

Chapter 11

Learning Through and from Drawing in Early Years Geometry



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Abstract This chapter focuses on the relations between spatial reasoning, drawing and mathematics learning. Based on the strong link that has been found in educational psychology between children's finished drawings and their mathematical achievement, and the central importance of diagramming in mathematics thinking and learning, we wanted to study children's actual drawing process in order to gain insight into how the movements of their hands and eyes can play a role in perceiving, creating, and interpreting geometric shapes and patterns. We pay particular attention to the interplay between children's drawings and their gestures, to the role of language in modulating children's perceptions, and to the back and forth that drawing seems to invite between two-dimensional and three-dimensional perceptions of geometric figures. We seek to forge new ways of including drawing as part of the teaching and learning of geometry and offer new ways of thinking about and analyzing the types of spatial/geometric reasoning young children are capable of.

Keywords Diagram · Gesture · Geometry · Verbal · Visual · Triangle · Square · Symmetry · Congruence · Dimensional deconstruction · Pointing · Tracing · Structure · Transformation · Array · Quick draw · Segment · Diagonal

From its earliest roots in Ancient Greece, drawing has been a significant practice of geometers. The drawings of geometers, whether in the sand, on papyrus, or on paper, have involved the use of tools that produce one-dimensional marks on two-dimensional surfaces. With the widespread use of textbooks and worksheets, drawings that used to be made by hand are now offered to students ready-made,

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thereby reducing the amount of drawing they do themselves. However, during the past decade, we have seen a renewed interest in promoting drawing and diagramming in school mathematics, both amongst educational psychologists and mathematics educators. This “return to drawing” is likely due in part to an increased awareness of and research on the importance of spatial reasoning and, more broadly, the role of embodied cognition in mathematics thinking and learning (Mix & Cheng, 2012). Indeed, a growing body of research points to sensorimotor activities, such as moving, gesturing and drawing, as being fundamental spatial processes in the learning and communication of mathematics (e.g., see Lakoff & Núñez, 2000).

In this chapter, we turn to children’s drawings—both the act and artefact—as a means of eliciting and working towards a better understanding of children’s geometric and spatial reasoning. More specifically, we examine how children can learn *through* and *from* drawing, with a particular focus on spatial processes and key concepts of primary school geometry. In examining the complexity of the drawing process, we aim to show how drawing is not a static or innate skill, but one that can be worked on and improved through the intermediaries of language, gestures and spatial visualisation. Broadly speaking, we aim to shed light on the mathematical learning and instructional opportunities afforded through the act of geometric drawing.

We begin our study by first providing a brief review of research on children’s drawings. Working within the theme of this book, we draw sharp parallels in the ways in which the disciplines of psychology and mathematics education (and mathematics more generally) have traditionally made children’s drawings a focus of their respective work. Our own work attempts to build on the longer tradition of research on drawing in educational psychology but in a way that is more sensitive to the disciplinary and pedagogical values of geometry education. In this way, we see our study as being mutually informed by research from psychology and mathematics education, and also as informing both disciplines going forward.

What we Know About Drawing: A Review of the Literature from Psychology and Mathematics Education

Early psychological studies of children’s drawings focused on stages of development and viewed drawings as markers of cognitive maturation. For example, one of the first papers on the subject was published by Ebenezer Cooke in 1885. He offered a qualitative analysis of children’s drawings in terms of age-related stages of representation. Beginning in the early twentieth century, researchers became increasingly interested in how children’s drawings were related to developmental change, and also, more specifically, how differences in children’s drawings were related to associated differences in general intelligence (Ivanoff, 1909; Kamphaus & Pleiss, 1992). Indeed, early research efforts indicated correlations between objective measures of children’s drawings and intelligence (Goodenough, 1926a; Ivanoff, 1909). Interestingly, it was during this time that the longstanding tradition began—and continues to this day—of assigning quantitative scores to children’s drawings and using these scores as metrics and correlates of intellectual functioning (Claparede, 1907; Ivanoff, 1909).

A major impetus in this movement was Goodenough's publication of *Measurement of Intelligence by Drawings* in 1926, where he introduced the Goodenough Draw-A-Man test, an assessment of children's intelligence based on how well they were able to represent and draw the human figure. Spanning from its development in 1926 and continuing to the late 1950s, the Draw-A-Man Test was consistently one of the most popular tests used by clinical psychologists in the United States (Sundberg, 1961). Although the popularity of this test waned thereafter, the central ideas about what children's drawings can tell us about their general cognitive development remained strong. For example, Piaget drew on this earlier work and incorporated the developmental stages of drawing into his own developmental framework. Children's drawings were taken as evidence of developmental shifts in core cognitive competencies and aligned with the belief that children's drawings follow a consistent, universal and sequential progression (Kellogg, 1970). Accordingly, children were only able to progress to a more complex representational stage once earlier, more basic representations were mastered.

According to this view, children's drawing behaviour unfolds naturally and the role of adults and education in this process is minimal to non-existent (e.g., see Brooks, 2009). Moreover, this view conceives of children's drawings as stable indicators of general cognitive development, or, said differently, artefacts or outcomes of general cognition. In the words of Goodenough (1926b), "the nature of the drawings made by children in their early years is conditioned by their intellectual development" (pp. 185). The psychological literature is bereft of studies looking at how the very act of drawing may provide an essential vehicle for cognitive development and learning. Given how spatial reasoning improves through practice, the act of drawing could also improve one's performance, both in drawing and spatial reasoning, especially if it was adequately supported, which would challenge the idea of using drawing as an indicator of intelligence.

The scientific study of children's drawings has changed over the past few decades and there is increasing acknowledgement of drawing as a driver of cognition rather than a mere indicator or outcome of cognition. More recently, researchers in the psychological sciences have begun to reveal more nuanced connections between children's drawings and domain-specific knowledge and performance (Brooks, 2009; Malanchini et al., 2016). Researchers have revealed especially strong connections between children's drawing skills and mathematics performance (e.g., see Carlson, Rowe, & Curby, 2013). For example, children's abilities to accurately draw human figures at 4½ years of age have been found to significantly correlate with teacher ratings of these same children's mathematics skills (i.e., numbers, geometry, measurement, data, and applied problems) at the age of 12.

The ability to draw a human figure has been theorised to implicate a number of underlying mathematical concepts (Case & Okamoto, 1996; Malanchini et al., 2016). For example, awareness of number of body parts, proportional reasoning, appropriate use of space (e.g., depth cues), and symmetry are all mathematically relevant features inherent in the drawing of human figures. In fact, fundamental to drawing most anything is the need to consider spatial relations within and between

objects. This is true of self-generated images, but is also, as we will see below, involved in the act of copying a static image or design. The close relations between drawing and mathematics might thus be explained by the geometrical and spatial reasoning involved in the drawing process.

Copying an image requires the drawer to attend to the geometric and spatial relations present. Researchers have consistently found correlations between children's abilities to copy simple geometric designs and mathematics performance (e.g., see Carlson, Rowe, & Curby, 2013; Grissmer, Grimm, Aiyer, Murrah, & Steele, 2010; Kulp, 1999). For example, Fig. 11.1 shows an example of the types of items from the Visual Motor Integration test (VMI). This assessment involves having children copy simple geometric designs and children are assigned a score based on accuracy. Children's performance on this assessment has been shown to be a reliable predictor of both concurrent and future mathematics achievement (Kulp, 1999; Kurdek & Sinclair, 2001; Pieters, Desoete, Roeyers, Vanderswalmen, & Van Waelvelde, 2012; Sortor & Kulp, 2003). For example, research by Pieters et al. (2012) found that 7- to 9-year-olds' performance on the VMI explained a substantial proportion of variance in both number fact retrieval and procedural calculation. Furthermore, children with mathematics learning disabilities performed significantly worse on the measure compared to their peers.

The relation between children's drawings and mathematics is even more telling when we consider how children use drawings and other marks to both represent and understand mathematics problems. We see examples of this when children make discrete marks on a piece of paper in order to keep track of and represent the two addends of an addition problem. Children use drawings as a means to represent and bring meaning to fractions problems (e.g., shading in $1/3$ of a rectangular array). Drawings also assist in the comprehension and solutions to mathematical word problems. In fact, this is an area of study where psychologists have made significant headway in recent years. Researchers have found that children's representational drawings of mathematical word problems provide important insight into individual

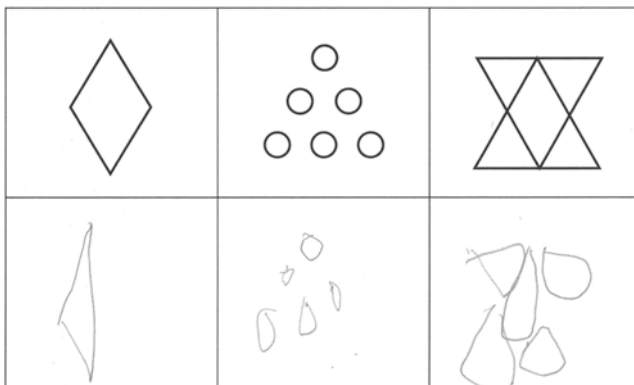


Fig. 11.1 Example of types of items and responses found on tests of Visual-Motor Integration (VMI; e.g., see Beery & Beery, 2010)

differences in solution accuracy (Boonen, van der Schoot, van Wesel, de Vries, & Jolles, 2013; Boonen, van Wesel, Jolles, & van der Schoot, 2014; Hegarty & Kozhevnikov, 1999). Children who focus their drawing efforts on detailing the visual-spatial relations of the word problem tend to outperform children who focus their drawings on the more pictorial aspects of the problem (e.g., including extraneous details rather than just the essential mathematical relations). This suggests that drawing is linked to mathematical abstraction.

From a mathematics education point of view, there has been a growing focus on drawing, and especially on the use and creation of diagrams. Perhaps because of the seminal nature of Polyà's (1957) work on problem solving (one of the heuristics in the first stage of "Understanding the Problem" is "Draw a diagram"), much of the literature has focussed on the diagrams that students create while solving problems (Bremigan, 2005; Diezmann & English, 2001; Nunokawa, 2006; Yancey, Thompson, & Yancey, 1989).

Increasingly, researchers have also realised that diagrams are not always transparent for learners, who might perceive and interpret them in different ways, especially diagrams that carry a significant amount of cultural encoding, such as the Cartesian coordinate system. For example, Steenpaß and Steinbring (2014) focus on students' subjective interpretations of mathematical diagrams, offering the distinction between object-oriented (where the focus is on the visible elements of the diagram) and system-oriented (where the focus is on the relation between the elements of the diagram) as two different ways that students may interpret diagrams.

Some mathematics education research has begun to attend to the interplay between diagramming and gestures in mathematical activity—an interest fuelled in part by recent theories of embodied cognition. Although most often studied separately, there is a natural relation between the two as they both involve actions with the hand. Indeed, using the work of the philosopher of mathematics Gilles Châtelet, who studied the pivotal role of diagramming in mathematical inventions, de Freitas and Sinclair (2012) examined the interplay of gesturing and diagramming in undergraduate students' drawings, highlighting the way in which these drawings can be seen as gestures in "mid-flight" and thus capturing on the page the mobile actions of the hand. Also with an attention to the interplay between gestures, diagrams, and speech, Chen and Herbst (2013) compared the interactions of two groups of high school students: one working with a diagram that contained relevant labels (for vertices and angles) and another working with a diagram that contained no labels. While the students in the first group only used pointing gestures, those in the other group made gestures that extended the existing diagram (by extending a segment, for example) and thus created new geometric elements. These students talked about hypothetical objects and properties, which led the authors to conclude that the "gestures played a crucial role in engaging students in reasoned conjecturing" (p. 303) because the hypothetical objects enabled the students to make and justify conjectures.

Theoretical Considerations

Given that we are working in the context of geometry, it is important to underscore the fact that geometry is a complex activity that does not just involve shapes or the use and creation of visual images, but instead centrally involves the interplay between seeing and saying; that is, between visualisation and the language for stating and deducing properties. Indeed, Duval (2005) argues that much of the school geometry curriculum encourages prototypical thinking, in which shapes are recognised based on their visual properties. The shape becomes an iconic representation. This approach fails to develop coordination between the visual and discursive registers of geometry. Duval proposes that children should engage in more construction tasks, which would better support the crucial process of *dimensional decomposition*. This process involves the passage from one dimension to the other, for example, from the two-dimensional square to the one-dimensional line segments that constitute the square. When going from seeing a shape to describing a shape, there is usually a reduction of dimension. Dimensional deconstruction involves both seeing a basic shape (such as a square) as being constructed from a network of lines and points and seeing that different shapes can emerge from that network of lines (such as two triangles).

While drawing can be a powerful way of engaging in dimensional deconstruction, since it involves the creation of lines and points that form two-dimensional shapes, we also follow Duval in stressing the importance of the coordination between the visual and the discursive, which we pursue in our tasks by asking children to describe what they see and how they would draw. To this verbal/visual interplay, we follow also recent work in embodied cognition in focussing on the gestural as an important means through which people think, communicate and invent. Gestures can act as effective replacements for speech, and can therefore participate in the interplay between the visual and the discursive. Gestures are also effective at communicating the spatial and temporal aspects of mathematical concepts (e.g., Núñez, 2003; Sinclair & Gol Tabaghi, 2010). Finally, drawing itself can be seen as a kind of manual gesture (Streeck, 2009), which can easily transform into a gesture “in the air” (sometimes called a drawing gesture, when people gesture as if they are holding a pen and drawing on paper—therefore not leaving a visible trace). Indeed, the temporal nature of drawing connects it strongly to the temporal nature of gesturing.

Methods

In this chapter, we report on a study in which we examined the interplay of gestures, diagrams and speech as young children were engaged in various drawing tasks. The tasks we chose were based on Wheatley’s Quick Draw Program. Accounts of this program in classrooms, and past research conducted on the effectiveness of the program (Hanlon, 2010; Tzuriel & Egozi, 2010; Weckbacher & Okamoto, 2015;

Wheatley, 1997; Yackel & Wheatley, 1990), have consistently shown that it encourages spatial thinking, improves students' recognition of and ability to name shapes and improves their general geometric vocabulary. Given their value in prior research, we wanted to build on these tasks and adapt them to more complex situations that would involve both gesturing and talking as well as drawing.

Participants and Setting

Eleven kindergarten students (ages 5.9–6.2 years) from a classroom of 22 children participated in the study. The children were attending a fee-paying laboratory school in Toronto that is open to all learners serving students from nursery to sixth grade. Seven percent of the population receive financial assistance and up to 15% of students receive some form of special academic assistance. A central mission of the school is to serve as a model for inquiry-based teaching and learning. The 11 students selected for the study were chosen by their classroom teacher (fourth author) because, as a group, they offered a range of mathematics ability.

The students in this kindergarten classroom had all been exposed to an enriched geometry curriculum involving composition and decomposition, shape, spatial transformations and symmetry along vertical, horizontal and diagonal lines. In addition, and directly relevant to this study, early in January all of the children in the class, as part of their regular mathematics program, had participated in two half-hour lessons of quick image drawings, which involved them in drawing, from memory, geometric shapes embedded in squares. Specifically, in each of these two lessons the teacher held up for 3 s an image of a geometric shape embedded in a square (e.g., a diamond (rhombus) with a horizontal line through the midpoint), and asked the children to “take a picture in their minds” and only once the image was removed from view, they were instructed to draw what they remembered of the image. To facilitate the process for the young children, each child had been provided with a response sheet with four squares, thereby allowing them to draw the image(s) directly into the square(s). The letters L and R were written on the outer edges of each square to indicate left and right sides respectively to help the children describe the images they created. In the first of the two lessons, the students did not receive any feedback on their drawings but were given a second quick look at the target image to make any changes they thought necessary.¹ In the second lesson, after the students had drawn their image they were encouraged to describe what they drew—e.g., how they got started, where they placed their pencils, which lines they drew first. Over the two lessons, the children were shown a total of 12 geometric drawings in squares, which were either part of, or adapted from Wheatley's (2007) most recent Quick Draw program. Full descriptions of the quick image drawing lessons can be found as part of a new spatial geometry curriculum (see Moss et al., 2016, pp. 126–129).

¹In the original Quick Images, students were then shown the image and asked to comment on it and their drawings. As stated below, this was done in the interviews as well.

Materials: The Interviews

Two different exploratory open-ended clinical interviews were designed for the present study: Every child in the study participated in one or the other of the two interviews. The design for the interviews was based both on the way the kindergarten children responded to the two short lessons taught to them earlier in the school year, and also on our observations over the years of the many teachers and some hundreds of students participating in *Can You Draw This* lessons. These lessons have been part of our Math for Young Children (M4YC) project, an ongoing spatial and geometry professional development program that began in 2011 (e.g. Hawes, Tepylo, & Moss, 2015)

Thus, for Interview 1 (see Fig. 11.2 for images) we designed an exploratory, open-ended protocol to probe the potential of *Can You Draw This* activities. While we continued to use images of geometric shapes embedded in squares, rather than presenting them as quick image, as outlined in the first of the two pilot lessons we describe above, we followed Duval's work in varying the activities to promote more visual/discursive synergies. For example, we described images for children to draw instead of presenting the image visually. We also did the reverse: asking the children to describe an image for the interviewer to draw; in addition we also had the children study an image and consider its symmetries, congruence, composition and structure prior to their drawing.

One of the things we noticed in the two pilot lessons conducted in January with all of the kindergarten children was the difficulty a number of students had in reproducing an isosceles triangle in the square. As part of the earlier lessons in the kindergarten class, we included two isosceles triangle challenges. In the first, the triangle was oriented upwards with the "point" touching the top edge of the square. In the second, the triangle was oriented towards the left with the point to the left edge of the square. In trying to reproduce the first of the two triangles, many children's drawings did not go into the corners of the square or touch the top lines. The drawings of a number of the children produced for the isosceles triangle oriented to the left often did not resemble any kind of triangle. We were intrigued by the challenges faced by many children in their attempt to reproduce the triangles. Thus, Interview 2 (see Fig. 11.3 for images) was designed with a specific focus on triangles. For example, the first task asked the students to comment on, and compare

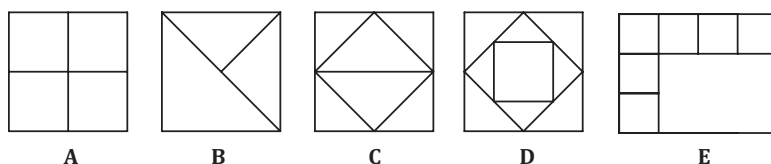


Fig. 11.2 Items used in Interview Protocol 1. For each item, children were asked the following (abbreviated) questions: (A) Can you draw this?; (B) Can you draw this from what I tell you?; (C) Can you describe how to draw it?; (D) Can you draw this?; (E) Can you complete the grid?

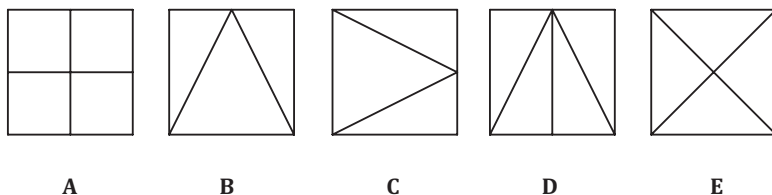


Fig. 11.3 Items used in Interview Protocol 2. For each item, children were asked the following (abbreviated) questions: (A) Can you draw this?; (B) Can you spot the difference between the two pictures?; (C) Can you draw this?; (D) Can you draw this?; (E) Can you draw this?

an incorrectly drawn triangle (incorporating typical errors by students) with the correct image (see Fig. 11.3B). Other questions required students to draw triangles in a different orientation or with a vertical line through the midpoint. Interview 2 was administered to two of the 11 students, both of whom had had significant trouble with the pilot lesson drawings.

Procedure

The interviews were conducted one-on-one in March 2017. The second, third, and fourth authors each conducted three or four interviews, each for about 30 min, in small rooms adjacent to children’s classrooms. Children were provided with pencils and a small booklet, each page containing a drawing of a single square. All interviews were videotaped using one camera to capture the drawing sequences and the gestures, which helped facilitate the classification of children’s responses. Administration followed a semi-structured individual interview protocol, with the task order and procedures remaining constant.

Analysis

This was an exploratory qualitative study and the analysis had a number of steps. To analyse the videotaped interviews each author looked at all of the interview videos and firstly attended closely not only to the actual drawings that the children were making (both the process of drawing and the final product), but also the language they were using (and being offered by the interviewer) and the gestures they made. We each highlighted particular segments of the videos in which we noticed shifts in the way that the children were seeing the drawings. Second, we each identified places in each video where different mathematical concepts arose. Some of these were planned in developing the interview protocol (symmetry, congruence) but others emerged from our attending to the unplanned aspects of the videos such as the back and forth between global and local features of the images. Once we had

generated a list of excerpts that were related to these two main categories, we met again to discuss the excerpt and select which ones to use for this chapter. In the next two sections, we therefore focus on the two aspects of drawings alluded to in our title, which are the ways that children learn *through* the process of drawing (and talking and gesturing) and the concepts they can learn *from* the drawings they have produced.

Figure 11.2 provides the images on which Interview Protocol 1 was based: 9 of the 11 children received this interview. In the first task, shown in Fig. 11.2A, the children were shown the image for a few seconds, then the image was taken away and the children were asked to draw it. For the second task, the interviewer described the image shown in Fig. 11.2B in a step-by-step manner and the children had to draw it accordingly. For example, the interviewer said “Start at the top left corner and draw a line segment to the bottom right corner. Then draw a line segment from the top right segment to the middle of the line segment you just drew.” For the third task, the children were shown the image in Fig. 11.2C and asked to describe how to draw it, as if to another person who could not see it. The image in Fig. 11.2D was shown to the children, who were asked questions about what they saw, about symmetry and about congruence. Then the image was taken away and the children were asked to draw it. Finally, in Fig. 11.2E, the children were given the unfinished grid and asked to complete it. Although this instruction could be interpreted in many ways, all of the children assumed they had to produce an array.

All the children succeeded in tasks 1, 2 and 3. Eight of the nine children who were asked to draw Fig. 11.2D did so correctly, though one child drew the diamond and then was prompted to continue with the interviewer who asked “do you remember how many squares you said you saw?” For the final task, five of the children completed the grid using lines and the four others completed it using squares or parts of squares.

Two children were given interview protocol 2, with the tasks based on the images shown in Fig. 11.3.

Learning Through Drawing: Focus on Verbal, Gestural and Visual Interplay

Across all the interviews (for both interview protocols 1 and 2), the interviewers offered and probed for spatial language, introducing words such as middle, top, side, bottom and diagonal. The children all made extensive use of gestures, though this was not explicitly requested by the interviewers. In this section, we are interested in how students used gestures and speech, and how this might help us better understand what they are perceiving and how they are visualising when they draw.

In this short episode (3:29–4:15), Neva was asked to describe how to draw Fig. 11.2C. She went around with her pencil. Carol asked her to, “Tell me where you would start?” She began by placing her pencil around the middle of the bottom side. Following the interview protocol, the interviewer intervened before Neva continued and asked for precision, thus eliciting the words “bottom” and “middle” from Neva.

Carol: So where is that?

Neva: On the bottom.

Carol: Anywhere on the bottom?

Neva: In the middle.

Carol: Bottom middle. And then where are you going to go? Tell me with your words before you do it.

Neva then traced out a segment with her right thumb before responding to Carol:

Neva	Neva	Neva
<i>Puts thumb on bottom middle</i>	<i>Traces segment with thumb</i>	“here”
		

Again, Carol elicited more words, this time “diagonal” and tried to get Neva to describe how she would draw her segment.

Carol: What kind of line will that be?

Neva: A diagonal (*starts to draw, from bottom middle*).

Carol: Where are you going to stop?

Neva: Right here (*places RH thumb on midpoint of side and lifts pencil*).







Carol: How would you describe that point?

Neva: Right at the R.²

Carol: R? Okay! Great! (*Neva finishes drawing the segment*). And now where would you go? What would you tell people to do next?

Carol seemed to have expected the use of words such as “middle” and “side” but accepted Neva’s description of “how would you describe that point?” as sufficiently precise. When Neva was asked to describe where she would go next, she responded with the deictic “right here” and put the eraser end of her pencil on the top line, but decidedly not in the middle.

²Neva says “R” because the letters R and L had been placed near the middle of the right and left sides of the square, respectively, to help provide orientation for the children.

Neva	Carol	Neva	Neva	Carol	Neva
Right here	Where's that?		At the top	Anywhere at the top?	No
					
Carol	Neva	Carol	Carol		Neva
Where at the top	Right here.	Okay. Let's see that.	And what kind of line did you just make?		Another diagonal.
					

We highlight this excerpt because it shows two places where the use of language, of gestures and of drawing was imbricated. In the first place, the gesture Neva used to trace out the desired segment is followed by the oral description, which suggests that the gesture enables Neva to describe the drawing, perhaps by actualising it on the paper. She only said “here” once she had completed tracing the segment with her thumb. In the second place, Carol’s question about “anywhere in the middle” prompted Neva to make a pointing gesture with her RH to the middle of the top side, whereas she had previously been pointing (both with the eraser end of her pencil and then with the pencil tip) towards the right side of the square. We hypothesise that although Neva didn’t say “middle”, Carol’s question oriented her to showing the middle with her finger, after which she drew the correct segment. The middle of the side had become an anchor point around which gesturing, talking and drawing occurred, even though there was no midpoint actually visible on the sheet of paper.

Without the gesturing (tracing, pointing) and the talking (using words introduced by Carol), Neva would have had a more difficult time describing what she was seeing in the original drawing and would likely have not drawn the diagonal segment correctly. We find this significant because it displaces drawing tasks from being uniquely about seeing, remembering and reproducing, and shows how gestures and language can change how children see and draw. In line with the findings of Hu, Ginns, and Bobis (2015), it would seem that the acts of tracing and pointing, particularly when it is close to the paper, can enhance learning in the context of geometry. Of interest in our study, unlike that of Hu et al., is that Neva pointed and traced spontaneously, without being asked to do so by the interviewers.

The second example also relates to the third task (Fig. 11.2C), but we focus on a slightly different phenomenon, which involves the interplay between seeing, drawing/gesturing and saying. John had already described the image as “a square that’s tilted with a line in the square” when Zack asked him to explain how to draw the image “step by step”.

Zack: Okay, where would I put my pencil first (*Zack places his pencil on the page and indicates he is ready to follow the instructions*)?

John: Start at the (*places pencil at the top middle*) mmmiddle (*turns to Zack*) of the top, middle of the page.

Zack: Okay.

John: At the top of the square, start there, and then draw a diagonal (*traces pencil along diagonal of Fig. 11.2C and places pencil on the middle of the left side*) line to the top (*looks down at his booklet, where the R and L legend can be seen*) left corner.

As with Neva, John traced the line first, before he announced where it would end. Though it was evident through gestures, John struggled to describe the location of the endpoint of the first diagonal, which should have been the left middle (and not the left corner), referring back to the booklet for help. His choice of the word “corner” is difficult to interpret, but it could be that in looking at the booklet, he saw that a kind of corner was formed at the left middle where two diagonals meet. Zack then asked for clarification.

Zack: Top left corner? Can you tell me a little more what you mean by the top left?

Cause I started at the top middle and then where does my pencil go?

John: To the top, to the bottom (*looks down*) to the middle (*eraser end down, traces from top middle to the left middle, then looks down*) left.

Zack: Okay, gotcha. Now what?



John: Then you draw (*eraser end on the left middle*) a line (*moves eraser end to top middle and retraces segment from top middle to left middle*) to the bottom of the (*looks at booklet and puts the eraser end on the bottom middle*) square.

Zack: Okay, where does the...

John: Middle

In this subsection of the transcript, John was struggling to describe the drawing of the second diagonal. Before he did so, he began by re-tracing (with the eraser end of the pencil) the first diagonal, as if the second diagonal was a continuation of the first one. This suggests that John was seeing the square in terms of four repeated actions so that the drawing of the first diagonal was a rehearsal for the drawing of the second one, a rehearsal that also included the verbal descriptions of top, left, bottom and middle. He might also have been seeing a continuous path consisting of a sequence of segments that needed to be drawn from the beginning. In either case, it would seem that John was thinking of the square as a whole, and not seeing four independent segments. Certainly, the opportunity to perform the drawing gesture seemed important to John’s way of seeing and describing the image. Across all the interviews, we saw this phenomenon repeatedly, where the children would gesture-draw a segment before describing it and/or before saying where the segment would end. We see in this phenomenon the dual nature of the perception of a segment, which can be seen as a single object to be apprehended all at once or as the process of moving from one point to another (in a straight line). The images can privilege the former, while the act of drawing privileges the latter.

A final, third example of children gesturing/drawing in response to a request to describe an image involved Mathias, on the fourth task of interview protocol 2 (Fig. 11.3D). The interviewer asked Mathias how he had drawn the image and he began by putting his pencil down on the table and placing his right index finger on the bottom left corner of the square, then tracing his finger along the diagonal line up to the top middle. Then the interviewer asked him, “How would you say this? How would you describe it [to your teacher]?”

Interviewer	Mathias	Interviewer	Mathias
The bottom left?		And then where did you go?	
	<i>Nods</i>		 <i>Takes pencil and traces it along diagonal</i>
Interviewer	Mathias	Interviewer	Mathias
You went up to the?	Top	Where at the top was it? Was it on the side?	In the middle top

Here again we see the act of tracing the drawing occurring first, with the child perhaps taking the question literally (what did you do?) but also perhaps using the gesture-drawing with his index finger to bring to focal attention the segment that he wants to talk about, the destination that the segment will arrive at and to movement between the two endpoints. Mathias thus makes the diagonal line three times, once by drawing it, once by tracing it with his index finger and a third time by gesture-tracing it with his pencil. It is only after the third time, with prodding of the interviewer, that Mathias describes the segment in terms of the location in the square, and where we see the beginning of the visual and language registers working together.

Learning Mathematics from Drawing

Instead of seeing the act of drawing as an end in itself, we want to highlight the various ways in which the kinds of drawing tasks that we used in the interview can give rise to significant mathematical ideas, especially in geometry, in which the actual act of drawing (instead of using given images) plays a pivotal role. We have already shown how the invitations to describe the images enabled the children to develop more geometric language, not so much in terms of the names of shapes, but in terms of position (on, in, middle), property (straight) and parts (side, corner, etc.). In this

section, we focus on particular mathematical concepts that were pursued through the different tasks. These are: congruence, symmetry and structure.

Congruence

The interviewers asked about congruence on several occasions (the language used was “are these the same?” or, when considering similar shapes, “which of these are the same size?”), never as a way to guide drawing, and always as a way to describe a given drawing. All but three of the children correctly identified shapes that were congruent as being “the same” (in all cases reflected shapes). We found this interesting especially in relation to some of the early fraction work that some teachers and researchers have done in which it is taken for granted that the different parts of a whole are of the same size, but children are clearly not always seeing it that way.

Sometimes, the interviewers asked the children how they might “prove” that two given shapes were the same, which the children did in a variety of ways involving both transformations and measurement. For example, Leo said, in reference of Fig. 11.2D, that you could cut the four outer triangles up and then “pile them up” one on top of the other (making a gesture as if holding a deck of cards that needs to be lined up). Christine also suggested cutting the four triangles in Fig. 11.2D up and making a pile. Neva suggested cutting the triangles up and putting them beside each other. Diana and Sara both suggested that you could fold the piece of paper to show that the triangles were the same. In terms of measurement, Elka referred to the fact that the two triangles in Fig. 11.3B were the same by showing that each of the corresponding sides were the same size. Maya used her fingers—using her finger width as a unit of measure—to count the lengths of the different sides for Fig. 11.2D. In relation to Fig. 11.2D, Diana moved her pencil along the sides of the diamond asserting that “all of the lines are the same” and then moved her pencil around the inner square asserting that “all of the lines are the same”. When asked how that helped her see that the smaller, inner triangles were the same, she again traced the sides of the inner square (which are the hypotenuses of the inner triangles) saying they were the same, and then traced the two other sides of one inner triangle, asserting that they were the same and then repeating for the other three triangles. When asked whether the two triangles making up the rectangle in Fig. 11.3D were the same Leo asserted they are because there is a line dividing them in half (gesturing a cutting action).

The use of transformations (cutting and flipping or rotating or piling up) focusses on the shapes as a whole, while the measuring strategies engage in dimensional deconstruction in that the children are attending to the lengths of the segments that make up the shapes. We hypothesise that the shifting of attention to the segments and their lengths arose out of the drawings that the children made, where they had to attend to the one-dimensional properties of the image more than the two-dimensional properties.

Symmetry

The children were frequently asked, after having drawn or described an image, to say whether the image was symmetrical. Symmetry had already been part of the children's classroom activities in which the children were challenged to both identify and create symmetrical images in a variety of ways over the course of 3 weeks, so it is perhaps not surprising that they were all able to identify at least one line of symmetry in the images they were shown. Each child used a gesture to indicate the lines of symmetry (for Fig. 11.2D, for example, first starting with a vertical one, then horizontal and, for some, also diagonal). While some children used their whole hand to indicate a line of symmetry, most used a drawing gesture to do so. For example, Maya used a drawing gesture (with her RH index on the page) (Fig. 11.4A) and said "if you put it down this way" and did so for each other line of symmetry. In another case using Fig. 11.2C, Sandro raised his hand in the air, which was holding a pencil, oriented the pencil so that it was pointing towards the top middle of the picture and moved it down by several centimetres (Fig. 11.4B), saying "I think so" in a very tentative manner. When the interviewer provided encouraging feedback, he shifted the pencil to the top middle of the square and moved it downwards—but not all the way, a similar short amount as he had done previously "in the air". Christine was asked to "use your pencil to show me where a line of symmetry might be" and placed the eraser end of her pencil on the top middle of the paper, moving it down to the bottom middle (Fig. 11.4C)—saying nothing as she did this. Also saying nothing, Sarah used her whole pencil (Fig. 11.4D) to indicate the horizontal line of symmetry. One student, when asked whether the square had symmetry or was symmetrical, responded affirmatively, explaining that "if you fold it that way it would work" and making a whole hand horizontal gesture starting at the top and then moving to the bottom of the square (Fig. 11.4E).

As in the discussion of segments in the previous section, the line of symmetry was seen both as an object (with a whole hand gesture or a whole pencil one) and a process of moving from one point to another (with finger or pencil). It may be that the presence of the pencil in their hands encouraged more process conceptions of lines of symmetry.

There were also several instances in which the students made use of symmetry in their drawings, without being prompted. For example, when describing to the

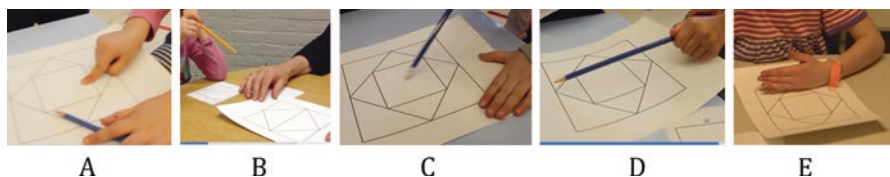


Fig. 11.4 (A) Maya tracing line of symmetry with index finger; (B) Sandro pointing to line of symmetry with pencil tip; (C) Christine gesture-drawing the line of symmetry; (D) Sarah using the pencil as a line of symmetry; (E) Diana using the whole hand to indicate folding over a line of symmetry

interviewer how to draw Fig. 11.2C, Leo told the interviewer to draw the first diagonal line going from the top middle to the left middle and then to “jump back” and “do the same thing on the other side”, which suggests that he was seeing symmetry in the image. One other child, Diana, did a similar thing, telling the interviewer that he could “do the same” for the bottom part of the diamond, after having drawn the two segments making up the top part, adding that “it doesn’t matter which side you do” and “you can start from either side”. This leads nicely to the next type of geometric thinking that we observed, which was the children’s movement back and forth from the local to the global, that is, from focusing on particular one-dimensional objects (lines, points, corners) to two-dimensional shapes (squares, diamonds, triangles).

Intrafigural Structure

In the first task for interview protocol 1, when the children were asked what they saw, they all referred to global shapes such as a T or a cross, and a window or four boxes. The latter two descriptions include the outer square and are more two-dimensional in nature whereas the former two ignore the outer square and thus focus more on the relation of the lines inside the outer square. In general, upon being prompted to describe what they saw, the children used a global approach (e.g., “I see a cross”), but when asked to deconstruct the image or draw the image based on a description, their attention shifted to one-dimensional parts. This is perhaps unsurprising, given that the pencil is a tool for creating one-dimensional objects and even when drawing a triangle or a square, these 2D shapes have to be enclosed in segments. Following Duval (1998), as well as Whiteley (2002), what is important in geometry is the ability to move back and forth between the local and the global, depending on the context of the problem. The geometer must be able to see the image in Fig. 11.2B both in terms of three triangles, but also in terms of midpoints, diagonals and angles. When Duval encourages educators to place more emphasis on one-dimensional objects in early geometry education, it’s because most curricula focus children’s attention on identifying prototypical two-dimensional shapes. While the move from global to local was common in the interviews, we also saw more complex and dynamic shifts in the children’s ways of seeing.

For example, Leo (interview protocol 1), who was first asked to draw the image shown in Fig. 11.2A, did so by drawing two line segments (first horizontal and then vertical). Then, when he was asked, “what do you see?” he responded, “four boxes”. In this case, even the act of drawing the lines did not shift his attention away from the two-dimensional shapes. However, 2 min later, after he had followed the instructions of the interviewer to draw the image shown in Fig. 11.2B (which he did correctly) and was asked “what do you see, in shapes?”, he used his pencil to point to and then gesture-trace the two line segments he had drawn in his own booklet, saying “diagonal, diagonal”. He then gesture-traced the right side of the square, the short diagonal and half of the long diagonal, and when he got to the top right corner said, “triangle”. Then he moved his pencil to the bottom side, gesture-tracing out the

congruent triangle on the bottom and said “triangle” again. (Given that neither triangle is in its prototypical orientation or shape, this identification is not insignificant.) From the first task to the second, Leo thus drew one-dimensional objects, described two-dimensional objects, drew one-dimensional objects, described one-dimensional objects then described two-dimensional shapes, thereby going back and forth between seeing the image in terms of one-dimensional objects and seeing it in terms of two-dimensional shapes.

A different shift from one to two dimensions, and one that is highly relevant to the prototypical tendencies that children have in two-dimensional shape identification, arose in the discussion of the images shown in Fig. 11.2C and D. Most of the children began by referring to the inscribed squares as diamonds. For example, when shown the image in Fig. 11.2C, Maya describes it as “a diamond and a straight line” (thus naming both a two-dimensional shape and a one-dimensional object). However, several of the children also referred to the same shape as a square, either after having rotated the booklet around or after having been asked what different shapes they noticed (this was especially true for the image shown in Fig. 11.2D, which the children described as having three squares). For example, when Diana was asked, “What do you see?” after being shown the image in Fig. 11.2C, she leaned forward, put her pencil on the page and said “there’s a diamond” then traced her pencil along the horizontal line and said, “and split in two, which means two triangles, so there’s one, two, three, four, five, six triangles.” When asked what other shape the diamond could make, Diana responded, “square”, then turned the paper around.

Given the tenacious way in which children identify shapes through prototypical means (what does it look like?), we hypothesise that the flexibility that these children showed may have arisen out of their drawing activities and, in particular, their attending to line segments through drawing, gesturing and describing, here and in previous related activities in class. This would be consistent with Duval’s (2005) theory.

Thus far, we have considered structure in terms of the shifts between local and global—and particularly between one- and two-dimensional geometric objects—but the theme of structure arose quite intentionally with the fifth task of interview protocol 1, when students were asked to complete a grid (see Fig. 11.2E). This question has previously been used by Mulligan and Mitchelmore (2009) as a way of assessing young children’s understanding of mathematical array structure. They found a strong correlation with general mathematics performance. Children who performed poorly on this task produced laborious and mathematically inefficient drawings. For example, a child who completed the grid by creating individual squares, as seen in Fig. 11.5a, was seen as having only partial structural awareness (Mulligan and Mitchelmore identified two classes of drawings that showed even less structural awareness). On the other hand, children who complete the grid by drawing three lines to extend the columns and rows, as seen in Fig. 11.5b, were seen as demonstrating fully developed structural awareness. In their study, 27% of the 103 grade 1 students tested produced drawings like the one in Fig. 11.5a while 24% produced a drawing like the one in Fig. 11.5b. The remaining students produced

Fig. 11.5 (a) Demonstration of partial structural awareness; (b) demonstration of full structural awareness

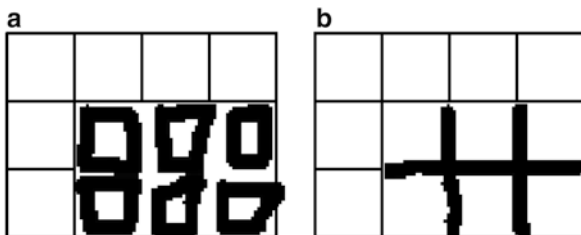
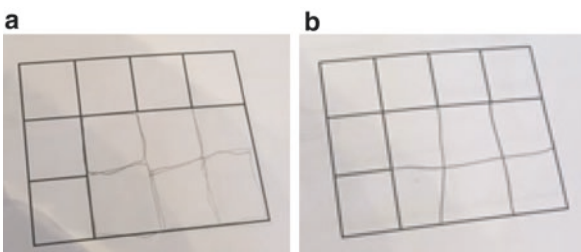


Fig. 11.6 (a) Example of partial/full structural awareness; (b) example of full structural awareness



drawings with comparatively less structure and were classified as demonstrating either pre-structural (11%) or emerging structural (38%) awareness in their grid completion. Other studies have found that it is not until about fourth grade that most children learn to construct the row-by-column structure of rectangular arrays (Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000).

Against these findings, we were interested in seeing how children in the current study would perform on the task and what might be revealed about their structural awareness. Given Carol's extensive and continued focus on geometrical structures in her teaching, we predicted that her students might demonstrate a relatively high level of structural awareness. Of the nine children who were given this task in our study, none produced drawings like the one in Fig. 11.5a; four produced drawings that we consider partial/full (see Fig. 11.6a); five produced drawings like the one in Fig. 11.5b (see Fig. 11.6b). Thus, all students appeared to demonstrate at least some level of structural awareness, indicating higher levels of performance than previously reported despite being much younger in age (e.g., see Mulligan & Mitchelmore, 2009). Given the small sample size and selected population, it is difficult to interpret these findings and more research is needed to understand the effects of learning conditions on children's development of structural awareness. However, this finding does raise the intriguing possibility that Carol's extensive focus on geometrical structure (through building, graphing, drawing and grid activities—as well as *Can You Draw This* activities)—may have positively influenced her students' performance.

Some evidence that performance on the task is flexible and prone to immediate improvements can be seen in the case of Elka who was asked to complete the task both with and without prompts. When first asked to finish completing the grid, Elka

did so by drawing individual squares (see Fig. 11.7a). Although her drawing was accurate in that it produced a 4×3 grid of adjoining cells, her approach to the task was inefficient and imprecise (i.e., her cells were of varying proportions and drawn in piecemeal). Here we see that when unprompted, Elka did not demonstrate full structural awareness. However, as revealed next, Elka's second attempt at the task provides insight into her performance and potentially demonstrates the immediate impact of instruction.

Zack: What if you did it like this? So, watch my finger. What if you just took your pencil here and went, zoom (*uses finger to gesture the drawing of a horizontal line across the page*). And then you took your pencil and you went, zoom (*uses finger to gesture drawing vertical line that create new row*). And then you took your pencil and you went, zoom (*uses finger to gesture drawing of another vertical line to create new rows*). Do you think that would work too?

Elka: (*Nods.*)

Zack: What if you try it? What if you try to do it with long lines, as long as you possibly can?

Even before the interviewer has finished the question, Elka had completed a horizontal line that created the second and third rows. She then quickly and efficiently completes the grid by drawing two vertical lines that complete the grid (see Fig. 11.7b). In comparing Elka's first and second attempts at this task, it is clear that the second attempt was much more accurate and according to Mulligan and Mitchelmore (2009), was representative of a more sophisticated understanding of mathematical structure. In this example we see how a few short prompts may have been enough to facilitate drawing performance. However, it is also possible that Elka's improvements are not indicative of increases in structural awareness per se, but a result of simply copying the interviewer's gestures or even as a result of repeated practice on the same task. This finding, along with the general finding of high levels of structural awareness amongst Carol's students, is deserving of further research as it addresses the important question of the extent to which children's structural awareness, and more broadly geometric drawing performance, is influenced by developmental constraints but also malleable and subject to improvements given proper instruction. Research of this sort will help reconcile the

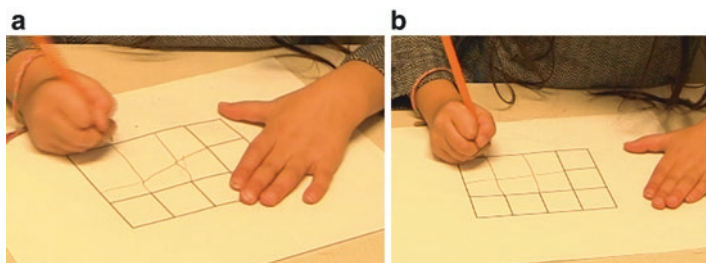


Fig. 11.7 (a) Elka's first attempt. (b) Elka's second attempts at the grid task

presumably false dichotomy evident in the larger research literature; that is, the contrast in views of children's drawings as relatively stable outcomes of cognitive development versus the perspective of children's drawings as a dynamic process that not only presents an artefact *of* learning but also an act *for* learning. More broadly, a better understanding of the impact that instruction has on children's geometric drawing behaviour and related mathematical insights will go a long way in helping us better understand the potential role of drawing in early mathematics education.

Discussion

In this chapter, we have drawn on educational psychology research to motivate investigation into children's spatial reasoning more broadly, and their drawing in particular. Instead of studying children's finished drawings for how they indicate cognitive development or intelligence or even correlation to mathematical ability, we have focussed on their drawing processes in order to gain insight into how children might learn through and from drawing, and how this might relate to the development of their spatial reasoning. We developed tasks in which the children's drawing processes involved instructional prompts and included both language and gesture. This was done in part to investigate how language and gesture might interplay with drawing and in part in order to pursue drawing as a geometrical activity, following the work of Duval. This latter point signals a shift away from the more commonly used van Hiele (1986), in which there is less emphasis both on the coordination of visual and language registers and on the dimensional deconstruction involved in drawing and seeing. It also differs from the focus that is found in Clements and Sarama's (2011) learning trajectories of composing and decomposing, particularly in relation to the centrality of drawing and of using/naming one-dimensional geometric objects.

In analysing the drawing processes of the children participating in our study, we were able to identify a significant number of situations in which the use of language or gesture changed the way that the students saw, drew and described (that is, the way they spatially reasoned) the images shown in Figs. 11.2 and 11.3. Gestures were used extensively to mark out future line segments, including their endpoints either at the corner or in the middle of the sides of existing line segments. The use of words such as "middle" and "top" or "side" also oriented the children's drawings and enabled them to successfully describe images to the interviewers. Based on our analyses, we propose that providing children with opportunities to draw while also talking and gesturing can improve their performance on Quick draw-type tasks and increase their spatial reasoning.

Instead of focussing solely on finished drawings, our interview protocols also included questions that enabled us to probe mathematical concepts that could be relevant to the drawings the children had produced. As we described above, these concepts include congruence, symmetry and structure. While these concepts can be

taught and assessed without a drawing component, we suggest that, based on our analyses, the very act of drawing, and of gesturing and tracing, changes the way that students see (spatially reason about) geometric images. The actual drawing of the segments that make up a triangle or square seems to prompt children to attend to two-dimensional properties of shapes, which they can use in order to reason about whether or not the shapes are the same. The act of drawing may also help children become aware of symmetry (of drawing the same thing on both sides), as well as structures such as grids.

Despite increased interest in, and current research findings showing the importance of spatial reasoning for mathematics education, and despite the call for increased focus and time spent on geometry in the mathematics curricula (e.g. NCTM) very little progress has been made either within mathematics education research or in instructional material design (apart from Moss et al., 2016). In particular, in current standards and curricula for early school geometry, drawing has had extremely limited attention. Our chapter seeks to forge new ways of including drawing as part of the teaching and learning of geometry and offers new ways of thinking about and analysing the types of spatial/geometric reasoning young children are capable of. In this regard, our study takes a different view of the potential of drawing and identifies productive ways in which drawing could support spatial reasoning in the context of geometry. Our study specifically looks at both gesturing and drawing together, a focus arising from the study of mathematical activity (Châtelet, 2000), and one that we see as very productive in future research in spatial reasoning—and drawing in particular—in both educational psychology and mathematics education.

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