{r}$.

Proposition 6. Let $C\underline{r}$ and $C\overline{r}$ be sets of combined rules obtained from $C\underline{apr}_{a_i}(\mathcal{O})$ and $C\overline{apr}_{a_i}(\mathcal{O})$, respectively. When $\mathcal{O}$ is specified by restriction W, if $(a_i = [l, u] \rightarrow W) \in C\underline{r}$, then $\exists l' \leq l, \exists u' \geq u$ $(a_i = [l', u'] \rightarrow W) \in C\overline{r}$.

Proposition 7. Let $P\underline{r}$ and $P\overline{r}$ be sets of combined rules obtained from $P\underline{apr}_{a_i}(\mathcal{O})$ and $P\overline{apr}_{a_i}(\mathcal{O})$, respectively. When $\mathcal{O}$ is specified by restriction W, if $(a_i = [l, u] \rightarrow W) \in P\underline{r}$, then $\exists l' \leq l, \exists u' \geq u$ $(a_i = [l', u'] \rightarrow W) \in P\overline{r}$.

Example 2. Let $\mathcal{O}$ be specified by restriction $a_4 = b$ in IT of Fig. 2.

IT

U	a_1	a_2	a_3	a_4
1	$\{3.06, 3.11\}$	2.98	$[3.02, 3.17]$	$\{b, c\}$
2	2.94	$\{3.64, 3.65\}$	3.44	b
3	2.33	$[3.69, 3.72]$	3.28	f
4	4.78	$[2.98, 3.12]$	3.52	a
5	3.42	2.35	2.67	b
6	3.03	4.52	4.07	c
7	2.81	2.95	2.91	c
8	4.36	3.11	3.49	a
9	$\{2.97, 3.22\}$	4.63	4.21	b
10	3.07	3.78	3.57	c
11	$[2.96, 2.97]$	3.98	3.68	b
12	2.63	4.81	4.16	c
13	3.91	3.71	3.77	a
14	3.12	2.78	2.88	b
15	3.05	3.29	3.22	c
16	2.95	$\{3.35, 3.65\}$	3.44	b
17	$[2.89, 2.92]$	3.51	$[3.32, 3.40]$	$\{b, c\}$
18	$[2.45, 2.55]$	3.96	$\{3.49, 3.51\}$	f
19	$[3.86, 3.92]$	3.44	3.57	$\{a, b\}$

Fig. 2. Information table IT with incomplete information

$$C\mathcal{O}_{a_4=b} = \{o_2, o_5, o_9, o_{11}, o_{14}, o_{16}\},$$
$$P\mathcal{O}_{a_4=b} = \{o_1, o_2, o_5, o_9, o_{11}, o_{14}, o_{16}, o_{17}, o_{19}\}.$$

Each $C[o_i]_{a_1}$ for $i = 1, \ldots, 19$ is, respectively,

$$C[o_1]_{a_1} = \{o_1, o_{10}\}, C[o_2]_{a_1} = \{o_2, o_{11}, o_{16}, o_{17}\}, C[o_3]_{a_1} = \{o_3\},$$
$$C[o_4]_{a_1} = \{o_4\}, C[o_5]_{a_1} = \{o_5\}, C[o_6]_{a_1} = \{o_6, o_{10}, o_{15}\}, C[o_7]_{a_1} = \{o_7\},$$
$$C[o_8]_{a_1} = \{o_8\}, C[o_9]_{a_1} = \{o_9\}, C[o_{10}]_{a_1} = \{o_1, o_6, o_{10}, o_{14}, o_{15}\},$$
$$C[o_{11}]_{a_1} = \{o_2, o_{11}, o_{16}\}, C[o_{12}]_{a_1} = \{o_{12}\}, C[o_{13}]_{a_1} = \{o_{13}, o_{19}\},$$
$$C[o_{14}]_{a_1} = \{o_{10}, o_{14}\}, C[o_{15}]_{a_1} = \{o_6, o_{10}, o_{15}\}, C[o_{16}]_{a_1} = \{o_2, o_{11}, o_{16}\},$$
$$C[o_{17}]_{a_1} = \{o_2, o_{17}\}, C[o_{18}]_{a_1} = \{o_{18}\}, C[o_{19}]_{a_1} = \{o_{13}, o_{19}\}.$$

Each $P[o_i]_{a_1}$ for $i = 1, \ldots, 19$ is, respectively,

$$P[o_1]_{a_1} = \{o_1, o_6, o_{10}, o_{14}, o_{15}\}, P[o_2]_{a_1} = \{o_2, o_9, o_{11}, o_{16}, o_{17}\}, P[o_3]_{a_1} = \{o_3\},$$
$$P[o_4]_{a_1} = \{o_4\}, P[o_5]_{a_1} = \{o_5\}, P[o_6]_{a_1} = \{o_1, o_6, o_{10}, o_{15}\}, P[o_7]_{a_1} = \{o_7\},$$
$$P[o_8]_{a_1} = \{o_8\}, P[o_9]_{a_1} = \{o_2, o_9, o_{11}, o_{16}, o_{17}\}, P[o_{10}]_{a_1} = \{o_1, o_6, o_{10}, o_{14}, o_{15}\},$$
$$P[o_{11}]_{a_1} = \{o_2, o_9, o_{11}, o_{16}, o_{17}\}, P[o_{12}]_{a_1} = \{o_{12}\}, P[o_{13}]_{a_1} = \{o_{13}, o_{19}\},$$
$$P[o_{14}]_{a_1} = \{o_1, o_{10}, o_{14}\}, P[o_{15}]_{a_1} = \{o_1, o_6, o_{10}, o_{15}\},$$
$$P[o_{16}]_{a_1} = \{o_2, o_9, o_{11}, o_{16}, o_{17}\}, P[o_{17}]_{a_1} = \{o_2, o_9, o_{11}, o_{16}, o_{17}\},$$
$$P[o_{18}]_{a_1} = \{o_{18}\}, P[o_{19}]_{a_1} = \{o_{13}, o_{19}\}.$$

Four approximations are:

$$C\underline{apr}_{a_1}(\mathcal{O}) = \{o_5\},$$
$$P\underline{apr}_{a_1}(\mathcal{O}) = \{o_2, o_5, o_9, o_{11}, o_{16}, o_{17}\},$$
$$C\overline{apr}_{a_1}(\mathcal{O}) = \{o_2, o_5, o_9, o_{10}, o_{11}, o_{14}, o_{16}, o_{17}\},$$
$$P\overline{apr}_{a_1}(\mathcal{O}) = \{o_1, o_2, o_5, o_6, o_9, o_{10}, o_{11}, o_{13}, o_{14}, o_{15}, o_{16}, o_{17}, o_{19}\}.$$

$$U^C_{a1} = \{o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_{10}, o_{12}, o_{13}, o_{14}, o_{15}, o_{16}\},$$
$$U^I_{a1} = \{o_1, o_9, o_{11}, o_{17}, o_{18}, o_{19}\}$$

Objects in U^C_{a1} are aligned in ascending order of values of attribute a_1 as follows:

$$o_3, o_{12}, o_7, o_2, o_{16}, o_6, o_{15}, o_{10}, o_{14}, o_5, o_{13}, o_8, o_4$$

A series of superscripts is attached to these objects:

$$o^1, o^2, o^3, o^4, o^5, o^6, o^7, o^8, o^9, o^{10}, o^{11}, o^{12}, o^{13},$$

where $o^1 = o_3, o^2 = o_{12}, \ldots, o^{13} = o_4$. Using objects with the superscript, the four approximations are expressed as follows:

$$C\underline{apr}_{a_1}(\mathcal{O}) = \{o^{10}\},$$
$$P\underline{apr}_{a_1}(\mathcal{O}) = \{o^4, o^5, o^{10}, o_9, o_{11}, o_{17}\},$$
$$C\overline{apr}_{a_1}(\mathcal{O}) = \{o^4, o^5, o^8, o^9, o^{10}, o_9, o_{11}, o_{17}\},$$
$$P\overline{apr}_{a_1}(\mathcal{O}) = \{o^4, o^5, o^6, o^7, o^8, o^9, o^{10}, o^{11}, o_1, o_9, o_{11}, o_{17}, o_{19}\}.$$

where objects with a superscript and with a subscript have complete and incomplete information for attribute a_1, respectively; namely,

$C\underline{apr}_{a_1}(\mathcal{O}) \cap U^C_{a_1} = \{o^{10}\}, C\underline{apr}_{a_1}(\mathcal{O}) \cap U^I_{a_1} = \emptyset,$
$P\underline{apr}_{a_1}(\mathcal{O}) \cap U^C_{a_1} = \{o^4, o^5, o^{10}\}, P\underline{apr}_{a_1}(\mathcal{O}) \cap U^I_{a_1} = \{o_9, o_{11}, o_{17}\},$
$C\overline{apr}_{a_1}(\mathcal{O}) \cap U^C_{a_1} = \{o^4, o^5, o^8, o^9, o^{10}\}, C\overline{apr}_{a_1}(\mathcal{O}) \cap U^I_{a_1} = \{o_9, o_{11}, o_{17}\},$
$P\overline{apr}_{a_1}(\mathcal{O}) \cap U^C_{a_1} = \{o^4, o^5, o^6, o^7, o^8, o^9, o^{10}, o^{11}\}, P\overline{apr}_{a_1}(\mathcal{O}) \cap U^I_{a_1} = \{o_1, o_9, o_{11}, o_{17}, o_{19}\}.$

From these expressions, four types combined rules are derived. For certain and consistent rules,

$$a_1 = 3.42 \rightarrow a_4 = b.$$

For possible and consistent rules,

$$a_1 = [2.89, 2.97] \rightarrow a_4 = b,\ a_1 = [3.22, 3.42] \rightarrow a_4 = b.$$

For certain and inconsistent rules,

$$a_1 = [2.89, 2.97] \rightarrow a_4 = b,\ a_1 = [3.07, 3.42] \rightarrow a_4 = b.$$

For possible and inconsistent rules,

$$a_1 = [2.89, 3.92] \rightarrow a_4 = b.$$

Last, we describe the case where $o \in \mathcal{O}$ is specified by numerical attribute a_j with incomplete information. $o \in U^C_{a_j}$ is aligned in ascending order of $a_j(o)$ and is attached with the serial superscript with 1 to N^C_j where $|U^C_{a_j}| = N^C_j$. We specify $\mathcal{O}$ by $a_j(o^m) \in U^C_{a_j}$ and $a_j(o^n) \in U^C_{a_j}$ with $m \leq n$.

$$C\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid P[o]_{a_i} \subseteq C\mathcal{O}_{[a_j(o^m), a_j(o^n)]}\}, \tag{24}$$
$$P\underline{apr}_{a_i}(\mathcal{O}) = \{o \mid C[o]_{a_i} \subseteq P\mathcal{O}_{[a_j(o^m), a_j(o^n)]}\}, \tag{25}$$
$$C\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid C[o]_{a_i} \cap C\mathcal{O}_{[a_j(o^m), a_j(o^n)]} \neq \emptyset\}, \tag{26}$$
$$P\overline{apr}_{a_i}(\mathcal{O}) = \{o \mid P[o]_{a_i} \cap P\mathcal{O}_{[a_j(o^m), a_j(o^n)]} \neq \emptyset\}, \tag{27}$$

where

$$C\mathcal{O}_{[a_j(o^m), a_j(o^n)]} = \{o \in \mathcal{O} \mid a_j(o) \subseteq [a_j(o^m), a_j(o^n)]\}, \tag{28}$$
$$P\mathcal{O}_{[a_j(o^m), a_j(o^n)]} = \{o \in \mathcal{O} \mid a_j(o) \cap [a_j(o^m), a_j(o^n)] \neq \emptyset\}. \tag{29}$$

$o \in U^C_{a_j}$ is aligned in ascending order of $a_j(o)$ and is attached the serial superscript with 1 to N^C_j. Now, $\mathcal{O}$ is specified by attribute values $a_j(o^m)$ and $a_j(o^n)$ with $o^m \in U^C_{a_j}$ and $o^n \in U^C_{a_j}$. $o \in U^C_{a_i}$ is aligned in ascending order of $a_i(o)$ is attached the serial superscript with 1 to N^C_i. Also, four types of combined rules with $a_i = [l, u] \rightarrow a_j = [a_j(o^m), a_j(o^n)]$ are obtained: certain and consistent, certain and inconsistent, possible and consistent, and possible and inconsistent combined rules.

These types of combined rules are obtained in incomplete information table IT in Fig. 2. For example, let $\mathcal{O}$ be specified by numerical attribute a_3 with incomplete information. When $\mathcal{O}$ is approximated on numerical attribute a_2 with incomplete information, the four types of combined rules are derived.

4 Conclusions

We have described rough sets and rule induction from them in information tables with continuous domains. First, we have dealt with complete information tables. Rough sets are obtained from indiscernible classes. Individual objects that belongs to the rough sets support single rules. The single rules are short of applicability. To improve the applicability of rules, we have put a series of single rules derived from the rough sets into one combined rule. The combined rule is expressed by using intervals.

Second, we have dealt with incomplete information tables. Incomplete information is depicted in a disjunctive set of values or an interval of values. We have dealt with it from viewpoints of certainty and possibility, as was introduced by Lipski in the field of incomplete databases. Lots of indiscernible classes are derived. The family of indiscernible classes is expressed by a lattice having the minimal and maximal elements. The number of indiscernible classes increases exponentially as the number of attribute values with incomplete information grows. However, approximations are obtained by using the minimal and the maximal indiscernible classes. Therefore, we have no difficulty of computational complexity. By using the minimal and the maximal indiscernible classes, four types approximations: certain lower, certain upper, possible lower, and possible upper approximations are obtained, as is so in incomplete information tables with nominal attributes. From these approximations, we have derived four types of combined rules that are expressed by using interval values: certain and consistent, certain and inconsistent, possible and consistent, and possible and inconsistent combined rules. The combined rules are more applicable than single ones.

References

1. Grzymala-Busse, J.W.: Mining numerical data – a rough set approach. In: Peters, J.F., Skowron, A. (eds.) Transactions on Rough Sets XI. LNCS, vol. 5946, pp. 1–13. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-11479-3_1
2. Hu, M.J., Yao, Y.Y.: Rough set approximations in an incomplete information table. In: Polkowski, L., et al. (eds.) IJCRS 2017. LNCS (LNAI), vol. 10314, pp. 200–215. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-60840-2_14
3. Jing, S., She, K., Ali, S.: A universal neighborhood rough sets model for knowledge discovering from incomplete hetergeneous data. Expert Syst. **30**(1), 89–96 (2013). https://doi.org/10.1111/j.1468-0394.2012.00633_x
4. Kryszkiewicz, M.: Rules in incomplete information systems. Inf. Sci. **113**, 271–292 (1999)
5. Lipski, W.: On semantics issues connected with incomplete information databases. ACM Trans. Database Syst. **4**, 262–296 (1979)
6. Lipski, W.: On databases with incomplete information. J. ACM **28**, 41–70 (1981)
7. Lin, T.Y.: Neighborhood systems: a qualitative theory for fuzzy and rough sets. In: Wang, P. (ed.) Advances in Machine Intelligence and Soft Computing, vol. IV, pp. 132–155. Duke University (1997)

8. Nakata, M., Sakai, H.: Rough sets handling missing values probabilistically interpreted. In: Ślęzak, D., Wang, G., Szczuka, M., Düntsch, I., Yao, Y. (eds.) RSFDGrC 2005. LNCS (LNAI), vol. 3641, pp. 325–334. Springer, Heidelberg (2005). https://doi.org/10.1007/11548669_34
9. Nakata, M., Sakai, H.: Applying rough sets to information tables containing missing values. In: Proceedings of 39th International Symposium on Multiple-Valued Logic, pp. 286–291. IEEE Press (2009). https://doi.org/10.1109/ISMVL.2009.1
10. Nakata, M., Sakai, H.: Twofold rough approximations under incomplete information. Int. J. Gener. Syst. **42**, 546–571 (2013). https://doi.org/10.1080/17451000.2013.798898
11. Nakata, M., Sakai, H.: Describing rough approximations by indiscernibility relations in information tables with incomplete information. In: Carvalho, J.P., Lesot, M.-J., Kaymak, U., Vieira, S., Bouchon-Meunier, B., Yager, R.R. (eds.) IPMU 2016. CCIS, vol. 611, pp. 355–366. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-40581-0_29
12. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers, Dordrecht (1991). https://doi.org/10.1007/978-94-011-3534-4
13. Sikora, M.: Decision rule-based data models using TRS and NetTRS – methods and algorithms. In: Peters, J.F., Skowron, A. (eds.) Transactions on Rough Sets XI. LNCS, vol. 5946, pp. 130–160. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-11479-3_8
14. Skowron, A., Stepaniuk, J.: Tolerance approximation spaces. Fundamenta Informaticae **27**, 245–253 (1996)
15. Stefanowski, J., Tsoukiàs, A.: Incomplete information tables and rough classification. Comput. Intell. **17**, 545–566 (2001)
16. Yang, X., Zhang, M., Dou, H., Yang, Y.: Neighborhood systems-based rough sets in incomplete information system. Inf. Sci. **24**, 858–867 (2011). https://doi.org/10.1016/j.knosys.2011.03.007
17. Zenga, A., Lia, T., Liuc, D., Zhanga, J., Chena, H.: A fuzzy rough set approach for incremental feature selection on hybrid information systems. Fuzzy Sets Syst. **258**, 39–60 (2015). https://doi.org/10.1016/j.fss.2014.08.014
18. Zhao, B., Chen, X., Zeng, Q.: Incomplete hybrid attributes reduction based on neighborhood granulation and approximation. In: 2009 International Conference on Mechatronics and Automation, pp. 2066–2071. IEEE Press (2009)