Chapter 15
Part II Commentary 1: Mathematics Educators' Perspectives on Spatial Visualization and Mathematical Reasoning

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Researchers generally agree that spatial problem solving skills involve the ability to generate mental images as a strategy for solving mathematics problems, often in conjunction with maintaining and manipulating those images. Further, translating these mental images into physical representations/graphics through drawings or diagrams is advantageous for many mathematics problems. The chapters by Sinclair, Moss, Hawes, and Stephenson (this volume) and Lowrie and Logan (this volume), point out Polya's (1965) recommendation to "draw a diagram" as one of the first steps in understanding a mathematics problem. Students who use this heuristic may be more successful on a wide range of problems across mathematics content areas. Ho and Lowrie (2014) report that Singapore students are taught to use the model method, which is a visual problem-solving heuristic prevalently used in Singapore classrooms, and Murata (2008) reports on the use of the tape diagram approach as visual-spatial tool used to solve many types of mathematics problems in Japanese classrooms—both countries that score highly on standardized testing.

Generating Diagrams for Solving Mathematics Word Problems

One beneficial effect of applying spatial reasoning to mathematics problems is the ability to utilize spatial imagery to solve problems under circumstances that do not obviously require their use for problem solving. Thus, this spatial representation approach may be particularly beneficial when no graphic is available for children to depend upon. One clear example of this is the application of spatial skills to mathematics word problems. In recent reviews, researchers have investigated the benefits

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of using spatially based schematic representations to solve word problems and have found that it can be quite effective (see review by Kingsdorf & Krawec, 2016; Jitendra, Nelson, Pulles, Kiss, & Houseworth, 2016). Typically, this approach involves the use of diagrams to represent the mathematics problem. It also often incorporates the representation of connections between the different problem parts in order to link the different steps in the problem-solving process (Gonsalves & Krawec, 2014). Hegarty, Mayer, and Monk (1995) proposed that there is a spatial component to word problems when students construct a mental-model of a problem and plan the solution based on that model. Hegarty and Kozhevnikov (1999) found evidence that sixth grade students who used schematic spatial representations such as diagrams had better mathematical problem solving success than students using other approaches. Use of schematic representations was also shown to significantly correlate with spatial skills.

Boonen and associates (Boonen, van der Schoot, van Wesel, DeVries, & Jolles, 2013) found that a substantial proportion of the association between spatial skills and numerical word problem solving (21%) was explained through the indirect effects of strategies involving visual-schematic representations. For numerically-based mathematics reasoning problems, spatial reasoning may facilitate the ability to translate complex verbal and number problems into an appropriate spatial array or diagram representing the problem solution. Thus, research suggests that spatial problem solving can be a useful tool when solving mathematics problems unrelated to either geometry and measurement, and even under conditions in which no decoding of graphics is required.

As the research on use of spatial representations progresses, Lowrie and Logan (this volume) point out an important consideration: Do educators introduce spatial representations and problem-solving approaches through heuristic models such as the Singapore approach—involving teaching practices where spatial heuristics are explicitly taught and practiced involving "draw a diagram"—or do they use an approach in which students are exposed to a diverse variety of mathematics representations and are encouraged to use their own personal strategies to solve these tasks, as is more typical of Western educational systems? Lowrie and colleagues (Lowrie, Logan, & Ramful, 2016) compared sixth grade students from Singapore to students from Australia in terms of their use of spatial and non-spatial problem solving approaches to numerical word problems. These researchers found that "… the Singapore students are able to use these foundational approaches and skills in quite flexible ways. Consequently, the restricted development of problem solving strategies actually enhances their capacity to solve unfamiliar tasks…It may be the case that too much variety in strategy development and not enough explicit teaching does not equip Australian students with a sufficient skill set when faced with unfamiliar or challenging tasks. The demands of the Australian school system place great importance on an inquiry approach. However, to maximize student learning potential, intentional teaching still needs to take place, as is the case in Singapore." (p. 107). These instructional issues will have to be addressed as we move to greater integrations of spatial approaches to solving mathematics problems across the curriculum.

Early Introduction of Spatial Reasoning Approaches to Arithmetic Problems

We have shown strong longitudinal support for a spatial-numerical association in a recent study in which we examined spatial skills in first grade girls as predictors of two types of mathematics reasoning skills 4 years later in fifth grade (Casey et al., 2015). The results showed that spatial skills, assessed as early as first grade functioned as key long-term predictors for numeric/algebraic mathematics reasoning skills in fifth grade, as well as for geometry/measurement mathematics-reasoning skills (even when controlling for early verbal skills and arithmetic accuracy). In a follow-up study on the same students, we found a strong pathway leading from spatial skills at the outset of first grade to use of advanced decomposition strategies by the end of first grade, and then leading to higher level numeric and algebraic mathematics reasoning skills in fifth grade (Casey, Lombardi, Pollock, Fineman, & Pezaris, 2017). Though correlational, this pattern of associations suggests the possibility that levels of spatial reasoning may impact arithmetic strategy choices at the outset of arithmetic learning, which in turn may have long-term effects on later mathematics reasoning.

A recent study by Frick (2018) further reinforces the importance of emphasizing spatial approaches to arithmetic instruction starting at early ages. Using structural equation modeling, Frick found that mental rotation and spatial scaling in kindergarten showed their strongest relation to the component of the mathematics test tapping arithmetic operations in second grade, whereas mental transformations and cross-sectioning were more strongly related to geometry and magnitude estimation. Thus, a future goal of spatial-mathematics research should be to examine in greater depths how different types of spatial skills impact different types of mathematics skills when applying spatial reasoning strategies to mathematics problems.

Spatial Skills as Predictors of Geometry/Measurement Versus Numerical/Algebraic Mathematics

To further argue for greater emphasis on the importance of spatial skills extending to a wider range of mathematics content, I would like to present data from a recent study that made it possible to directly examine spatial skills—both as predictors of geometry and measurement reasoning problems that involved the use of graphics—and as predictors of numerical/algebraic problems in which no graphics were provided. We examined spatial skills, consisting of the Vandenberg Mental Rotation task (Peters et al., 1995) and the Water Levels Task (Piaget & Inhelder, 1956) at the beginning of seventh grade as predictors of two types of mathematics reasoning skills at the end of seventh grade. The mathematics assessment tools were designed to maximize the number of geometry and measurement items that addressed spatial mathematics reasoning and the number of numerical and algebra items that addressed analytical reasoning. We conducted regression analyses to determine the extent to which spatial skills predicted these two types of mathematics items. The specific goal was to examine the strength of these associations on the two types of mathematics problems—one type that would seem to maximize the association with spatial skills, while the other type might be expected to be less likely associated with spatial skills.

For the geometry/measurement items with graphics, the standardized coefficient for the composite spatial measure was 0.53. For the numeric/algebra problems with no graphics, the standardized coefficient for the composite spatial measure was 0.51. Thus, the spatial skills-mathematics associations for both types of mathematics problems were very similar—and substantial. Next, we controlled for students' mathematics fact fluency and verbal skills, because these skills might account for substantially greater variance in predicting numeric/algebra performance than for geometry/measurement. When these additional measures were included in the regression analyses, spatial skill still significantly contributed to both the geometry/measurement items (standardized coefficient = 0.42) and the numerical/algebraic items (standardized coefficient = 0.33). Although the standardized coefficient for spatial skills as a predictor dropped more for the numeric/algebraic items than for the geometry/measurement, the association between spatial skills and numeric-based mathematics performance were still the strongest predictors in the regression analyses for both types of mathematics items.

In conclusion, in my commentary I have made the argument that more research should be done by mathematics educators to identify specific strategies for teaching students how to approach a much wider range of mathematics content areas utilizing their spatial reasoning processes. Findings from many research studies suggest a greater potential role for applying spatial problem solving approaches across mathematics content than is typically applied in practice within schools in US and other Western countries (Mix & Cheng, 2012). Now our task is to conduct intervention research in order to figure out explicit ways of helping teachers incorporate spatial thinking successfully throughout these mathematics content areas.

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