Chapter 2
From Intuitive Spatial Measurement to Understanding of Units

Eliza L. Congdon, Marina Vasilyeva, Kelly S. Mix, and Susan C. Levine

Abstract The current chapter outlines children's transition from an intuitive understanding of spatial extent in infancy and toddlerhood to a more formal understanding of measurement units in school settings. In doing so, the chapter reveals that children's early competence in intuitive spatial thinking does not translate directly into success with standardized measurement units without appropriate scaffolding and support. Findings from cognitive science and education research are integrated to identify (a) the nature of children's difficulties with measurement units, (b) some effective instructional techniques involving spatial visualization, and (c) suggestions for how instruction could be further modified to address children's specific conceptual difficulties with standardized measurement units. The chapter ends by suggesting that the most effective instruction may be that which directly harnesses the power of children's early intuitive reasoning as those children navigate the transition into a deeper conceptual understanding of standardized units of measure.

Keywords Mathematical development · Spatial thinking · Spatial visualization · Units · Linear units of measure · Ruler measurement understanding · Spatial extent · Area · Angle · Misconceptions · Manipulatives · Gestures · Children · Infants · Cognitive development · Education · Instruction · Mathematics learning · Procedural understanding of measurement · Conceptual understanding of measurement

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The current chapter focuses on children's ability to understand and visualize spatial units of measurement, a foundational concept in mathematics. As stated by Gal'perin and Georgiev (1969), "Mastery of the initial concept of the unit is the most important step in the formation of elementary arithmetic concepts (they are all built on the unit or presuppose it)." In reviewing existing findings, we consider evidence of early measurement competence and evidence of later measurement struggles, and work to integrate and reconcile these seemingly disparate findings. We also outline some successful instructional techniques that have come out of basic cognitive science research. In doing so, we suggest that units of measure, an inherently spatial concept, are a fitting case study for understanding how children's learning outcomes are improved when spatial visualization techniques are employed during instruction.

Units are powerful because they allow us to meaningfully discretize continuous quantities, thereby allowing for extremely accurate comparisons across space and time. But the concept of "units" has important implications beyond this—it is also foundational to humans' understanding of quantity and numeracy more generally (e.g., Davydov, 1975; Gal'perin & Georgiev, 1969; Sophian, 2007). For example, when counting a set of shoes, one could count each shoe, or one could count each pair of shoes as one unit. Just as when measuring length, the numerosity obtained depends on the unit one adopts. Children under the age of 4 struggle with this idea. They tend to count any discrete physical object as a unit, even if the object is actually part of a larger unit (Brooks, Pogue, & Barner, 2011; Shipley & Shepperson, 1990). For example, when asked, "How many forks?", they might count one whole fork and one fork broken into two pieces as two forks, and respond that there are three forks altogether. At around age four, they instead respond that there are two forks, which is consistent with the way adults tend to answer this question. This suggests that with age, learners become increasingly sensitive to the unit-based information represented by nouns. In fact, when parts of objects have readily accessible names (e.g., wheels), children are able to focus on these part-of-object units at an earlier age than if the parts do not have labels (Shipley & Shepperson, 1990). These biases to attend to and count discrete physical entities that are readily labeled ultimately help children count different kinds of units (Shipley & Shepperson, 1990). This is critical since later in development, having an understanding that units are flexible and depend on the question one is addressing, becomes the backbone of children's understanding of topics such as place value, measurement, geometry, part-whole relations, and fractions (Piaget, Inhelder, & Szeminska, 1960).

Despite the importance of units in the ontogenetic development of mathematical thinking, there are well-documented challenges children face in understanding units of measure and how they are applied in problem-solving scenarios. Jean Piaget, a master observer of children's behavior, claimed that children were not capable of reasoning accurately about distance, length, or angle measure until middle childhood (Piaget et al., 1960). For example, children up until about 7 years of age were likely to fail a conservation of length test, stating that if one of two equal sticks was shifted with respect to the other, it had become "longer." In another classic experiment, Piaget showed children ages 3–7 years a tower of blocks and then asked them to construct a tower of equal height with smaller blocks on the other side of the room. Children did not create a metric of conversion (e.g., "two large blocks are equal to 4 smaller blocks") nor did they spontaneously use available resources to aid in transitive inference (wooden sticks and strips of paper) until at least 7 years of age. Similarly, children who were asked to replicate a drawing of an angle figure tended to approximate the drawing and did not spontaneously measure with available paper, string, or compasses until mid-to-late elementary school. These findings suggest that children may have a fundamental misunderstanding of the form and function of formal systems of measurement.

Even when children do receive instruction about the proper usage of measurement tools, they continue to demonstrate conceptual difficulties. Recent international assessments of children's mathematics performance indicate that children perform particularly poorly on measurement test items as compared to other mathematics assessment items at least through fourth grade (TIMSS, 2011), echoing similar patterns of findings reported decades ago (Carpenter et al., 1988; Lindquist & Kouba, 1989). Children also struggle with test items about angle measures through elementary school and even middle school (e.g., Clements & Battista, 1992; Mitchelmore & White, 2000).

In stark contrast to these dire assessments of children's understanding of formal spatial units, there is ample evidence that young children and infants are able to reason intuitively about continuous extent, length, and angle (e.g., Baillargeon, 1987; Lourenco & Huttenlocher, 2008; Slater, Mattock, Brown, & Bremner, 1991; Spelke, Lee, & Izard, 2010). For example, 2- to 4-month-old infants who are habituated to an angle figure will dishabituate to a change in angle measure (Cohen & Younger, 1984), and 5- to 7-month-old infants can encode an object's height and make subsequent predictions about its behavior even when the object is not visible (Baillargeon & Graber, 1987). What, then, can explain how children's intuitive understanding of spatial extent gets "lost in translation" when encountering similar concepts in formal schooling contexts? In the current chapter, we propose that a formal understanding of units requires children to overcome two challenges. First, they must integrate their intuitive understanding of continuous spatial extent with discrete, countable entities. In other words, one challenge of mastering units of measure is that they lie squarely at the intersection of intuitive spatial understanding and learned numerical representations. Secondly, children must connect intuitive, non-verbal understandings with the corresponding formal concepts that are referenced by newly acquired spatial language terms (e.g., units, angle, length, area, volume).

The present chapter reviews the literature related to these developmental achievements. In Part I, we review evidence that young infants have the perceptual capabilities to process and compare various dimensions of continuous extent. In Part II, we discuss how these perceptual abilities of infancy fail to directly translate to success with formal units of measure in school settings. In Part III, we end with some optimistic evidence from successful training interventions that help school-aged children to bridge the gap between intuitive understanding of extent and formal units of measure.

Part I: Intuitive Understanding of Extent

Before they are introduced to formal measurement and numerical systems, there is evidence to suggest that even infants can make judgments that reflect their sensitivity to continuous spatial extent, a developmental precursor of measurement skills. In a series of violation of expectation paradigms investigating infants' ability to reason about an occluded object, 5- to 7-month-old infants can encode an object's height and make subsequent predictions about its behavior when the object is not visible (Baillargeon, 1987; Baillargeon & Graber, 1987). For example, 7-month olds expected a rotating screen to stop sooner when a taller object was placed behind the screen than when a shorter object was placed behind the screen. In a separate study, 5-month olds were surprised when a taller rabbit's path of movement behind a barrier did not show the rabbit's head poking above the barrier (Baillargeon & Graber, 1987). Extensions of these findings show that infants as young as 5.5 months can simultaneously track the width of one object—in this case, a cylinder—and the displacement distance of a second object, a small bug toy, to reason appropriately about collision events (Kotovsky & Baillargeon, 1998). Further, 6.5-month-old infants can use proportional information about objects that are partially resting on a surface to predict when the object has sufficient support and when it will fall (Baillargeon, Needham, & DeVos, 1992).

In addition to this research evidencing infants' qualitative judgments about height, width, and distance, there has been research suggesting that young children can reason quantitatively about extent. That is, some researchers have proposed that infants may be able to encode and reason about the absolute size of objects. In one study, 6-month olds were habituated to a glass cylinder with a certain amount of red liquid (Gao, Levine, & Huttenlocher, 2000). At test, infants dishabituated to the same size cylinder with a novel amount of liquid, but not to the same size cylinder with the same amount of liquid. In an experiment where objects were hidden in a long, narrow rectangular sandbox, children as young as 5-months old were surprised when the object was revealed in a location 6 in. from where it was initially hidden (Newcombe, Huttenlocher, & Leamonth, 2000; Newcombe, Sluzenski, & Huttenlocher, 2005). These findings could indicate that infants are capable of encoding approximate absolute extent without the explicit presence of a measurement standard or comparison object.

However, subsequent evidence has called this conclusion into question and has shown that this early reasoning about height and length may be based on intuitive proportional reasoning rather than a true understanding of absolute extent. In all of the work described above, target stimuli were presented within some sort of container (e.g., sandbox, glass cylinder), next to another comparison object, or in relation to the salient frame of a computer screen. Because of this, the absolute height of stimuli (e.g., more liquid in a cylinder) was conflated with the relative proportion the stimulus occupied within a container or relative to a frame (e.g., the liquid fills a larger proportion of the cylinder).

Evidence for encoding of relative extent. To disentangle the question of whether infants encode absolute or relative spatial extent, several experiments were conducted. In one study, infants were habituated to a wooden dowel in one of three conditions: alone, within a glass cylinder, or next to a wooden stick (Huttenlocher, Duffy, & Levine, 2002). Infants only dishabituated to a novel dowel when the dowels during habituation and test were presented either inside the glass container or next to the wooden stick. The most parsimonious explanation of these data is that infants were using the container or the stick to encode relative height. Yet it remained a possibility that the mere presence of a second object or container heightened infants' awareness of absolute extent of the original object. To directly address this possibility, a second study directly compared infants' sensitivity to absolute versus relative extent in the presence of a container (Duffy, Huttenlocher, Levine, & Duffy, 2005). In this work, 6.5-month-old infants were habituated to a wooden dowel that was a specific height, say three inches, and filled a certain proportion, say three-fourths, of the clear cylinder in which it was placed. In the key test conditions, infants were shown a larger cylinder with a wooden dowel that either filled the same proportion (e.g., three-fourths), or was the exact same absolute height as the original dowel (e.g., three inches). Infants dishabituated to the latter display—the same size object as in the original display with a different proportional relation to the container. These findings indicate that infants were encoding the height of the dowel relative to its container, and not its absolute height.

Using a different experimental technique where 2- to 4-year-old children were asked to remember the height of a target object and then select the matching object in a two-option test trial, these investigators found that it was not until 4 years of age that children were able to accurately encode the height of the target objects. Even then, they were only able to make the correct selection at test in the presence of a salient comparison standard and a distractor that was substantially different in size from the target (Huttenlocher et al., 2002). By 8 years of age, the ability to focus on absolute extent was more refined and children could differentiate lengths that were closer in size, perhaps by imposing a mental unit, such as a mental inch (Vasilyeva, Duffy, & Huttenlocher, 2007).

Continued development of proportional reasoning. If young children are indeed encoding relative and not absolute extent as a kind of proportional reasoning, to do so still requires an impressive set of reasoning skills and an emerging ability to unitize, with resulting improvements in precision. In the sandbox search paradigm mentioned above, work with toddlers has shown that by the age of 24 months, most children can remember the location of a hidden object long enough to go retrieve it from the sandbox (Huttenlocher, Newcombe, & Sandberg, 1994). The patterns of children's errors were biased toward the center of the sandbox, suggesting a rudimentary unitizing of the continuous space into two equal parts (Huttenlocher et al., 1994). With increasing age, children's errors cluster around smaller division points (e.g., by dividing the space into quarters). This intuitive unitization of continuous extent sharpens over developmental time and has been hypothesized to represent a Bayesian combination of categorical and continuous information, and may be a necessary precursor to children's understanding of later unit-based concepts (Mix, Levine, & Newcombe, 2016; Newcombe, Levine, & Mix, 2015). Similar patterns have been identified in number line estimation tasks, where children's improvement over developmental time can be explained by improvements in proportional reasoning, rather than the previously proposed qualitative shift from logarithmic to linear representations of the number line (e.g., Barth & Paladino, 2011).

As children get older, their increasing ability to represent and reason about proportional relations of continuous quantities predicts success on more demanding spatial reasoning tasks, such as map reading (Huttenlocher, Newcombe, & Vasilyeva, 1999) and symbolic fraction tasks (Möhring, Frick, Newcombe, & Levine, 2015). Though toddlers struggle with map reading tasks and tend to rely on matching of object features, by the age of 4, they are able to read a simple map that indicates the location of a hidden object. Researchers have argued that success on this task relies, in part, on the same skills young infants use when coding relative extent on simpler object comparison tasks (Duffy, Huttenlocher, & Levine, 2005; Huttenlocher et al., 2002; Vasilyeva & Lourenco, 2012). In this case, the scaling of distance from the map to the object search space is akin to encoding the relative extent between, say, a dowel and its container, and being able to identify this same proportional relation in a test trial. This emerging map reading ability in children, while impressive, remains quite fragile through many more years of developmental change. Children struggle with reading maps when the referent space is misaligned or shifted in orientation (Liben & Downs, 1993), when the scale of the referent space becomes too large (Davis & Uttal, 2007), or when the space becomes too complex or includes distracting but salient landmark features (Liben & Yekel, 1996).

Understanding of angles. Just as with judgments of length, young infants show an intuitive understanding of angle well before they learn about angles in formal school settings. For example, 2- to 4-month-old infants who are habituated to an angle figure composed of two line segments will dishabituate to a change in angle, or a change in the relative position of the lines composing the angle figure, but do not dishabituate to a change in orientation of the entire figure (Cohen & Younger, 1984). In addition, infants differentiate between acute and obtuse angles (Cohen & Younger, 1984; Lourenco & Huttenlocher, 2008), and even newborn infants are capable of tracking the relation between two components or features of an angle figure (Slater, Mattock, Brown, Burnham, & Young, 1991). This work suggests that the perceptual skills needed to encode angles are present very early in development, and indeed, may be innate (Izard, O'Donnell, & Spelke, 2014).

Children approaching kindergarten age begin to make explicit decisions and judgments based on these early percepts of angle. For example, 4-year-old children can accurately identify which of six figures drawn on a card looks different from the others when the key dimension of difference is angular measure (Izard & Spelke, 2009). And though performance is more variable, some 4-year-olds can match fragments of geometric figures from two-dimensional to three-dimensional space when the only informative dimension available in the fragments is angle measure (Izard et al., 2014).

Navigation tasks, which require children to use angle on a 3-D scale, follow a more protracted developmental trajectory than 2-D tasks. For example, 4-year-old children are much more likely to spontaneously use distance cues to succeed on a map task than they are to use angle or orientation (Shusterman, Ah Lee, & Spelke, 2008). However, by the age of 5 or 6, children can successfully use angular relations in map reading and navigation tasks (Spelke, Gilmore, & McCarthy, 2001).

Similar to what has been found for other forms of measurement, the early sensitivity to angle does not confer immediate success in understanding more formal systems of angular measure, which children struggle with until much later in development. Indeed, as we will discuss, they are often confused by irrelevant information—such as the absolute the lengths of the lines composing the angle figure—in making judgments about which of two angles is larger.

Interim conclusions. These and other findings show that before children are exposed to any explicit training in formal systems for linear measurement or angle measure, they are sensitive to continuous spatial extent. This sensitivity is largely confined to reasoning about relative or proportional rather than absolute quantities, but there is some indication that successive divisions of continuous space—which can be regarded as nascent measurement units—might help young children gain greater precision through the first 5 years of development. Taken together, this work suggests that the foundations for reasoning about spatial units emerge quite early.

Part II: Transition to Understanding of Conventional Measurement

Infants' and young children's intuitive reasoning and perceptual sensitivity to differences in length, distance, and rotational measure are necessary but not sufficient for success on unit-based tasks in formal school settings. In this section, we begin by reiterating the key concepts children must learn to navigate the transition from intuitive reasoning about continuous extent to more formal reasoning that involves the application of discrete units to gain precision about these extents. We then briefly discuss how educators assist children in this transition by looking at common teaching techniques for unit-based measurement topics in mathematics. Finally, we identify some potential shortcomings in current instructional practices, and in doing so, strive to characterize some of the common misconceptions children develop regarding unit-based tasks.

Key concepts for children to master. One reason that measurement may prove difficult for young children is that it requires them to integrate their preexisting imprecise intuitions about quantity and continuous extent with conventional, number-based measurement tools such as rulers. When using a simple tool like the ruler, children must understand a set of not-so-simple conventional rules such as what to count, where to start (and stop) counting, that the beginning of the ruler is the zero-point, and the significance of the hatch marks and numbers on the ruler (Solomon, Vasilyeva, Huttenlocher, & Levine, 2015). Simultaneously, children must master several key ideas about spatial units more generally—they are consistent in size within a given measurement instance; a single unit can be iterated to determine length; and units follow an inverse relation rule. That is, as the unit size increases, the number of units needed decreases.

Beyond learning the conventions of measurement tools and units, a true conceptual understanding of measurement requires that children can make transitive inferences (e.g., if the length of A = B and B = C then A = C). More concretely, children must understand that to compare the length of two objects, one can measure the first object and then the second, providing a way to compare the sizes of two objects even when the measurements are separated by time, physical distance, or both (Sophian, 2007). This kind of transitive inference is not intuitive, and there is evidence that children are not capable of this kind of thinking until at least 4 or 5 years of age (e.g., Bryant & Kopytynska, 1976; Miller, 1989; Piaget et al., 1960).

Cognitive biases can inhibit learning. There are several cognitive biases that may inhibit children's ability to master these important conventional rules about measurement. First, children have a tendency to attend to and count bounded objects (e.g., Sophian, 2007) or "countable entities" (Shipley & Shepperson, 1990). Yet the units on conventional measurement tools are spatial intervals, which are, in a sense, "non-objects." The numbers and hatch-marks on rulers serve as countable distractors, obscuring the link between ruler markings and the spatial interval units they represent. Indeed, children who fail on unaligned ruler measurement problems (see Fig. 2.1a) succeed on measurement problems where the to-be-measured object is unaligned with respect to a set of discrete, adjacent circles, which are more readily countable objects than spatial interval units on a ruler (see Fig. 2.1b) (Solomon et al., 2015). On unaligned measurement problems with rulers, young children tend to make one of two kinds of errors (Lehrer, Jenkins, & Osana, 1998; Solomon et al., 2015). They either read off the number on the ruler that aligns with the rightmost edge of the object (i.e., read-off error) irrespective of where the object begins, or count the hatch marks rather than the intervals of space that fall between an object's left-most and right-most edges (i.e., hatch-mark counting error). This later strategy likely reflects an object counting bias because children are drawn to count the object (i.e., lines or nubers) rather than the spaces.

In addition to their documented bias to count objects, children also show a bias to estimate continuous quantities based on perceptual spatial cues alone even when a salient, helpful discrete cue is present. For example, in one experiment by Huntley-Fenner (2001), preschool children were presented with two boxes. One box had three clear glasses full of sand, and one had two. When asked which box had more glasses, children could easily say that the box with three glasses full of sand had more glasses than one with two glasses. In this first case, participants were asked to make a judgment about discrete quantities using discrete units. But when asked which of the two boxes had more *sand*, children's performance dropped significantly and was no better than when they were asked to compare piles of sand consisting of

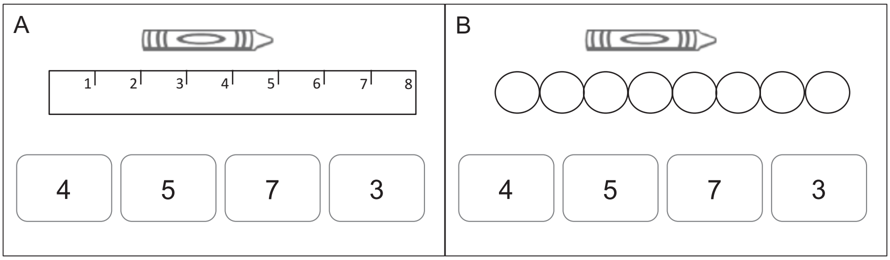


Fig. 2.1 Sample measurement items: (**a**) unaligned ruler with inch units and (**b**) circles representing units. Question: "How many units long is the crayon?" The correct answer is 3. Common answers for the ruler item are 4 and 5, for the circles item the most common answer is 3

these same amounts. In this second case, children were asked to make a judgment about a continuous dimension using a discrete unit—the glasses. Thus, although a discrete unit was readily available, children did not spontaneously use this option, and performance reflected the noisy guesses one would expect from a task asking children to compare continuous quantities based on approximate perceptual estimation. Together, these cognitive biases likely interfere with children's ability to grasp the function of spatial units, how they can be used, and how they are incorporated into conventional tools such as the ruler.

Traditional classroom instruction. Children in American schools are typically given two different types of measurement instruction to help them understand and visualize spatial units. In the first, they are provided with unconventional units (e.g., paperclips, shoes, coins) and asked to measure an object or distance by lining up the units, end to end. While the goal of such an exercise is ostensibly to teach children about the importance of utilizing same-size spatial interval units—a key measurement concept—there is research suggesting that children do not spontaneously make the link between objects and spatial intervals. Children may see such an activity as a game in which the goal is to count objects, not to measure. Indeed, children often leave gaps between objects, overlap objects during an iterative procedure (Bragg & Outhred, 2004; Lehrer, 2003) or select units of differing, non-standard sizes to line up along an object's edge (Lehrer et al., 1998). Such errors indicate that children do not understand a fundamental aspect of measurement—that it requires the use of adjacent equal-size units. Moreover, even if they execute the measurement correctly, they may not grasp that the objects used represent underlying spatial extents and may instead view the exercise as an object counting task (Solomon et al., 2015).

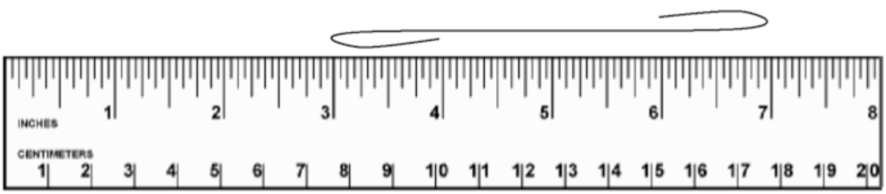
A second common classroom activity is to ask children to measure objects with a ruler by aligning the object with the leftmost edge (zero-point) of the ruler and reading off the number at the rightmost edge of the object (Smith, Males, Dietiker, Lee, & Mosier, 2013). Such a procedure is effective when children perform it properly. Yet there is evidence that this type of instruction leaves children with a relatively shallow, procedural understanding of measurement. The measurement performance of children in early elementary school is particularly poor on test items where objects to be measured are not aligned with the "0" point on the ruler, such as the problem depicted in Fig. 2.1a above (Clements, 2003; Lehrer et al., 1998; Solomon et al., 2015; Wilson & Rowland, 1993) or when they are a given a "broken" ruler that does not begin at 0, but rather at some other non-0 starting point (Nunes & Bryant, 1996). Importantly, both the read-off and hatch-mark counting strategies, described above, consistently result in correct answers when the object to-be-measured is properly aligned at the 0-point of the ruler. In other words, classroom instruction that does not challenge children with difficult, shifted-object problems may allow misconceptions to go unnoticed by educators.

Different instructional needs. Performance on these difficult shifted ruler problems raises questions about whether children may require different types of instruction depending on their specific misconceptions. Children who use the read-off strategy tend to be younger or from lower socio-economic status backgrounds (Kwon, Levine, Ratliff, & Snyder, 2011; Solomon et al., 2015) and therefore may have less experience with measurement problems. They also tend to have lower spatial working memory than their peers who make hatch-mark counting errors, despite equal scores on a verbal working memory task (Congdon & Levine, 2017). Irrespective of the specific cause of their misconception, children who use the read-off strategy have no trouble perceiving that an object does not begin at the start of the ruler, but do not know how to adapt their strategy to account for this unusual arrangement when asked how long an object is. This suggests that there may be something about how these questions are typically asked (i.e., "How long is the X?") that leads children to assume they are being asked to determine the end-point of an object no matter where that object begins. Children of this age, around 5 years old, are also likely to say that two walking paths, one straight and one with a large bend in it are the same length if they have the same starting and ending points (Clements, 1999).

By contrast, children who count hatch marks are aware that determining length involves counting something, but appear to be distracted by an object-counting bias that draws their attention to the lines rather than the spaces. These children may have a firmer understanding of the pragmatics of the problem, but have not yet mastered the ways in which rulers represent discrete spatial units. In other words, these children do not understand the relation between a single unit and the whole object that is being measured, and how that relation is represented by the ruler (Lehrer, 2003). Overall, neither the read-off nor the hatch-mark counting strategy indicates an understanding of the concept of a unit as a measure of that involves uniform spatial intervals (Kamii, 2006; Martin & Strutchens, 2000).

International performance. Alarmingly, children in the United States score lower on test items assessing measurement skills than on items assessing most other mathematical topics (Carpenter et al., 1988; Clements & Bright, 2003; Lindquist & Kouba, 1989; Mullis, Martin, Gonzalez, & Chrostowski, 2004; National Center for Educational Statistics, 2009). Specifically, when given a multiple choice test item akin to the one shown in Fig. 2.2, only 20% of US fourth grade students answered correctly, a rate that is lower than chance and that was significantly lower than the international average (TIMSS, 2011). These struggles could potentially be due to limitations in current classroom instructional practices in the United States. Though there is little published research comparing specific instructional methods in measurement across countries, there is some work suggesting that in the countries where children are generally more successful on measurement test items, like Japan (where 52% of fourth grade children answered that same problem correctly), children are given more opportunities to engage in creative problem-solving and critical thinking than they are in the US (Kawanaka, Stigler, & Hiebert, 1999). These children spend less time practicing memorized procedures, and more time discussing and exploring ideas with the teacher. Though a causal link cannot be drawn between general cultural differences in teaching practices and differences in understanding of units, the parallels are suggestive of the idea that children in the US would benefit from deeper engagement in exploring the conceptual underpinnings of measurement.

Higher order measurement skills. Even if children master the basic procedures of linear measurement, they continue to struggle with unit-related concepts later into childhood. For example, many children find it difficult to understand the inverse relation of units—that you will need more units to measure something if the units are smaller, and fewer units if they are larger (Hiebert, 1984). First grade children overwhelmingly rely on the number of units in a task, and will attempt to keep that number constant when re-measuring an object, even if the experimenter has changed the unit size. Hiebert argued these inverse relations are difficult for young learners because understanding them requires both (1) an understanding of conservation (i.e., the idea that an object does not change length even when moved in space or measured a second time), and (2) an understanding of transitivity (i.e., the idea that two objects can be compared with a standardized measurement tool).



If the string above was pulled straight, which of the following would be closest to its length?

a) 4 inches
b) 6 inches
c) 7 inches
d) 8 inches

Fig. 2.2 A test item similar to one included in the 2011 version of the Trends in International Mathematics Fourth Grade Assessment. The correct answer is b

Hiebert claimed that children who have not yet mastered these two ideas tend to fall back on counting strategies to compare object lengths, irrespective of changes in unit size. Some research suggests that when task complexity is decreased, children as young as 7 years can succeed with unit conversion (Sophian, Garyantes, & Chang, 1997), but other work suggests that children continue to struggle as late as fourth grade, or 9–10 years of age (Vasilyeva, Casey, Dearing, & Ganley, 2009). For example, when given the following scenario, "It took Marc 8 steps to cross the room and it took Peter 5 steps. Who has the longer step?," Fourth grade students tended to respond incorrectly that Marc had larger steps.

Another complex idea involved in a mature understanding of measurement is unit conversion. Even when given a short lesson to demonstrate that 5 centimeters is about the same as 2 in., children up through 8 years old make mistakes in judging the relative lengths of two objects that have been measured in different standard units (Nunes & Bryant, 1996). For example, they may state that a 3-cm stick is longer than a 2-in. stick, suggesting that they rely primarily on the number of units than on the number *and* size of units. Lastly, even after children understand the importance of standardized units, they continue to struggle with selecting the appropriate units for certain measuring tasks (Tipps, Johnson, & Kennedy, 2011). For example, first grade children might not know whether it is more appropriate to measure a computer screen with inches or feet. Currently, the Common Core State Standards do not explicitly suggest introducing these more advanced unit-related concepts until second or third grade, perhaps explaining why children show protracted understanding of these mathematical ideas (e.g., Clements, 1999; Tipps et al., 2011).

Area measurement. While children's understanding of linear measurement is important on its own terms, understanding units and how they represent equal parts of space also lays the groundwork for later understanding of mathematical concepts such as perimeter, area, and fractions. In linear measurement, there is only one dimension on which to compare two objects. In area measurement, there are two relevant dimensions, making this a more difficult concept. Research shows that children consistently struggle on area measurement problems through at least fifth grade, frequently confusing perimeter and area, for example (Lin & Tsai, 2003; Strutchens, Martin, & Kenney, 2003). Without a thorough understanding of units, children tend to fall back on visual comparison techniques that may have worked when comparing the linear extent of two objects, but are more difficult to apply successfully when comparing the area of two differently shaped objects (e.g., Yuzawa, Bart, & Yuzawa, 2000). Additionally, there is evidence that children rely heavily on memorized formulas to calculate the area of shapes without developing a conceptual understanding of why this procedure works (Barrantes & Blanco, 2006; Strutchens, Harris, & Martin, 2001).

Proportions and fractions. Despite the fact that infants and young children have an intuitive understanding of proportion, understanding conventional fractions is notoriously difficult. Fractions, unlike whole numbers, require children to keep track of the relative magnitude of two different sets of units—the denominator representing the number of partitions of a whole unit and the numerator, the number of these units. It also requires that children understand the non-intuitive inverse relation between the size of the denominator and the magnitude of the fraction, a skill that is fundamentally linked to the idea of units of measure and is parallel to understanding the inverse relation between the size of the unit and the number of units in measurement (Sophian, 2007; Sophian et al., 1997).

Even with simple fractions, such as one-half, 4-to-7-year-old children who show an intuitive understanding of one half of a continuous quantity have extreme difficulty when the quantity is discretized into units (Hunting & Sharpley, 1988). For example, they may be able to bisect a cookie roughly in half, but struggle to decide what constitutes half of a set of 12 blocks. Similarly, 6- to 10-year-olds succeed on proportional equivalence tasks when continuous quantities are used but not when these same quantities are divided into countable units. In this latter condition, instead of reasoning proportionally, children tend to rely on counting the number of "shaded" units, akin to attending only to the numerator of a fraction, or they count the total number of units (e.g., Boyer, Levine, & Huttenlocher, 2008).

These findings suggest that access to approximate proportional magnitudes is not sufficient to learn how to map the number words of fractions to their proper unit referents. With more complex fractions, older children will commonly apply a label like "three-fifths" to an image with, say, three shaded parts and five unshaded parts, rather than three shaded parts and two unshaded parts (e.g., Mix & Paik, 2008; Newcombe et al., 2015). Even middle school and high school students will try to add two fractions by simply adding both numerators and both denominators (Kerslake, 1986). These errors indicate that children have a fundamental misunderstanding about how the denominator of a fraction delineates unit size while the numerator indicates the number of units.

Angular measurement. Children's difficulties with angle measurement share some parallels with their misconceptions about linear measurement (Clements & Battista, 1992; Mitchelmore & White, 2000). For example, children must master the ideas of equal partitioning of space, and must understand unit iteration (Clements & Stephan, 2004). There is also evidence that young learners have difficulty understanding the proper referent of the word *angle* (Gibson, Congdon, & Levine, 2015). Because of a quirk of the English language, the word *angle* can actually be used to refer to both the figure of an angle, composed of two rays extending from the same point, and to the measure of rotation between the rays (Clements & Stephan, 2004). This linguistic ambiguity likely contributes to longstanding misconceptions for children in elementary and even middle school who will focus on irrelevant properties such as the length of an angle's sides in a figure, the area contained within the sides, or the absolute distance between the sides when making judgments about the size of angles (see Fig. 2.3; Clements & Battista, 1989; Lindquist & Kouba, 1989).

In school settings, angles are not typically introduced until second or third grade. Before that point, curricula tend to avoid proper spatial labels, instead calling angles "corners." In addition, there is some evidence from case study observations that protractors (i.e., tools for measuring angular rotation), are challenging for children to understand and may be imbuing them with a sense of angle as a static measure rather than allowing them to imagine angles as a dynamic measure of rotation

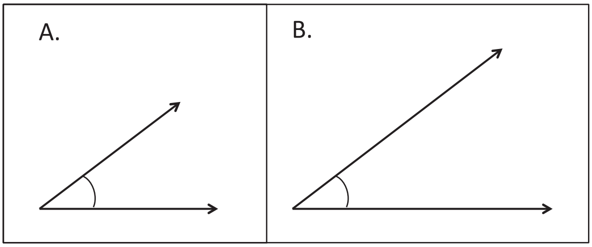


Fig. 2.3 These two angle measures are equivalent, but as late as middle school, children will assume B is a larger angle due to overall size, line length, or distance between the rays

(Clements & Burns, 2000). Given a paucity of research on the topic, it is currently unclear whether confusing use of the word "angle," late exposure, conventional tools, or a combination of the three are to blame for children's long-term misconceptions about angles. What is clear is that similar to other kinds of measurement, children struggle to make the transition from intuitive, perceptual reasoning about angles to a more formal understanding of angular rotation and angle size.

Interim conclusions. Taken together, this rich literature on children's difficulties with measurement reveals a few consistent patterns. First, the transition from reasoning non-verbally about continuous spatial extent to understanding and visualizing discrete units of spatial extent is challenging for children across many subdomains of measurement including linear measurement, angle, and higher-level skills like area and fraction understanding. The specific challenges include learning the proper referents of newly acquired spatial language (e.g., "length/long" does not always mean end-point; "units" on a ruler are not hatch marks or numbers; and "angle size" refers to a measure of rotation rather than the length of the lines that comprise the angle). Second, children must learn to use conventional unit-based tools and understand how those tools allow for transitive inference. Third, children must understand units themselves, which are a way to integrate intuitive understanding of continuous properties with exact numerical representations. Lastly, some of this work suggests that current instructional practices may be overemphasizing rote procedures or improper or ambiguous use of spatial language that could be leaving learners with poor conceptual understanding and thus, a shaky foundation for later mathematics success.

Part III: Training Interventions

It is clear that children face many challenges when making the transition from an intuitive understanding of continuous extent to a formal understanding of unit measures. In the final section of the chapter, we review interventions designed by researchers to scaffold children's learning in the domains of linear measurement, area measurement, and angle understanding. Our aim is to showcase proven instructional techniques, while further clarifying the nature of children's difficulties.

Improving spatial visualization of linear measurement units. To date, the majority of research on linear measurement has documented the nature of children's difficulties and misconceptions. Only a small number of studies have focused on correcting those misconceptions and helping children visualize how discrete units can comprise continuous lengths. One research group did an in-depth case study with eight students who were given a number of different measurement activities and who were continuously assessed across nearly a full year (Barrett et al., 2012). Based on their findings, the authors proposed several instructional tasks that could move children from one conceptual stage of measurement understanding to the next. For example, having children draw their own rulers, having an instructor overlap units to get children to think about why that is problematic, or having an instructor explicitly teach about how to deal with fractions of a unit. At the highest level, the authors argue that learning about intervals as countable units was not sufficient to promote a full conceptual understanding of measurement, so they proposed a lesson to link the ruler, hatch marks, spaces, and numbers all at once. This work, while certainly valuable, used many instructional strategies at once, and did so over a long period of time, making it difficult to ascertain which specific features of the instruction might have driven children's improvement.

In a more recent study, researchers tested whether exposure to and training on measurement test items with objects shifted away from the start of the ruler (unaligned problems) might be beneficial to learning (Kwon, Ping, Congdon, & Levine, under revision). The children completed a brief training lesson with either unaligned ruler problems or more traditional aligned ruler problems, with the object starting at the 0-point on the ruler. The results showed that exposure to unaligned ruler problems during training was crucial for learning. The authors argued that the unaligned ruler problem training was powerful because it provided children with self-discovered evidence that disconfirmed their previous strategies, a technique that can lead to better learning outcomes (e.g., Ramscar, Dye, Popick, & O'Donnell-McCarthy, 2011; Rescorla & Wagner, 1972). For example, if a child who used the hatch-mark counting strategy initially believed an object to be five units long, they would generate a guess of 5, then count the spaces and quickly discover that they were only at the number 4 when reaching the end of the object (Kwon et al., under revision).

A second study used a similar procedure to test the relative efficacy of different ways of drawing attention to a spatial interval as a unit of measure. One group of children was given practice on shifted-object measurement problems with discrete plastic unit chips, and a second group was given the same instruction but was taught to use a thumb-and-forefinger "pinching" gesture instead of the unit chips (Congdon, Kwon, & Levine, 2018). Results showed that children who started the session by counting hatch-marks improved markedly after either type of instruction, whereas children who began the session with the read-off strategy improved much more after unit-chip instruction than gesture-based instruction. These findings suggest that even within a single age group and single domain, children at a lower level of conceptual understanding may need more concrete scaffolding to promote learning. Notably, children who used the read-off strategy and received unit chip training occasionally switched their strategy to a hatch-mark counting strategy after training.

This strategy shift suggests that the training helps by causing children to reevaluate their understanding of the referent of "unit"—a process that occasionally goes awry due to an object counting bias.

Teaching area measurement. A solid understanding of linear measurement can help children when they encounter more difficult problems, such as measuring the area of a two-dimensional figure. As discussed in Part II, a true understanding of area measurement requires children to coordinate multiple dimensions and to understand, conceptually, how formulas for calculating area represent two-dimensional space. In one training study, researchers tested what type of instruction best promoted this understanding (Huang & Witz, 2011). They taught three groups of fourth grade students. One group received practice with applying formulas (i.e., procedural instruction). Another group focused on the properties and features of 2-D geometric shapes and how those features conceptually related to surface area (i.e., conceptual instruction). The third group received both types of instruction simultaneously in an integrated lesson. The results revealed that children who received both types of instruction made better decisions about and more accurately explained challenging area calculation problems than children who received either procedural and conceptual instruction in isolation. The findings echo those of linear measurement training studies, and suggest that optimal interventions for unit-based tasks should target both procedural and conceptual understanding (e.g., Congdon, Kwon, & Levine, 2018; Kwon et al., under revision).

Spatial visualization of angular measurement. Another unit-based concept that is not typically introduced until later in school is that of angle measurement comparison. In one recent study, researchers tested whether children's word-learning biases might explain children's well documented misconceptions about angles (Gibson et al., 2015). The study focused on preschool aged participants who had not yet been introduced to angles in formal school settings. All children were taught about angles, but half of the children were given a second nonsense word to represent the angle figures (i.e., "toma"), while the control group heard the word "angle" used as it is in traditional instruction, ambiguously referring to both the angle figure and the measure of the angle. Children in the experimental condition improved significantly more than the control group after training. The finding was driven by improvement on trials in which the larger overall angle figure was not the figure with the larger angle measure (Fig. 2.4, panel c). These results suggested that children's early misconceptions about angle may stem, in part, from their propensity to apply novel labels to an entire object rather than a feature of that object (e.g., Hollich, Golinkoff, & Hirsh-Pasek, 2007; Landau, Smith, & Jones, 1988; Markman & Hutchinson, 1984). Only when given a label for the angle figure did children then search for another referent of their newly acquired spatial vocabulary. This study also offered some convincing evidence that children as young as 4 years old are capable of successfully learning about angles—a much younger age than is traditionally targeted for this type of lesson.

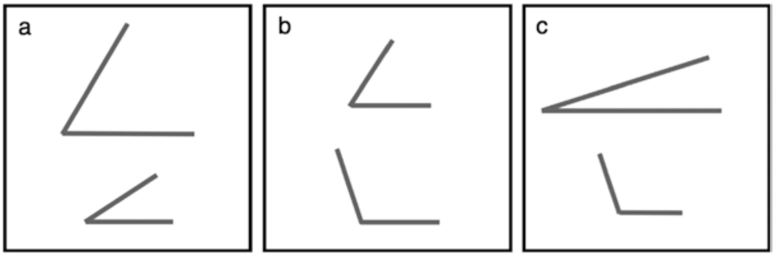


Fig. 2.4 Sample test trials from Gibson et al. (2015). Panel c was the type of trial that was most difficult for all children at pre-test, with children incorrectly selecting the larger figure significantly more than chance. After training, the experimental group selected the correct answer (i.e., the larger angle) at rates significantly above chance

In the 1990s, there was some promising research with older children using a computer programming language, Logo, which was adapted to help children to learn simultaneously about angle and linear measurement. In this platform, learners could direct a small, computerized turtle to turn a certain number of degrees left or right and move certain distances forward to accomplish simple goals (e.g., "go around the pond to get to the house" or "draw a rectangle"). Researchers argued that such a game accomplished two goals. First, it required children to apply numerical values to their perceptual intuitions, and second, it revealed the dynamic nature of mathematics, by, for example, emphasizing that degrees of an angle are really about rotation and the length of a side is about the distance it transverses (Clements & McMillen, 1996). Indeed, after playing with a game like Logo, children in middle school and high school age had more accurate, precise ideas about mathematical concepts like shapes, length and angle than those who followed more traditional instructional methods (Clements & Battista, 1989; Clements & Battista, 1992).

Conclusion

Despite young children's initial successes perceiving and processing continuous spatial properties, understanding how units represent those properties is a difficult transition, rife with the misconceptions. In this chapter, we have argued that to successfully make this leap, children must integrate continuous spatial properties with discrete representations of exact number, and they must identify the proper referents of newly acquired key spatial terms, including unit, angle, length, area, perimeter. Only in doing so can they begin to master higher-order unit-based concepts like transitivity, conservation of length/area/rotation, and the inverse relations between number of units and unit size. It may be helpful for researchers and educators interested in improving children's learning outcomes to be aware of the potential pitfalls children face as a result of their cognitive biases.

Studies aimed at teaching children the role and function of units have revealed several effective techniques. First, by exposing children to difficult exemplars of unit-based problems, such as shifted-object or "broken ruler" linear measurement problems, we can help children avoid applying a memorized procedure, and challenge them instead to reevaluate their preexisting strategies through self-discovered disconfirming evidence about their intuitive strategies. Second, we can make the referents of ambiguous spatial language more transparent by becoming aware of children's misconceptions and then explicitly pointing out through words and actions what spatial language does and does not mean. Third, hands-on, dynamic practice counting units, particularly in the presence of a conventional measuring tool, may help children use their intuitive reasoning about continuous properties to visualize and interpret discrete units in a structured way. Finally, a more radical suggestion is to augment existing mathematics curricula in a way that helps children establish a stronger foundation in proportional reasoning and relative comparisons of magnitude well before transitioning to numerical unit-based instruction. Such a modification could take advantage of children's natural propensity to reason about the intensive (e.g., proportional, comparative) properties of measurement problems before they are asked to master formal systems of extensive measurement (e.g., absolute extent, units). It is an open question as to whether this approach could ease the ultimate transition to formal systems of measurement. In the meantime, it seems that activities that link the intensive and extensive properties of measurement by using representations of units to help to concretize abstract labels and spatial properties of extent are maximally beneficial for improving student learning outcomes.

The lessons learned in this domain of mathematics, measurement, can likely be applied to many other areas. In this chapter we reiterate that the goal of a modern education is not for children to memorize tricks and procedures, but rather to develop a deep conceptual understanding of general principles, irrespective of the specific domain. We use units of measure as an example to outline some of the ways in which findings from cognitive science and psychology may assist in this goal by exploring the cognitive underpinnings of mathematical understanding in infants and young children, explaining the mechanisms that underlie some of the errors and misconceptions children face in formal schooling, and helping to promote crucial development beyond procedural knowledge to deeper conceptual understanding.

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