Chapter 4
What Kind of Opportunities Do Abstract Algebra Courses Provide for Strengthening Future Teachers' Mathematical Knowledge for Teaching?

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Introduction

This section of the volume is focused on the cognitive complexity of abstract algebra. The chapters highlight specific cognitive challenges around concepts that are closely connected to important concepts in secondary mathematics. Drawing on these connections, the authors leverage their research to develop insights about preservice teacher education. In order to frame our commentary, we begin by briefly reviewing the literature around students' understanding of abstract algebra. We situate the studies described in the two previous chapters within this literature. Then, to support our efforts to examine the proposed implications for the mathematical preparation for teachers, we introduce a framework (Ball, Thames, & Phelps, 2008) that we will use to characterize and critically analyze the connections the authors make between their research into students' understanding of abstract algebra and the mathematical preparation of teachers.

Students' Understanding of Abstract Algebra

The early research on the teaching and learning of abstract algebra was often focused on students' difficulties and misunderstandings. Much of this work relied

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on the APOS (Action, Process, Object Schema) framework (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Dubinsky & Leron, 1994) to develop and refine proposed cognitive models, called genetic decompositions, which were then used to investigate students' understanding of topics, such as groups (Dubinsky et al., 1994), binary operations (Brown, DeVries, Dubinsky, & Thomas, 1997), coset multiplication (Asiala, Dubinsky, Mathews, Morics, & Oktaç, 1997), and permutations (Asiala, Kleiman, Brown, & Mathews, 1998). For the most part, these studies consistently found that many students possessed a very limited understanding of these concepts. In particular, they struggled to coordinate the various objects and processes involved (see Leron, Hazzan, & Zazkis, 1995) and they struggled with the level of abstraction (see Hazzan, 1999) of the content. The research described in the chapter by Melhuish and Fagan represents an important new phase of this area of research. Drawing on previous work, Melhuish (2018) developed and validated an instrument for assessing students' conceptual understanding of group theory, producing the first true large-scale investigation of students' understanding of abstract algebra.

More recent research has moved away from a focus on student difficulties and utilized design research methodologies to explore alternative instructional treatments that build on students' thinking to support them in developing abstract algebra concepts through a process of mathematical inquiry (e.g., Cook, 2014; Larsen, 2009). In addition to supporting the creation of new curricular approaches (Larsen, Johnson, & Bartlo, 2013), this research has provided new insights into the complexity of concepts like group and isomorphism (e.g., Larsen, 2009), and has uncovered cognitive challenges that mirror previously reported student difficulties with secondary concepts like associativity and commutativity (e.g., Larsen, 2010). The research described in Cook's Chap. 3 is situated within this body of work and represents a first step in developing an inquiry approach to ring and field theory instruction. His approach builds on students' previous experiences solving equations to develop key abstract algebra concepts, such as zero divisors, units, integral domains, and fields.

While both of the studies described in the two chapters are best understood as fitting within (and advancing) the research on undergraduate students' understanding of abstract algebra concepts, the authors leverage these studies to draw attention to the potential value of an abstract algebra course as part of preservice teachers' mathematical preparation. In the case of Melhuish and Fagan, insights about undergraduate students' struggles with binary operations and functions in abstract algebra suggest important connections to preservice teachers' mathematical knowledge for teaching around the secondary versions of these core concepts. In the case of Cook, his qualitative analysis of students' reinvention of the integral domain concept highlights the relationship between the students' understanding of the zero-product property from secondary mathematics and their development of the abstract algebra notion of a zero divisor. To support our analysis of these two chapters, we now turn to the ongoing efforts of mathematics educators to understand the role of mathematical knowledge in supporting the practice of teaching mathematics.

Mathematical Knowledge in the Practice of Teaching

Likely spurred by Monk's (1994) discouraging finding that taking advanced mathematics courses did not seem to help secondary teachers support their students' learning, there has been much work done in the past couple of decades in order to more fully understand the nature of the mathematical knowledge needed by teachers. Ball et al. (2008) presented a framework that delineated six domains of mathematical knowledge for teaching (MKT). These were divided into two categories, pedagogical content knowledge and subject matter knowledge.

Pedagogical content knowledge (PCK) is the professional subject-specific knowledge for teaching mathematics. Within the MKT framework, PCK contains three subdomains that include such things as knowing which mathematical representations or examples to use in teaching, or being aware of the difficulties student typically encounter when engaging with particular mathematics—part of Shulman's (1986) original descriptions of PCK. *Knowledge of content and teaching* (KCT) includes knowledge required to see the relationship between mathematics and teaching and how instructional decisions are made based on these relationships. For example, strategically sequencing examples or deciding whether to explore or table student contributions, involves drawing on knowledge in the KCT subdomain. The subdomain of *knowledge of content and students* (KCS) combines knowledge of mathematics with knowledge of students' mathematics. Teachers draw on KCS when they anticipate how students are likely to approach a task, or when they interpret their students' solutions. The subdomain of *knowledge of content and curriculum* includes knowledge of available curricular materials and how those approach various mathematical topics. Teachers draw on knowledge of content and curriculum to ascertain how well specific curricular materials are aligned with educational goals and how effective they are likely to be in supporting their students' learning.

The Subject Matter Knowledge domain includes the kinds of knowledge needed for teaching that are more strictly mathematical, in that they do not explicitly involve coordination with pedagogical issues. This includes *common content knowledge* (CCK), which describes mathematics that can be used in a wide variety of settings outside of teaching and is knowledge that is considered common among other fields that use mathematics. In contrast, *specialized content knowledge* (SCK) is knowledge that is needed specifically for the work of teaching mathematics. Teachers draw on CCK when they recognize that a student answer is incorrect or nonstandard while they draw on SCK when they notice patterns in students' errors or determine the viability of a nonstandard approach. Lastly, and of particular importance to this chapter, is *horizon content knowledge* (HCK), which describes an awareness of the mathematical territory and "how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Ball and Bass (2009) point out that teachers draw on HCK when they reconcile pedagogical choices in relation to the larger mathematical landscape. An example of this is when a teacher makes an instructional decision based on knowledge that particular choices may (or may not) align with the mathematics that students' will encounter in later courses.

An important purpose of this framework is to tease apart the different kinds of knowledge that might be important for teachers in supporting their students' learning. The work of finding ways to measure these kinds of teacher knowledge and ascertain how they are (or could be) related to student success is ongoing. For our purposes, we will *assume* that developing knowledge in any of these domains can be useful to preservice teachers and we will draw on the framework in order to characterize the nature of the learning opportunities that are highlighted in the two previous chapters.

In the next section, we briefly summarize the way that each of the previous chapters call explicit attention to the potentially bidirectional relationship between preservice teachers' learning of abstract algebra and their knowledge of secondary mathematics. We then consider this relationship in light of Ball et al.'s (2008) framework in order to characterize in what ways this relationship could provide opportunities to strengthen teachers' MKT. In the subsequent sections, we critically consider the validity of the existence of such opportunities as an argument in support of requiring preservice teachers to study abstract algebra. Finally, we share our thoughts on how to leverage these opportunities along with some recommendations and directions for future research on the topic.

Knowledge of Secondary Mathematics and Learning Abstract Algebra

As Wasserman (this volume) notes in his introduction, there are a number of ways to think about how secondary teachers might benefit from studying abstract algebra. Perhaps the most obvious way to think about this is to focus on the mathematical connections between abstract algebra and school mathematics. Such a focus is not without its dangers, as indicated by the controversial New Math era reforms. Yet the idea behind that reform can still be seen in the CBMS (2012) recommendations for the mathematical preparation of teachers in the form of statements like, "it would be quite useful for prospective teachers to see how $\mathbb{C}$ can be built as a quotient of $\mathbb{R}[x]$" (CBMS, 2012, p. 59). A very reasonable question to ask in response to such a statement is "Why?" Abstract algebra certainly provides a highly sophisticated perspective on a variety of secondary mathematics topics, but it simply does not follow that a teacher's pedagogical practice would (or even could) benefit from studying abstract algebra. Or perhaps, rather, we should say it does not follow simply. In their chapters, both Cook, and Melhuish and Fagan take a much deeper (and focused) look at the connection between abstract algebra and secondary mathematics. In doing so, they are able to make considerably more convincing and specific conjectures about how studying abstract algebra could benefit preservice teachers.

Melhuish and Fagan report on a mixed methods study using the Group Concept Assessment (GCA). The quantitative results from a large-scale administration of the GCA tool demonstrated that abstract algebra students struggle with questions about binary operations and functions, and struggle even more when tasks require the coordination of these concepts with other abstract algebra concepts. The validation interviews conducted as part of the development of the instrument provided richer qualitative data that surfaced a number of critical ways in which students' preexisting understandings of binary operation and function supported and constrained their reasoning about the abstract algebra tasks. Further, these tasks appeared to provide opportunities for students to confront limitations in these understandings and to develop richer knowledge of binary operation and function. For example, students tended to have views of both concepts that limited the kinds of things they were able to perceive as functions or binary operations. Some students expected functions to be injective; others expected binary operations to be presented in E-O-E (element–operation–element) form. These limited conceptions constrained the students' abilities to construct the kernel of a non-injective homomorphism or to determine whether the averaging operation was associative.

Although these observations are relevant to those interested in the teaching and learning of abstract algebra for its own sake (they suggest that students' prior experiences, especially regarding functions and operations, may have an important impact on how they are able to engage with abstract algebra), the authors appropriately focus on the possibility of an influence in the opposite direction. As this volume is interested in the preparation of secondary mathematics teachers, it is worth considering whether grappling with these sorts of abstract algebra tasks might impact students' understanding of the concepts of function and operation (core K-12 concepts) in ways that can support their teaching practice. Melhuish and Fagan frame this in terms of Hohensee's (2014) notion of backward transfer. Ultimately, they claim that there are opportunities, while studying abstract algebra, for preservice teachers to deepen their understandings of secondary concepts, such as functions and binary operations. They then describe a number of different ways that preservice teachers' knowledge could be deepened by studying abstract algebra. For the most part, these are focused on increasing mathematical flexibility and disentangling related concepts.

In his chapter, Cook provides an even richer analysis of a connection between secondary mathematics and abstract algebra. He engages a pair of students in the process of reinventing fundamental concepts of ring and field theory in the context of a laboratory teaching experiment. This context gives him a unique opportunity to watch students grapple with a limited view of the zero-product property (ZPP) as they attempt to reinvent the related abstract algebra concept of a zero divisor, on the way to developing a productive way to classify different kinds of rings. In particular, Cook observed that one of the students came into the reinvention process with a view that the ZPP is a universal truth. As such, this student rejected solutions to an equation that involved two nonzero elements multiplying to produce zero. This perspective was eventually refined as he began to use the label "awkward" to describe products of nonzero elements that equaled zero. This refinement enabled him to build a new understanding, in which (for example) uniqueness of solutions is a property that is enjoyed by systems that don't have zero divisors (awkward ways to get zero). In this way, Cook's study provides an existence proof for the kind of backward transfer that Melhuish and Fagan propose in their chapter.

To What Kinds of MKT Are These Chapters Referring?

The first two chapters of this section make claims about the potential of learning abstract algebra to deepen preservice teachers' knowledge. Our aim in this commentary is to critically examine these claims and then, under the assumption that this potential exists, to consider how to capitalize on this potential in ways that can positively impact preservice teacher education. We start by attempting to characterize the nature of the learning opportunities the authors refer to in terms of the MKT framework.

We begin by observing that while the kinds of knowledge referenced in the two chapters might support preservice teachers in developing pedagogical content knowledge, they cannot be categorized as examples of KCS, KCT, or knowledge of content and curriculum. Each of these categories of MKT involves coordinating mathematics and an aspect of pedagogy (students, teaching, and curriculum). Later, when we discuss recommendations, we will briefly address the possibility of promoting the development of pedagogical content knowledge in the context of learning abstract algebra (drawing inspiration from Cook's Chap. 3). However, until then, we will focus on subject matter knowledge, which more accurately captures the kinds of knowledge the authors explore as it is focused on knowledge of mathematics itself.

Melhuish and Fagan delineate a number of aspects of knowledge related to functions and binary operations that could be developed in the context of studying abstract algebra. Here we will consider two specific aspects as we attempt to identify the kinds of subject matter knowledge that might be developed in an abstract algebra course: (1) symbolic expressions for binary operations; and (2) conceptual understanding of the associative property. Similarly, we will attempt to characterize the primary learning opportunity (involving the zero-product property) that was highlighted in Cook's Chap. 3. However, we first argue that none of these is properly characterized as exemplifying specialized content knowledge (SCK).

Specialized content knowledge refers to knowledge "not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400). This includes things like the ability to analyze a nonstandard student strategy to see if it is likely to work in general, and the ability to recognize patterns in students' errors. Johnson and Larsen (2012) observed that it is likely the case that mathematicians make use of this kind of knowledge in their research activity (e.g., reviewing research papers or collaborating with other mathematicians), which suggests that it is perhaps an overstatement to say that this kind of knowledge is not typically needed outside of teaching. Nevertheless, it is useful to think of this category of knowledge as being related closely to the activity of teaching.

Can one categorize the kinds of learning opportunities described in the previous two chapters as developing SCK? To help tease apart why we do not think so, we consider Melhuish and Fagan's observation that abstract algebra can help preservice teachers develop flexibility in moving between and leveraging alternate function representations. Certainly, when students generate nonstandard approaches, these approaches often come with nonstandard representations. Thus, developing flexibility with representations likely positions a preservice teacher well to engage in the activity of making sense of students' ideas in practice. So, why do we not consider developing such flexibility to be an example of developing SCK? First, as Melhuish and Fagan note, this kind of flexibility is important to mathematics majors in the context of abstract algebra, so it does not qualify as knowledge specific to teaching. Second (and more importantly), having such flexibility is not the same thing as having the ability to leverage it to make sense of student thinking. The argument, that flexibility with representations likely supports teachers in making sense of student thinking, actually suggests that a teacher's ability to develop and apply SCK is likely constrained by the nature of their common content knowledge (and perhaps also by their horizon content knowledge). Given the goals of this book, this is both an important connection and an important distinction to make between SCK and other kinds of subject matter knowledge. The connection suggests the potential usefulness of having teachers study abstract algebra, while the distinction suggests that realizing this potential is a nontrivial proposition. In our summary remarks, we pick up this thread again and consider the possibility of developing SCK in the context of an abstract algebra course. Until that time, we will focus on the potential of studying abstract algebra for supporting the development of preservice teachers' horizon content knowledge and the deepening of their common content knowledge, which we regard as the two categories of MKT most appropriately associated with the learning opportunities highlighted in these two chapters. We begin by considering two issues addressed by Melhuish and Fagan and argue that one exemplifies horizon content knowledge, while the other exemplifies common content knowledge (CCK).

We first consider the observation by Melhuish and Fagan that abstract algebra provides an opportunity to move students beyond limited views of binary operation that would require symbolic expressions to be in the E-O-E form. For a secondary teacher, horizon content knowledge would refer to knowledge of how ideas in secondary mathematics evolve as one moves into undergraduate mathematics and beyond. Certainly, binary operations are not assumed to be given by symbolic formulae or presented in E-O-E form in advanced mathematics. Thus, it is clear that this advance in knowledge about binary operations counts as an instance of developing horizon content knowledge. An examination of the secondary mathematics curriculum suggests that students are confronted infrequently by binary operations that are not expressed in terms of symbolic expressions and, while they may be exposed to binary operations (e.g., averaging) not expressed in E-O-E form, they are unlikely to need to think of them explicitly as binary operations (e.g., to prove averaging is non-associative). In this sense, this deepening of the binary operation concept is probably not best categorized as an instance of common content knowledge, which refers to knowledge that is not specific to teaching, but that would be expected of anyone with a working understanding of secondary mathematics.

However, when Melhuish and Fagan discuss opportunities to develop conceptual understanding of the associative property, they *are* referring to an opportunity to strengthen preservice teachers' common content knowledge (CCK). While anyone with a working knowledge of secondary mathematics would have some understanding of the associative property, research suggests that preservice teachers may still need to strengthen this knowledge by disentangling a number of related concepts (e.g., commutativity, order of operations, bracketing). This raises an important issue related to common content knowledge. While it is classified as knowledge that is not specific to teaching, the strength of a preservice teachers' common content knowledge in terms of connections is likely to be a significant factor in their ability to develop and utilize other kinds of MKT. For example, as Larsen (2010) notes, understanding *how* commutativity and associativity are related to *order* (order of operations versus order of operands) is a nontrivial matter, even for undergraduates. Such an understanding is likely important when teachers are making sense of students' struggles (Kieran, 1979) with bracketing (SCK) or making decisions about what kind of tasks will help their students in understanding associativity (KCT). So while this and other kinds of knowledge highlighted in the previous two chapters are best categorized as exemplifying common content knowledge (CCK), this should not be taken as an indication that they do not represent important potential ways that studying abstract algebra could support future teachers.

Cook focused on the relationship between the zero-product property (an important concept from secondary mathematics) and zero divisors (a foundational idea in ring theory). In his design experiment, Cook was able to identify a deepening of students' understanding of the zero-product property as they developed the concept of zero divisors in the context of exploring uniqueness (or rather nonuniqueness) of solutions to linear equations. We argue that the knowledge gained in this process included both horizon and common content knowledge. On the one hand, the students' understanding of the zero-product property was deepened in the sense that they realized that this is not a basic fact about the additive identity (zero), but rather is dependent on the nature of the operation. Furthermore, the connection between this property and the uniqueness of solutions was strengthened by the awareness that systems that do not have this property contain linear equations with multiple solutions. In this sense, the students were deepening their common content knowledge (CCK). On the other hand, the students were able to build direct connections between secondary algebra (ZPP, equation solving) and abstract algebra (zero-divisors, integral domains) to develop knowledge of the trajectory of these ideas across the mathematics curriculum. In this sense, the students were developing horizon content knowledge.

In summary, we consider the two previous chapters to be compelling arguments that abstract algebra courses provide numerous opportunities to develop preservice teachers' horizon content knowledge and deepen their common content knowledge. We also argue that the nature of a preservice teachers' common and horizon content knowledge likely provides important constraints on and affordances for their development of other kinds of mathematical knowledge for teaching. We now turn to the question of realizing these opportunities provided by abstract algebra courses.

Reasons to Be Skeptical About the Idea of Backward Transfer

Although both chapters make a credible case for the possibility that studying abstract algebra can provide opportunities to deepen preservice teachers' understanding of secondary mathematics, there are good reasons to be skeptical. Lobato (2006) notes that students construct their own connections between situations and that these do not necessarily reflect the mathematical invariants to which experts attend (like algebraic structure). It seems that the prospects for backward transfer are likely to be even worse, since in the context of an abstract algebra course there is little reason to expect preservice teachers to do the work of reconsidering their secondary mathematics knowledge in light of their new knowledge of abstract algebra. Certainly, the research demonstrates that students are very capable of maintaining unconnected and even contradictory ideas about concepts. For example, students may have concept images of the limit of a function that are in conflict with the formal definition of limit without perceiving any contradiction (Tall & Vinner, 1981). This suggests that a preservice teacher could develop a concept image of a secondary concept that contains the more sophisticated aspects developed in an abstract algebra course *without* these being coordinated with the aspects associated with the secondary version of the concept. As a result, while we (as experts) may see a more sophisticated version of a secondary mathematics concept when we observe a preservice teacher reasoning successfully in an abstract algebra context, it does not follow that the way the preservice teacher will reason about this concept in a secondary context will reflect this sophisticated understanding. For example, a preservice teacher who is able to construct an operation on the set $\{1, 2, 4\}$ that defines a group may still only bring to mind operations with explicit formulas when designing and teaching a high school algebra lesson.

Even in the case where preservice teachers do make connections between abstract algebra and secondary mathematics, it does not follow directly that as teachers they will be able (or should even try) to leverage them to support student learning. Some connections may hold more potential for positively impacting practice than others and some means of leveraging such knowledge in practice may be more beneficial than others. For example, it seems unlikely that it would be helpful for a high school teacher to start using terms like zero divisor and integral domain when teaching a lesson on polynomial equations. However, it could be quite helpful for a teacher to make extra time in such a lesson to help students make sense of *why* it is useful to manipulate the equation so that zero appears alone on one side of the equation, while a factored polynomial appears on the other.

In short, one can argue that even with the more focused and nuanced connections between abstract algebra and school mathematics that are elaborated in the two previous chapters, we are still in very much the same place we were when the

New Math reforms were initiated. As experts, we see a connection between abstract algebra and school mathematics. It seems reasonable to us that knowledge of those connections could somehow support teachers in their practice. However, even with the ability to articulate what kind of mathematical knowledge this backward transfer could support, we are still left to figure out *how* to actually promote the occurrence of this kind of learning and *how* to support teachers in productively leveraging these kinds of knowledge in their practice.

A Reason for Optimism about Intentionally Engineering Backward Transfer Opportunities

We start this section by acknowledging that the field of mathematics education is still working to understand connections between MKT, teaching practice, and student learning. It may very well be the case that the field eventually learns that some of the kinds of knowledge described in Ball et al.'s (2008) framework do not actually have a significant impact on teachers' practice or on their students' learning. We will not take on this question in this commentary. Instead, we will operate under the assumption that the domains of MKT are potentially valuable to teachers, and accept the claims of Meluish and Fagan, and Cook that abstract algebra provides an opportunity to promote these kinds of MKT. As such, we will focus for the remainder of the chapter on the question of how to capitalize on these opportunities.

We make two recommendations for teacher educators who find potential value in the idea of supporting the development of MKT in preservice teachers through a process of what Melhuish and Fagan refer to as backward transfer. The first recommendation is to not expect it to happen without intentional intervention. Here we note that preservice teachers sometimes take abstract algebra courses that are not specifically designed for teachers. In this case, there is little reason to hope that the preservice teachers in the course will make these connections for themselves. As Wasserman (this volume) notes in his introduction, research suggests that they are likely to see this course as having little or nothing to do with teaching secondary mathematics. However, it is possible to follow up such a course with an algebra course for teachers that could recreate the situations seen in Melhuish and Fagan's interviews, or Cook's teaching experiment, in order to explicitly design opportunities for preservice teachers to actively engage in actor-oriented (Lobato, 2006) backward transfer. We strongly recommend the inclusion of such courses as part of the mathematical preparation of teachers. Frankly, given the findings of Monk (1994) and others, we see no good argument for requiring preservice teachers to take a regular abstract algebra course if it is not followed up by experiences that are designed to leverage the MKT learning opportunities provided by the study of abstract algebra.

Our second recommendation stems from Cook's experience with his participants. His goal was to develop an approach to teaching ring and field theory featuring a process of guided reinvention. With this approach, the students are explicitly engaged in developing the formal abstract algebra concepts by building on the informal understandings they bring to the process (including their knowledge of secondary mathematics). This approach seems to have the potential to support backward transfer in that it engages the students in a process that requires them to navigate back and forth between their knowledge of secondary mathematics (e.g., ZPP) and abstract algebra (e.g., zero divisors). This kind of process seems well suited to support students in connecting these two domains. For this reason, we argue that engaging students in this type of reinvention process is a promising instructional approach for developing preservice teachers' horizon content knowledge and deepening their common content knowledge.

Additionally, we argue that such an approach also creates opportunities to expand on the potential of studying abstract algebra to develop MKT by also supporting the development of specialized content knowledge (SCK) and knowledge of content and students (KCS). Collectively engaging abstract algebra students—who, in this case, are preservice teachers—in guided reinvention requires an instructor to elicit and then build on students' informal understandings to develop formal concepts. Often these informal understandings represent students' common content knowledge of related secondary concepts. We saw this in Cook's Chap. 3, as Brian's understanding of the secondary concept of the zero-product property provided the starting point for the development of the concept of zero divisors (and eventually integral domains). In a classroom in which students are engaged in the guided reinvention of abstract algebra concepts, students will frequently share their informal understandings and be asked to make sense of and critique other students' strategies and ideas. This provides two kinds of learning opportunities that are not afforded by teacher-centered instructional approaches. First, this context provides them with opportunities to confront common struggles that students (themselves and others) have with core concepts (e.g., they can observe classmates struggle with the notion of a zero divisor or with an operation not given by a formula) and thus develop KCS (a category of knowledge that is focused on understanding typical student approaches and struggles related to a concept). Second, this context provides many opportunities to make sense of other students' ideas, thus providing an excellent opportunity to develop SCK. Essentially, the collective endeavor of guided reinvention has a unique feature in that the students are engaged as participants in what is typically a teaching activity (analyzing student thinking), which affords a rare opportunity to engage in the kind of mathematical thinking that is (typically) uniquely required in teaching. Notice that this offers a second argument in support of providing preservice teachers with experiences, as learners, in the kinds of classrooms (interactive and focused on student thinking) that they are encouraged to create for their own students. Not only do such courses provide important models of teaching for preservice teachers (as noted by Wasserman in his introduction to this volume), but they also engage preservice teachers in the kinds of mathematical activity that can promote the development of SCK and KCS.

Summary and Directions for Future Research

In the previous two chapters, the authors make a credible case that learning abstract algebra can provide opportunities to strengthen preservice teachers' common content knowledge and to develop their horizon content knowledge. However, we cannot assume that these opportunities will be realized simply by requiring preservice teachers to take a traditional abstract algebra course. Instead, we should design experiences for preservice teachers that actively engage them in making connections between their newly acquired knowledge of abstract algebra and secondary mathematics. Furthermore, preservice teachers should be actively engaged in thinking about the consequences of these connections for their practice and how their new and deeper knowledge of secondary mathematics could help them support their students' learning. Finally, we argue that actively engaging preservice teachers in developing the concepts of abstract algebra by building on their understandings of the secondary concepts (perhaps through guided reinvention) is likely to be more effective than traditional approaches in terms of building connections between secondary mathematics and abstract algebra. Additionally, such an approach has the potential to develop aspects of MKT beyond horizon and common content knowledge by actively engaging preservice teachers in making sense of and critiquing the mathematics of others. In summary, Melhuish and Fagan, and Cook provide us with productive ways to think about *why* preservice mathematics teachers should study abstract algebra, but much work will need to be done to learn *how* to consistently realize the potential of abstract algebra courses in preservice teacher education.

We conclude by calling for research and development focused on ascertaining and realizing the true potential of abstract algebra courses to develop preservice teachers' MKT. First, research is needed to determine the extent to which the kinds of learning opportunities described in the two previous chapters are, in fact, realized by preservice teachers who take standard courses in abstract algebra. A study modeled after that of Melhuish and Fagan (but focused more closely on preservice and/or inservice teachers, and with a more intentional emphasis on core concepts of secondary mathematics) could address this issue nicely. We strongly suspect that the results of such research may be discouraging because traditional abstract algebra courses are not typically designed to support these specific kinds of learning. For this reason, we also call for instructional design research focused on developing means of intentionally engineering these kinds of opportunities in algebra courses for teachers. Such research could profitably build on the work described in Cook's Chap. 3.

Finally, it is important that the field continue to take on the very difficult questions regarding how and to what extent different kinds of MKT can have a positive impact on teachers' practice and students' learning. We strongly suspect that it is necessary to explicitly and intentionally support preservice teachers in developing these kinds of knowledge *and* to explicitly and intentionally support them in translating this knowledge to their teaching practice. For this reason, we encourage projects that bring together researchers primarily focused on the mathematical preparation of teachers with researchers primarily focused on the pedagogical training of teachers. Such collaborations hold the most promise for realizing the potential of abstract algebra courses as an important part of preservice teacher education.

References

Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S., & Oktaç, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *Journal of Mathematical Behavior, 16*(3), 241–309.

Asiala, M., Kleiman, J., Brown, A., & Mathews, D. (1998). The development of students' understanding of permutations and symmetries. *International Journal of Computers for Mathematical Learning, 3*(1), 13–43.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Ball, D. L., & Bass, H. (2009). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures*. Paper presented at the 43rd Jahrestagung der Gesellschaft für Didaktik der Mathematik, Oldenburg. Retrieved from www.mathematik.uni-dortmund.de/ieem/BzMU/BzMU2009/BzMU2009-Inhalt-fuer-Homepage.htm

Brown, A., DeVries, D. J., Dubinsky, E., & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *Journal of Mathematical Behavior, 16*(3), 187–239.

Conference Board of Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II*. Providence, RI: American Mathematical Society and Mathematical Association of America.

Cook, J. P. (2014). The emergence of algebraic structure: Students come to understand units and zero-divisors. *International Journal of Mathematical Education in Science and Technology, 45*(3), 349–359.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267–305.

Dubinsky, E., & Leron, U. (1994). *Learning abstract algebra with ISETL*. New York, NY: Springer.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71–90.

Hohensee, C. (2014). Backward transfer: An investigation of the influence of quadratic functions instruction on students' prior ways of reasoning about linear functions. *Mathematical Thinking and Learning, 16*(2), 135–174.

Johnson, E. M., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *Journal of Mathematical Behavior, 31*(1), 117–129.

Kieran, C. (1979). Children's operation thinking within the context of bracketing and the order of operations. In D. Tall (Ed.), *Proceedings of the third international conference for the psychology of mathematics education* (pp. 128–133). Coventry: Warwick University, Mathematics Education Research Centre.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *Journal of Mathematical Behavior, 28*(2–3), 119–137.

Larsen, S. (2010). Struggling to disentangle the associative and commutative properties. *For the Learning of Mathematics, 30*(1), 37–42.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in abstract algebra. *Journal of Mathematical Behavior, 32*(4), 693–711.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *Journal of the Learning Sciences, 15*(4), 431–449.

Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*(1), 9–38.

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*(2), 125–145.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.