Nonlinear Transverse and In-Plane Vibrations of a Thin Rotating Disk

Xiang-lin Wu, Ying-hou Jiao(✉), Zhao-bo Chen, and Wen-sheng Ma

Harbin Institute of Technology, Harbin, China
jiaoyh@hit.edu.cn

Abstract. An analytical method is presented to investigate nonlinear transverse and in-plane vibrations of a thin rotating disk by using a theory of geometrically nonlinear thin plate. The nonlinear wave solutions of the rotating disk are obtained by Galerkin analysis. The disk is assumed to be isotropic and rotating at the constant speed. The influence of amplitude ratios and rotating speed on natural frequency is studied. Natural frequency and static waves for different nodal-diameter numbers are also calculated. This analytical method not only takes into account the vibration perpendicular to the middle surface of the disk but also the vibration in the middle surface of the disk. In addition, this analytical method provides a more accurate way to solve the severe vibration problems in rotating disks of turbine engine rotors.

Keywords: Nonlinear vibration · Rotating disk · Dimensionless speed Dimensionless natural frequency · Amplitude ratios · Nodal diameters

1 Introduction

Thin rotating disks are frequently applied in engineering, from gas turbine rotors to computer memory disks. Since the turbine disks are important components of gas turbine rotors, the vibrations of turbine disks have an important effect on the behavior of the entire rotors. This kind of periodic motion of rotating disks has been investigated widely.

von Karman [1] first established a nonlinear plate theory when the nonlinear stretch effects in the transverse, equilibrium balance were considered. The first nonlinear analysis of transverse vibration in a spinning disk is due to Nowinski [2], he analyzed the large amplitude vibrations of a spinning disk by using the von Karman field equations. But he only analyzed the transverse vibration of the rotating disk without analyzing the in-plane vibration of the disk. Later Nowinski [3] analyzed the thermal stability of the rotating membrane disk. Maher and Adams [4] investigated the influence of coupling between in-plane displacements and transverse deflections considering the effects of bending stiffness and of the air flow between the disk. The von Karman equations have also been used to investigate the nonlinear vibration of a spinning disk by Renshaw and Mote [5], Hamidzadeh [6, 7] and Luo [8]. It should be noted that professor Hamidzadeh's work was based on the research of Nowinski, he expanded Nowinski's research and got some meaningful results. Luo [9, 10] developed a more accurate theory of thin plates. In his theory, the exact geometry of the deformed middle surface is used to derive the physical strains of plates and equilibrium equations

K. L. Cavalca and H. I. Weber (Eds.): IFToMM 2018, MMS 62, pp. 135–147, 2019.
https://doi.org/10.1007/978-3-319-99270-9_10

in the plate was established based on the exact geometry of the deformed middle surfaces. By using his own theory, he analyzed the response and natural frequencies for the nonlinear vibrations of a rotating thin disk. Koo and Lesieutre [11] analyzed the transverse vibration of a composite-ring disk for data storage, they calculated its natural frequencies and critical speeds. Maretic, Glavardanov, Milosevic-Mitic [12] studied the frequencies of transverse vibrations of a disk assembled from two rings of two different materials, they analyzed the influence of angular velocity, moduli of elasticity, the volume densities of the materials and the radius of the connection on the vibration frequencies of the rotating disk. Pei, Wang and Yang [13] analyzed the natural frequency, dynamic stability, critical speeds and steady state response amplitude of a rotating disk under several boundary conditions.

This research work is based on the work of Nowinski and Hamidzadeh, the presented work get the solutions of the nonlinear transverse and in-plane vibrations of a thin rotating disk and the static waves for different nodal-diameter numbers are presented, also, the variations of dimensionless natural frequency versus dimensionless speed and amplitude ratio are analyzed.

2 Equations of Motions

The vibration of a thin elastic rotating disk of radius a and thickness h is considered. The disk rotates about its central axis at a constant angular velocity Ω. The thin rotating disk is shown in the following Fig. 1.

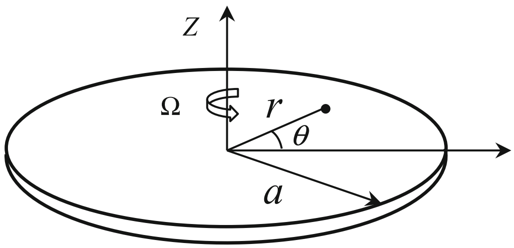

Fig. 1. A thin rotating disk

The transverse deflection of the rotating disk is large compared with its thickness h. According to the nonlinear plate theory, the strain-displacement relationship in polar coordinate system is as follows:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - z\frac{\partial^2 w}{\partial r^2} \tag{1a}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}u + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 - z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \tag{1b}$$

$$\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{1}{r}v + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} - 2z\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \tag{1c}$$

Where ε_{rr}, $\varepsilon_{\theta\theta}$ and $\varepsilon_{r\theta}$ are radial, hoop, and shear strains. u, v and w are the displacements in cylindrical coordinates. The stress-strain relation is expressed as follows:

$$\sigma_{rr} = \frac{E}{1-\mu^2}(\varepsilon_{rr} + \mu\varepsilon_{\theta\theta}) \tag{2a}$$

$$\sigma_{\theta\theta} = \frac{E}{1-\mu^2}(\varepsilon_{\theta\theta} + \mu\varepsilon_{rr}) \tag{2b}$$

$$\sigma_{r\theta} = \frac{E}{2(1+\mu)}\varepsilon_{r\theta} \tag{2c}$$

Where σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are radial, hoop, and shear stress. Also E and μ are Young's modulus and Poisson ratio. The unit thickness membrane forces of the disk can be calculated by using the following equations:

$$\begin{array}{lll} N_r = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{rr} dz & N_\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{\theta\theta} dz & N_{r\theta} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{r\theta} dz \\ M_r = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{rr} z dz & M_\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{\theta\theta} z dz & M_{r\theta} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{r\theta} z dz \end{array} \tag{3}$$

By substituting Eqs. (1a), (1b), (1c) and (2a), (2b), (2c) in (3), one can get membrane forces which are presented by displacements:

$$N_r = \frac{E}{1-\mu^2}\left\{\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \mu\left[\frac{1}{r}u + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]\right\} \tag{4a}$$

$$N_\theta = \frac{E}{1-\mu^2}\left\{\frac{1}{r}u + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 + \mu\left[\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right]\right\} \tag{4b}$$

$$N_{r\theta} = \frac{E}{2(1+\mu)}\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{1}{r}v + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) \tag{4c}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \tag{4d}$$

$$M_\theta = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \mu\frac{\partial^2 w}{\partial r^2}\right) \tag{4e}$$

$$M_{r\theta} = -(1-\mu)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \tag{4f}$$

Where D is the stiffness for the disk, $D = E/12(1-\mu^2)$.

Because the in-plane vibration displacement amplitudes are much smaller than that of transverse vibration, so the inertia terms in equations of in-plane motions are ignored. The equilibrium equations of motions in terms of membrane forces for the disk can be written as:

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{1}{r}(N_r - N_\theta) + \rho\Omega^2 r = 0 \tag{5a}$$

$$\frac{\partial N_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{2}{r}N_{r\theta} = 0 \tag{5b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(N_r\frac{\partial w}{\partial r}\right) + N_{r\theta}\cdot\frac{\partial w}{\partial \theta} + r\cdot Q_r\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\left(N_\theta\frac{\partial w}{\partial \theta}\right) + N_{\theta r}\frac{\partial w}{\partial r} + Q_\theta\right] + q = 0 \tag{5c}$$

$$\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial \theta} + \frac{1}{r}(M_r - M_\theta) - Q_r = 0 \tag{5d}$$

$$\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} + \frac{2}{r}M_{r\theta} - Q_\theta = 0 \tag{5e}$$

The in-plane stress function ϕ is introduced in order to satisfy Eqs. (5a) and (5b) by introducing the following expressions [12]:

$$N_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{2}\rho\Omega^2 r^2 \tag{6a}$$

$$N_\theta = \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{2}\rho\Omega^2 r^2 \tag{6b}$$

$$N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \tag{6c}$$

The von Karman equation of the rotating disk is obtained by inserting Eqs. (4a), (4b), (4c), (4d), (4e), (4f) and (5d–5e) into Eq. (5c), Under the hypothesis of free vibration, the governing equation of the rotating disk in the polar coordinate system becomes:

$$\begin{aligned} \frac{D}{h}\nabla^4 w + \rho\frac{\partial^2 w}{\partial t^2} = {} & \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\frac{\partial^2 \phi}{\partial r^2} \\ & - 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) - \frac{1}{2}\rho\Omega^2 r^2\nabla^2 w - \rho\Omega^2 r\frac{\partial w}{\partial r} \end{aligned} \tag{7}$$

The compatibility equation is also obtained:

$$\frac{1}{E}\left[\nabla^4\phi - 2(1-\mu)\rho\Omega^2\right] = -\frac{1}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 \tag{8}$$

3 Approximate Solution

An approximate solution was proposed by Nowinski [2], Hamidzadeh [7] analyzed the case of no nodal circles but any number of nodal diameters, but they only analyzed the transverse vibration of the disk and ignored the in-plane vibration of the disk, this research work expands their work to analyze the transverse and in-plane coupling vibrations of the disk. According to the work of Hamidzadeh [7], the displacement of transverse direction is:

$$w(r,\theta,t) = W_0 T(t) r^n \cos(n\theta + \varphi) \tag{9}$$

Where $w(r,\theta,t)$ is the transverse deflection of the disk in polar coordinates, 'W_0' is a constant, 'φ' is the phase constant, '$T(t)$' is a time function respecting that 'w' varies with time, and 'n' is the number of nodal diameters.

The stress function 'ϕ' is obtained by substituting Eq. (9) into (8) according to Nowinski [2]. The stress function is as follows:

$$\phi = k\frac{a_1}{c_1}r^{2n} + \frac{1-\mu}{32}\Omega^2\rho r^4 + Ar^2 + \left(Cr^{2n} + Dr^{2(n+1)}\right)\cos 2(n\theta + \varphi) \tag{10}$$

Where A, C, and D are constants and

$$k = \frac{E}{2}W_0^2 T^2 \tag{11}$$

$$a_1 = 2n^2(n-1)^2 \tag{12a}$$

$$c_1 = 16n^2(n-1)^2 \tag{12b}$$

Substitute (10) into (6a), (6b), (6c), one can get the following equations:

$$\begin{aligned} N_r = {} & \frac{2kna_1}{c_1}r^{2(n-1)} - \frac{\mu+3}{8}\Omega^2\rho r^2 + 2A \\ & + \left[2Cn(1-2n)r^{2(n-1)} + 2D\left(n - 2n^2 + 1\right)r^{2n}\right]\cos 2(n\theta + \varphi) \end{aligned} \tag{13}$$

$$\begin{aligned} N_\theta = {} & \frac{2kn(2n-1)a_1}{c_1}r^{2(n-1)} - \frac{1+3\mu}{8}\Omega^2\rho r^2 + 2A \\ & + \left[2Cn(2n-1)r^{2(n-1)} + 2D(n+1)(2n+1)r^{2n}\right]\cos 2(n\theta + \varphi) \end{aligned} \tag{14}$$

$$N_{r\theta} = 2n\left[C(2n-1)r^{2(n-1)} + D(2n+1)r^{2n}\right]\sin 2(n\theta + \varphi) \tag{15}$$

Constants A, C, and D can be determined by satisfying the stress boundary conditions at $r = a$, which will be presented in the later analysis. According to Nowinski [2], apply the procedure of Galerkin to the Eq. (7), then substitute Eqs. (9) and (10) into (7) and integrate the result over the domain of the disk result in the following second-order non-linear time equation:

$$\frac{d^2T}{dt^2} + \alpha T + \beta T^3 = 0 \tag{16}$$

The α and β are given by

$$\alpha = -\frac{s_4}{2s_1}\Omega^2 \tag{17a}$$

$$\beta = -\frac{EW_0^2 a^{2(n-2)}}{\rho s}\left(s_2 C' + s_3 a^2 D' + \frac{1}{2}s_5\right) \tag{17b}$$

Where

$$C' = \frac{C}{EW_0^2 T^2} \tag{18a}$$

$$D' = \frac{D}{EW_0^2 T^2} \tag{18b}$$

and

$$s_1 = \frac{1}{2(n+1)} \tag{19a}$$

$$s_2 = -2n^2(n-1) \tag{19b}$$

$$s_3 = -n(n-1)(2n+1) \tag{19c}$$

$$s_4 = -\frac{n}{n+1}\left[1 + \frac{n-1}{4}(1-\mu)\right] \tag{19d}$$

$$s_5 = -\frac{n^2(n-1)^2}{4(2n-1)} \tag{19e}$$

The solution to Eq. (13) is a Jacobian elliptical function:

$$T(t) = cn(qt, \lambda) \tag{20}$$

Where

$$q = \sqrt{\alpha + \beta} \tag{21a}$$

$$\lambda = \sqrt{\frac{\beta}{2(\alpha + \beta)}} \tag{21b}$$

Obviously, $cn(qt, \lambda)$ is a periodic function with the period $T_0 = 4K/q$, and K is the first kind of complete elliptic integral [16].

4 Free Nonlinear Vibration

In order to identity unknown constants A, C, and D, two stress boundary conditions need to be satisfied. The two boundary conditions are that the radial and tangential stresses on the outer radius of the disk are zero:

$$N_r(r = a, \theta) = 0 \tag{22a}$$

$$N_{r\theta}(r = a, \theta) = 0 \tag{22b}$$

By satisfying the stress boundary conditions, ones yield:

$$Cn(1 - 2n) + D\left(n - 2n^2 + 1\right)a^2 = 0 \tag{23a}$$

$$C(2n - 1) + D(2n + 1)a^2 = 0 \tag{23b}$$

Solve Eqs. (23a), (23b), ones obtain:

$$C = D = 0 \tag{24}$$

Substitute (24) into (13), A becomes:

$$A = \frac{\mu + 3}{16}\Omega^2 \rho a^2 - kn\frac{a_1}{c_1}a^{2(n-1)} \tag{25}$$

Since C and D are zero, Eq. (15) yields $N_{r\theta} = 0$.
Thus, with (11), ones obtain:

$$N_r = \frac{En}{8}W_0^2T^2\left[r^{2(n-1)} - a^{2(n-1)}\right] + \frac{\mu + 3}{8}\Omega^2\rho\left(a^2 - r^2\right) \tag{26}$$

$$N_\theta = \frac{En}{8}W_0^2T^2\left[(2n - 1)r^{2(n-1)} - a^{2(n-1)}\right] - \frac{1 + 3\mu}{8}\Omega^2\rho r^2 + \frac{\mu + 3}{8}\Omega^2\rho a^2 \tag{27}$$

$$N_{r\theta} = 0 \tag{28}$$

In order to calculate in-plane vibration displacements u and v, the relationship between u, v and N_r, N_θ are obtained, subtracting Eq. (4b) multiplied by μ from Eq. (4a) yields:

$$\frac{\partial u}{\partial r} = \frac{N_r - \mu N_\theta}{E} - \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \tag{29a}$$

Similarly, subtracting Eq. (4a) multiplied by μ from Eq. (4b) yields:

$$\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{N_\theta - \mu N_r}{E} - \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \tag{29b}$$

Substitute (27), (28) to (29a), (29b), ones obtain:

$$\begin{aligned}\frac{\partial u}{\partial r} = & \frac{nW_0^2 T(t)^2}{8}\left\{[1-(2n-1)\mu]r^{2(n-1)} + (\mu-1)a^{2(n-1)}\right\} + \frac{3(\mu^2-1)}{8E}\Omega^2\rho r^2 \\ & + \frac{(\mu+3)(1-\mu)}{8E}\Omega^2\rho a^2 - \frac{1}{2}[nW_0T(t)\cos(n\theta+\varphi)]^2 r^{2n-2}\end{aligned} \tag{30}$$

$$\begin{aligned}u + \frac{\partial v}{\partial \theta} = & \frac{nW_0^2 T(t)^2}{8}\left[(2n-1-\mu)r^{2n-1} + (\mu-1)a^{2(n-1)}r\right] + \frac{\mu^2-1}{8E}\Omega^2\rho r^3 \\ & + \frac{-\mu^2-2\mu+3}{8E}\Omega^2\rho a^2 r - \frac{1}{2}n^2W_0^2T(t)^2 r^{2n-1}\sin^2(n\theta+\varphi)\end{aligned} \tag{31}$$

Integrate (30) and (31), ones obtain:

$$\begin{aligned}u = & \frac{nW_0^2 T(t)^2}{8}\left\{\frac{1-(2n-1)\mu}{2n-1}r^{2n-1} + (\mu-1)a^{2(n-1)}r\right\} + \frac{(\mu^2-1)}{8E}\Omega^2\rho r^3 \\ & + \frac{(\mu+3)(1-\mu)}{8E}\Omega^2\rho a^2 r - \frac{n^2}{2(2n-1)}W_0^2T(t)^2 r^{2n-1}\cos^2(n\theta+\varphi) + f(\theta, t)\end{aligned} \tag{32}$$

$$v = \frac{n^2}{4(2n-1)}W_0^2T(t)^2 r^{2n-1}\sin 2(n\theta+\varphi) - \int f(\theta, t)d\theta + R(r, t) \tag{33}$$

To calculate unknown functions $f(\theta, t)$ and $R(r, t)$, the displacement boundary conditions need to be satisfied. the displacement boundary conditions are:

$$u(r=0, \theta) = 0 \tag{34a}$$

$$v(r=0, \theta) = 0 \tag{34b}$$

$$\left.\frac{\partial u}{\partial \theta}\right|_{r=0} = 0, \left.\frac{\partial v}{\partial r}\right|_{r=0} = 0 \tag{34c}$$

By imposing the above conditions on Eqs. (32) and (33), and with Eq. (9), the transverse deflection and in-plane displacements of the non-linear vibration rotating disk are finally obtained:

$$u = \frac{nW_0^2 T(t)^2}{8}\left\{\frac{1-(2n-1)\mu}{2n-1} r^{2n-1} + (\mu-1)a^{2(n-1)}r\right\} + \frac{(\mu^2-1)}{8E}\Omega^2 \rho r^3 + \frac{(\mu+3)(1-\mu)}{8E}\Omega^2 \rho a^2 r - \frac{n^2}{2(2n-1)} W_0^2 T(t)^2 r^{2n-1}\cos^2(n\theta+\varphi) \tag{35}$$

$$v = \frac{n^2}{4(2n-1)} W_0^2 T(t)^2 r^{2n-1} \sin 2(n\theta+\varphi) \tag{36}$$

$$w = W_0 T(t) r^n \cos(n\theta+\varphi) \tag{37}$$

5 Results and Discussion

In order to do the analysis and show the results, the following dimensionless parameters are introduced

$$\text{Amplitude ratio:} \quad W = \frac{W_0 a^n}{h} \tag{38}$$

$$\text{Dimensionless rotating speed:} \quad \Omega_1 = \frac{\Omega a}{\sqrt{E/\rho}} \tag{39}$$

$$\text{Dimensionless period:} \quad T^* = \frac{4K}{qa}\sqrt{\frac{E}{\rho}} \tag{40}$$

$$\text{Dimensionless frequency:} \quad \Omega_2 = \frac{2\pi}{T^*} \tag{41}$$

The presented results in this research work are for the disk with the following parameters: Young's modulus $E = 2.1 \times 10^{11}$ Pa, Poisson ratio $\mu = 0.33$, density $\rho = 7.85 \times 10^3$ kg/m^3, outer radius $a = 0.5$ m, thickness $h = 0.02$ m, rotating speed $\Omega = 100\pi$ rad/s.

Let time-relative terms vanish, for $W = 0.2$, $n = 3$, the displacements of static waves in the three-directions from Eqs. (35), (36) and (37) are plotted in Fig. 2(a)–(c).

Figure 2 shows that the in-plane displacements are much smaller than the deflection in the transverse direction. The results also show that the nodal diameters number of circumferential mode is always twice that of the transverse vibration. The frequency associate to this mode is 1438.3 rad/s.

From Eqs. (35), (36) and (37), we can see that the in-plane vibrations of the rotating disk are affected by rotating speed and nodal diameters number. So the variation of radial displacement amplitude u on the outer radius versus rotating speed for different numbers of nodal diameters is presented in Fig. 3 for a dimensionless amplitude ratio of $W = 0.2$.

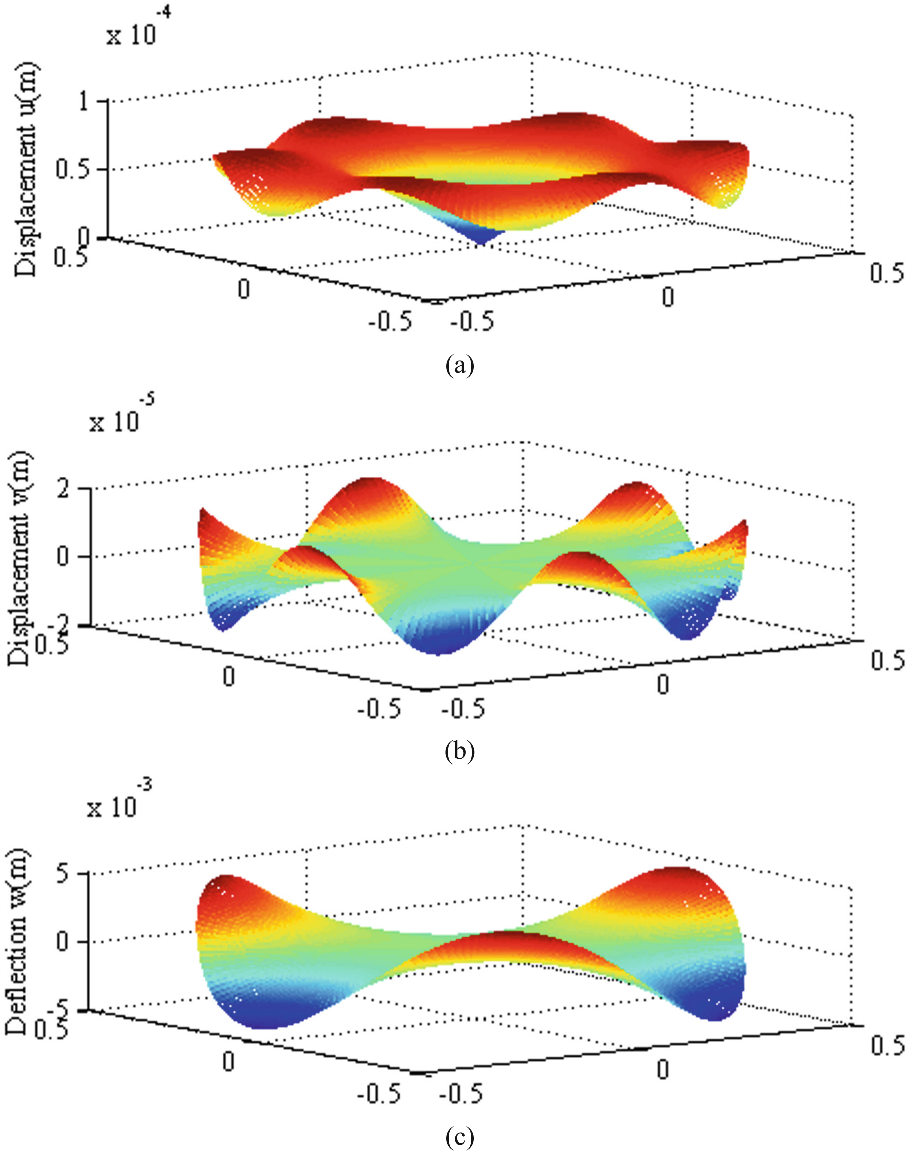

Fig. 2. Static waves in rotating disk: (a) radial displacement u, (b) circumferential displacement v, (c) transverse deflection w

The results indicate that the value of the radial displacement amplitude on the outer radius is negative when the rotating speed is zero and increases with rotating speed for different numbers of nodal diameters. The radial vibration disappears at a certain rotating speed. The radial displacement amplitude on the outer radius is also increases with the number of nodal diameters.

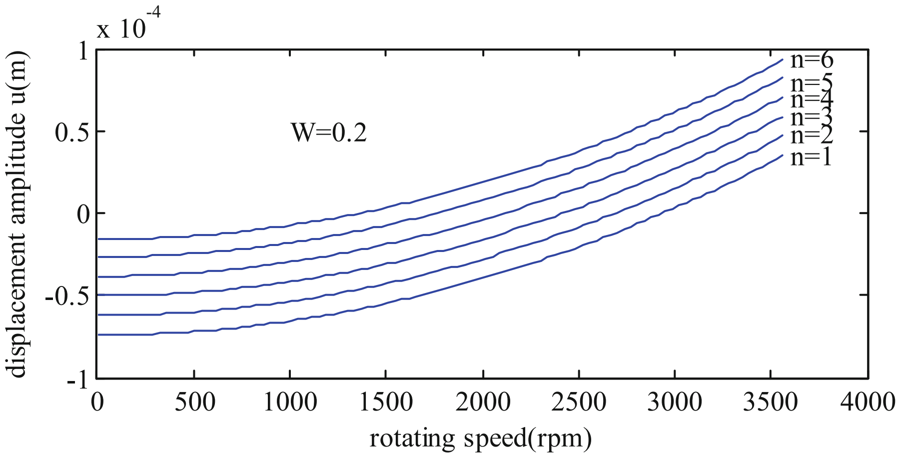

Fig. 3. Variation of radial displacement amplitude u on the outer radius versus rotating speed for different numbers of nodal diameters

The nonlinear and linear dimensionless natural frequencies of the rotating disks versus a wide range of rotating speeds are calculated. For a dimensionless amplitude ratio of $W = 2$, the variations of dimensionless natural frequencies for different numbers of nodal diameters is presented in Fig. 4. Presented results show that the natural frequencies in both the nonlinear analysis and linear analysis depend on nodal diameter, and have no difference when $n = 1$. Nonlinear natural frequencies and linear natural frequencies are mainly distinguished at lower speed, at higher speed, the nonlinear dimensionless frequencies of different nodal diameters numbers approach the corresponding linear dimensionless frequencies.

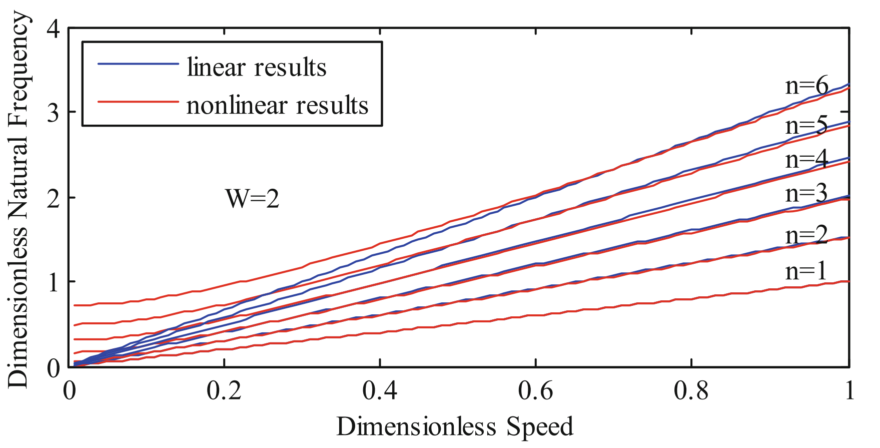

Fig. 4. Variation of dimensionless natural frequency versus dimensionless speed for different numbers of nodal diameters

The nonlinear and linear dimensionless natural frequencies of the rotating disks versus amplitude ratios at different dimensionless speeds for $n = 6$ are also calculated and presented in Fig. 5. The results show that the natural frequencies in the nonlinear analysis are dependent of amplitudes, and the effect of speeds on natural frequency at small amplitudes is higher than that at large amplitudes, and the relationship between natural frequencies and amplitudes gradually become linear at large amplitudes. But the dimensionless natural frequencies in the linear analysis are independent of amplitudes for all speeds.

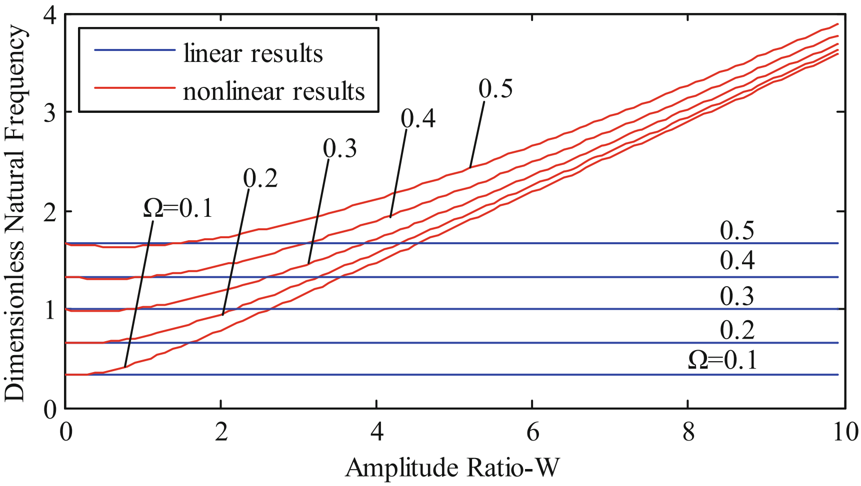

Fig. 5. Dimensionless natural frequency versus amplitude ratio at different dimensionless speeds for $n = 6$

6 Conclusion

An analytical method is presented to investigate nonlinear transverse and in-plane vibrations of a thin rotating disk, the solutions of the nonlinear transverse and in-plane vibrations of the thin rotating disk are finally obtained, the static waves, natural frequency for nonlinear transverse vibrations of the rotating disk are also determined. The provided modal analysis is valid for thin rotating disks with any number of nodal diameters without nodal circles. The results show that the in-plane displacements of the vibration are much smaller than the deflection in the transverse direction. Analysis indicates that the natural frequencies provided by nonlinear analysis are different from that of linear analysis. The nonlinear natural frequencies are highly dependent on amplitude of vibration and nodal diameters. The presented results provide the designer an analytical method for analyzing vibrations in three directions of a thin rotating disk.

Acknowledgements. This research is funded by the National Natural Science Foundation of China (Grant No. 11672083).

References

1. von Karman, T.: Festigkeitsprobleme im mashinenbau. Encyklopadie der Mathematischen Wissenschaften **4**(4), 348–352 (1910)
2. Nowinski, J.L.: Nonlinear transverse vibrations of spinning disk. J. Appl. Mech. **31**, 72–78 (1964)
3. Nowinski, J.L.: Stability of nonlinear thermos elastic waves in membrane-like spinning disks. J. Therm. Sci. **4**, 1–11 (1981)
4. Maher, J.F., Adams, G.G.: Effect of displacement dependent membrane stresses on the axisymmetric configuration of a spinning flexible disk. ATLE Tribol. Trans. **34**(4), 597–603 (1991)
5. Renshaw, A.A., Mote Jr., C.D.: A perturbation solution for the flexible rotating disk: nonlinear equilibrium and stability under transverse loading. J. Sound Vib. **183**, 309–326 (1995)
6. Hamidzadeh, H.R., Nepal, N., Dehghani, M.: Transverse vibration of thin rotating disks–nonlinear modal analysis. In: ASME International Mechanical Engineering Congress and Exposition, Anaheim, California, pp. 219–225 (1998)
7. Hamidzadeh, H.R.: Non-linear free transverse vibration of thin rotating discs. In: 20th Biennial Conference on Mechanical Vibration and Noise, Long Beach, CA, vol. 221, pp. 467–473 (2005)
8. Luo, A.C.J., Mote Jr., C.D.: An analytical solution of the non-linear vibration of rotating disks. In: ASME 17th Biennial Conference on Mechanical Vibration and Noise, Las Vegas, Nevada, DETC99/VIB-8150 (1999)
9. Luo, A.C.J.: Approximate theory for geometrically-nonlinear, thin plates. Int. J. Solids Struct. **37**, 7655–7670 (2000)
10. Luo, A.C.J., Mote Jr., C.D.: Nonlinear vibration of rotating, thin disks. ASME J. Vib. Acoust. **122**(4), 376–383 (2000)
11. Koo, K.-N., Lesieutre, G.A.: Vibration and critical speeds of composite-ring disks for data storage. J. Sound Vib. **329**, 833–847 (2010)
12. Maretic, R., Glavardanov, V., Milosevic-Mitic, V.: Vibration and stability of rotating annular disks composed of different materials. Arch. Appl. Mech. **85**(1), 117–131 (2015)
13. Pei, Y., Wang, J., Yang, F.: Dynamics comparison of rotating flexible annular disk under different edge boundary conditions. Int. J. Mech. Sci. **137**, 121–132 (2018)
14. Novozhilov, V.V.: Foundations of the Nonlinear Theory of Elasticity, 3rd edn. Graylock Press, Rochester (1953)
15. Timoshenko, S.: Theory of Plates and Shells. McGraw-Hill Book Company, New York (1940)
16. Ye, Q., Shen, Y.: Handbook of Mathematical Functions, 2nd edn. Science Press, Beijing (2013)