Sinai's Dynamical System Perspective on Mathematical Fluid Dynamics

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Abstract We review some of the most remarkable results obtained by Ya.G. Sinai and collaborators on the difficult problems arising in the theory of the Navier–Stokes equations and related models. The survey is not exhaustive, and it omits important results, such as those related to "Burgers turbulence". Our main focus in on acquainting the reader with the application of the powerful methods of dynamical systems and statistical mechanics to this field, which is the main original feature of Sinai's contribution.

1 Introduction

One of the fundamental unsolved problems in mathematical fluid dynamics is whether smooth solutions to the three-dimensional incompressible Navier–Stokes System (NSS) can develop singularities in finite time. Sinai has a remarkable intuition that the formation of finite time singularities is possible for the 3D Navier–Stokes system: NSS without external forcing can be regarded a reasonable approximation to the dynamics of a dry air in a big desert, and in deserts such phenomena as tornados are possible due to purely kinematic mechanisms. Mathematically speaking, the most notable difficulties of NSS are its non-locality and super-criticality. The system is nonlocal due to the incompressibility constraint and supercritical with respect to the basic energy conservation law. Super-criticality can also be derived through a scaling analysis on the life-span of solutions.

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Over the years, Sinai and his collaborators have developed several original and powerful methods to tackle many difficult wellposedness and regularity questions in hydrodynamics. Unlike the usual practitioners of PDEs, his approach to these problems is highly original, and his incredible technical power and remarkable insight from dynamical systems has led to substantial progress on the understanding of NSS at fine scales, which is the key to the global regularity conjecture.

The list of results surveyed below is certainly not exhaustive and only represents a small fraction of his many important works. For example, we do not discuss Dinaburg–Sinai's Fourier space model of the NSS and Euler systems (see [15, 16] and see also Friedlander–Pavlovic [22] for further developments), and we do not include a detailed survey on Sinai's ground-breaking work on Burgers turbulence, stochastic hydrodynamics and further developments. Nevertheless, we hope that what we report reflects his unique dynamical system perspective on mathematical fluid dynamics. The topics selected here include: a geometric trapping method for wellposedness and regularity of solutions to NSS [35], power series and diagrams [36–38], complex solutions and renormalization group for the three-dimensional NSS [32], bifurcation of solutions for two-dimensional NSS [33, 34] and stochastic dynamics of two-dimensional NSS [18].

2 A Geometric Trapping Method for NSS

Consider the d-dimensional incompressible Navier–Stokes system on the periodic torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$,

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u = -\nabla p + \nu \Delta u, \quad (t,x) \in (0,\infty) \times \mathbb{T}^d, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1}$$

Here $u = u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$ represents the velocity of the fluid and $p = p(t,x)$ denotes the pressure. When $\nu = 0$ the system (1) becomes the incompressible Euler equation. The first equation in (1) is just the usual Newton's law: the left-hand side describes the acceleration of the fluid in Eulerian frame, whereas the right-hand side represents the force. The second equation in (1) is the usual incompressibility (divergence-free) condition. It can also be regarded a constraint through which the pressure gradient term emerges as a Lagrange multiplier. To reduce the complexity of the system one can use the vorticity formulation. In two dimensions, define $w = \nabla^{\perp} \cdot u = -\partial_{x_2} u_1 + \partial_{x_1} u_2$. Then the equation governing w takes the form

$$\partial_t w + (u \cdot \nabla)\, w = \nu \Delta w, \tag{2}$$

where, under suitable regularity assumptions, u is connected to w by the Biot–Savart law:

$$u = \Delta^{-1}\nabla^{\perp}w = \left(-\Delta^{-1}\partial_{x_2}w, \Delta^{-1}\partial_{x_1}w\right).$$

It is evident from the vorticity form that for smooth solutions the L^p-norm $\|w\|_p$ is preserved in time for all $1 \le p \le \infty$ in 2D. On the other hand, in three dimensions, one can introduce the vorticity vector $w = \nabla \times u$ for which the vorticity equation takes the form:

$$\partial_t w + (u \cdot \nabla)\, w = (w \cdot \nabla)\, u + \nu \Delta w, \tag{3}$$

with

$$u = -\Delta^{-1}\nabla \times w.$$

Compared with two dimensions, the vorticity stretching term $(w \cdot \nabla u)$ is the main obstruction to global wellposedness in three dimensions. In the whole plane $\mathbb{R}^2$ case, the first existence and uniqueness results for weak solutions of (1) were obtained in Leray's thesis in 1933. For the three-dimensional whole space case Leray [30] proved the existence of weak solutions. Hopf in [23] then obtained the existence of weak solutions in arbitrary open subsets Ω of $\mathbb{R}^n$, $n \ge 2$. Ladyzenskaya [26] in 1962 proved existence and uniqueness of solutions for two-dimensional domains. Since then many other strong methods were developed in [10, 39, 40, 42], providing deep insights into the fine behavior of solutions to (1).

In [35], Mattingly and Sinai developed a novel geometric trapping method for proving existence, uniqueness and regularity of solutions to the Navier–Stokes system. To describe this method, consider the two-dimensional vorticity equation (2). Expand the vorticity w in Fourier series:

$$w(x,t) = \sum_{k \in \mathbb{Z}^2} w_k(t) e^{2\pi i k \cdot x}, \quad x = (x_1, x_2)$$

where w_k denote the Fourier coefficients. Since w is real-valued, we have $w_{-k} = \overline{w_k}$. One can then write a coupled ODE-system for the modes $w_k(t)$ as

$$\frac{d}{dt} w_k + 2\pi i \sum_{l_1 + l_2 = k} w_{l_1} w_{l_2} \frac{k \cdot l_2^{\perp}}{|l_2|^2} = -4\pi^2 \nu |k|^2 w_k, \tag{4}$$

where $|k| = \sqrt{k_1^2 + k_2^2}$, $l^{\perp} = (l^{(1)}, l^{(2)})^{\perp} = (-l^{(2)}, l^{(1)})$.

A more general version of (2) is the case where the Laplacian is replaced by the fractional Laplacian $|\nabla|^{\alpha}$ with $\alpha > 0$. Correspondingly, (4) can be generalized as:

$$\frac{d}{dt} w_k + 2\pi i \sum_{l_1 + l_2 = k} w_{l_1} w_{l_2} \frac{k \cdot l_2^{\perp}}{|l_2|^2} = -4\pi^2 \nu |k|^{\alpha} w_k. \tag{5}$$

Without loss of generality one can assume $w_0 = 0$ since the mean value of w is preserved by the dynamics.

The results obtained in [35] can be formulated as follows.

Theorem 1 ([35]) *Let* $\alpha > 1$ *in* (5). *Suppose for some constant* $0 < D_1 < \infty$, $1 < r < \infty$,

$$|w_k(0)| \leq \frac{D_1}{|k|^r}, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}.$$

Then one can find a finite constant $D_1' > 0$, *depending only on* (D_1, ν), *such that any solution to* (5) *with these initial conditions satisfies*

$$|w_k(t)| \leq \frac{D_1'}{|k|^r}, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}$$

for all $t > 0$.

A few remarks are now in order. First, the main theorems stated in [35] are more general and include the case with external forcing under suitable decay assumptions on the Fourier modes which are uniform in time. By using some refined estimates, Mattingly and Sinai also proved that the solutions become real analytic for $t > 0$ (i.e., $|w_k(t)| \leq \mathrm{const} \cdot e^{-\mathrm{const} \cdot |k|}$, for $t > t_0 > 0$). Statements close to these were also proved in [17, 21, 24], but the methods are quite different and more function analytic in nature.

In the three-dimensional setting, one can introduce

$$u(x,t) = \sum_{k \in \mathbb{Z}^3} u_k(t) e^{2\pi i k \cdot x},$$
$$w(x,t) = \sum_{k \in \mathbb{Z}^3} w_k(t) e^{2\pi i k \cdot x}.$$

By using (3), we obtain

$$\begin{aligned} \frac{d}{dt} w_k(t) &= -2\pi i \sum_{l_1 + l_2 = k} \left[(u_{l_1} \cdot l_2) w_{l_2} - (w_{l_1} \cdot l_2) u_{l_2} \right] - 4\pi^2 \nu |k|^2 w_k \\ &= -2\pi i \sum_{l_1 + l_2 = k} \left[(u_{l_1} \cdot k) w_{l_2} - (w_{l_1} \cdot k) u_{l_2} \right] - 4\pi^2 \nu |k|^2 w_k, \end{aligned}$$

where the second equality follows from the incompressibility condition. Similar to the two-dimensional case, one can replace the Laplacian with the fractional Laplacian $|\nabla|^\alpha$, and obtain

$$\frac{d}{dt} w_k(t) = -2\pi i \sum_{l_1 + l_2 = k} \left[(u_{l_1} \cdot k) w_{l_2} - (w_{l_1} \cdot k) u_{l_2} \right] - 4\pi^2 \nu |k|^\alpha w_k. \tag{6}$$

For this nonlocal system, the following theorem was proved in [35].

Theorem 2 ([35]) *Consider* (6) *with* $\alpha > \frac{5}{2}$. *If the initial data* $\{w_k(0)\}$ *are such that for some* $0 < D < \infty$, $r > \frac{3}{2}$,

$$|w_k(0)| \le \frac{D}{|k|^r}, \quad \forall k \in \mathbb{Z}^3 \setminus \{0\},$$

then there exists a constant D' *depending only on* (D, r, α), *such that for any* $t \ge 0$,

$$|w_k(t)| \le \frac{D'}{|k|^r}, \quad \forall k \in \mathbb{Z}^3 \setminus \{0\}.$$

Remark One should note that $\alpha = 2$ corresponds to the usual Navier–Stokes case. Analogous statements can also be proved for that situation, provided the constant D is sufficiently small, which will become a typical small data global wellposedness result for 3D NSS. For large data global wellposedness, one can lower the constant $\alpha > 2.5$ to $\alpha = 2.5$ or even with some logarithmic damping of the symbol. All of these difficulties are ultimately connected with the lack of globally coercive quantities stronger than energy.

We now focus on the two-dimensional case and describe in more detail the geometric trapping method of Mattingly and Sinai. Roughly speaking, the idea is to consider a finite Galerkin system of coupled ODEs for the Fourier coefficients. One can write a finite approximation of (5) abstractly as

$$\frac{d}{dt} w_k(t) = B_k(w, w) - 4\pi^2 \nu |k|^\alpha w_k.$$

By using the basic enstrophy inequality

$$\sum_k |w_k(t)|^2 \le \mathscr{E}_0, \quad \forall t > 0,$$

one can trap the low modes, i.e., for any $K_0 > 0$, there exists $D_1(K_0)$, such that

$$|w_k(t)| \le \frac{D_1}{|k|^r}, \quad \forall |k| \le K_0.$$

One then defines a trapping region for all modes as

$$\Omega = \left\{ (w_k) : |w_k| \le \frac{D_1}{|k|^r}, \forall 0 \ne k \in \mathbb{Z}^2 \right\}$$

It is evident that the low modes $\{|k| \le K_0\}$ are already in the trapping region, and the boundary of the trapping region is given by

$$\partial\Omega = \left\{(w_k) \;:\; |w_k| \le \frac{D_1}{|k|^r},\ \forall\, 0 \ne k \in \mathbb{Z}^2,\ \text{and equality holds for some } k = k^*\right\}.$$

By choosing D_1 large, Ω contains the initial data in its interior. Then one endeavors to show that the dynamics will always trap the sequence of Fourier modes inside $\overline{\Omega}$. Geometrically speaking, it amounts to showing that the vector field on the boundary $\partial\Omega$ always points into the interior of Ω. More precisely one checks that for K_0 sufficiently large, if there are $|k^*| > K_0$, with $w_{k^*} = \frac{D_1}{|k^*|^r}$ (the case $w_{k^*} = -\frac{D_1}{|k^*|^r}$ is similar), then

$$\frac{d}{dt} w_k(t)\Big|_{k=k^*} < 0.$$

By using the enstrophy estimate together with the trapping estimate, one can estimate the nonlinear term as

$$|B_k(w,w)(t)| \le \text{const} \cdot \sqrt{\mathscr{E}_0} \cdot \frac{D_1}{|k^*|^{r-1}} \cdot \log|k^*|.$$

Thus

$$\frac{d}{dt} w_k(t)\Big|_{k=k^*} \le \text{const} \cdot \sqrt{\mathscr{E}_0} \cdot \frac{D_1}{|k^*|^{r-1}} \cdot \log|k^*| - 4\pi^2 \nu \frac{1}{|k^*|^{r-\alpha}} < 0, \qquad (7)$$

if K_0 is chosen sufficiently large.

This concludes the trapping argument. One should note from (7) that the restriction $\alpha > 1$ is purely technical, and due to the fact that only enstrophy conservation and L_t^∞-type breakthrough scenario enter the argument. By using more time integrability, one can obtain analyticity also for $\alpha = 1$ (for global wellposedness we do not need any constraint on α since 2D Euler is globally wellposed by using $\|w\|_{L_x^\infty}$).

One can also rephrase in typical PDE language the trapping argument of Mattingly and Sinai, as a sort of maximum principle in Fourier space. It is a beautiful geometric dynamical system proof, which has since been generalized and developed to many other situations (cf. [2, 4, 11–14] and the references therein).

3 Power Series and Diagrams

In the seminal works [36–38], Sinai developed a power series and diagram representation for the Navier–Stokes system. These works can be viewed as a precursor to the renormalization group approach developed later. Consider the three-dimensional Navier–Stokes system (1), with viscosity $\nu = 1$ and on the whole space $\mathbb{R}^3$. After the Fourier transform

$$v(k,t) = \int_{\mathbb{R}^3} u(x,t)e^{-ik\cdot x}\,dx,$$

it becomes a nonlinear non-local equation:

$$v(k,t) = e^{-|k|^2 t}v(k,0) + i\int_0^t e^{-(t-s)|k|^2}\int_{\mathbb{R}^3}\left\langle k, v(k-k',s)\right\rangle P_k v(k',s)dk'ds. \tag{8}$$

The incompressibility condition enforces $v(k,t) \perp k$ for any $k \neq 0$. The operator P_k is the orthogonal projection to the subspace orthogonal to k. In this way the pressure does not appear and we consider the space of functions $\{v(k)\colon v(k) \perp k\}$ as the main phase space of the dynamical system defined by (1).

Classical (strong) solutions to (8) on the time interval $[0,t_0]$ are functions $v(k,t),\ 0 \le t \le t_0$, such that the integrals

$$\int_0^t e^{-(t-s)|k|^2}\int_{\mathbb{R}^3}|v(k-k',s)|\cdot|v(k',s)|\,dk'ds,$$

are bounded for any $0 \le t \le t_0$ and the left-hand side is equal to the right-hand side. A more convenient (easily checkable), but stronger condition, is to require the integrals

$$\int_{\mathbb{R}^3}|v(k-k',s)|\cdot|v(k',s)|\,dk'$$

to be uniformly bounded in s. The latter definition was adopted in [38].

Sinai considered (8) in the space of functions which can have singularities near $k = 0$ or $k = \infty$. The following space $\Phi(\alpha, w)$ was introduced in [38].

Definition 4 $\{v(k),\ k \in \mathbb{R}^3\} \in \Phi(\alpha, w)$ if for some constants $0 < C, D < \infty$,

$$|v(k)| \le \begin{cases} \frac{C}{|k|^\alpha}, & \text{if } |k| \le 1,\\ \frac{D}{|k|^w}, & \text{if } |k| > 1. \end{cases}$$

The cut-off "1" for $|k|$ can be replaced by any positive number. The parameters α and w satisfy the inequalities $\alpha \ge 2$, $w < 3$. One can endow the space $\Phi(\alpha, w)$ with a norm by taking the infimum of all possible $C + D$.

In [38], Sinai proved a short-time local existence theorem in the space $\Phi(\alpha, w)$, $\alpha > 2$, $w < 3$. Namely, for any initial data (in the Fourier space) $v(k,0) \in \Phi(\alpha, w)$, there exists $T_0 > 0$ sufficiently small, such that (8) admits a unique solution on $[0, T_0]$ in the space $\Phi(\alpha, w)$. One should note that in this theorem, $v(k,0)$ is allowed to be an arbitrary complex ($\mathbb{C}^3$-valued) vector function. When $v(k,0) = \overline{v(-k,0)}$ for any $k \in \mathbb{Z}^3$, the corresponding velocity $u(x,0)$ is a $\mathbb{R}^3$-valued vector function.

In the space $\Phi(2,2)$ one can prove a small data global wellposedness result. Namely, let $v(k,0) = \frac{C(k,0)}{|k|^2}$, with $\sup_k |C(k,0)| \le C_0$ and C_0 is sufficiently small. Then there exists a unique solution $v(k,t)$ of (8) defined for all $t > 0$.

One can see the references [8, 28, 38] for short proofs of this theorem. Recently, Lei and Lin [29] discovered a remarkable fact, that for Eq. (1) with $\nu > 0$ and on $\mathbb{R}^3$ one can have global wellposedness as long as $\sup_k |C(k,0)| \le C\nu$, where C is an absolute constant.

In [36], Sinai considered the space $\Phi(\alpha,\alpha)$ with $\alpha = 2+\epsilon$ and $\epsilon > 0$ sufficiently small. Denote $v(k,0) = \frac{C(k,0)}{|k|^\alpha}$ where $C(k,0)$ is continuous everywhere outside $k=0$, and $\|C(k,0)\|_{L^\infty_k} = \sup_{k\neq 0} |C(k,0)| = 1$. Introduce a one-parameter family of initial conditions $v_A(k,0) = \frac{AC(k,0)}{|k|^\alpha}$, where A is a complex-valued parameter. For given A, the time of existence for the local solution will depend on A. More precisely, the following theorem was proven in [36].

Theorem 5 ([36]) *There exists a constant $\lambda_0 = \lambda_0(\alpha) > 0$ depending only on α such that if $|\lambda| = |AT^{\frac{\epsilon}{2}}| \le \lambda_0$, then there exists a unique local solution in the space $\Phi(\alpha,\alpha)$ on the time interval* $[0,T]$.

To prove this theorem Sinai used the method of iterations. In terms of the unknown $C_A(k,t) = |k|^\alpha v_A(k,t)$, one can define the iterations $C_A^{(n)}(k,t)$ via the formula

$$\begin{aligned} C_A^{(n)}(k,t) &\\ &= Ae^{-|k|^2 t} C(k,0) \\ &\quad + i|k|^\alpha \int_0^t e^{-|k|^2(t-s)} \int_{\mathbb{R}^3} \frac{\langle k, C_A^{(n-1)}(k-k',s)\rangle P_k C_A^{(n-1)}(k',s)}{|k-k'|^\alpha |k'|^\alpha} dk' ds, \qquad n \ge 1, \end{aligned}$$

with

$$C_A^{(0)}(k,t) = Ae^{-|k|^2 t} C(k,0).$$

By splitting into low and high frequencies, Sinai showed that if $|\lambda| \le \lambda_0(\alpha) \ll 1$, then $\|C^{(n)}\|_\infty \le 2A$ for all $n \ge 1$, and the sequence of iterations $(C^{(n)})$ is a contraction. From the point of view of dynamical systems, the scalar λ is a ruling parameter in the current situation. In the same paper, Sinai then went on to construct a power series for the solution $C_A(k,t)$, namely:

$$C_A(k,t) = AC(k,0)e^{-t|k|^2} + \sum_{p\ge 1} A^p \int_0^t e^{-(t-s)|k|^2} s^{\frac{p\epsilon}{2}} h_p(k,s)\, ds, \tag{9}$$

where

$$s^{\frac{\epsilon}{2}} h_1(k,s) = i|k|^\alpha \int_{\mathbb{R}^3} \frac{\langle k, C(k-k',0)\rangle P_k C(k',0) e^{-s|k-k'|^2 - s|k'|^2}}{|k-k'|^\alpha \cdot |k'|^\alpha} dk',$$

$$s^\epsilon h_2(k,s)$$
$$= i|k|^\alpha \cdot \Big[\int_0^s s_1^{\frac{\epsilon}{2}} ds_1 \int_{\mathbb{R}^3} \frac{\langle k, h_1(k-k',s_1)\rangle P_k C(k',0) \cdot e^{-(s-s_1)|k-k'|^2 - s|k'|^2} dk'}{|k-k'|^\alpha |k'|^\alpha}$$
$$+ \int_0^s s_2^{\frac{\epsilon}{2}} ds_2 \int_{\mathbb{R}^3} \frac{\langle k, C(k-k',0)\rangle P_k h_1(k',s_2) e^{-s|k-k'|^2 - (s-s_2)|k'|^2} dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \Big],$$

and

$$s^{\frac{p\epsilon}{2}} h_p(k,s)$$
$$= i|k|^\alpha \cdot \Big[\int_0^s s_1^{\frac{p-1}{2}\epsilon} ds_1 \cdot \int_{\mathbb{R}^3} \frac{\langle k, h_{p-1}(k-k',s_1)\rangle P_k C(k',0) e^{-(s-s_1)|k-k'|^2 - s|k'|^2} dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}$$
$$+ \int_0^s s_2^{\frac{p-1}{2}\epsilon} ds_2 \cdot \int_{\mathbb{R}^3} \frac{\langle k, C(k-k',0)\rangle P_k h_{p-1}(k',s_1) e^{-s|k-k'|^2 - (s-s_2)|k'|^2} dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}$$
$$+ \sum_{\substack{p_1,p_2\geq 1 \\ p_1+p_2=p-1}} \int_0^s s_1^{\frac{p_1\epsilon}{2}} ds_1 \int_0^s s_2^{\frac{p_2\epsilon}{2}} ds_2$$
$$\times \int_{\mathbb{R}^3} \frac{\langle k, h_{p_1}(k-k',s_1)\rangle P_k h_{p_2}(k',s_2) e^{-(s-s_1)|k-k'|^2 - (s-s_2)|k'|^2} dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \Big].$$

Now use the ansatz $h_p(k,s) = s^{\frac{\epsilon}{2}} |k|^\alpha g_p(k\sqrt{s}, s)$ and make the change of variables: $s_1 = s\widetilde{s_1}$, $s_2 = s\widetilde{s_2}$, $k\sqrt{s} = \widetilde{k}$, $k'\sqrt{s} = \widetilde{k'}$. Then $h_p(k,s) = s^{\frac{\epsilon}{2}} |k|^\alpha g_p(\widetilde{k}, s)$. The system of recurrent relations governing the functions $g_p(\widetilde{k}, s)$ then takes the form:

$$g_1(\widetilde{k}, s) = i \int_{\mathbb{R}^3} \frac{\langle \tilde{k}, C(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}}, 0)\rangle P_{\tilde{k}} C(\frac{\tilde{k}'}{\sqrt{s}}, 0) e^{-|\tilde{k}-\tilde{k}'|^2 - |\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha},$$

$$g_2(\widetilde{k}, s)$$
$$= \int_0^1 \tilde{s}_1^\epsilon d\tilde{s}_1 \int_{\mathbb{R}^3} \frac{\langle \tilde{k}, g_1((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1)\rangle \cdot P_{\tilde{k}} C(\frac{\tilde{k}'}{\sqrt{s}}, 0) e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - |\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha}$$
$$+ \int_0^1 \tilde{s}_2^\epsilon d\tilde{s}_2 \int_{\mathbb{R}^3} \frac{\langle \tilde{k}, C(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}}, 0)\rangle P_{\tilde{k}} g_1(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2) e^{-|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha},$$

and for $p \geq 3$

$$
\begin{aligned}
&g_p(\widetilde{k}, s) \\
&= i\Big[\int_0^1 \tilde{s}_1^{\frac{p\epsilon}{2}} d\tilde{s}_1 \int_{\mathbb{R}^3} \frac{\langle \tilde{k}, g_{p-1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\tilde{s}_1)\rangle P_{\tilde{k}} C(\frac{\tilde{k}'}{\sqrt{s}}, 0) e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - |\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha} \\
&\quad + \int_0^1 \tilde{s}_2^{\frac{p\epsilon}{2}} d\tilde{s}_2 \int_{\mathbb{R}^3} \frac{\langle \tilde{k}, C(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}}, 0)\rangle P_{\tilde{k}} g_{p-1}(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2) e^{-|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha} \\
&\quad + \sum_{\substack{p_1, p_2 \geq 1 \\ p_1+p_2=p-1}} \int_0^1 \tilde{s}_1^{\frac{p_1\epsilon}{2}} d\tilde{s}_1 \int_0^1 \tilde{s}_2^{\frac{p_2\epsilon}{2}} d\tilde{s}_2 \int_{\mathbb{R}^3} \langle \tilde{k}, g_{p_1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1)\rangle \cdot \\
&\qquad\qquad \cdot P_{\tilde{k}} g_{p_2}(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2) \cdot e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2 - (1-\tilde{s}_2)|\tilde{k}'|^2} d\tilde{k}'\Big]. \qquad (10)
\end{aligned}
$$

It follows from these recurrent relations that each $g_p(\widetilde{k}, s)$ depends on the initial conditions $C(k, 0)$ via the sum of not more than b^p $4p$-dimensional integrals where b is some constant. The main assumption is that $C(k, 0)$ is compactly supported in $\{|k| \leq R_0\}$, where R_0 is a positive constant.

By using a sophisticated inductive analysis together with some combinatorics, Sinai proved the following theorem.

Theorem 6 ([36]) *The functions $g_p(\widetilde{k}, s)$ satisfy the inequality:*

$$|g_p(\widetilde{k}, s)| \leq C_p f(|\widetilde{k}|) e^{-\frac{|\widetilde{k}|^2}{p+1}},$$

where $f(x) = \min\{x, \frac{1}{x}\}$ for $x > 0$, and $C_p \leq b_1 b_2^p$ for some constants $0 < b_1, b_2 < \infty$ depending only on α.

It follows that if $At^{\frac{\epsilon}{2}} < b_2^{-1}$, then the series (9) converges for every $0 \neq k \in \mathbb{R}^3$.

In [37], Sinai analyzed in more detail the recurrent system (10) and introduced diagrams, corresponding to each multi-dimensional integral in the series. Each diagram is determined by a scheme, and any scheme is a sequence of partitions of the set starting from $[1, 2, \ldots, p+1] = \Delta^{(0)}$. By using a deep analogy with statistical mechanics, Sinai then estimated several classes of diagrams and showed that the partition functions of short diagrams decay exponentially. In [37], one can find a systematic approach to study and estimate short diagrams for large p. This approach has a striking resemblance of the renormalization group method in statistical mechanics.

4 Complex Valued Solutions and Renormalization Group

Consider the Navier–Stokes system (1) on $\mathbb{R}^3$ with viscosity $\nu = 1$. By using the Fourier transform

$$\widetilde{v}(k,t) = \int_{\mathbb{R}^3} u(x,t) e^{-ik\cdot x}\, dx,$$

one obtains an equivalent non-local nonlinear system

$$\widetilde{v}(k,t) = e^{-|k|^2 t}\widetilde{v}(k,0) + i\int_0^t e^{-(t-s)|k|^2}\int_{\mathbb{R}^3}\left\langle \widetilde{v}(k-k',s),k\right\rangle P_k \widetilde{v}(k',s)dk'ds, \tag{11}$$

where P_k is the solenoidal projection operator

$$P_k\widetilde{v} = \widetilde{v} - \frac{\langle \widetilde{v}, k\rangle}{|k|^2}k,$$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product

$$\langle a,b\rangle = a\cdot b, \quad \text{if } a,\ b\in\mathbb{C}^3.$$

Introduce the change of variable

$$\widetilde{v}(k,t) = -iv(k,t).$$

Then in terms of $v(k,t)$, the integral equation (11) now takes the form

$$v(k,t) = e^{-|k|^2 t}v(k,0) + \int_0^t e^{-(t-s)|k|^2}\int_{\mathbb{R}^3}\left\langle v(k-k',s),k\right\rangle P_k v(k',s)dk'ds. \tag{12}$$

This non-local integral equation is the main object of study. In general, $\mathbb{R}^3$-valued solutions to (12) will correspond to complex solutions $u(x,t)$ in (1). If one restricts to the class of $v(k,0)$ such that $v(k,0) = -v(k,0)$ for all $k\in\mathbb{Z}^3$, then $v(k,t)$ will also be odd in k and such solutions correspond to $\mathbb{R}^3$-valued real (and physical) fluid flows.

In [32], a Renormalization Group type method was developed to show that there exists a class of $\mathbb{R}^3$-valued initial data $v(k,0)$ which are compactly supported such that the corresponding solution to (12) blows up in finite time. The velocity field $u(x,0)$ corresponding to $v(k,0)$ is, however, $\mathbb{C}^3$-valued. As such, these solutions do not obey energy conservation and correspond to non-physical flows. Nevertheless the behavior of these solutions in some sense resemble the forward cascade of

Fourier modes and they are a show-case of some important fine structures of the Navier–Stokes system.

We now review in more detail the results of [32].

Consider a one-parameter family of initial data in the form $v_A(k,0) = Av_0(k)$, where $v_0(k)$ will be a fixed profile and A is a positive parameter. The corresponding solution to (12) can then be represented as a power series

$$v_A(k,t) = Ae^{-t|k|^2}v_0(k) + \int_0^t e^{-|k|^2(t-s)}\left[\sum_{p=2}^{\infty} A^p g^{(p)}(k,s)\right] ds. \tag{13}$$

Set $g^{(1)}(k,s) = e^{-s|k|^2}v_0(k)$. Substituting (13) into (12), we then obtain

$$g^{(2)}(k,s) = \int_{\mathbb{R}^3} \langle v_0(k-k'), k\rangle P_k v_0(k')e^{-s|k-k'|^2-s|k'|^2}\, dk',$$

and for $p > 2$

$$\begin{aligned} g^{(p)}(k,s) = &\int_0^s ds_2 \int_{\mathbb{R}^3} \langle v_0(k-k',k\rangle P_k g^{(p-1)}(k',s_2)e^{-s|k-k'|^2-(s-s_2)|k'|^2}dk' \\ &+ \int_0^s ds_1 \int_{\mathbb{R}^3} \langle g^{(p-1)}(k-k',s_1),k\rangle P_k v_0(k')e^{-(s-s_1)|k-k'|^2-s|k'|^2}dk' \\ &+ \sum_{\substack{p_1+p_2=p \\ p_1,p_2>1}} \int_0^s ds_1 \int_0^s ds_2 \langle g^{(p_1)}(k-k',s_1),k\rangle \\ &\times P_k g^{(p_2)}(k',s_2)e^{-(s-s_1)|k-k'|^2-(s-s_2)|k'|^2}dk'. \end{aligned} \tag{14}$$

The initial data v_0 will be assumed to have support localized in a sphere around some $K^{(0)} = (0,0,k_0)$, $k_0 \gg 1$. The radius of the sphere is much smaller than k_0. By a deep analogy with probability theory, the support of the functions $g^{(p)}$ is then expected to be localized about the point $pK^{(0)} = (0,0,pk_0)$ with a fattened size $\sqrt{p}$ for large p. From these considerations, one can then introduce the change of variable and ansatz:

$$k = pK^{(0)} + \sqrt{p}Y, \quad h^{(p)}(Y,s) = g^{(p)}(pK^{(0)} + \sqrt{p}Y, s),$$

where the new variable Y typically takes values $O(1)$. In all integrals over s_1, s_2 in (14), make another change of variables $s_j = s(1 - \frac{\theta_j}{p_j^2})$, $j = 1, 2$. Instead of the integration over k', we introduce Y' such that $k' = p_2k_0 + \sqrt{pk_0}Y'$. Denote $\gamma = \frac{p_1}{p}$. Then we obtain from (14) the recurrent relation

$$h^{(p)}(Y,s) = p^{5/2} \sum_{\substack{p_1+p_2=p \\ p_1,p_2>\sqrt{p}}} \frac{1}{p_1^2 p_2^2} \int_{\mathbb{R}^3} P_{e_3+\frac{Y}{\sqrt{p}}} h^{(p_2)}(\frac{Y'}{\sqrt{1-\gamma}}, s)\cdot$$

$$\times \langle h^{(p_1)}(\frac{Y-Y'}{\sqrt{\gamma}}, s), e_3 + \frac{Y}{\sqrt{p}} \rangle dY' \cdot (1+o(1)),$$

where $e_3 = (0,0,1)$. In coordinates one can write

$$h^{(p)}(Y,s) = \left(h_1^{(p)}(Y,s), h_2^{(p)}(Y,s), \frac{F^{(p)}(Y,s)}{\sqrt{p}} \right). \tag{15}$$

For large p the incompressibility condition $\langle h^{(p)}(Y,s), k \rangle = 0$ enforces

$$Y_1 h_1^{(p)}(Y,s) + Y_2 h_2^{(p)}(Y,s) + F^{(p)}(Y,s) = O(p^{-1/2}).$$

It follows that $F^{(p)} = O(1)$ and the vector $h^{(p)}(Y,s)$ is almost orthogonal to the k_3-axis for large p.

Make the ansatz

$$h^{(p)}(Y,s) = p\Lambda(s)^p \prod_{j=1}^{3} g^{(3)}(Y) \left(H(Y) + \delta^{(p)}(Y,s) \right), \tag{16}$$

where $\Lambda(s)$ is a positive function, $g^{(3)}(Y) = (2\pi)^{-3/2} e^{-|Y|^2/2}$ is the standard Gaussian density, and the remainder term $\delta^{(p)}$ tends to zero as $p \to \infty$. The vector function

$$H(Y) = (H_1(Y_1,Y_2), H_2(Y_1,Y_2), 0)$$

will correspond to the fixed point of the renormalization group. The fact that it is two-dimensional and depends only on (Y_1, Y_2), can be traced back to (15), which is a consequence of the divergence-free condition.

As we take the limit $p \to \infty$, the discrete sum over p_1 in the recurrent relation becomes an integral over $\gamma = \frac{p_1}{p}$. The fixed point equation for the renormalization group then takes the form

$$g_1^{(2)}(Y)H(Y) = \int_0^1 d\gamma \int_{\mathbb{R}^2} g_\gamma^{(2)}(Y-Y') g_{1-\gamma}^{(2)}(Y') \mathscr{L}(H;\gamma,Y,Y') \tag{17}$$

$$\times H\left(\frac{Y'}{\sqrt{1-\gamma}}\right) dY',$$

where, by abuse of notation, $H(Y) = (H_1(Y_1, Y_2), H_2(Y_1, Y_2))$, $g_0^{(2)}(Y) = \frac{1}{2\pi\sigma} e^{-\frac{Y_1^2+Y_2^2}{2\sigma}}$, and

$$\mathscr{L}(H;\gamma, Y, Y') = -(1-\gamma)^{3/2} \left\langle \frac{Y-Y'}{\sqrt{\gamma}}, H\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right\rangle + \gamma^{1/2}(1-\gamma) \left\langle \frac{Y'}{\sqrt{1-\gamma}}, H\left(\frac{Y'}{\sqrt{1-\gamma}}\right)\right\rangle.$$

In Eq. (17), the Y_3-variable was integrated out since it is just the usual convolution. By using the theory of Hermite polynomials, one can classify the solutions to the functional equation (17). Amongst all such solutions, a particular simple one is

$$H^{(0)}(Y_1, Y_2) = C(Y_1, Y_2),$$

where the pre-factor $C > 0$ can be determined from the equation. One can then linearize around this fixed point and study the spectrum of the linearized operator. As it turns out, there are 6 unstable directions and 4 neutral directions. The following theorem was proven in [32].

Theorem 7 ([32]) *For $K^{(0)} = (0, 0, k_0)$ and k_0 large enough, there exists a 10-parameter family of initial data and a time interval $[s_-, s_+]$ such that the ansatz (16) holds for $H = H^{(0)}$ and $s \in [s_-, s_+]$.*

As observed in [5, 6], the recurrent relations and the fixed point equation remain unchanged if $h^{(p)}$ is replaced by $(-1)^p h^{(p)}$. This consideration then leads to two types of solutions, with type I corresponding to the solution described before and type II corresponding to $(-1)^p h^{(p)}$. Note that if the initial data v_0 leads to a type I solution with the fixed point $H^{(0)}$, then $-v_0$ leads to a type II solution with the same fixed point.

In [5], it was shown that the solutions corresponding to type I and type II will have energy and enstrophy diverging as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2\, dx = \frac{(2\pi)^3}{2}\int_{\mathbb{R}^3} |v(k,t)|^2\, dk \sim \frac{C_E^{(\alpha)}}{(\tau - t)^{\beta_\alpha}},$$

$$S(t) = \int_{\mathbb{R}^3} |\nabla u(x,t)|^2\, dx = (2\pi)^3 \int_{\mathbb{R}^3} |k|^2 |v(k,t)|^2\, dk \sim \frac{C_S^{(\alpha)}}{(\tau - t)^{\beta_\alpha + 2}},$$

where τ is the blowup time, $\alpha = \mathrm{I}, \mathrm{II}$ denotes the type of function, $\beta_{\mathrm{I}} = 1$, $\beta_{\mathrm{II}} = \frac{1}{2}$ and $C_E^{(\alpha)}$, $C_S^{(\alpha)}$ are constants depending on the initial data.

Numerical simulations of the complex-valued singular solutions reveal very interesting features [5, 6] some of which are similar to those of related real-valued energy-preserving solutions.

5 Bifurcations of Solutions to Two-Dimensional Navier–Stokes Systems

The usual bifurcation theory in dynamical systems deals with one-parameter families of smooth maps or vector fields. In that situation fixed points or periodic orbits become functions of this parameter. Bifurcations appear when their linearized spectrum changes its structure. The classical approach is to use versal deformations, i.e., special families such that arbitrary families can be represented as some projections of versal deformations [3]. In such kind of approach the positions of the bifurcating orbits and their dependence on the parameter are known. In [33, 34] a new approach is developed to study deformations produced by solutions of a PDE system and construct bifurcations using properties of the dynamical flow. The construction is nonlinear and does not rely on any knowledge of special fixed points. As a model case, one can study the bifurcation of critical points for a stream function driven by a two-dimensional incompressible viscous flow. Unlike the usual scenario the profile of the function can display quite disparate patterns at different time intervals due to the nonlocal nature of the dynamics.

Consider the Cauchy problem for the two-dimensional Navier–Stokes System written for the stream function $\psi = \psi(t, x, y)$:

$$\begin{cases} \frac{\partial \psi}{\partial t} + \Delta^{-1}\left(\frac{\partial \psi}{\partial x} \cdot \frac{\partial \Delta \psi}{\partial y} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial \Delta \psi}{\partial x}\right) = \Delta \psi, \\ \psi(t, x + 2\pi, y) = \psi(t, x, y + 2\pi) = \psi(t, x, y), \qquad \forall\, (x, y) \in \mathbb{T}^2, \end{cases} \tag{18}$$

where $\mathbb{T}^2$ is the two-dimensional periodic torus with period 1 in each directions. The velocity u of the fluid is given by $u = \nabla^{\perp}\psi = (-\partial_y \psi, \partial_x \psi)$. For general initial data the global wellposedness and regularity of solutions to (1) is well-known by using Mattingly–Sinai's geometric trapping method or energy type estimates. The main problem is to study the dynamics of critical points of the stream function ψ. In [33] it was proposed that if the critical points of the stream function (i.e., stagnation points of the velocity field) are points of maxima or minima, then these points are called viscous vortices because near these points the velocity u is tangent to the level sets of ψ which is a closed curve. The nonlocal operator Δ^{-1} in front of the nonlinear term in (18) is of prime importance (i.e., used in an essential way) in the construction of the bifurcation. On the other hand, such construction does not seem to carry over directly to the vorticity formulation. This is deeply connected with the fact that vorticity only obeys a transport equation and during such processes the local maxima or minima of the vorticity function are simply transported.

The following theorem establishes in some sense the splitting (bifurcation) of vortices. It was first proved in [33] under a symmetry assumption and then in [34] for the general case.

Theorem 8 ([33, 34] Existence of bifurcations) *There exists an open set $\mathscr{A}$ in the space of stream functions such that the following holds true: For each stream* *function* $\psi_0 \in \mathscr{A}$, *there is an open neighborhood* U *of the origin, two moments of time* $0 < t_1 < t_2$ *such that the corresponding stream function* $\psi = \psi(t, x, y)$ *solves* (18) *with initial data* ψ_0 *and has critical points which bifurcate from 1 to 2 on* $[0, t_1]$, *and 2 to 3 on* $(t_1, t_2]$ *in the neighborhood* U.

Although the Navier–Stokes equation is not time-reversible, by using a different construction one can reverse the above scenario and also show the merging of vortices (see [33, 34] for more details). The bifurcation method devised in [33, 34] is quite robust and has been generalized to a number of other situations (cf. [31, 43]). In general the behavior of the critical points is not well studied in multi-dimensional situations. For parabolic equations, one can show that the number of critical points decreases as a function of time (see [1]), and estimate the size of critical points (see [9]).

6 Stochastic Hydrodynamics

Stochastic fluid mechanics is an important tool in the study of real fluid flows, and a huge physical literature is devoted to it. The traditional approach deals with space or time averages of some relevant physical quantities. For a deeper insight one needs information on the typical behavior of the solutions, such as can come from the knowledge of the invariant measures and their space-time properties.

A brilliant contribution of Sinai and collaborators in this sense is given by the paper [18], which deals with the two-dimensional Navier–Stokes equations on the 2D torus $\mathbb{T}^2$ with random forcing on a finite set of modes:

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u + \nabla p - \nu \Delta u = \frac{\partial}{\partial t} W(x, t), \quad (t, x) \in (0, \infty) \times \mathbb{T}^2, \\ \nabla \cdot u = 0. \end{cases} \tag{19}$$

$$W(x, t) = \sum_{0 \neq |k| \leq N} \sigma_k w_k(t, \omega) e_k(x), \quad k \in \mathbb{Z}^2, \quad e_k(x) = i \frac{k^{\perp}}{|k|}.$$

Here the $\{w_k\}$'s are standard i.i.d. complex Wiener processes such that $w_{-k}(t) = \overline{w_k(t)}$ and $\sigma_{-k} = \overline{\sigma_k}$, $|\sigma_k| > 0$. Let $u(x) = \sum_k u_k e_k(x)$ with $u_0 = 0$, be the Fourier expansion, and consider the space $\mathbb{L}^2 = \{\sum_{k \in \mathbb{Z}^2} |u_k|^2 < \infty\}$. Projecting on $\mathbb{L}^2$ we get a system of Ito stochastic equations

$$du(x, t) + \nu \Lambda^2 u(x, t) dt = B(u, u) dt + dW(x, t) \tag{20}$$

where, denoting by P the projection on the subspace of the divergence-free functions, we write $\Lambda^2 u = -P \Delta u$, $B(u, v) = -P(u \cdot \nabla) v$. Equation (20) defines a Markovian stochastic semi-flow $\varphi^{\omega}_{s,t}$, $s < t$, on $\mathbb{L}^2$, for all $\omega \in \Omega$, the canonical space generated by $\{dw_k(t)\}$. A measure μ on $\mathbb{L}^2$ is said to be invariant if for any bounded continuous function F on $\mathbb{L}^2$ and $t > 0$ we have

$$\int_{\mathbb{L}^2} F(u)\mu(du) = \int_{\mathbb{L}^2} \mathbb{E}F(\varphi^{\omega}_{0,t}u)\mu(du) \tag{21}$$

were $\mathbb{E}$ denotes expectation with respect to the measure $\mathbb{P}$ on Ω.

The existence of stationary measures was established by compactness in [19, 41]. Uniqueness was proved, under restrictive assumptions, when all modes are forced, as in the papers by Kuksin and Shirikyan [25] and by Bricmont, Kupiainen and Lefevere [7]. The main result of E, Mattingly and Sinai is the following theorem.

Theorem 9 ([18]) *There is an absolute constant* $\mathscr{C}$ *such that if* $N^2 \geq \mathscr{C}\frac{\mathscr{E}_0}{\nu^3}$*, where* $\mathscr{E}_0 = \sum_{|k|\leq N} |u_k|^2$ *then Eq.* (20) *has a unique stationary measure on* $\mathbb{L}^2$.

Some comment is here in order. Following the seminal work of Ladyzhenskaya [27] we know that the 2-dimensional Navier–Stokes equations in a bounded domain, with no forcing, or with a bounded finite-dimensional force, has a finite-dimensional attractor, of dimension depending on the Reynolds number [20]. There is a finite number of "determining" modes, and for large times the other modes are determined by the past history of the determining ones. The main theorem of [18] states that uniqueness of the stationary measure holds under the condition that all determining modes are forced, and is a natural extension of the above results.

A main step in the proof is a representation of the high modes as functionals of the time-history of the low modes. Let $\mathbb{L}^2_\ell = \text{span}\{e_k \colon |k| \leq N\}$, $\mathbb{L}^2_h = \text{span}\{e_k \colon |k| > N\}$ define the subspaces of low and high modes, and denote by P_ℓ, P_h the corresponding projectors in $\mathbb{L}^2$. Setting $\ell(t) = P_\ell u$, $h(t) = P_h u$, Eq. (20) becomes

$$\begin{aligned} d\ell(t) = &\left[-\nu \Lambda^2 \ell + P_\ell B(\ell,\ell)\right] dt \\ &+ \left[P_\ell B(\ell,h) + P_\ell B(h,\ell) + P_\ell B(h,h)\right] dt + dW(t), \end{aligned} \tag{22}$$

$$\frac{dh(t)}{dt} = \left[-\nu \Lambda^2 h + P_h B(h,h)\right] + P_h B(\ell,h) + P_h B(h,\ell) + P_h B(\ell,\ell). \tag{23}$$

If $\ell(t)$ is assigned, Eq. (23) can be solved for h, and let $\Phi_{s,t}(\ell, h_0)$ be the solution of (23) at time t with initial condition h_0 at time s and fixed ℓ.

By stationarity, one can represent the initial data as coming from a distant past. Let $C((-\infty,0],\mathbb{L}^2)$ be the path space of the past and $\psi^{\omega}_t u \in C((-\infty,t],\mathbb{L}^2)$ the evolution of $u \in C((-\infty,0],\mathbb{L}^2)$ induced by the semi-group: $(\psi^{\omega}_t u)(s) = u(s)$ for $s \leq 0$ and $(\psi^{\omega}_t u)(s) = \varphi_{0,s}u(0)$ for $s \in [0,t]$.

There is an obvious measure μ_p on $C((-\infty,0],\mathbb{L}^2)$, induced by the product measure $\mathbb{P}\times\mu$ on $\Omega\times\mathbb{L}^2$. Defining the shift on the trajectories as $(\theta_t v)(s) = v(s+t)$, the operator $\theta_t\psi^{\omega}_t$ maps $C((-\infty,0],\mathbb{L}^2)$ into itself. If μ is stationary, then μ_p is also stationary in the sense that for any bounded function $F(u)$ on $C((-\infty, 0], \mathbb{L}^2)$ we have

$$\int_{C((-\infty,0],\mathbb{L}^2)} F(u)d\mu_p(u) = \mathbb{E}\int_{C((-\infty,0],\mathbb{L}^2)} F(\theta_t\psi_t^\omega u)d\mu_p(u).$$

Moreover, it is clear that if μ and ν are two stationary measures for the stochastic flow (20), then $\mu_p = \nu_p$ implies $\mu = \nu$.

The proof further shows that there is a subset $U \subset C((-\infty, 0], \mathbb{L}^2)$ of full measure consisting of functions $v\colon (-\infty, 0] \to \mathbb{H}$ where $\mathbb{H} = \{u \in \mathbb{L}^2\colon \sum_k k^2|u_k|^2 < \infty\}$, and moreover the energy has the correct average in time and the fluctuations are typical.

The reconstruction of the high modes as a function of the past stretching to $-\infty$ is given by the following lemma.

Lemma 10 ([18]) *There is some absolute constant $\mathscr{C}$ such that if $N^2 \geq \mathscr{C}\frac{\mathscr{E}_0}{\nu^3}$ then the following holds*

(i) If there are two solutions $u_1(t) = (\ell(t), h_1(t))$, $u_2(t) = (\ell(t), h_2(t))$ corresponding to some (maybe different) realization of the forcing and such that $u_1, u_2 \in U$, then $h_1 = h_2$.
(ii) Given a solution $u(t) = (\ell(t), h(t)) \in U$, any $h_0 \in \mathbb{L}_h^2$ and $t < 0$, the limit $\lim_{t_0\to-\infty} \Phi_{t_0,t}(\ell, h_0) = h^*$ exists and $h^* = h$.

The lemma implies that there is a map Φ_t giving the high modes at time t in terms of the past trajectory of the low modes $L^t = \{\ell(s)\colon s \in (-\infty, t]\} \in C((-\infty, t], \mathbb{L}^2)$: $h(t) = \Phi_t(L^t)$. Equation (22) then becomes

$$d\ell(t) = \Big[-\nu\Lambda^2\ell + P_\ell B(\ell, \ell) + G(\ell(t), \Phi_t(L^t))\Big]dt + dW(t) \tag{24}$$

where $G(\ell, h) = P_\ell B(\ell, h) + P_\ell B(h, \ell) + P_\ell B(h, h)$. Equation (20) is thus reduced to a dynamics of the low modes: it is a finite-dimensional process with memory extending back to $-\infty$, which is not Markovian, but rather Gibbsian.

In the final part of the proof one shows that the memory is not so strong as to violate ergodicity. A crucial fact is that for a set of full measure of "nice" past histories of the low modes $L \in C((-\infty, t], \mathbb{L}_\ell^2)$ and for any $t > 0$, the conditional distribution of $\ell(t) \in \mathbb{L}_\ell$ has a component equivalent to the Lebesgue measure. This fact is shown to imply that the assumption that the corresponding stationary measures on the path space of the past $\mu_{p,i}, i = 1, 2$ are different, leads to a contradiction.

We remark that Kuksin and Shirikyan [25] who deal with a forcing given by a bounded kicked noise acting on all modes, did also introduce a Gibbs construction in their proof of uniqueness.

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