A Circle Packing Problem and Its Connection to Malfatti's Problem

D. Munkhdalai and R. Enkhbat

Abstract We have analytically solved the problem how to split a given triangle's two sides by a line such that a total area of inscribed two circles embedded in each side of the line reaches the maximum. We also show that Malfatti's problem for $n = 2$ is a particular case of our problem.

Keywords Geometry · Circle packing · Malfatti's problem

1 Introduction

1.1 Problem Definition

We have analytically solved the problem how to split a given triangle's two sides by a line such that a total area of inscribed two circles embedded in each side of the line reaches the maximum. We also show that Malfatti's problem for $n = 2$ is a particular case of our problem.

As in Figure 1, the cutting line l splits the $\triangle ABC$ into two areas. There are two embedded circles O_1 and O_2, respectively, in both areas. The cutting line l has to go through both AB and AC. We denote the radius of circle O_1 as r_1 and the radius of circle O_2 as r_2.

The global optimization problem is defined to maximize $r_1^2 + r_2^2$ subject to all positions of line l.

We call the problem as Enkhbat problem named after the last author, because the split line acts as an auxiliary line to collect the maximum points and leads to simple objective function with regard to the original Malfatti's problem for $n = 2$. We will discuss the maximization $r_1^n + r_2^n (n \geq 1)$, subject to all positions of line l in the next article.

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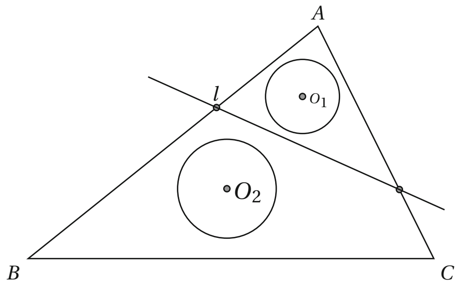


Fig. 1 The definition of the problem

1.1.1 The Problem of Malfatti

In [5], the authors describe the Malfatti's problem: Given a triangle, to find three non-overlapping circles inside it with total maximum area. This problem is also referred to as Malfatti's marble problem in [5]. They mentioned Malfatti's construction problem as well. Malfatti's construction problem is to construct Malfatti's arrangement using ruler and compass. Malfatti's arrangement is that three mutually tangent circles such that each of them is also tangent to two edges of the triangle. In [1], the authors geometrically solved the Malfatti's problem with two circles. In [6], the authors solved Malfatti's problem by the greedy algorithm. Also, in [7–9], Malfatti's problem as well as high dimensional Malfatti's problems were solved by use of global optimization theory and algorithms.

1.1.2 The Special Case of Malfatii's Problem

In [4] and Appendix, R. Enkhbat formulates the following problem: the cutting line l which goes through A and across BC divides the $\triangle ABC$ into two areas. There are two embedded circles I_1 and I_2, respectively, in both areas. We also denote the radius of circle I_1 as r_1 and the radius of circle I_2 as r_2. The global optimization problem is defined to maximize $r_1^2 + r_2^2$. This problem is a special case of our problem which is discussed in this article. Luvsanbyamba Buyankhuu gave an analytic solution of the problem in Appendix using Lagrange multiplier method with an equality constraint.

1.1.3 Optimization Problem of the Inscribed Ball in Polyhedral Set

In [2], R. Enkhbat and B. Barsbold considered the problem for optimal inscribing of two balls into bounded polyhedral set, so that sum of their radii is maximized. The authors formulate this problem as a bilevel programming problem. The gradient based method for solving it has been proposed. In [3], R. Enkhbat and A. Bayarbaatar considered a problem of finding the maximum radius of inscribed ball and minimum radius of circumscribed ball defined over a polyhedral set and proposed some optimization algorithms to solve them. They formulate linear programming model for maximum radius of inscribed ball (from Page 24 of [3]):

$$D = \{x \in \mathbb{R}^n | \langle a^i, x\rangle \leq b_i, i = 1, 2, \ldots, m\}, \text{int}D \neq \emptyset,$$

$$\text{where } a^i \in \mathbb{R}, b_i \in \mathbb{R}, i = 1, 2, \ldots, m$$

denoting x_{n+1} as radius of an inscribed ball in D.

$$x_{n+1} \to \max$$

$$\text{subject to}: \begin{cases} \frac{|\langle a^i, x\rangle - b_i|}{\|a^i\|} \geq x_{n+1} \\ \langle a^i, x\rangle \leq b_i, i = 1, 2, \ldots, m \end{cases}$$

which is equivalent to the following linear programming problem :

$$x_{n+1} \to \max$$

$$\text{subject to : } \langle a^i, x\rangle + \|a^i\| x_{n+1} \leq b_i, i = 1, 2, \ldots, m$$

This is a linear programming solution of Chebyshev center of polyhedral set.

2 Problem Formulation

Let us start to formulate the problem.

As shown in Figure 2a, for maximization purpose, the circle O_1 and circle O_2 both should be inscribed (two dashed circles which are all tangent to sides and the lines). Especially inscribed circle O_2 means it is the biggest one and tangent to PT, PB, and BC assuming T is vertically below P.

We denote the inradius of $\triangle ABC$ as R, so $r_1 \leq R$ and $r_2 \leq R$ are held geometrically.

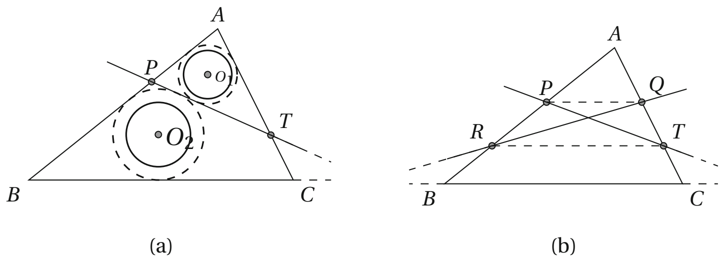


Fig. 2 The formulation. (**a**) The biggest embedded one is inscribed. (**b**) Two types of the cutting line

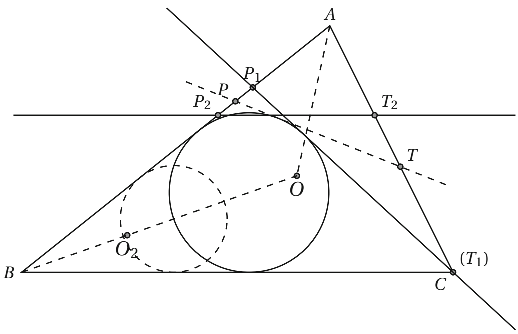


Fig. 3 Solving of Type 1 PT case

In Figure 2b, we have the cutting line PT, Point P on AB, and Point T on AC. We build $PQ||RT||BC$ to have line QR, Point Q on AC, and Point R on AB.

The cutting line has two types:

1. Line PT: for any fixed P, T moves from Q to C, T is always below the P in the vertical direction.
2. Line QR: for any fixed Q, R moves from P to B, R is always below the Q in the vertical direction.

First, we consider only type 1 PT case. Then we will get the result of type 2 QR case in Section 5.

As shown in Figure 3, we can imagine that the circle O_2 is from vertex B to grow away to the end to become the inscribed circle of the $\triangle ABC$, so r_{PTCB} is from 0 to R (inradius of $\triangle ABC$). For any fixed r_{PTCB}, tangent line PT of circle O_2 goes around from P_1T_1 to P_2T_2.

3 Solution

3.1 Some Notations

R is inradius of $\triangle ABC$, r_1 is radius of inscribed circle of $\triangle APT$ and r_2 is radius of inscribed circle of quadrilateral BPTC, $p = \frac{1}{2}(a+b+c)$ is semiperimeter of the triangle.

$$\angle A = 2\alpha, \angle B = 2\beta, \angle C = 2\gamma(\alpha, \beta, \gamma \in (0, \frac{\pi}{2}))$$

$$AB = c, BC = a, AC = b$$

$$\angle APT = 2\theta, \angle BPT = 2(\frac{\pi}{2} - \theta)$$

without loss of generality, we assume $AC \leq AB \leq BC$.

3.2 A Few Preliminary Results

The following results come from basic rules of a triangle such as law of sines, Heron's formula, and feature of half angle of the triangle. We will often use them implicitly or explicitly by reference number later.

$$b \le c \le a$$

$$c = R\cot\alpha + R\cot\beta, b = R\cot\alpha + R\cot\gamma$$

$$\cot\alpha = \frac{p-a}{R}, \cot\beta = \frac{p-b}{R}$$

$$S_{\triangle ABC} = pR = \sqrt{p(p-a)(p-b)(p-c)}(\text{Heron's formula})$$

$$pR^2 = (p-a)(p-b)(p-c)$$

$$\begin{aligned} \frac{R}{a\sin^2\beta} &= \frac{pR}{pa\sin^2\beta} = \frac{S_{\triangle ABC}}{pa\sin^2\beta} = \frac{\frac{1}{2}\sin 2\beta \cdot a \cdot c}{pa\sin^2\beta} \\ &= \frac{c sin\beta\cos\beta}{p\sin^2\beta} = \frac{c \cdot cot\beta}{p} \\ \frac{a\sin\beta\cos\beta - R}{a\sin^2\beta} &= \cot\beta(1-\frac{c}{p}) = \frac{p-b}{R}\cdot\frac{p-c}{p} = \frac{(p-a)(p-b)(p-c)}{pR(p-a)} \\ &= \frac{pR^2}{pR(p-a)} = \frac{R}{p-a} \\ &= \tan\alpha \end{aligned} \tag{1}$$

$$\frac{R}{asin^2\beta} = \cot\beta - \tan\alpha$$

$$asin^2\beta = \frac{R}{\cot\beta - \tan\alpha}$$

$$0 < \alpha < \frac{\pi}{2}, 0 < \beta < \frac{\pi}{2}$$

$$\cot\alpha > 0, \cot\beta > 0$$

$$\cot(\alpha+\beta) > 0(\alpha+\beta+\gamma = \frac{\pi}{2} \implies \alpha+\beta < \frac{\pi}{2})$$

$$\cot\alpha\cot\beta - 1 > 0(\cot\alpha\cot\beta - 1 = \cot(\beta+\alpha)(\cot\beta+\cot\alpha) > 0),$$

$$\cot\alpha\cot\gamma - 1 > 0$$

$$\cot\beta > \tan\alpha(\cot\alpha\cot\beta - 1 > 0 \implies \cot\alpha\cot\beta > 1), \cot\gamma > \tan\alpha \tag{2}$$

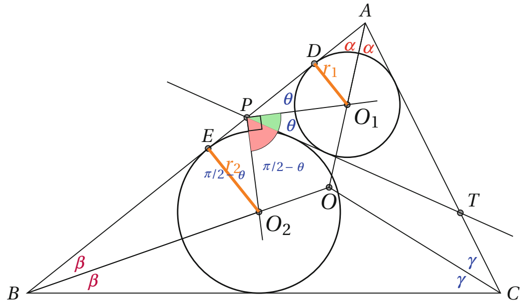


Fig. 4 Express r_1 using r_2 as parameter

3.3 Expression of r_1 via r_2

From Figure 4, we have the following results geometrically

$$AP = AD + PD = r_1(\cot\alpha + \cot\theta)$$

$$PE = r_2 \cot(\frac{\pi}{2} - \theta) = r_2 \tan\theta = r_2 \frac{1}{\cot\theta}$$

$$AE = AB - BE = c - r_2 \cot\beta = AP + PE$$

$$c - r_2 \cot\beta = r_1(\cot\alpha + \cot\theta) + r_2 \frac{1}{\cot\theta}$$

$$r_1 = \frac{c - r_2 \cot\beta - r_2 \frac{1}{\cot\theta}}{\cot\alpha + \cot\theta}$$

Let $x = r_2$ and $y = \cot\theta$, then we have

$$r_1 = \frac{c - x\cot\beta - x\frac{1}{y}}{\cot\alpha + y} = \frac{cy - xy\cot\beta - x}{y^2 + y\cot\alpha}$$

$$r_2 = x (x \in [0, R])$$

Now we can build the objective function of the problem. For further purpose to show the convexity of the objective function, we also need to calculate the partial differentiation of the objective function.

$$f(x, y) = r_1^2 + r_2^2 = (\frac{cy - xy\cot\beta - x}{y^2 + y\cot\alpha})^2 + x^2$$

$$\frac{\partial f}{\partial x} = 2\frac{cy - xy\cot\beta - x}{(y^2 + y\cot\alpha)^2}(-y\cot\beta - 1) + 2x$$

$$\frac{\partial^2 f}{\partial x^2} = 2\frac{(-y\cot\beta - 1)(-y\cot\beta - 1)}{(y^2 + y\cot\alpha)^2} + 2 = 2\frac{(y\cot\beta + 1)^2}{(y^2 + y\cot\alpha)^2} + 2 > 0 \quad (3)$$

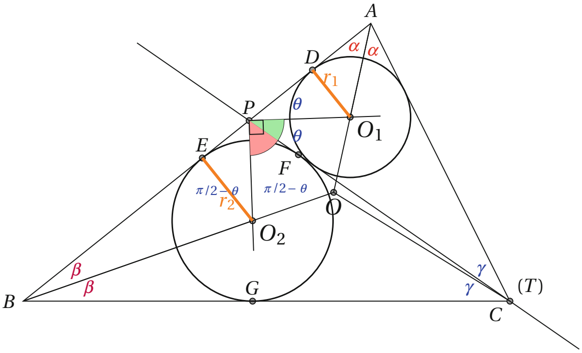


Fig. 5 $y(\cot\theta)$ goes to one edge position: T duplicates with C

The range of x is known as $[0, R]$. we show the range of y now.

θ changes continuously when PT goes around circle $O2$. If PT is parallel to BC , $\theta = \beta$, so $y = \cot\theta = \cot\beta$. If T coincides with C as shown in Figure 5:

In $\triangle PBC$

$$PE = PF, BE = BG, CF = CG$$

$$S_{\triangle PBC} = r_2 \cdot \frac{PB + PC + BC}{2}$$

$$= r_2 \cdot (PE + BG + GC) = r_2 \cdot (PE + BC) = r_2 \cdot (PE + a)$$

$$S_{\triangle PBC} = \frac{1}{2}\sin(2\beta) \cdot PB \cdot BC$$

$$= \frac{1}{2}\sin(2\beta) \cdot (PE + BE) \cdot a = a \cdot \sin\beta \cdot \cos\beta \cdot (PE + r_2 \cot\beta)$$

$$r_2 \cdot (PE + a) = a \cdot \sin\beta \cdot \cos\beta \cdot (PE + r_2 \cot\beta)$$

$$PE = \frac{ar_2 - ar_2\cos^2\beta}{a\sin\beta\cos\beta - r_2} = \frac{ar_2\sin^2\beta}{a\sin\beta\cos\beta - r_2}$$

$$PE = r_2\cot(\frac{\pi}{2} - \theta) = r_2\tan\theta$$

$$r_2\tan\theta = \frac{ar_2\sin^2\beta}{a\sin\beta\cos\beta - r_2}$$

$$\tan\theta = \frac{a\sin^2\beta}{a\sin\beta\cos\beta - r_2}$$

$$\cot\theta = \frac{a\sin\beta\cos\beta - r_2}{a\sin^2\beta}$$

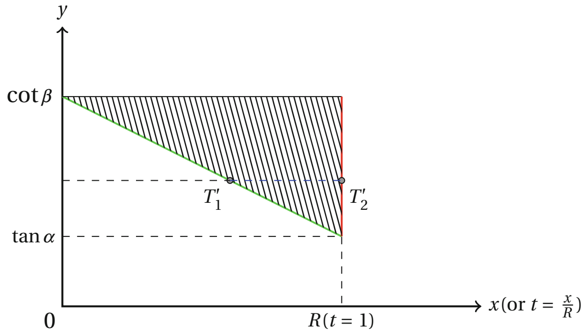


Fig. 6 Domain of (x, y) or (t, y)

$x \in [0, R]$

$$x = 0 \implies \cot\theta = \frac{a \sin\beta \cos\beta}{a \sin^2\beta} = \cot\beta$$

$$x = R \implies \cot\theta = \frac{a \sin\beta \cos\beta - R}{a \sin^2\beta} = \tan\alpha, \text{ according to}(1)$$

(2) $\implies \cot\beta > \tan\alpha$, so $y = \cot\theta \in [\tan\alpha, \cot\beta]$. Figure 6 shows domain of (x, y).

3.4 Existence of Global Maximum of $r_1^2 + r_2^2$

Theorem 1 *Domain of* (x, y) *showed in Figure 6 is convex and compact.*

Proof For any points (x_1, y_1) and (x_2, y_2) of domain of (x, y)

$$x_1, x_2 \in [0, R] \text{ and } y_1, y_2 \in [\frac{a \sin\beta \cos\beta - x}{a \sin^2\beta}, \cot\beta]$$

$$\text{For any constant } \theta \in [0, 1]$$

$$\text{We denote } (\bar{x}, \bar{y}) = \theta(x_1, y_1) + (1-\theta)(x_2, y_2) = (\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$$

$$[0, R] \text{ is convex } \implies \bar{x} = \theta x_1 + (1-\theta)x_2 \in [0, R]$$

$$y_1 \geq \frac{a \sin\beta \cos\beta - x_1}{a \sin^2\beta} \text{and} y_2 \geq \frac{a \sin\beta \cos\beta - x_2}{a \sin^2\beta} \implies$$

$$\theta y_1 + (1-\theta)y_2 \geq \theta \frac{a \sin\beta \cos\beta - x_1}{a \sin^2\beta} + (1-\theta)\frac{a \sin\beta \cos\beta - x_2}{a \sin^2\beta}$$

$$= \frac{a\sin\beta\cos\beta + (\theta x_1 + (1-\theta)x_2)}{a\sin^2\beta} = \frac{a\sin\beta\cos\beta + \bar{x}}{a\sin^2\beta}$$

$$\text{also } y_1 \le \cot\beta \text{ and } y_2 \le \cot\beta$$

$$\theta y_1 + (1-\theta)y_2 \le \theta\cot\beta + (1-\theta)\cot\beta = \cot\beta$$

$$\bar{y} \in [\frac{a\sin\beta\cos\beta - \bar{x}}{a\sin^2\beta}, \cot\beta]$$

$$(3.4) \text{ and } (3.4) \implies (\bar{x}, \bar{y}) \in \text{domain of } (x, y)$$

$$\text{So domain of } (x, y)\text{is convex}$$

$$\text{Domain of } (x, y) \text{ is closed and bounded in } \mathbb{R}^2.$$

$$\text{According to Heine} - \text{Borel theorem the domain of (x, y) is compact.}$$

According to Weierstrass extreme value theorem and continuity of the objective function $f(x, y) = r_1^2 + r_2^2$, we have

Theorem 2 $r_1^2 + r_2^2$ *reach its global maximum in domain of (x,y), showed in Figure 6.*

3.5 Green Candidate and Red Candidate

According to (3), $f(x, y)$ is convex about x, it reaches its maximum at the edge of x. In Figure 6, we can observe, for any fixed y, T_1' and T_2' are the edge points of x, so $f(x, y)$ will reach its maximum on green line and red line.

In Section 4, we will show that one point on green line and another point on red line together make the global maximum point set of $f(x, y)$. We call the two points as green candidate (or GC) and red candidate (or RC) of the global maximum point set of $f(x, y)$. Before we calculate the GC and RC in Section 4, we use the following theorem to show that the GC and RC are both in the domain of $f(x, y)$, because the middle results are useful in Section 4.

Also because we are considering the type 1 PT case , we use GC$_{PT}$ and RC$_{PT}$ notations.

Theorem 3 GC_{PT}:$(R\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1}, \sqrt{\frac{\cot\beta}{\cot\alpha}})$,$RC_{PT}$:$(R, \tan\alpha + \sec\alpha)$ *are both in domain of (x,y) showed in Figure 6.*

Proof First we prove $(R\frac{\cot\beta\cot\alpha-\sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha-1}, \sqrt{\frac{\cot\beta}{\cot\alpha}})$ in domain of (x,y)

$$y = \sqrt{\frac{\cot\beta}{\cot\alpha}} = \sqrt{\cot\beta\tan\alpha}$$

$$y = \frac{a\sin\beta\cos\beta - x}{a\sin^2\beta} \implies$$

$$x = asin^2\beta(\cot\beta - \sqrt{\cot\beta\tan\alpha}) = R\frac{\cot\beta - \sqrt{\cot\beta\tan\alpha}}{\cot\beta - \tan\alpha}$$

$$= R\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1} = R\frac{1 - \sqrt{\frac{1}{\cot\beta\cot\alpha}}}{1 - \frac{1}{\cot\beta\cot\alpha}}$$

$$\frac{1}{\cot\beta\cot\alpha} = \frac{R^2}{(p-b)(p-a)} = \frac{pR^2}{p(p-b)(p-a)} = \frac{p-c}{p}$$

$$x = R\frac{1 - \sqrt{\frac{p-c}{p}}}{1 - \frac{p-c}{p}} = R\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1} \implies$$

$$\left(R\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1}, \sqrt{\frac{\cot\beta}{\cot\alpha}}\right) \text{ on the line}$$

$$y = \frac{a\sin\beta\cos\beta - x}{a\sin^2\beta}$$

It is suffice to show $y \in [\tan\alpha, \cot\beta]$

$\tan^2\alpha < \cot\beta\tan\alpha < \cot^2\beta,\ y^2 = \cot\beta\tan\alpha,\ y > 0 \implies$

$y \in [\tan\alpha, \cot\beta]$

Figure 6 showed, $x = R \implies y \in [\tan\alpha, \cot\alpha]$, so we just need to prove $\tan\alpha + \sec\alpha \in [\tan\alpha, \cot\alpha]$

$\tan\alpha + \sec\alpha > \tan\alpha(\sec\alpha > 0)$

We have an auxiliary function

$$g(y) = y^2 - 2y\tan\alpha - 1$$

$$g(\tan\alpha + \sec\alpha) = (\tan\alpha + \sec\alpha)^2 - 2(\tan\alpha + \sec\alpha)\tan\alpha - 1$$

$$= \tan^2\alpha + 2\tan\alpha\sec\alpha + \sec^2\alpha - 2\tan^2\alpha - 2\tan\alpha\sec\alpha - 1$$

$$= \sec^2\alpha - \tan^2\alpha - 1 = 0$$

$$g(\cot\beta) = cot^2\beta - 2\tan\alpha\cot\beta - 1$$

$$= \frac{(p-b)^2}{R^2} - 2\frac{R}{p-a}\frac{p-b}{R} - 1$$

$$= \frac{(p-b)^2}{\frac{(p-a)(p-b)(p-c)}{p}} - 2\frac{p-b}{p-a} - 1$$

$$= \frac{p(p-b)}{(p-a)(p-c)} - 2\frac{(p-b)(p-c)}{(p-a)(p-c)} - \frac{(p-a)(p-c)}{(p-a)(p-c)}$$

$$= \frac{p^2 - bp - 2p^2 + 2bp + 2cp - 2bc - p^2 + ap + cp - ac}{(p-a)(p-c)}$$

$$= \frac{-2p^2 + bp + ap + cp + 2cp - 2bc - ac}{(p-a)(p-c)}$$

$$= \frac{-2p^2 + (a+b+c)p + 2p \cdot c - 2bc - ac}{(p-a)(p-c)}$$

$$= \frac{-2p^2 + 2p^2 + c(a+b+c) - 2bc - ac}{(p-a)(p-c)}$$

$$= \frac{ac + bc + c^2 - 2bc - ac}{(p-a)(p-c)}$$

$$= \frac{c^2 - bc}{(p-a)(p-c)} = \frac{c(c-b)}{(p-a)(p-c)} \geq 0 (c \geq b)$$

$g\ (y)$ increase in $[\tan\alpha, +\infty)$, $g\ (\tan\alpha + \sec\alpha) = 0$,

$g\ (\cot\beta) \geq 0 \implies \tan\alpha + \sec\alpha \leq \cot\beta$

4 Maximum of $r_1^2 + r_2^2$ Type 1 PT Case

We denote $t = \frac{x}{R}$ and show that r_1 and r_2 are functions of variables t and y.

$x \in [0, R] \implies t \in [0, 1]$

$\tan\alpha \leq y \leq \cot\beta$

$r_2 = x = tR$

$$r_1 = \frac{cy - xy\cot\beta - x}{y^2 + y\cot\alpha} = \frac{cy - tRy\cot\beta - tR}{y^2 + y\cot\alpha}$$

$$= \frac{(R\cot\alpha + R\cot\beta)y - tRy\cot\beta - tR}{y^2 + y\cot\alpha} = \frac{y\cot\alpha + y\cot\beta - ty\cot\beta - t}{y^2 + y\cot\alpha} R$$

$$\frac{a\sin\beta\cos\beta - x}{a\sin^2\beta} = \frac{a\sin\beta\cos\beta - tR}{a\sin^2\beta}$$

$$= \cot\beta - t\frac{R}{a\sin^2\beta} = \cot\beta - t(\cot\beta - \tan\alpha) \implies$$

$$\cot\beta - t(\cot\beta - \tan\alpha) \le y \le \cot\beta \implies$$

$$\frac{\cot\beta - y}{\cot\beta - \tan\alpha} \le t \le 1$$

Because $r_1^2 + r_2^2 = R^2 Q(t, y)$, we have objective function as $Q(t, y)$

$$Q(t, y) = (\frac{y\cot\alpha + y\cot\beta - ty\cot\beta - t}{y^2 + y\cot\alpha})^2 + t^2$$

$$\frac{\partial Q}{\partial t} = 2\frac{y\cot\alpha + y\cot\beta - ty\cot\beta - t}{(y^2 + y\cot\alpha)^2}(-y\cot\beta - 1) + 2t$$

$$\frac{\partial^2 Q}{\partial t^2} = 2\frac{(-y\cot\beta - 1)(-y\cot\beta - 1)}{(y^2 + y\cot\alpha)^2} + 2 = 2\frac{(y\cot\beta + 1)^2}{(y^2 + y\cot\alpha)^2} + 2 > 0$$

$\frac{\partial^2 Q}{\partial t^2} > 0$ means Q(t,y) is convex about parameter t, so for every fixed y,$Max\,Q(t, y) = Max\{Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y), Q(1, y)\}$. When y goes through $[\tan\alpha, \cot\beta]$, we also have $Max\,Q(t, y) = Max\{Max\ Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y), Max\ Q(1, y)\}$.

Geometrically $Max\ Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y)$ will be searched on the green line of the domain and $Max\ Q(1, y)\}$ will be searched on the red line of the domain.

4.1 $Max\ Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y)$: *Searching on the Green Line of the Domain*

We denote $T(y) = Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y)$

$y \in [\tan\alpha, \cot\beta] \implies Max\ T(y) = Max\{T(\tan\alpha), T(\cot\beta), T(\frac{dT}{dy} = 0)\}$

$$\frac{y\cot\alpha + y\cot\beta - ty\cot\beta - t}{y^2 + y\cot\alpha}$$

$$= \frac{y\cot\alpha + y\cot\beta - (\frac{\cot\beta - y}{\cot\beta - \tan\alpha})y\cot\beta - \frac{\cot\beta - y}{\cot\beta - \tan\alpha}}{y^2 + y\cot\alpha}$$

$$= \frac{y\cot\beta\cot\alpha - y + y\cot^2\beta - y\cot\beta\tan\alpha - y\cot^2\beta + y^2\cot\beta - \cot\beta + y}{(y^2 + y\cot\alpha)(\cot\beta - \tan\alpha)}$$

$$= \frac{y\cot\beta\cot\alpha - y\cot\beta\tan\alpha + y^2\cot\beta - \cot\beta}{(y^2 + y\cot\alpha)(\cot\beta - \tan\alpha)}$$

$$= \frac{y\cot\beta(y + \cot\alpha) - \cot\beta\tan\alpha(y + \cot\alpha)}{y(y + \cot\alpha)(\cot\beta - \tan\alpha)}$$

$$= \frac{y\cot\beta - \cot\beta\tan\alpha}{y(\cot\beta - \tan\alpha)}$$

$$T(y) = [\frac{y\cot\beta - \cot\beta\tan\alpha}{y(\cot\beta - \tan\alpha)}]^2 + (\frac{\cot\beta - y}{\cot\beta - \tan\alpha})^2$$

$$= (\frac{\cot\beta\cot\alpha - \frac{\cot\beta}{y}}{\cot\beta\cot\alpha - 1})^2 + (\frac{\cot\beta\cot\alpha - y\cot\alpha}{\cot\beta\cot\alpha - 1})^2$$

$$= \frac{1}{(\cot\beta\cot\alpha - 1)^2}[(\cot\beta\cot\alpha - \frac{\cot\beta}{y})^2 + (\cot\beta\cot\alpha - y\cot\alpha)^2]$$

$$T(\tan\alpha) = \frac{1}{(\cot\beta\cot\alpha - 1)^2}[(\cot\beta\cot\alpha - \frac{\cot\beta}{\tan\alpha})^2 + (\cot\beta\cot\alpha - \tan\alpha\cot\alpha)^2] = 1$$

$$T(\cot\beta) = (\frac{1}{\cot\beta\cot\alpha - 1})^2[(\cot\beta\cot\alpha - \frac{\cot\beta}{\cot\beta})^2 + (\cot\beta\cot\alpha - \cot\beta\cot\alpha)^2] = 1$$

$$\frac{dT}{dy} = 0$$

$$\implies 2(\cot\beta\cot\alpha - \frac{\cot\beta}{y})\frac{\cot\beta}{y^2} + 2(\cot\beta\cot\alpha - y\cot\alpha)(-\cot\alpha) = 0$$

$$\implies \cot^2\beta\cot\alpha\frac{1}{y^2} - \frac{\cot^2\beta}{y^3} - \cot\beta\cot^2\alpha + y\cot^2\alpha = 0$$

$$\implies \cot\beta\cot\alpha\frac{1}{y^2}(\cot\beta - y^2\cot\alpha) - \frac{1}{y^3}(\cot^2\beta - y^4\cot^2\alpha) = 0$$

$$\implies \cot\beta\cot\alpha\frac{1}{y^2}(\cot\beta - y^2\cot\alpha) - \frac{1}{y^3}(\cot\beta - y^2\cot\alpha)(\cot\beta + y^2\cot\alpha) = 0$$

$$\implies (\cot\beta - y^2\cot\alpha)\frac{1}{y^3}(y\cot\beta\cot\alpha - \cot\beta - y^2\cot\alpha) = 0$$

$\frac{1}{y^3} > 0$

$\cot\beta - y^2\cot\alpha = 0$ or $y\cot\beta\cot\alpha - \cot\beta - y^2\cot\alpha = 0$

$$\cot\beta - y^2\cot\alpha = 0$$

$$\implies y = \sqrt{\frac{\cot\beta}{\cot\alpha}}$$

$$y\cot\beta\cot\alpha - \cot\beta - y^2\cot\alpha = 0$$

$$\implies \cot\beta\cot\alpha - \frac{\cot\beta}{y} = y\cot\alpha \text{and}$$

$$y^2\cot\alpha - y\cot\beta\cot\alpha = -\cot\beta$$

$$T(y) = \frac{1}{(\cot\beta\cot\alpha - 1)^2}[(\cot\beta\cot\alpha - \frac{\cot\beta}{y})^2$$

$$+ (\cot\beta\cot\alpha - y\cot\alpha)^2]$$

$$= \frac{1}{(\cot\beta\cot\alpha - 1)^2}(2y^2\cot^2\alpha - 2y\cot^2\alpha\cot\beta$$

$$+ \cot^2\beta\cot^2\alpha)$$

$$= \frac{1}{(\cot\beta\cot\alpha - 1)^2}[2\cot\alpha(y^2\cot\alpha - y\cot\alpha\cot\beta)$$

$$+ \cot^2\beta\cot^2\alpha]$$

$$= \frac{\cot^2\beta\cot^2\alpha - 2\cot\beta\cot\alpha}{(\cot\beta\cot\alpha - 1)^2} < 1$$

$$Max\ T(y) = Max\{T(\tan\alpha), T(\cot\beta), T(\frac{dT}{dy} = 0)\}$$

$$= Max\{1, T(\sqrt{\frac{\cot\beta}{\cot\alpha}})\} \implies$$

$$Max\ Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y)$$

$$= Max\{1, 2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1})^2\}$$

4.2 *Max Q*(1, *y*): *Searching on the Red Line of the Domain*

We denote $S(y) = Q(1, y)$
$y \in [\tan\alpha, \cot\beta] \implies Max\ S(y) = Max\{S(\tan\alpha), S(\cot\beta), S(\frac{dS}{dy} = 0)\}$

$$S(y) = (\frac{y\cot\alpha - 1}{y^2 + y\cot\alpha})^2 + 1$$

$$S(\tan\alpha) = 1$$

$$S(\cot\beta) = (\frac{\cot\beta\cot\alpha - 1}{\cot^2\beta + \cot\beta\cot\alpha})^2 + 1 > S(\tan\alpha)$$

$$\frac{dS}{dy} = 0$$

$$\implies (\frac{y\cot\alpha - 1}{y^2 + y\cot\alpha})[\frac{y^2\cot\alpha + y\cot^2\alpha - (y\cot\alpha - 1)(2y + \cot\alpha)}{(y^2 + y\cot\alpha)^2}] = 0$$

$$\implies (\frac{y\cot\alpha - 1}{y^2 + y\cot\alpha})[\frac{-y^2\cot\alpha + 2y + \cot\alpha)}{(y^2 + y\cot\alpha)^2}] = 0$$

$y\cot\alpha - 1 = 0 \implies y = \tan\alpha$ is already calculated
Let us calculate $-y^2\cot\alpha + 2y + \cot\alpha = 0$

$$\begin{aligned}
&- y^2\cot\alpha + 2y + \cot\alpha = 0 \implies y^2 - 2y\tan\alpha - 1 \\
&= 0 \implies y = \tan\alpha + \sec\alpha \\
\frac{y\cot\alpha - 1}{y^2 + y\cot\alpha} &= \frac{(\tan\alpha + \sec\alpha)\cot\alpha - 1}{(\tan\alpha + \sec\alpha)^2 + (\tan\alpha + \sec\alpha)\cot\alpha} \\
&= \frac{\sec\alpha\cot\alpha}{\tan^2\alpha + \sec^2\alpha + 2\sec\alpha\tan\alpha + 1 + \sec\alpha\cot\alpha} \\
&= \frac{\sec\alpha\cot\alpha}{2\sec^2\alpha + 2\sec\alpha\tan\alpha + \sec\alpha\cot\alpha} \\
&= \frac{1}{2\sec\alpha\tan\alpha + 2\tan^2\alpha + 1} \\
&= \frac{1}{2\sec\alpha\tan\alpha + \tan^2\alpha + \sec\alpha} \\
&= \frac{\sec^2\alpha - \tan^2\alpha}{(\sec\alpha + \tan\alpha)^2} \\
&= \frac{\sec\alpha - \tan\alpha)}{\sec\alpha + \tan\alpha} \qquad (4) \\
S(\tan\alpha + \sec\alpha) &= (\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha})^2 + 1 \\
&Theorem\ 3 \implies \tan\alpha + \sec\alpha \le \cot\beta \\
&\text{in } [\tan\alpha + \sec\alpha, \cot\beta], y\cot\alpha - 1 > 0,\ y^2 + y\cot\alpha > 0, \\
&- y^2\cot\alpha + 2y + \cot\alpha \le 0
\end{aligned}$$

$$\frac{dS}{dy}(y) = (\frac{\cot\beta\cot\alpha - 1}{y^2 + y\cot\alpha})[\frac{-y^2\cot\alpha + 2y + \cot\alpha)}{(y^2 + y\cot\alpha)^2}] \leq 0$$

$S(y)$ does not increase in $[\tan\alpha + \sec\alpha, \cot\beta] \implies S(\cot\beta) \leq S(\tan\alpha + \sec\alpha)$
$Max\ S(y) = S(\tan\alpha + \sec\alpha)$

$Max\ Q(1, \tan\alpha + \sec\alpha) = S(\tan\alpha + \sec\alpha)$

4.3 $Max(r_1^2 + r_2^2)$ *in Type 1 PT Case*

We can combine the above two results as
$Max\, Q(t, y) = Max\{Max\ Q(\frac{\cot\beta - y}{\cot\beta - \tan\alpha}, y), Max\ Q(1, y)\}$
$= Max\{1, 2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1})^2, Q(1, \tan\alpha + \sec\alpha)\}$
Because $Q(1, \tan\alpha + \sec\alpha)\} = S(\tan\alpha + \sec\alpha) \geq S(\cot\beta) \geq S(\tan\alpha) = 1$
Therefore $Max\, Q(t, y) = Max\{2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1})^2, (\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha})^2 + 1\}$
And so, $Max(r_1^2 + r_2^2) = Max\{2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1})^2, (\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha})^2 + 1\}R^2$

5 Maximum of $r_1^2 + r_2^2$ Type 2 QR Case

We just need to exchange b and c , exchange γ and β in all formula in Section 4, for example,

$$f(x, y) = r_1^2 + r_2^2 = (\frac{by - xy\cot\gamma - x}{y^2 + y\cot\alpha})^2 + x^2$$

Because the relation $c \geq b$ is not symmetrical , we have

$$\begin{aligned} g(\cot\gamma) &= cot^2\gamma - 2\tan\alpha\cot\gamma - 1 \\ &= \frac{(p-c)^2}{R^2} - 2\frac{R}{p-a}\frac{p-c}{R} - 1 = \frac{(p-c)^2}{\frac{(p-a)(p-b)(p-c)}{p}} - 2\frac{p-c}{p-a} - 1 \\ &= \frac{p(p-c)}{(p-a)(p-b)} - 2\frac{(p-b)(p-c)}{(p-a)(p-b)} - \frac{(p-a)(p-b)}{(p-a)(p-b)} \\ &= \frac{p^2 - cp - 2p^2 + 2cp + 2bp - 2bc - p^2 + ap + bp - ab}{(p-a)(p-b)} \end{aligned}$$

$$= \frac{-2p^2 + bp + ap + cp + 2bp - 2bc - ab}{(p-a)(p-b)}$$
$$= \frac{-2p^2 + (a+b+c)p + 2p \cdot b - 2bc - ab}{(p-a)(p-b)}$$
$$= \frac{-2p^2 + 2p^2 + b(a+b+c) - 2bc - ab}{(p-a)(p-b)}$$
$$= \frac{ab + bc + b^2 - 2bc - ab}{(p-a)(p-b)}$$
$$= \frac{b^2 - bc}{(p-a)(p-b)} = \frac{b(b-c)}{(p-a)(p-b)} \leq 0 (c \geq b)$$

$g(y)$ increase in $[\tan\alpha, +\infty)$, g $(\tan\alpha + \sec\alpha) = 0$, g $(\cot\gamma) \leq 0 \implies$ $\tan\alpha + \sec\alpha \geq \cot\gamma$

$\tan\alpha + \sec\alpha$ beyond $[\tan\alpha, \cot\gamma]$, so RC_{QR} become $(R, \cot\gamma)$, we have the following theorem in type 2 QR case.

Theorem 4 *GC_{QR}:$(R\frac{\cot\gamma\cot\alpha - \sqrt{\cot\gamma\cot\alpha}}{\cot\gamma\cot\alpha - 1}, \sqrt{\frac{\cot\gamma}{\cot\alpha}})$,$RC_{QR}$:$(R, \cot\gamma)$ are both in the domain of (x,y)*

6 Maximum of $r_1^2 + r_2^2$

The candidate point set are $\{\mathrm{GC}_{PT}, \mathrm{RC}_{PT}, \mathrm{GC}_{QR}, \mathrm{RC}_{QR}\}$.

$r_1^2 + r_2^2$ is geometrically the same at RC_{QR}:$(R, \cot\gamma)$ point and at $(R, \cot\beta)$) of type 1 PT case, so less than $r_1^2 + r_2^2$ at RC_{PT}:$(R, \tan\alpha + \sec\alpha)$.

The candidate point set becomes $\{\mathrm{GC}_{PT}, \mathrm{RC}_{PT}, \mathrm{GC}_{QR}\}$.

$$Max(r_1^2 + r_2^2) = Max\{2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1})^2, (\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha})^2 + 1, 2(\frac{\cot\gamma\cot\alpha - \sqrt{\cot\gamma\cot\alpha}}{\cot\gamma\cot\alpha - 1})^2\}R^2$$

$$\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1}$$
$$= \frac{\sqrt{\cot\beta\cot\alpha}(\sqrt{\cot\beta\cot\alpha} - 1)}{(\sqrt{\cot\beta\cot\alpha} + 1)(\sqrt{\cot\beta\cot\alpha} - 1)}$$

$$= \frac{\sqrt{\cot\beta\cot\alpha}}{\sqrt{\cot\beta\cot\alpha}+1} =$$

$$\frac{1}{1+\sqrt{\tan\beta\tan\alpha}} > \frac{1}{1+\sqrt{\tan\gamma\tan\alpha}}(\beta < \gamma) \tag{5}$$

$$\text{also,} \frac{\cot\gamma\cot\alpha - \sqrt{\cot\gamma\cot\alpha}}{\cot\gamma\cot\alpha - 1}$$

$$= \frac{\sqrt{\cot\gamma\cot\alpha}(\sqrt{\cot\gamma\cot\alpha}-1)}{(\sqrt{\cot\gamma\cot\alpha}+1)(\sqrt{\cot\gamma\cot\alpha}-1)}$$

$$= \frac{\sqrt{\cot\gamma\cot\alpha}}{\sqrt{\cot\gamma\cot\alpha}+1} = \frac{1}{1+\sqrt{\tan\gamma\tan\alpha}}$$

$$2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha - 1})^2$$

$$> 2(\frac{\cot\gamma\cot\alpha - \sqrt{\cot\gamma\cot\alpha}}{\cot\gamma\cot\alpha - 1})^2$$

The candidate point set becomes $\{\text{GC}_{PT}, \text{RC}_{PT}\}$.
$Max(r_1^2 + r_2^2) = Max\{2(\frac{\cot\beta\cot\alpha - \sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha-1})^2, (\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2 + 1\}R^2$
we have the following theorem about Max$(r_1^2 + r_2^2)$

Theorem 5 *When the cutting line goes through AB, AC sides the $r_1^2 + r_2^2$ reach its maximum at $(R\frac{\cot\beta\cot\alpha-\sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha-1}, \sqrt{\frac{\cot\beta}{\cot\alpha}})$ or $(R, \tan\alpha+\sec\alpha)$, the maximum value is $2(\frac{\cot\beta\cot\alpha-\sqrt{\cot\beta\cot\alpha}}{\cot\beta\cot\alpha-1})^2R^2$ or $[(\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2+1]R^2$, respectively.*

7 Proof of Malfatti's Problem for $n = 2$

7.1 The Cutting Line Goes Through All Three Sides of the Triangle

If the cutting line goes through AB, AC, andBC, according to Theorem 5 and (5) we have
$Max(r_1^2 + r_2^2) = Max\{2(\frac{1}{1+\sqrt{\tan\beta\tan\alpha}})^2, (\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2 + 1, 2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2, (\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2 + 1, 2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2, (\frac{\sec\gamma-\tan\gamma}{\sec\gamma+\tan\gamma})^2 + 1\}R^2$
BC is the long one, the $2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2$ duplicate we have

$Max(r_1^2 + r_2^2) = Max\{2(\frac{1}{1+\sqrt{\tan\beta\tan\alpha}})^2, 2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2, (\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2 + 1, (\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2 + 1, (\frac{\sec\gamma-\tan\gamma}{\sec\gamma+\tan\gamma})^2 + 1\}R^2$
and $\alpha \geq \gamma$, we get
$Max(r_1^2 + r_2^2) = Max\{2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2, (\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2 + 1, (\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2 + 1, (\frac{\sec\gamma-\tan\gamma}{\sec\gamma+\tan\gamma})^2 + 1\}R^2$

$$(\frac{\sec\beta - \tan\beta}{\sec\beta + \tan\beta})^2 + 1 = (\frac{1-\sin\beta}{1+\sin\beta})^2 + 1 = \frac{2(1+\sin^2\beta)}{(1+\sin\beta)^2} \tag{6}$$

$$(\frac{\sec\gamma - \tan\gamma}{\sec\gamma + \tan\gamma})^2 + 1 = (\frac{1-\sin\gamma}{1+\sin\gamma})^2 + 1 = \frac{2(1+\sin^2\gamma)}{(1+\sin\gamma)^2} \tag{7}$$

if $\beta \leq \gamma \implies \tan\beta \leq \sqrt{\tan\beta\tan\gamma}$

$$2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2 = \frac{2}{(1+\sqrt{\tan\beta\tan\gamma})^2} < \frac{2(1+\sin^2\beta)}{(1+\sqrt{\tan\beta\tan\gamma})^2}$$
$$\leq \frac{2(1+\sin^2\beta)}{(1+\tan\beta)^2} < \frac{2(1+\sin^2\beta)}{(1+\sin\beta)^2}$$
$$(6) \implies 2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2 < (\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2 + 1 \tag{8}$$

And if $\beta > \gamma \implies \tan\gamma < \sqrt{\tan\beta\tan\gamma}$

$$2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2 = \frac{2}{(1+\sqrt{\tan\beta\tan\gamma})^2} < \frac{2(1+\sin^2\gamma)}{(1+\sqrt{\tan\beta\tan\gamma})^2}$$
$$\leq \frac{2(1+\sin^2\gamma)}{(1+\tan\gamma)^2} < \frac{2(1+\sin^2\gamma)}{(1+\sin\gamma)^2}$$
$$(7) \implies 2(\frac{1}{1+\sqrt{\tan\beta\tan\gamma}})^2 < (\frac{\sec\gamma-\tan\gamma}{\sec\gamma+\tan\gamma})^2 + 1 \tag{9}$$

$Max(r_1^2 + r_2^2) = Max\{(\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2 + 1, (\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2 + 1, (\frac{\sec\gamma-\tan\gamma}{\sec\gamma+\tan\gamma})^2 + 1\}R^2$
From (8) and (9), we have

Theorem 6 *When the cutting line goes through AB, AC, and BC sides, the* $r_1^2 + r_2^2$ *reach its maximum at* $(R, \tan\alpha + \sec\alpha)$ *or* $(R, \tan\beta + \sec\beta)$ *or* $(R, \tan\gamma + \sec\gamma)$*, the maximum value is* $[(\frac{\sec\alpha-\tan\alpha}{\sec\alpha+\tan\alpha})^2 + 1]R^2$ *or* $[(\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2 + 1]R^2$ *or* $[(\frac{\sec\gamma-\tan\gamma}{\sec\gamma+\tan\gamma})^2 + 1]R^2$*, respectively.*

$$(\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha})^2 + 1 = (\frac{1-\sin\alpha}{1+\sin\alpha})^2 + 1 = [\frac{(1-\sin\alpha)^2}{1-\sin^2\alpha}]^2$$
$$+1 = (\frac{1-\sin\alpha}{\cos\alpha})^4 + 1 = [\tan(\frac{\pi}{4} - \frac{\alpha}{2})]^4 + 1$$

We have assumed $\alpha \geq \gamma \geq \beta$, so
$[\tan(\frac{\pi}{4} - \frac{\alpha}{2})]^4 + 1 \leq [\tan(\frac{\pi}{4} - \frac{\gamma}{2})]^4 + 1 \leq [\tan(\frac{\pi}{4} - \frac{\beta}{2})]^4 + 1$
and we have $Max(r_1^2 + r_2^2) = (\frac{\sec\beta - \tan\beta}{\sec\beta + \tan\beta})^2 R^2$
Finally, we have

Theorem 7 *When the cutting line goes through AB, AC, and BC sides, if β is smallest half angle of $\triangle ABC$, the $r_1^2 + r_2^2$ reach its maximum at $(R, \tan\beta + \sec\beta)$, the maximum value is $[(\frac{\sec\beta - \tan\beta}{\sec\beta + \tan\beta})^2 + 1]R^2$.*

7.2 Malfatti n = 2 Problem as Corollary

Theorem 8 *At $(R, \tan\alpha + \sec\alpha)$ point, r_1 and r_2 go to tangent position.*

Proof we just need to proof $|O_1O_2| = r_1 + r_2$

$$r_1 = \frac{cy - xy\cot\beta - x}{y^2 + y\cot\alpha} = \frac{R(\cot\alpha + \cot\beta)y - Ry\cot\beta - R}{y^2 + y\cot\alpha}$$
$$= \frac{Ry\cot\alpha - R}{y^2 + y\cot\alpha} = R\frac{y\cot\alpha - 1}{y^2 + y\cot\alpha}$$

$$(4) \implies \frac{y\cot\alpha - 1}{y^2 + y\cot\alpha} = \frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha}$$
$$= \frac{1-\sin\alpha}{1+\sin\alpha}$$
$$r_1 + r_2 = R(1 + \frac{1-\sin\alpha}{1+\sin\alpha})$$
$$= R\frac{2}{1+\sin\alpha}$$

$$(\frac{r_1}{\sin\theta})^2 + (\frac{r_2}{\cos\theta})^2 =$$
$$(\frac{R}{\cos\theta})^2 + (\frac{cy - xy\cot\beta - x}{(y^2 + y\cot\alpha)\sin\theta})^2 =$$
$$(\frac{R}{\cos\theta})^2 + (\frac{R(\cot\alpha + \cot\beta)y - Ry\cot\beta - R}{(y^2 + y\cot\alpha)\sin\theta})^2 =$$

$$(\frac{R}{\cos\theta})^2 + (\frac{R}{\sin\theta})^2(\frac{y\cot\alpha - 1}{y^2 + y\cot\alpha})^2$$

$$(4) \implies (\frac{r_1}{\sin\theta})^2 + (\frac{r_2}{\cos\theta})^2 =$$

$$(\frac{R}{\cos\theta})^2 + (\frac{R}{\sin\theta})^2(\frac{\sec\alpha - \tan\alpha}{\sec\alpha + \tan\alpha})^2 =$$

$$(\frac{R}{\cos\theta})^2 + (\frac{R}{\sin\theta})^2(\frac{\sec\alpha - \tan\alpha}{\cot\theta})^2 =$$

$$(\frac{R}{\cos\theta})^2 + (\frac{R}{\sin\theta\cot\theta})^2(\sec\alpha - \tan\alpha)^2 =$$

$$(\frac{R}{\cos\theta})^2(1 + \sec^2\alpha - 2\sec\alpha\tan\alpha + \tan^2\alpha) =$$

$$(\frac{R}{\cos\theta})^2(2\sec^2\alpha - 2\sec\alpha\tan\alpha) =$$

$$(R^2\sec^2\theta)2\sec\alpha(\sec\alpha - \tan\alpha)$$

$$\sec^2\theta = 1 + \tan^2\theta = 1 + (\frac{1}{\sec\alpha + \tan\alpha})^2 =$$

$$\frac{\sec^2\alpha + 2\sec\alpha\tan\alpha + \tan^2\alpha + 1}{(\sec\alpha + \tan\alpha)^2} =$$

$$\frac{2\sec^2\alpha + 2\sec\alpha\tan\alpha}{(\sec\alpha + \tan\alpha)^2} =$$

$$2\sec\alpha\frac{\sec\alpha + \tan\alpha}{(\sec\alpha + \tan\alpha)^2}$$

$$(\frac{r_1}{\sin\theta})^2 + (\frac{r_2}{\cos\theta})^2 =$$

$$(R^2\sec^2\theta)2\sec\alpha(\sec\alpha - \tan\alpha) =$$

$$4R^2\sec\alpha\frac{\sec\alpha + \tan\alpha}{(\sec\alpha + \tan\alpha)^2}\sec\alpha(\sec\alpha - \tan\alpha) =$$

$$4R^2(\frac{\sec\alpha}{\sec\alpha + \tan\alpha})^2(\sec^2\alpha - \tan^2\alpha) =$$

$$4R^2(\frac{1}{1 + \sin\alpha})^2$$

$$O_1O_2 = \sqrt{(\frac{r_1}{\sin\theta})^2 + (\frac{r_2}{\cos\theta})^2} = R\frac{2}{1 + \sin\alpha} = r_1 + r_2$$

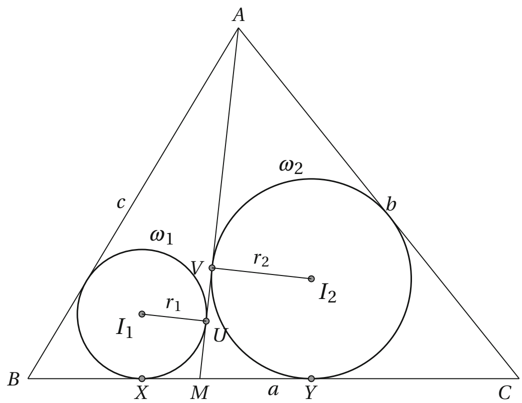
The following theorem also holds symmetrically.

Theorem 9 *At* $(R, \tan\beta + \sec\beta)$ *point,* r_1 *and* r_2 *go to tangent position.*

According to Theorem 9, the following theorem is the solution of Malffati two circle problem as corollary of Theorem 7. It is also the greedy arrangement of the two circles in the triangles.

Corollary 10 *For all tangent circle* O_1 *and circle* O_2 *inside* $\triangle ABC$*, if* β *is the smallest half angle of* $\triangle ABC$ *, the* $r_1^2+r_2^2$ *reach its maximum value* $[(\frac{\sec\beta-\tan\beta}{\sec\beta+\tan\beta})^2+1]R^2$.

Appendix



Find a point M on the side BC, for which sum of the areas of the inscribed circles of ABM and ACM reaches maximum value. (R. Enkhbat)

Solution is given by Luvsanbyamba [4]: Let us denote inscribed circles of ABC,ABM,AMC as $\omega = C(I, r)$, $\omega_1 = C(I_1, r_1)$, $\omega_2 = C(I_2, r_2)$, respectively, and the area of triangle ABC by S, the height from the vertex A by h.

Lemma $pr^2 + ar_1r_2 = pr(r_1 + r_2)$.

Proof $BC \cap \omega_1 = X$, $BC \cap \omega_2 = Y$, $AM \cap \omega_1 = U$, $AM \cap \omega_2 = V$. Without loss of generality $b > c$.

Let us denote $BX = m$, $CY = n$. Then $UV = b - c + m - n$, $\tan\frac{\angle B}{2} = \frac{r}{p-b} = \frac{r_1}{m}$,$\tan\frac{\angle C}{2} = \frac{r}{p-c} = \frac{r_2}{n}$,$XY = a - m - n$.

$UV^2 + (r_1 + r_2)^2 = I_1I2^2 = XY^2 + (r_2 - r_1)^2 \implies UV^2 + 4r_1r_2 = XY^2 \iff [(b-c)+(m-n)]^2 + 4r_1r_2 = [a-(m+n)]^2 \iff 2a(m+n) + 2(b-c)(m-n)+4r_1r_2 = a^2-(b-c)^2+4mn \implies (p-c)m+(p-b)n+r_1r_2 = (p-b)(p-c)+mn \implies (p-c)\frac{(p-b)r_1}{r}+(p-b)\frac{(p-c)r_2}{r}+r_1r_2 = (p-b)(p-c)+\frac{(p-b)r_1}{r}\frac{(p-c)r_2}{r} \implies (p-b)(p-c)(r-r_1)(r-r_2) = r^2r_1r_2; (p-a)(p-b)(p-c) = pr^2 \implies p(r-r_1)(r-r_2) = (p-a)r_1r_2 \implies pr^2 + ar_1r_2 = pr(r_1+r_2)$. Q.E.D.

$F(r_1, r_2) = r_1^2 + r_2^2$, using lemma, $G(r_1, r_2) = r_1 + r_2 - \frac{2r_1r_2}{h} = r = const.$

Let us consider $H(r_1, r_2) = F(r_1, r_2) - \lambda G(r_1, r_2)$

$$\begin{cases} \frac{\partial H}{\partial r_1} = 2r_1 - \lambda(1 - \frac{2r_1}{h}) = 0 \\ \frac{\partial H}{\partial r_2} = 2r_2 - \lambda(1 - \frac{2r_2}{h}) = 0 \end{cases}$$

$\iff \frac{r_1}{r_2} = \frac{h-2r_1}{h-2r_2} \iff h(r_1 - r_2) = 2(r_1^2 - r_2^2) \iff r_1 = r_2$ or $2(r_1 + r_2) = h$

If $r = r_1, r_2 = 0$, then $F_0 = F(r_1, r_2) = r_1^2 + r_2^2 = r^2$.

If $2(r_1+r_2) = h$, then the lemma implies $\frac{h^2}{2} - 2r_1r_2 = rh$ and $F_2 = F(r_1, r_2) = r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1r_2 = \frac{h^2}{4} - (\frac{h^2}{2} - rh) = rh - \frac{h^2}{4}$.

$F_0 > F_2 \iff r_2 > rh - \frac{h^2}{4} \iff (r - \frac{h}{2})^2 > 0$

If $r_1 = r_2$, by the lemma we have $2r_1 - \frac{2r_1^2}{h} = rr_1 = \frac{h-\sqrt{h^2-2rh}}{2}$. $F_1 = F(r_1, r_2) = 2r_1^2 = 2r_1h - rh = h^2 - h\sqrt{h^2 - rh} - rh$.

$F_0 < F_2 \iff r^2 < h^2 - h\sqrt{h^2 - rh} - rh \iff h\sqrt{h^2 - rh} < h^2 - r^2 - rh \iff 0 < r^2(r^2 + 2rh - h^2) \iff 0 < \frac{s^2}{p^2} + \frac{4s^2}{pa} - \frac{4s^2}{a^2} \iff (b+c)^2 < (a\sqrt{2})^2 \iff b + c < a\sqrt{2}$

Thus, in case of $b + c < a\sqrt{2}$, $r_1^2 + r_2^2$, our sum gets its maximum value iff $r_1 = r_2$, and in other case it gets the maximum value iff $r_1 = 0$ or $r_2 = 0$.

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