Chapter 3
Spatial Reasoning: A Critical Problem-Solving Tool in Children's Mathematics Strategy Tool-Kit

Beth M. Casey and Harriet Fell

Abstract This chapter reviews the spatial literature from the perspective of potential mechanisms for widening the range of spatially-based strategies available when solving math problems. We propose that teaching generalized spatial skills, disconnected from specific math content, may not be the best direction to go in future spatial interventions. Students who do not start out with strong spatial skills may need to learn to develop different types of "spatial sense" specific to each content area. Thus, acquiring and applying spatial strategies may depend in part on developing spatial sense within these specific math domains. In this chapter, we present an overview of evidence for different types of spatial sense that may serve as a prerequisite for effectively applying spatial strategies within these math content areas. The chapter also provides examples of math activities designed to help children acquire spatial sense and apply spatial strategies when solving diverse types of math problems.

Keywords Spatial skills · Math achievement · Math strategies · Decomposition · Visualization · Elementary school · Fractions · Geometry · Arithmetic · Mapping · Word problems · Gender differences · 2- and 3-dimensional representations · Mental images · Spatial sense · Image generation · Surfaces · Solids · Common Core Math Standards · Numerical concepts

B. M. Casey (✉)
Lynch School of Education, Boston College, Boston, MA, USA
e-mail: caseyb@bc.edu

H. Fell
College of Computer and Information Science, Northeastern University, Boston, MA, USA
e-mail: fell@ccs.neu.edu

The strong association between spatial skills and mathematics achievement has been demonstrated across a wide range of mathematics content areas and ages (Mix & Cheng, 2012; Wai, Lubinski, & Benbow, 2009). One dynamic in this association may be that students with good spatial skills have access to a unique subset of strategies utilizing spatial sense and spatial reasoning, which enables them to draw upon critical tools in their problem-solving tool set that are not available to students with poorer spatial skills. We would argue that this advantage is not just due to higher fluid reasoning ability, i.e., the ability to draw on novel and effective problem solving strategies in general, but instead is based on the unique association between spatial and mathematical processing (Li & Geary, 2017). We further propose that spatial reasoning can be improved through the development of mathematics activities that facilitate the acquisition of spatial sense and the use of spatially based strategies within different content areas. This approach should eventually result in greater potential by students to draw on spatially based strategies when approaching difficult mathematics problems across a range of mathematics content areas.

In the first part of the chapter, we start by examining the literature on early arithmetic strategy use in relation to spatial processing. We will present an overview of research on the relation between early spatial processing and the early use of higher-level mental arithmetic strategies as predictors of mathematics performance. We briefly address the literature on early spatial skills and mathematics, and then focus mainly on the role of spatial skills in acquiring the use of advanced mental strategies, and how use of these spatially based mental strategies is beneficial to greater strategy-choice flexibility and mathematics achievement at later points in time.

In the second part of the chapter, we address the question of how students may develop the ability to apply spatially based strategies when solving diverse types of mathematics problems. We consider the importance of developing "spatial sense" within particular mathematics content areas as a prerequisite to applying spatially based strategies in these areas. In particular, we briefly review the literature on the ability to *generate images* as a critical component of developing spatial sense within the following mathematics content areas: (1) visual representations of magnitude with respect to fractions, (2) the ability to translate verbal descriptions into visual representations, and (3) the ability to generate and maintain images of 2-d and 3-d visualizations in relation to one another when solving mathematics problems. Each content area is followed by instructional examples for ways of teaching these different types of image generation/visualization activities.

Part 1: Spatial Processing and Arithmetic Strategy Choices

This section focuses on spatial sense and strategy-choice related to early arithmetic skills. The goal is to discuss the link between spatial skills and the use of high-level mental arithmetic strategies as a possible mechanism for encouraging young learners to apply spatially based strategies when approaching arithmetic problems.

Relation Between Early Spatial Skills and Mathematics Achievement

Early spatial skills are related to the development of mathematical skills via multiple pathways, involving both early number line and calculation skills (Gunderson, Ramirez, Beilock, & Levine, 2012; LeFevre et al., 2013). Moss and her associates recently implemented an extensive spatial intervention for 4-to-8 year olds, consisting of approximately 46 h of in-class time over the academic year. The geometry intervention involved carrying out lessons and activities designed to primarily target spatial visualization skills (i.e., forming, maintaining, and manipulating visual-spatial information) (Hawes, Moss, Caswell, Naqvi, & MacKinnon, 2017; Moss, Bruce, Caswell, Flynn, & Hawes, 2016). Results revealed that compared to an active control group, children in the spatial intervention demonstrated gains on three separate measures of spatial thinking; spatial language, visual-spatial geometrical reasoning, and 2-d mental rotation. Interestingly, while there were no group differences on a non-symbolic magnitude comparison task or a number knowledge test, children in the intervention group demonstrated significant gains relative to the control group on a symbolic magnitude comparison task with a substantial effect size. More of this type of intensive spatial intervention research needs to be conducted in the future to experimentally examine the effects of early spatial skills on mathematics achievement.

Spatial reasoning depends in part on good visuospatial memory resources (Miyake, Friedman, Rettinger, Shah, & Hegarty, 2001), and visuospatial memory pathways have been shown to be critical predictors of numerical skills (Geary, 2011; Li & Geary, 2013, 2017; Nath & Szücs, 2014). After controlling for non-verbal IQ and verbal memory measures, Nath and Szücs (2014) found that the association between Lego block-building skills and numerical achievement in 7-year olds was mediated by visuospatial memory. Further, controlling for the central executive, IQ, and phonological memory, Li and Geary (2013, 2017) found that growth in visuospatial memory skills from first-to-fifth grade was predictive of later numerical operations, but not reading achievement. This visuospatial memory/mathematics association extends into high school with visuospatial memory becoming even more important to numerical operations across successive grades (Li & Geary, 2017).

In kindergarten and first grade, the association between visuospatial memory and mathematics achievement is not as strong as in later grades, with phonological and linguistic processing showing a greater relation with numerical development at this age (LeFevre et al., 2010; Li & Geary, 2013). This may be in part because the type of mathematics assessed at this early age (e.g., number words and numerals) may require less spatial processing (Li & Geary, 2013). In fact, Krajewski and Schneider (2009) found that in kindergarten, phonological awareness had a stronger impact on *lower* numeric competencies in first grade (i.e., when number words were isolated from quantities) than for *higher* numerical competencies (i.e., when number words needed to be linked with quantities). The reverse was true for kindergarten

visuospatial memory skills, which predicted the higher level numeric competencies at first grade.

Stronger early spatial-mathematics associations are found when more complex spatial reasoning processes (such as forming, maintaining, and manipulating visual-spatial information to solve mathematics problems) are examined in the literature. Mix and her colleagues (Mix et al., 2016) found that at kindergarten, mental rotation performance was a better predictor of mathematical performance than visuospatial memory performance. Studies have shown that as early as preschool, kindergarten, and first grade, these more complex spatial reasoning skills (e.g., mental rotation, spatial visualization, and block building) relate to numeracy and addition and subtraction skills (Casey, Dearing, Dulaney, Heyman, & Springer, 2014; Gunderson et al., 2012; Mix et al., 2016; Nath & Szücs, 2014; Verdine et al., 2014).

In summary, there is critical evidence for an association between early use of spatial skills and numeracy and addition and subtraction skills. The next step is to consider the pathways through which early spatial skills might impact these mathematics skills. One such pathway is the application of spatial skills when solving addition and subtraction problems by using mental arithmetic strategies that draw upon visuospatial memory processes and spatial reasoning.

Spatial Processing and Use of Higher Level Arithmetic Strategies

What are the mechanisms by which spatial reasoning skills might impact early addition and subtraction? One connection may occur through developing ability to visualize quantity along the mental number line. Researchers found that children's spatial skill (i.e., mental transformation ability) at the beginning of first and second grades predicted improvement in linear number line knowledge over the course of the school year. Second, in a separate sample, children's spatial skill at age 5 predicted their performance on an approximate symbolic calculation task at age 8 and this relation was mediated by children's linear number line knowledge at age 6 (Gunderson et al., 2012).

Arithmetic strategy choices. Another mechanism connecting spatial skills to early mathematics may be through the strategies that children use to solve addition and subtraction problems (Foley, Vasilyeva, & Laski, 2017; Laski et al., 2013; Siegler & Shrager, 1984). There are a variety of strategies that children use to solve arithmetic problems. When it comes to solving basic addition and subtraction problems, children generally choose from among four different strategies: count-all, count-on, decomposition, and retrieval (Laski et al., 2013). The count-all strategy involves counting out the first number, then counting out the second number, and then finally counting the total of the two numbers (e.g., to solve 7 + 5, one would first count to 7, then count to 5, and then finally count from 1 to 12). A slightly more sophisticated strategy is count-on, which involves counting up from one number the

value of the other number (e.g., to solve 7 + 5, one would start from 7 and count 8, 9, 10, 11, 12).

Higher-level mental strategies involve retrieval and decomposition. The retrieval strategy involves recalling the solution purely from memory (Laski et al., 2013). The decomposition strategy involves breaking the numbers down into simpler mathematics facts that the child knows, and then adding or subtracting the value that remains. For example, to solve 7 + 5, one might first add 5 + 5 to get 10 and then add the remaining 2 to arrive at 12. A retrieval strategy is typically applied to single digit addition and subtraction. When a mathematics fact is not known, use of a decomposition strategy is often applied. Decomposition strategies are typically applied to addition and subtraction problems involving mixed digit (one single digit and one double digit number) or two double digit arithmetic problems.

Decomposition and retrieval are generally considered more advanced strategies for a number of reasons. Decomposition and retrieval are more efficient because they require less time to arrive at an answer than do concrete counting strategies (Ashcraft & Fierman, 1982). In addition, decomposition and retrieval are more sophisticated strategies because they avoid use of concrete counting strategies with fingers and other manipulatives, and instead depend on mental processes, drawing on memory-based mental representations of numerical information that depend on prior knowledge of mathematics facts (Geary, 2011).

Furthermore, frequency of use of decomposition and retrieval in solving arithmetic problems tends to be linked with higher mathematics performance. For example, Carr, Hettinger Steiner, Kyser, and Biddlecomb (2008) examined the association between strategy use and mathematics competence as measured by standardized test scores in a group of second grade students. They found that uses of higher-level cognitive strategies (retrieval and/or decomposition) were stronger predictors of mathematics competence above and beyond fluency and accuracy of solving basic mathematics facts. Both attempted and accurate uses of higher-level cognitive strategies were the strongest predictors of mathematics competency.

A number of longitudinal investigations of the relation between arithmetic strategy use in early grades and later mathematics achievement suggest that mental arithmetic strategies used in first and second grades, such as decomposition and retrieval, positively predicted mathematics performance in third, fourth, and fifth grades (Carr & Alexeev, 2011; Fennema, Carpenter, Jacobs, Franke, & Levi, 1998; Geary, 2011). Carr and Alexeev (2011) followed a group of children longitudinally from second grade through fourth grade. They found better mathematics outcomes in fourth grade for students who had attempted to use higher-level mental strategies previously when solving the basic arithmetic problems in second grade. As measured by standardized test scores in fourth grade, those children who attempted to use and those who correctly used mental strategies in second grade had an increased probability that they would meet or exceed the standards set for the mathematics competency test in fourth grade. These findings suggest that early preferences for these types of mental strategies may have long-term influences on mathematics

competency. Studies also show that both retrieval and decomposition strategy-use increase with age (Bjorklund & Rosenblum, 2001; Siegler, 1987).

Spatial skills and arithmetic strategy choice. Both decomposition and retrieval depend on retrieving mathematics facts, and may draw on visuospatial memory processes and spatial reasoning. In a study of the association between spatial reasoning skills (mental rotation and spatial visualization) and arithmetic strategy preferences, Laski and Casey and associates (Laski et al., 2013) found that among first grade girls, these early spatial reasoning skills were positively related to use of both retrieval and decomposition. Furthermore, spatial skills were negatively associated with the count-all strategy and unrelated to the count-on strategy. Verbal skills were related to decomposition, but not retrieval. In a longitudinal study, M. Carr (personal communication, May 19, 2017) also found evidence of an association between 2-d spatial visualization skills and use of the decomposition strategy from first through fourth grades.

When Geary used measures from the Working Memory Test Battery for Children as predictors of frequency of decomposition strategy choice in first grade, he found a visuospatial measure (the Mazes Memory task) to be the strongest predictor at that age (D. C. Geary, personal communication, May 10, 2017). Foley et al. (2017) propose that visuospatial memory can serve as a mental sketchpad for storing problem information. They suggest that this may be particularly important for decomposition as, "a child's capacity for holding information in short-term memory—the storage component of working memory (Baddeley & Hitch, 1974; Gathercole, Pickering, Ambridge, & Wearing, 2004)—may contribute to his or her selection of decomposition because it requires maintenance of intermediate solutions and procedures" (p. 4). They found that short-term visuospatial memory was positively related to the frequency of children's decomposition use in second and fourth graders, while verbal memory was not. Most importantly, frequency of use of decomposition mediated the relation between visuospatial memory and arithmetic accuracy.

Use of decomposition. Decomposition differs from other arithmetic strategies because it does not just involve implementation of rote procedures, such as counting with fingers or recalling mathematics facts, but also requires active problem solving and a more complex series of reasoning processes. Thus, using decomposition for solving arithmetic problems in the early grades may provide a foundation for later mathematics problem solving.

Mix and her associates in a detailed analysis of kindergarten, third, and sixth grade spatial-mathematics associations proposed that across grade levels, the spatial-mathematics association is stronger when students are encountering novel problems and decrease as skills become more automatic or procedural (Mix et al., 2016). Therefore, at the start, as children begin to learn to apply complex decomposition strategies when trying to solve addition and subtraction problems, the ability to draw on spatial processes may be particularly beneficial.

Other longitudinal research has shown that the early use of decomposition arithmetic strategies (specifically in first grade) is a strong predictor of later numerical

mathematics performance when controlling for intelligence, working memory, and processing speed (Geary, 2011). In a recent longitudinal study on first grade girls (Casey, Pezaris, Fineman, Pollock, Demers, & Dearing, 2015), we compared early spatial, verbal, and arithmetic skills as predictors of two types of mathematics reasoning skills 4 years later in fifth grade: (1) geometry and measurement problems specifically selected and designed to tap spatial mathematics reasoning skills, and (2) numerical and algebraic problems specifically selected and designed to tap analytical logical deductive mathematics reasoning (Casey et al., 2015). As expected, we found that early spatial skills predicted later geometry and measurement, but were more surprised that first grade spatial skills were also the strongest predictors of performance on later numerical and algebra problems in fifth grade. Furthermore, the pathway between early spatial skills and later numerical/algebraic mathematics reasoning was both a direct and an indirect pathway. Of importance here, is that the indirect pathway led from first grade spatial skills to greater frequency of use of first grade decomposition strategies and then to mathematics reasoning in fifth grade. Thus, it was the early relation between spatial skills and decomposition (not the early relation between spatial skills and retrieval) that predicted later numerical and algebraic mathematics reasoning in fifth grade (Casey, Lombardi, Pollock, Fineman, Pezaris, & Dearing, 2016).

There are a number of studies indicating that early use of decomposition is a particularly strong predictor of mathematics performance (Casey et al., 2016; Foley et al., 2017; Geary, 2011). In a longitudinal study that followed children from first grade through third grade, Fennema et al. (1998) investigated the relation between children's use of the invented algorithm (a form of decomposition) in first and second grade and performance on "extension" problems, or more advanced mathematics problems that involved money and three-digit numbers, in third grade. They found that by third grade, the second graders who had preferred the invented algorithm showed better performance than children who had used the standard algorithm. A recent study found that the frequency with which first graders use a decomposition strategy predicted their accuracy on complex addition problems and mediated cross-national differences in accuracy on these complex arithmetic problems (Vasilyeva, Laski, & Shen, 2015).

Gender differences. There is an interesting parallel between the early development of spatial skills and the early development of decomposition use in solving arithmetic problems—both show evidence of early gender differences. In a recent review of the literature on early gender differences in spatial skills, Levine and her colleagues (Levine, Foley, Lourenco, Ehrlich, & Ratliff, 2016) found that there is evidence for gender differences favoring boys in young children's mental transformation and mental rotation skills, but the gender effects are not obtained as consistently when compared to older ages, where strong support for gender differences on mental rotation and transformation tasks has been found (Casey, 2013; Wai et al., 2009). Early evidence for spatial gender differences can vary depending on age and type of task, with most of the evidence for gender differences occurring at age 5 or above and involving mental rotation of abstract shapes (Casey et al., 2008; Cronin, 1967;

Ehrlich, Levine, & Goldin-Meadow, 2006; Frick, Ferrara, & Newcombe, 2013; Levine, Huttenlocher, Taylor, & Langrock, 1999; Tzuriel & Egozi, 2010). Two studies with kindergarten and first grade students (Casey, Erkut, Ceder, & Mercer Young, 2008; Tzuriel & Egozi, 2010) have used extended spatial interventions to improve spatial skills. Both studies found that the boys in the control groups improved in their spatial skills without the intervention, while the girls improved only with the intervention.

Starting as early as kindergarten and first grade, evidence for gender differences in arithmetic strategy use has been found as well, such that boys are more likely than girls to use the more advanced strategies of retrieval and/or decomposition, while girls are more likely than boys to use concrete manipulatives such as counters or fingers to solve arithmetic problems (Carr & Davis, 2002; Carr & Jessup, 1997; Fennema et al., 1998; Shen, Vasilyeva, & Laski, 2016). Carr et al. (2008) reported similar findings among a group of second-grade students. They found that boys were more likely to use cognitive strategies (e.g., mental count-on, decomposition) and girls were more likely to use manipulative strategies. Carr and Davis (2002) found that in a free-choice condition, girls were more likely than boys to correctly use and attempt to use counting strategies, whereas boys were more likely than girls to correctly use retrieval. When children were forced to use retrieval strategies, boys were more likely than girls to use them correctly.

The well-documented link between higher-level mental strategies and later mathematics performance suggests that girls' early preference for counting strategies may put them at risk for poorer mathematics achievement in later grades, and it may possibly have socio-emotional effects as well. Research suggests that a male mathematics gender stereotype is acquired quite early, and that it influences emerging mathematics self-concepts prior to ages at which there are actual gender differences in mathematics achievement (Cvencek, Metzoff, & Greenwald, 2011). One obvious and observable difference among students in classrooms is whether they are still counting on their fingers or doing the addition mentally. Even if boys and girls are equally accurate on arithmetic problems, gender differences in use of higher level mental strategies may well impact gender-based early mathematics self-concepts.

A recent international study found evidence for gender differences in arithmetic strategy use in Russian and US first graders, but not in Taiwanese first graders (Shen et al., 2016). Among the Taiwanese students, there were no gender differences in accuracy, and girls used decomposition more than boys, while both genders outperformed students from the other two countries. In both Russia and the US, boys were more likely to use decomposition on complex arithmetic problems and have higher accuracy scores on the arithmetic problems (Shen et al., 2016). Most importantly, the researchers found that it was the preferred use of a decomposition strategy in boys that mediated the gender difference in accuracy for the US and Russian students. Thus, early gender differences in strategy choice may have long-term impacts for later gender differences.

In conclusion, evidence of early gender differences in both spatial skills and use of decomposition for solving arithmetic problems suggest that use of spatially based instruction tools (such as the number line) to teach and represent decomposition strategy procedures may be a fruitful approach for girls, in particular. This can begin the process of introducing spatial visualization and problem solving at the outset of arithmetic instruction.

Developmental changes in strategy choice. A major theoretical analysis of developmental changes in arithmetic strategy choice was proposed by Siegler (Lemaire & Siegler, 1995). According to Siegler's Adaptive Strategy Choice Model, as children develop over time in strategy use, they: (1) acquire a wider range of strategies, (2) make more adaptive choices among strategies, (3) increase frequencies of more efficient strategies, and (4) make more efficient use of pre-existing strategies. Siegler argues that strategy *preference* is as important as whether the children are *accurate* in using a particular strategy. Even low frequency use of higher level strategies is considered beneficial, and may be a positive early indicator of later effective flexible strategy-use. According to Siegler (2005) "…children usually choose adaptively among strategies; that is, they choose strategies that fit the demands of problems and circumstances and that yield desirable combinations of speed and accuracy, given the strategies and available knowledge that children possess." (p. 771).

A number of researchers have proposed that a major role of educators is to nurture children's adaptive strategy choices, i.e., the ability to solve mathematical tasks flexibly by being able to draw on a range of strategies when approaching mathematics problems (Siegler, 2007; Torbeyns, Verschaffel, & Ghesquière, 2005). Children with a wider range of strategies available to them may have a later advantage in terms of greater flexibility in strategy choice when task demands favor some strategies over others (Siegler, 1987). Thus, at first grade, even those children who have low frequency of use of decomposition may eventually show greater adaptive choices relating to this strategy later on as they encounter more complex problems.

Summary of Part 1. We propose here that having strong spatial skills at an early age may result in more arithmetic strategy-choice flexibility later on. Children with higher spatial skills have greater ability to draw on spatial as well as analytical strategies when solving mathematical problems. Researchers studying use of mental imagery in solving word problems have argued that use of advanced spatial skills should enable children to more easily generate and manipulate mental representations (Boonen, van der Schoot, van Wesel, DeVries, & Jolles, 2013; Hegarty & Kozhevnikov, 1999; Krawec, 2012). A relation between spatial skills and decomposition is likely to occur because students may use spatial representation as an element of their decomposition strategy when solving mathematics problems.

These findings suggest the possibility that the use of a decomposition strategy may depend in part upon the application of effective spatial skills and memory processes, and that interventions involving use of decomposition strategies may be one way of teaching students how to reason mathematically by drawing on their spatial reasoning and memory abilities. Clearly, this needs to be tested empirically through

intervention research, but the present review suggests a possible direction for future studies to examine the benefits of using spatially based mathematics strategies at an early age.

Part 2: Development of Visualization Strategies Across Mathematics Content Areas

Before students can apply spatially based strategies to mathematics problems they need to first acquire sufficient spatial sense in relation to a wide range of mathematics content areas in order to be able to draw on their spatial problem-solving skills. Image generation/visualization is an important component of spatial sense. The concept of spatial sense is linked most frequently to geometry problem solving (NCTM, 2000) involving the ability to: spatially visualize and represent geometrical relations, hold images in spatial working memory, and mentally transform geometric shapes. Measurement sense means that students have a conceptual understanding of the processes underlying measurement procedures (Joram, 2003; Shaw & Pucket-Cliatt, 1989). For example, in terms of measurement sense, Battista (2003) describes the underlying processes in gaining competence in measuring area and volume as understanding how to enumerate arrays of squares and cubes. He identifies two mental processes essential to meaningful structuring of arrays: (1) forming and using mental models and (2) spatial structuring. Thus, he proposes that effective measurement performance involves an understanding of the underlying spatial nature of measurement, as well as the numerical and procedural competence to use measuring tools and apply formulas. Number sense also requires an understanding of magnitude through the generation of mental visual arrays displaying the relation of numbers to one another in terms of relative magnitude along a continuum. Recent research has documented the importance of generating a mental number line as a mechanism for spatially representing this relation (Gunderson et al., 2012; LeFevre et al., 2013). Thus, there is substantial literature documenting the key role of image generation/visualization skills as an important underpinning for conceptual understanding within these mathematics content areas.

In the last half of this chapter, we give a brief overview of the literature on visualization skills within three mathematics content areas. These mathematics areas include: (1) visual representations of magnitude with respect to fractions, (2) the ability to translate verbal descriptions into visual representations with respect to mathematics word problems and geometry, and (3) the ability to translate images from one type of representation into another when solving geometry problems, i.e., moving from 3-d representations to 2-d images, or vice versa.

In the present chapter, we have made the argument that rather than spatial skills being taught in isolation as abstract concepts, they should be taught within the framework of specific mathematics content areas. Although individuals with strong spatial skills may be able to apply spatial strategies across mathematics content

areas, this may well not be true for those students who are gradually acquiring spatial reasoning skills. Within each mathematics area, the focus should be on the type of spatial sense that is key for that content area, drawing on this spatial reasoning to teach effective spatial strategies in approaching these mathematics problems.

Within each of these mathematics content areas, we will provide examples of mathematics activities that can be used to develop image generation skills likely to develop both spatial sense and increase use of spatially based mathematics strategies. Many of these examples are similar to ideas introduced already into mathematics curricula and into the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2011). Other examples are less common.

Many of the examples here, connecting spatial skills to mathematics within the different mathematics content areas, depend in part on physical instantiations/concrete models or manipulatives, which are arguably also spatial models (Mix, 2010). Mix proposed that rather than depending on abstract number and language-based symbolic representations and known algorithms, these objects construe mathematics as spatial relations and provide spatial tools with which to reason about mathematics. There is disagreement in the literature as to how effective manipulatives are, and Mix (2010) points out that even those who advocate the use of manipulatives caution that it depends on the way they are implemented. In particular, it depends on whether the children can move beyond a dependence on manipulatives. Researchers have not yet identified the critical elements that make manipulative effective. Nevertheless, in the present chapter, we continue to use a number of examples that employ concrete manipulatives, in order to teach children about spatial representations and spatial strategic approaches to mathematics problem solving.

Representing Fraction Magnitudes by Generating Mental Visual Arrays

Siegler and his colleagues have conducted recent research showing that fractions have magnitudes that can be ordered and assigned specific locations on a mental number line just as whole numbers can (Siegler, Thompson, & Schneider, 2011). Recently, Hurst and Cordes (2016) found that fractions, decimals, and whole numbers can be used to represent the same rational-number values, and that adults conceive of these rational-number magnitudes as lying along the same ordered spatial mental continuum. Thus, development of successful number sense and fraction sense and decimal sense in students depends in part on the ability to successfully represent magnitude spatially in terms of generating mental visual arrays.

It is proposed here that once students acquire more experience generating images involving spatial arrays across different types of magnitude estimations, they will be able to more effectively draw on spatially based strategies for solving a wider range of mathematics problems. Understanding of fractions is particularly important
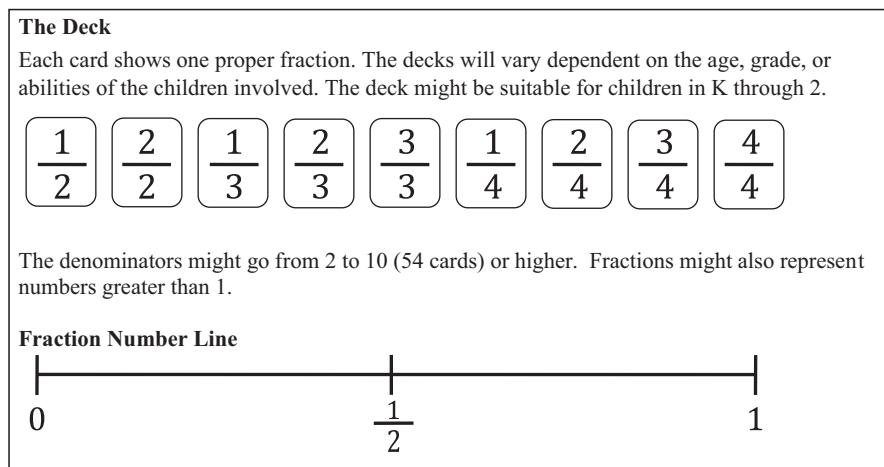
because elementary students' fraction knowledge has been found to uniquely predict their knowledge of algebra and overall mathematics achievement 5–6 years later (even after controlling for other key variables) (Siegler et al., 2012).

A brief overview of the theory and research related to *fraction magnitude representations*, along with suggested mathematics activities to further develop fraction magnitude sense will be presented next. Fraction sense involves conceptualizing fractions as a unit rather than perceiving them as separate numerators and denominators (Schneider & Siegler, 2010). A recent study by Hamdan and Gunderson (2017) found that training using a number line estimation of unit fractions showed significant transfer to an untrained fraction magnitude task, whereas equivalent training using an area model estimation task (unit fractions within a pie chart) did not. Research on fraction magnitude representations has shown that performance on these types of linear spatial representations correlates strongly with fraction arithmetic proficiency and general mathematics achievement scores (Siegler et al., 2011). This is somewhat surprising, as fraction magnitude knowledge is reported to be assessed little if at all on typical school mathematics assessments (Siegler et al., 2011).

In addition, fraction magnitude representations account for substantial variance in achievement scores above and beyond that explained by fraction arithmetic proficiency (Siegler et al., 2011). Furthermore, recent intervention research on at-risk fourth graders (Fuchs et al., 2013; Fuchs et al., 2014) found that fraction magnitude interventions, involving placing fractions on number lines as well as comparing and ordering magnitudes, resulted in both greater improvement in fraction arithmetic proficiency and conceptual understanding when compared to control group taught fractions through a school-based mathematics textbook. It was found that the gap between at-risk and low-risk students narrowed for the intervention group, but not the control group.

Examples of linear mental fraction line fraction games and activities. In this next section, a series of examples of mathematics games and activities are described that can be used to teach students how to generate images of fractions in order to develop fraction sense. The idea is for students to learn to represent fractions as a unit, and to compare them to other fractions along a mental number line. The curriculum, *Everyday Mathematics* (University of Chicago School Mathematics Project, 2007) includes lots of activities with fraction cards—including using them along a number line. The goal for presenting these particular games in the present chapter is to provide examples of how to apply this type of spatial sense as a basis for generating effective spatial strategies when solving fraction arithmetic problems.

Use of fraction card games. These games are based on the traditional card game "War." They encourage children to recognize common fractions, where they fit on the number line, and how they compare with each other. The materials are shown in Fig. 3.1.

The Deck

Each card shows one proper fraction. The decks will vary dependent on the age, grade, or abilities of the children involved. The deck might be suitable for children in K through 2.

$$\frac{1}{2} \quad \frac{2}{2} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{3}{3} \quad \frac{1}{4} \quad \frac{2}{4} \quad \frac{3}{4} \quad \frac{4}{4}$$

The denominators might go from 2 to 10 (54 cards) or higher. Fractions might also represent numbers greater than 1.

Fraction Number Line

Fig. 3.1 Materials for the fraction card games

Card game 1: Solitaire. Children can get used to the materials by playing solitaire games, e.g., shuffle the deck and turn up cards one at a time then do one of the following:

- place the card where it goes on the number line,
- place the card to the left, right, or on ½ as appropriate, or
- put the cards in order (no number line needed)

Card game 2: Two-person practice game. The deck is divided as evenly as possible with the cards dealt one at a time face down. Each player places his stack of cards face down, in front of him. Players each turn up a card at the same time and the one who places it correctly on the number line first gets both cards. If they place the cards correctly at the same time, they each keep their own card. The play can proceed with or without a number line. Whoever has the most cards at the end of some time period is the winner.

Alternate play—non-competitive. Proceed as with solitaire but taking turns, show where the fraction goes on the number line (see Fig. 3.2), or arrange the cards in order (see Fig. 3.3).

Card game 3: Two-person simpler practice game. The deal proceeds as above but the goal is to place the cards, as quickly as possible, into piles: <1/2, =1/2, >1/2.

Card game 4: Fraction Game WAR. Players each turn up a card at the same time and the player with the higher card takes both cards and puts them, face down, on the bottom of his stack.

If the cards have the same value, e.g., 1/2 and 2/4, it is War. Each player turns up one card face down and one card face up. The player with the higher cards takes both piles (six cards). If the turned-up cards are again the same rank, each player places another card face down and turns another card face up. The player with the
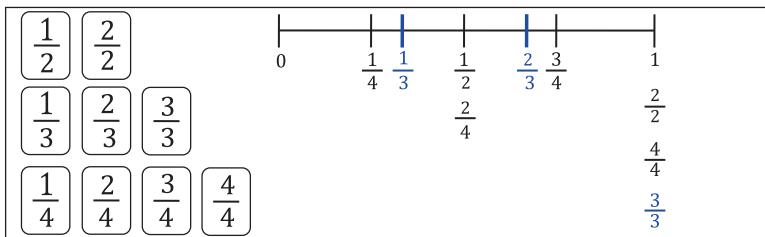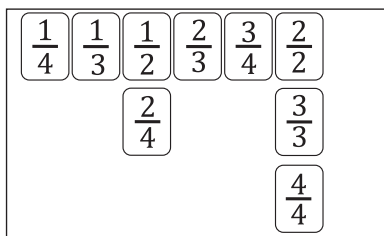
Fig. 3.2 Show where the cards go on the number line

Fig. 3.3 Arrange the cards in order

higher card takes all 10 cards, and so on. The game ends when one player has won all the cards.

Adding fractions using graphing. Another way to develop fraction sense is to use spatial representations through graphing. In these exercises, students use graph paper in two ways. They use a grid, e.g. 10×10 or 12×12, to color in regions to represent each fraction in a sum. They can also make a standard x vs. y graph to show a running sum of fractions, the x-axis showing the number of fractions added and the y-axis showing the sum so far.

Use of graphing to add fractions: convergent series. Though infinite series sounds pretty advanced, the idea of adding up fractions that follow a pattern is really pretty simple and keeping track of the sums visually provides another playful way for children to think about fractions. Note, this material is based on ideas from Cohen (1989).

Convergent series: First example. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$ The three dots mean "and so on." Each 8×8 square represents one whole (see Fig. 3.4).

- Shade in the squares to show the numbers above them.
- Show the sum as a fraction.
- Show the sum as a decimal.
- Can you tell what the sum would be if we kept adding similar terms?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$
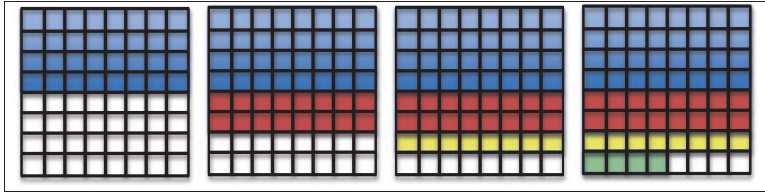
Fig. 3.4 Graphic representation of successive sums of 1/2 + 1/4 + 1/8 + 1/16
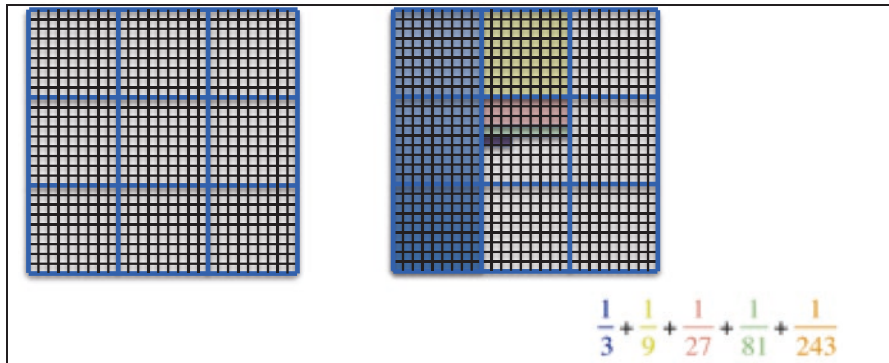
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$$

Fig. 3.5 Graphic representation of successive sums of 1/3 + 1/9 + 1/27 + 1/81 + 1/243

- What is the next term?
- What is the pattern of the denominators?

$$\frac{1}{2} \qquad \frac{1}{2}+\frac{1}{4} \qquad \frac{1}{2}+\frac{1}{4}+\frac{1}{8} \qquad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$$

Convergent series: Second example. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}\cdots$ This time, each denominator is three times the last one. Shade the box below (see Fig. 3.5) to show the sum of $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$. Notice that the blue lines divide the box into thirds horizontally and vertically. There are nine small boxes across and down in each blue box. Use a different color for each fraction you add on.

- What fraction represents one small black box? (answer 1/729)
- Express each of the partial sums as a single fraction, e.g., $\frac{1}{3}+\frac{1}{9}=\frac{4}{9}$ and $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}=\frac{13}{27}$.
- How big do you think the sum will get if you keep on adding more similar terms?
- Will it ever get bigger than 1?
- Will it ever get bigger than 4/9?

Convergent series: Third example. For something really different, think about this simple looking sum $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\cdots$

- Does it get bigger than 1? (Yes, just add the first 3 terms.)
- Does it get bigger than 2? (Yes, if you keep adding on 1/n it grows to infinity.)
- This is not a convergent series. It does not add up to a finite sum.

Translating Verbal Descriptions into Visual Representations

In the everyday world, visualization often involves translation of information you hear or read into a mental image, and in mathematics it also involves using that mental image to reason about a solution to a problem. Recently, there has been a focus on research addressing the benefits of translating words into images for mathematics word problems. In the initial research, Hegarty and Kozhevnikov (1999) clarified that the effectiveness of image generation depends on the type of imagery used; while schematic spatial representations were associated with success in mathematical word problems, use of pictorial representations was negatively correlated with success. Schematic representations encode the spatial relations described in a word problem, while pictorial representations encode the visual appearance of the objects described in the problem. Use of schematic representations was also significantly correlated with spatial ability, while pictorial representations were not. van Garderen (2006) obtained similar findings when comparing gifted, average, and learning disabled students, with gifted students tending to use more schematic representations. When examining pathways between spatial skills and word problems in sixth graders, Boonen and associates (Boonen et al., 2013) found that 21% of the association between spatial skills and word problem solving was explained through the indirect effects of strategies involving visual-schematic representations. Thus, spatial skills can be translated into useful spatially based mathematics strategies to solve word problems through the use of visual representations.

Jitendra and colleagues (Jitendra, Nelson, Pulles, Kiss, & Houseworth, 2016) conducted a review of studies examining the benefits of using visual representation models for teaching mathematics problems with at-risk mathematics students and found a substantial benefit of using this approach. In their review of the literature, Kingsdorf and Krawec (2016) also concluded that instructional methods involving representing word problems visually in third graders have been proven effective by organizing the problem information. However, they suggest further that effectively using graphic representations requires visually representing connections between the problem parts in order to effectively link various phases of the problem-solving process.

Examples for translating verbal descriptions into visual representations with early geometry learners. As indicated earlier, the National Council of Teachers of Mathematics (NCTM, 2000) conceives of geometry problem solving as involving

the ability to: spatially visualize and represent geometrical relations, hold images in spatial working memory, and mentally transform geometric shapes. This aspect of the geometry curriculum has been extensively developed within the mathematics curriculum series, *Investigations in Data, Number, and Space* (TERC, 2008). For example, throughout their curriculum, they make use of Quick Images where children are briefly shown images of quantities and shapes and asked to recognize them or reconstruct them in order to practice building and retaining such mental images. In *Taking Shape*, Joan Moss and co-authors (Moss et al., 2016) present a K-2 geometry curriculum that draws heavily on developing spatial visualization skills as a major component of mathematics education in early elementary school.

One element of these visualization skills is the ability to translate verbal into visual representations. If translating verbal information into visual representations is beneficial to later mathematics problem solving, how can educators start to develop this translation skill in young children at the outset of learning geometry? In this section, we provide examples of geometry activities and games that may be helpful in developing young students' ability to translate verbal descriptions into visual images that represent those descriptions. This proposed emphasis on translations of verbal representations into visual representations in early geometry activities may be useful later on when the spatial strategy instructional focus may be on translation of word problems into spatial representations through diagrams for older mathematics learners (e.g., Boonen et al., 2013; Jitendra et al., 2016; Kingsdorf & Krawec, 2016).

Geometry games for turning verbal or written descriptions into visual representations. Below, we present some two-person games that involve one partner giving verbal instructions to the other partner on how to draw or construct a particular representation. The objective is for the partner hearing the description to create a 2-d or 3-d representation from that description. Another set of examples uses written descriptions that have to be followed on a grid or map. These lead into learning coordinate geometry.

The Barrier Game. In the *Taking Shape* geometry curriculum, Joan Moss and co-authors (Moss et al., 2016) describe a series of "barrier games" for early elementary school that fit into this category, going from a verbal description to a spatial layout. In each game, there is a designer and a builder. There is a barrier between the two so they cannot see each other's creations as they describe and build (see Fig. 3.6). In these barrier games, children are encouraged to "Visualize" (create a visual image of what the partner is describing), "Verbalize" (ask your partner questions about where to place the tiles), and "Verify" (compare the two designs to see where they are similar and different). This visualize/verbalize/verify strategy can be useful for many types of geometry activities.

- In the simplest game, each child is given identical 7 square tiles.
- The designer creates a design with the 7 tiles.
- The designer then gives instructions to the builder so he or she can recreate the design on the other side of the barrier.

Fig. 3.6 A sample shape
with 7 tiles

George 5 directed by
Tova 7

Tova 7 directed by
George 5
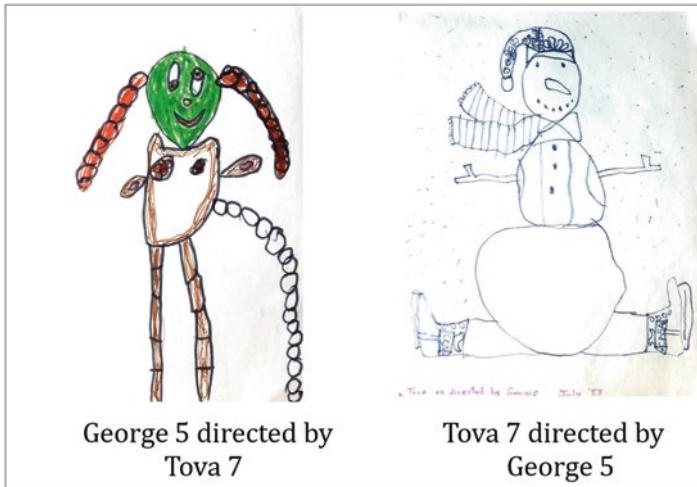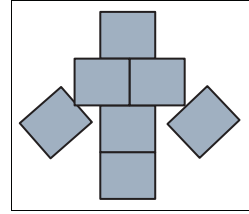
Fig. 3.7 Pictures from the Monster Game

- When the builder has finished, the children compare their designs.
- The children then swap roles and play again.
- The designer will have to use words and phrases like: besides, above, touching, rotate, turn, slide, left, right, etc.
- The game can be easily made more difficult by increasing the number of tiles or adding tiles with different shapes and colors.
- It could be turned into a 3-dimensional exercise by using Lego blocks.

The Monster Game. In a more open-ended activity that only requires pencil and paper, this is another two-person game to promote translation of verbal descriptions into visual representations. It even provides entertainment on long car trips. As with the barrier games, there is a designer and a builder and the children should not see each other's work until the game is finished (see Fig. 3.7).

- Each child has paper and a pencil with an eraser. Colored pencils or crayons can be used too.
- The designer draws a monster. A monster might be humanoid but they usually end up with strange sizes and numbers of arms and eyes. They can have claws in place of some hands or feet. There really are no constraints.

- The designer then gives instructions to the builder so he or she can recreate the monster on their own paper.
- When the builder has finished, the children compare their monsters.
- The children then swap roles and play again.

This is clearly very much like the barrier games, and children can also use the verbal description/drawing game to design buildings or parks or clothing as an alternative.

A mapping game based on verbal instructions. In the mathematics curriculum series, *Think Mathematics* (Educational Development Center, Inc., 2008) in grade 1, students are asked to explore direction on a map as part of their geometry section on maps, grids, and geometric figures. In one component they are asked to draw maps from verbal instructions. The example used here is from Harriet Fell's son's French Kindergarten class. The students were given a simple map of their village and then asked to draw routes, creatures, and other features according to the teacher's description. For example, after identifying their school and the village swimming pool on their maps, they drew a curve marked with arrows from school to pool. They put a duck in the pond, a goat on the farm, and waves in the pool (see Fig. 3.8). The purpose of this activity is to learn to generate visual images of maps in terms of spatial location, direction, and features on the map based on verbal directions from others. This should enable them to be more likely to use spatial strategies to solve these types of problems, such as generating mental images of maps rather than using landmarks and words.

Orienteering, drawing, and mapping routes on a grid based on verbal instructions. As part of the same unit, the *Think Mathematics* curriculum also focused on finding and following paths on a grid. By placing maps on a grid, the concept of graphing is also introduced as well as concepts involving spatial location and direction. This type of geometry activity is similar to orienteering in some ways. Orienteering is an outdoor sport where players use maps and a compass to find their way. Players may see on the map that the next goal is ¼ mile to the northeast of their current position and then use their compass to get there. In Orienteering Drawing, the player(s) is given a set of drawing directions to follow that are similar, e.g., head 3″ north. For young children, the step-by-step instructions can be given verbally and for older children it can be presented in writing (see Fig. 3.9).

Or the teacher could give a list of 2-d coordinates to make a picture. Here, the lower left-hand corner is the origin (0, 0). The child is instructed to go from coordinate to coordinate as in follow-the-dots.

$$(4,2)(4,4)(5,4)(5,5)(4,5)(4,6)(5,6)(5,7)(3,7)(3,2)(4,2)$$

For each of these methods of description, a teacher could provide the description or a child could create the description from the image.

1) Color the river (Yvette) blue.

2) Put a green "X" on the L-shaped building. That's our school.

3) With your green marker, let's make a route to the swimming pool.

    a. Go along the path from the school.

    b. Turn left when you get to the road.

    c. Turn right at the stop light.

    d. Follow the road around until you can turn right.

    e. Turn right as soon as you can.

    f. Head to the left. That's the swimming pool.

4) Put waves in the swimming pool.

5) ...

Fig. 3.8 Following directions to annotate a map

Assume the boxes are 1" on a side.
1. Start at the blue dot.
2. Head 2" North.
3. Head 1" East.
4. Head 1" North.
5. Head 1" West.
6. Head 1" North.
7. Head 1" East.
8. Head 1" North.
9. Head 2" West.
10. Head 5" South.
11. Head 1" East

(0,0)

Fig. 3.9 Orienteering Drawing

Translating 2-d Representations into 3-d Mental Images and Vice Versa

In the sciences, engineering, and in medicine, in particular, a critical skill involves being able generating mental images by moving back and forth between 2-d and 3-d representations. As reported in a literature review by Harris, Hirsh-Pasek, and

Newcombe (2013), students with high dynamic spatial transformation abilities (such as paper folding and mental rotation) were found to have greater ability to read graphs (Kozhevnikov, Motes, & Hegarty, 2007; Kozhevnikov & Thorton, 2006) and interpret diagrams (Höffler, 2010). Spatial research has been done on dynamic spatial transformations, such as mental rotation, in which the images have to be transformed and manipulated as well as generated (Mix & Cheng, 2012). A number of mathematics curricula for young children, such as *Investigations in Data, Number, and Space* (TERC, 2008) and *Taking Shape* (Moss et al., 2016) have incorporated 2-d to 3-d transformations that also require mental folding, such as identifying which 2-dimensional nets make cubes and rectangular prisms. This type of transformation is also used in origami. Boys, in particular, receive many hours of informal instruction doing these types of activities with Legos and model kits. These types of construction activities often require the ability to examine complex 2-d drawings, translate them into 3-d mental images, and ultimately produce 3-d structures.

Examples of instructional activities to develop translations of 2-d representations into 3-d images and vice versa. In the next section, we provide examples of geometry activities involving the simple translation of 2-d representations into 3-d structures and vice versa—without any dynamic transformations. Again, these activities are designed to develop spatial visualization skills and visuospatial memory, and are part of the NCTM (2000) and Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, Council of Chief State School Officers State Standards for Mathematics, 2011) relating to analyzing, comparing, creating, and composing 2-d and 3-d shapes in relation to modeling shapes in the world. In order to apply dynamic transformations strategies to solve many types of geometry problems, an important first step is to be able to visualize the relation between 2-d and 3-d representations of the same figures within static images.

However, when visualizing complex shapes, simple 2-d to 3-d transformations can be quite difficult for children with poorer spatial skills, even before adding on the requirement of manipulating the stimuli dynamically. For example, it can be argued that one of the initial difficulties in solving 3-d mental rotation tasks, such as the Vandenberg task (Vandenberg & Kuse, 1978), is that the stimuli consist of complex, unfamiliar, abstract shapes that project 2-d representations of 3-d stimuli. Just transforming that 2-d picture and projecting it as a 3-d image is difficult enough, in addition to the increased cognitive processing load required by holding it working memory, and then manipulating the image in order to mentally rotate it. In an analysis of the Vandenberg items, it was found that the most difficult items on the Vandenberg were ones in which the drawings of the 3-d figures had occluded parts (Voyer & Hou, 2006).

Thus, the simple translation of 2-d representations into 3-d mental images can be quite difficult to visualize, especially when stimuli are complex. Students need practice using their visualization skills to complete this simple translation process,

as they are important components of many geometry tasks that also require mental manipulation and rotations and are an important prerequisite to the mental transformations and manipulations that are needed to apply spatial strategies when solving many geometry problems.

   2-d representations of the swimming pool. The first example is for younger children, and involves exercises from an École Maternelle class in France for 5-year olds where Harriet Fell's son went for kindergarten, the year the family lived in France. Since the village had a pool, swimming lessons were part of the public schools' curriculum (see map of the village shown previously). Every week, the pool was divided into different sections for the three levels of classes and different types of floatation toys were placed in the pool in each section, e.g., foam barbells, kickboards, or doughnuts. The sections and the objects within the sections varied each week. When the children returned to the school after swim class, they were asked to draw how the pool looked that week from memory (see Fig. 3.10). When asked what he did that day, George would show a different drawing of the pool, and say, "It's a drawing of the swimming pool." What was being taught here? It involves not only the ability to translate a 3-d scene into a 2-d drawing, but accurately remembering the details of the different parts, as well as showing how the different parts relate to the whole as a gestalt. We have not found examples of similar activities in US kindergarten classrooms, and this approach may be idiosyncratic to France for this age group. However, the mathematics educators in the book may be able to provide other examples.

   Using modeling clay to explore surface, solids and cross-sections. Because one component of geometry, based on the NCTM Standards (2000), involves visualizing relationships between 2-d drawings and 3-d objects, clay and plasticine are great materials for exploring surfaces and solids, and have been used frequently in

Fig. 3.10 Drawings of the swimming pool
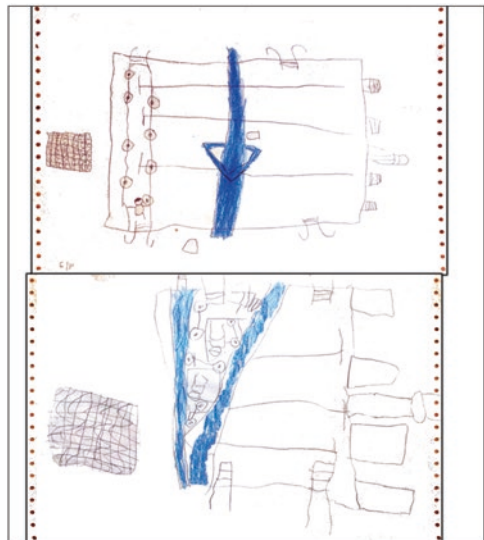
geometry classrooms. Clay can be used to study ways of describing 3-dimensional objects in 2-dimensions. Though there is wonderful software available for visualizing three-dimensional objects, we think it is important for children to have the experience of visualizing and manipulating three-dimensional objects as in the examples that follow to fully understand the two-dimensional projections they see.
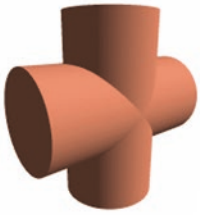
Build convex solids, given 2-d front, side, and top. Students were asked to build convex solids given the top, side, and front views (see Fig. 3.11).

Using plasticine to build objects and slicing through to see side views. Within the Common Core Standards for Mathematics at seventh grade (National Governors Association Center for Best Practices, Council of Chief State School Officers State Standards for Mathematics, 2011), students work with 3-d objects, relating them to 2-d figures by examining cross-sections. Students are expected to identify the shapes of 2-d cross-sections of 3-d object and identify 3-d objects generated by rotations of the 2-d figures. We think these types of cross-section activities can be done successfully with younger students. In the present example, given a contour map, the children are asked to draw the side views they would see if the scene were sliced along the lines, A, B, C, D, and E. They do this twice, first just from the contour map, e.g., by "walking" along the lines with their fingers, and then by using plasticine to build a model and slicing the model along the lines to see the side view (see Fig. 3.12).

Summary for Part 2. In this last half of the chapter, we provided examples of a variety of methods for teaching children to *generate images* as a critical component of developing spatial sense within the content areas of fractions, word problems, and geometry. We considered the importance of developing "spatial sense" within these mathematics content areas as a prerequisite to applying spatially based strategies in these areas.

Fig. 3.11 Front, side, and top views of solids

This is a contour map of a terrain with two hills and a lake.

Without building a model, sketch the side view contours on the grid below. Label the curves with the letters A B C D and E.

Now build a plasticene model.

When your mdel is finished, place it on the map and slice it along the lines marked
                A B C D E.
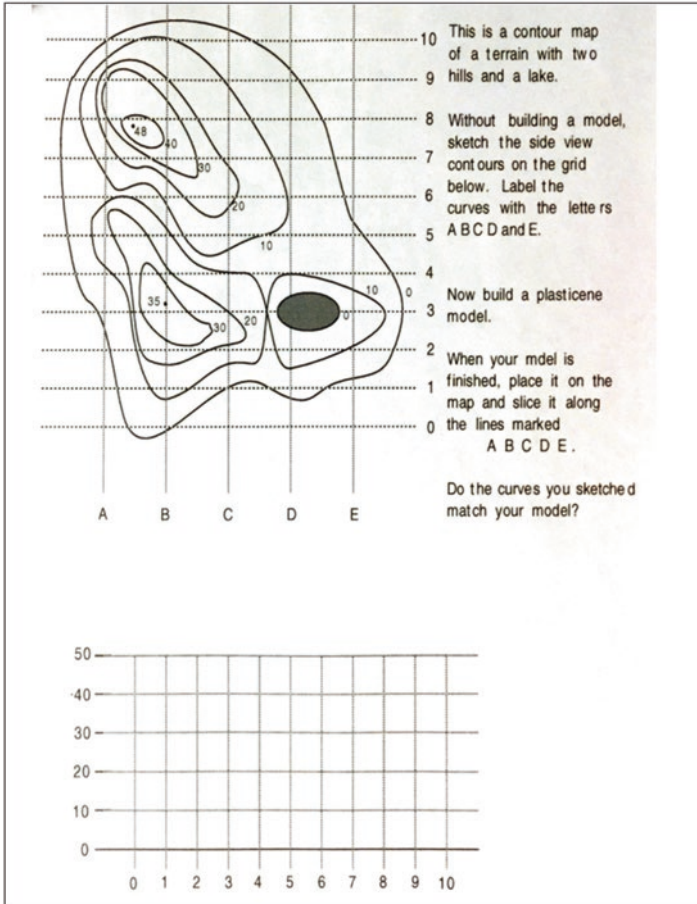
Do the curves you sketched match your model?

Fig. 3.12 Contour map and grid for plotting the cross-sections of the model

It is very clear from extensive research and reviews of the literature that strong spatial skills and visuospatial memory are predictive of mathematics achievement across a wide range of mathematics content areas and ages, and individuals with high spatial skills are likely to excel in mathematics (Li & Geary, 2017; Mix et al., 2016; Mix & Cheng, 2012; Wai et al., 2009). What we do not yet know is how malleable spatial skills are in terms of being able to teach students without initial high spatial ability how to: (1) acquire spatial sense, and (2) apply this knowledge by drawing on spatial strategies for mathematics problem solving. Future research on the spatial-mathematics association needs to focus on empirical research and spatial interventions to identify specific mechanisms for this association (Bailey, 2017).

Conclusion

In conclusion, we would like to propose that teaching generalized spatial skills disconnected from specific mathematics content areas may not be the best direction to go in future spatial intervention studies. Students who do not start out with strong spatial skills may need to learn to develop different types of spatial sense, specific to each type of content area, and to learn how to utilize spatial strategies based on developing spatial sense within these specific mathematics domains. Thus, the best strategy for future spatial-mathematics research may be: (1) understanding what types of spatial sense are required for different mathematics content areas, (2) conducting focused interventions for developing each type of spatial sense, (3) along with encouraging use of spatial strategies that draw upon them. If started early and done extensively across a range of mathematics content areas, students without initial spatial reasoning skills may eventually be able develop a wider mathematics spatial sense in order to approach a diverse range of mathematics problems with the added benefit of being able to draw upon critical spatial strategies from within in their problem-solving tool kit.

References

Ashcraft, M. H., & Fierman, B. A. (1982). Mental addition in third, fourth, and sixth graders. *Journal of Experimental Child Psychology, 33*, 216–234. https://doi.org/10.1016/0022-0965(82)90017-0

Baddeley, A. D., & Hitch, G. J. (1974). Working memory. In G. A. Bower (Ed.), *Recent advances in learning and motivation* (Vol. 8, pp. 47–90). New York: Academic Press.

Bailey, D. H. (2017). Causal inference and the spatial-math link in early childhood. *Monographs of the Society for Research in Child Development, 82*(1), 127–136. https://doi.org/10.1111/mono.12288

Battista, M. T. (2003). Understanding students' thinking about area and volume measurement. In D. H. Clements & G. Bright (Eds.), *Learning and teaching measurement: 2003 yearbook* (pp. 122–142). Reston, VA: National Council of Teachers of Mathematics.

Bjorklund, D. F., & Rosenblum, K. E. (2001). Children's use of multiple and variable addition strategies in a game context. *Developmental Science, 4*(2), 184–194. https://doi.org/10.1207/S15327647JCD0302_5

Boonen, A. J. H., van der Schoot, M., van Wesel, F., DeVries, M. H., & Jolles, J. (2013). What underlies successful word problem solving? A path analysis in sixth grade students. *Contemporary Educational Psychology, 38*, 271–279. https://doi.org/10.1016/j.cedpsych.2013.05.001

Carr, M., & Alexeev, N. (2011). Fluency, accuracy, and gender predict developmental trajectories of arithmetic strategies. *Journal of Educational Psychology, 103*, 617–631. https://doi.org/10.1037/a0023864

Carr, M., & Davis, H. (2002). Gender differences in mathematics strategy use: The influence of temperament. *Learning and Individual Differences, 13*, 83–95. https://doi.org/10.1006/ceps.2000.1059

Carr, M., Hettinger Steiner, H. H., Kyser, B., & Biddlecomb, B. A. (2008). Comparison of predictors of early emerging gender differences in mathematics competency. *Learning and Individual Differences, 18*, 61–75. https://doi.org/10.1016/j.lindif.2007.04.005

Carr, M., & Jessup, D. L. (1997). Gender differences in first-grade mathematics strategy use: Social and metacognitive influences. *Journal of Educational Psychology, 89*, 318–328. https://doi.org/10.1037/0022-0663.89.2.318

Casey, B. M. (2013). Spatial abilities and individual differences. In D. A. Waller & L. Nadel (Eds.), *Handbook of spatial cognition* (pp. 117–134). Washington, DC: American Psychological Association.

Casey, M. B., Andrews, N., Schindler, H., Kersh, J., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block-building activities. *Cognition and Instruction, 26*, 269–309. https://doi.org/10.1080/07370000802177177

Casey, M. B., Erkut, S., Ceder, I., & Mercer Young, J. (2008). Use of a storytelling context to improve girls' and boys' geometry skills in kindergarten. *Journal of Applied Developmental Psychology, 28*, 29–48. https://doi.org/10.1016/j.appdev.2007.10.005

Casey, B. M., Dearing, E., Dulaney, A., Heyman, M., & Springer, R. (2014). Young girls' spatial and arithmetic performance: The mediating role of maternal support during spatial problem solving. *Early Childhood Research Quarterly, 29*, 636–648. https://doi.org/10.1016/j.ecresq.2014.07.005

Casey, B. M., Lombardi, C., Pollock, A., Fineman, B., Pezaris, E., & Dearing, E. (2017). Girls' spatial skills and arithmetic strategies in first grade as predictors of fifth grade analytical math reasoning. *Journal of Cognition and Development, 18*(5), 530–555. https://doi.org/10.1080/15248372.2017.1363044

Casey, B. M., Pezaris, E., Fineman, B., Pollock, A., Demers, L., & Dearing, E. (2015). A longitudinal analysis of early spatial skills compared to arithmetic and verbal skills as predictors of fifth-grade girls' math reasoning. *Learning and Individual Differences, 40*, 90–100. https://doi.org/10.1016/j.lindif.2015.03.028

Cohen, D. (1989). *Calculus by and for young people*. Champaign/Urbana, IL: Don Cohen-The Mathman.

Cronin, V. (1967). Mirror-image reversal discrimination in kindergarten and first grade children. *Journal of Experimental Child Psychology, 5*, 577–585.

Cvencek, D., Metzoff, A. N., & Greenwald, A. C. (2011). Math gender stereotypes in elementary school children. *Child Development, 82*(3), 766–779. https://doi.org/10.1111/j.1467-8624.2010.01529.x

Educational Development Center, Inc. (2008). *Think math: Teacher guide, first grade*. Orlando, FL: Harcourt.

Ehrlich, S., Levine, S., & Goldin-Meadow, S. (2006). The importance of gestures in children's spatial reasoning. *Developmental Psychology, 42*(6), 1259–1268. https://doi.org/10.1037/0012-1649.42.6.1259

Fennema, E., Carpenter, T. P., Jacobs, V. R., Franke, M. L., & Levi, L. W. (1998). A longitudinal study of gender differences in young children's mathematical thinking. *Educational Researcher, 27*, 6–11.

Foley, A. E., Vasilyeva, M., & Laski, E. (2017). Children's use of decomposition strategies mediates the visuospatial memory and arithmetic accuracy relation. *British Journal of Developmental Psychology, 35*(2), 303–309. https://doi.org/10.1111/bjdp.12166

Frick, A., Ferrara, K., & Newcombe, N. S. (2013). Using a touch screen paradigm to assess the development of mental rotation between 3 ½ and 5 ½ years of age. *Cognitive Processes, 14*, 117–127. https://doi.org/10.1007/s10339-012-0534-0

Fuchs, L. S., Schumacher, R. R., Long, J., Jordan, N., Gersten, R., Cirino, P. T., … Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology, 105*(3), 683–700. https://doi.org/10.1037/a0032446

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., … Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude–treatment interaction. *Journal of Educational Psychology, 106*, 499–514. https://doi.org/10.1037/a0034341

Gathercole, S. E., Pickering, S. J., Ambridge, B., & Wearing, H. (2004). The structure of working memory from 4 to 15 years of age. *Developmental Psychology, 40*(2), 177–190. https://doi.org/10.1037/0012/1649.40.2.177

Geary, D. C. (2011). Cognitive predictors of achievement growth in mathematics: A 5-year longitudinal study. *Developmental Psychology, 47*(6), 1539–1552. https://doi.org/10.1037/a0025510

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: The role of the linear number line. *Developmental Psychology, 48*(5), 1229–1241. https://doi.org/10.1037/a0027433

Hamdan, N., & Gunderson, E. A. (2017). The number line is a critical spatial-numerical representation: Evidence from a fraction intervention. *Developmental Psychology, 53*(3), 587–596. https://doi.org/10.1037/dev0000252

Harris, J., Hirsh-Pasek, K., & Newcombe, N. S. (2013). Understanding spatial transformations: Similarities and differences between mental rotation and mental folding. *Cognitive Processing, 14*(1), 105–115.

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing children's spatial and numerical skills through a dynamic spatial approach to early geometry instruction: Effects of a 32-week intervention. *Cognition and Instruction, 35*(3), 236–264. https://doi.org/10.1080/07370008.2017.1323902

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684–689. https://doi.org/10.1037/0022-0663.91.4.684

Höffler, T. N. (2010). Spatial ability: Its influence on learning with visualizations—A meta-analytic review. *Educational Psychology Review, 22*(3), 245–269. https://doi.org/10.1007/s10648-010-9126-7

Hurst, M., & Cordes, S. (2016). Rational-number comparison across notation: Fractions, decimals, and whole numbers. *Journal of Experimental Psychology: Human Perception and Performance, 42*(2), 281–293. https://doi.org/10.1037/xhp0000140

Jitendra, A. K., Nelson, G., Pulles, S. M., Kiss, A. J., & Houseworth, J. (2016). Is mathematical representation of problems an evidence-based strategy for students with mathematical difficulties? *Exceptional Children, 83*(1), 8–25. https://doi.org/10.1177/0014402915625062

Joram, E. (2003). Benchmarks as tools for developing measurement sense. In D. H. Clements & G. Bright (Eds.), *Learning and teaching measurement: 2003 yearbook* (pp. 57–67). Reston, VA: National Council of Teachers of Mathematics.

Kingsdorf, S., & Krawec, J. (2016). A broad look at the literature on math word problem-solving interventions for third graders. *Cogent Education, 3*(1), 1–12. https://doi.org/10.1080/2331186X.2015.1135770

Kozhevnikov, M., Motes, M. A., & Hegarty, M. (2007). Spatial visualization in physics problem solving. *Cognitive Science, 31*, 549–579. https://doi.org/10.1080/15326900701399897

Kozhevnikov, M., & Thorton, R. (2006). Real-time data display, spatial visualization ability, and learning force and motion concepts. *Journal of Science Education and Technology, 15*(1), 111–132. https://doi.org/10.1007/s10956-006-0361-0

Krajewski, K., & Schneider, W. (2009). Exploring the impact of phonological awareness, visual-spatial working memory, and preschool quantity-number competencies on mathematics achievement in elementary school: Findings from a 3-year longitudinal study. *Journal of Experimental Child Psychology, 103*(4), 516–531. https://doi.org/10.1016/j.jecp.2009.03.009

Krawec, J. L. (2012). Problem representation and mathematical problem solving of students of varying math ability. *Journal of Learning Disabilities, 4*(2), 1–13. https://doi.org/10.1177/0022219412436976

Laski, E. V., Casey, B. M., Qingyi, Y., Dulaney, A., Heyman, M., & Dearing, E. (2013). Spatial skills as a predictor of first grade girls' use of higher-level arithmetic strategies. *Learning and Individual Differences, 23*, 123–130. https://doi.org/10.1016/j.lindif.2012.08.001

LeFevre, J., Fast, L., Skwarchuk, S., Smith-Chant, B. L., Bisanz, J., Kamawar, D., & Penner-Wilger, M. (2010). Pathways to mathematics: Longitudinal predictors of performance. *Child Development, 81*, 1753–1757. https://doi.org/10.1111/j.1467-8624.2010.01508.x

LeFevre, J., Jimenez Lira, C., Sowinski, C., Cankaya, O., Kamawar, D., & Skwarchuk, S. L. (2013). Charting the role of the number line in mathematical development. *Frontiers in Psychology, 4*(641), 1–9. https://doi.org/10.3389/fpsyg.2013.00641

Lemaire, P., & Siegler, R. S. (1995). Four aspects of strategic change: Contributions to children's learning of multiplication. *Journal of Experimental Psychology: General, 124*(1), 83–97. https://doi.org/10.1037/0096-3445.124.1.83

Levine, S. C., Foley, A., Lourenco, S., Ehrlich, S., & Ratliff, K. (2016). Sex differences in spatial cognition: Advancing the conversation. *WIREs Cognitive Science, 7*, 127–155. https://doi.org/10.1002/wcs.1380

Levine, S. C., Huttenlocher, J., Taylor, A., & Langrock, A. (1999). Early sex differences in spatial skill. *Developmental Psychology, 35*, 940–949. https://doi.org/10.1037/0012-1649.35.4.940

Li, Y., & Geary, D. C. (2013). Developmental gains in visuospatial memory predict gains in mathematics achievement. *PLoS One, 8*(7), e70160. https://doi.org/10.1371/journal.pone.0070160

Li, Y., & Geary, D. C. (2017). Children's visuospatial memory predicts mathematics achievement through early adolescence. *PLoS One, 12*(2), e0172046. https://doi.org/10.1371/journal.pone.0172046

Mix, K. S. (2010). Spatial tools for mathematical thought. In K. S. Mix, L. B. Smith, & M. Gasser (Eds.), *The spatial foundations of language and cognition*. New York: Oxford University Press.

Mix, K. S., & Cheng, U. L. (2012). The relation between space and math: Developmental and educational implications. In J. B. Benson (Ed.), *Advances in child development and behavior* (Vol. 42, pp. 199–243). Waltham, MA: Academic Press.

Mix, K. S., Levine, S. C., Cheng, Y., Young, C., Hambrick, D. Z., … Konstantopolous, S. (2016). Separate but correlated: The latent structure of space and mathematics across development. *Journal of Experimental Psychology: General, 145*(9), 1206–1227.

Miyake, A., Friedman, N. P., Rettinger, D. A., Shah, P., & Hegarty, M. (2001). How are visuospatial working memory, executive functioning, and spatial abilities related? A latent-variable analysis. *Journal of Experimental Psychology: General, 130*(4), 621–640. https://doi.org/10.1037/0096-3445.130.4.621

Moss, J., Bruce, C. D., Caswell, B., Flynn, T., & Hawes, Z. (2016). *Taking shape: Activities to develop geometric and spatial thinking grades k-2*. Toronto, ON, Canada: Pearson, Canada.

Nath, S., & Szücs, D. (2014). Construction play and cognitive skills associated with the development of mathematical abilities in 7-year-old children. *Learning and Instruction, 32*, 73–80. https://doi.org/10.1016/j.learninstruc.2014.01.006

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2011). *Common Core State Standards: Preparing America's Students for College and Career*. Retrieved from http://www.corestandards.org/.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*, 1227–1238. https://doi.org/10.1037/a0018170

Shen, C., Vasilyeva, M., & Laski, E. V. (2016). Here, but not there: Cross-national variability of gender effects in arithmetic. *Journal of Experimental Child Psychology, 146*, 50–65. https://doi.org/10.1016/j.jecp.2016.01.016

Shaw, J., & Pucket-Cliatt, M. (1989). Developing measurement sense. In P. Trafton & A. Schule (Eds.), *New directions for elementary school mathematics: 1989 yearbook* (pp. 149–155). Reston, VA: National Council of Teachers of Mathematics.

Siegler, R. S. (1987). The perils of averaging data over strategies: An example from children's addition. *Journal of Experimental Psychology: General, 116*(3), 250–264. https://doi.org/10.1037/0096-3445.116.3.250

Siegler, R. S. (2005). Children's learning. *American Psychologist, 60*(8), 769–778. https://doi.org/10.1037/0003-066X.60.8.769

Siegler, R. S. (2007). Cognitive variability. *Developmental Science, 10*(1), 104–109. https://doi.org/10.1111/j.1467-7687.2007.00571.x

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., … Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Siegler, R. S., & Shrager, J. (1984). A model of strategy choice. In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 229–293). Hillsdale, NJ: Erlbaum.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*, 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

TERC. (2008). *Investigations in number, data, and space* (2nd ed.). Boston, MA: Pearson Education.

Tzuriel, D., & Egozi, G. (2010). Gender differences in spatial ability of young children: The effects of training and processing strategies. *Child Development, 81*(5), 1417–1430. https://doi.org/10.1111/j.1467-8624.2010.01482.x

Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2005). Simple addition strategies in a first-grade class with multiple strategies. *Cognition and Instruction, 23*(1), 1–21. https://doi.org/10.1207/s1532690xci2301_1

University of Chicago School Mathematics Project. (2007). *Everyday math* (3rd ed.). Boston, MA: McGraw-Hill Education.

Vandenberg, S. G., & Kuse, A. R. (1978). Mental rotations, a group test of three-dimensional spatial visualization. *Perceptual and Motor Skills, 47*(2), 599–604.

van Garderen, M. (2006). Spatial visualization, visual imagery, and mathematical problem solving of students with varying abilities. *Journal of Learning Disabilities, 39*, 496–506. https://doi.org/10.1177/00222194060390060201

Van Helsing (2007, August 7). *Bicylinder Steinmetz solid.gif*. Retrieved from https://commons.wikimedia.org/wiki/File:Bicylinder_Steinmetz_solid.gif

Vasilyeva, M., Laski, E., & Shen, C. (2015). Computational fluency and strategy choice predict individual and cross-national differences in complex arithmetic. *Developmental Psychology, 51*(1), 1489–1500. https://doi.org/10.1037/dev0000045

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2014). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. *Child Development, 85*(3), 1062–1076. https://doi.org/10.1111/cdev.12165

Voyer, D., & Hou, J. (2006). Type of items and the magnitude of gender differences on the Mental Rotations Test. *Canadian Journal of Experimental Psychology, 60*(2), 91–100. https://doi.org/10.1037/cjep2006010

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*(4), 817–835. https://doi.org/10.1037/a0016127