# Chapter 4

# The Maximum Principle: Pure State and Mixed Inequality Constraints

In Chap. 2 we addressed optimal control problems having constraints only on control variables. We extended the discussion in Chap. 3 to treat mixed constraints that may involve state variables in addition to control variables.

Often in management science and economics problems there are nonnegativity constraints on state variables, such as inventory levels or wealth. These constraints do not include control variables. Also, there may be more general inequality constraints only on state variables, which include constraints that require certain state variables to remain nonnegative. Such constraints are known as *pure state variable inequality constraints* or, simply, *pure state constraints*.

Pure state constraints are more difficult to deal with than mixed constraints. We can intuitively appreciate this fact by keeping in mind that only control variables are under the direct influence of the decision maker. This enables the decision maker, when a mixed constraint becomes tight, to choose from the controls that would keep it tight for as long as required for optimality. Whereas with pure state constraints, the situation is different and more complicated. That is, when a constraint becomes tight, it does not provide any direct information to the decision maker on how to choose values for the control variables so as not to



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violate the constraint. Hence, considerable changes in the controls may be required to keep the constraint tight if needed for optimality.

This chapter considers pure state constraints together with mixed constraints. In the literature there are two ways of handling pure state constraints: *direct* and *indirect*. The direct method associates a multiplier with each constraint for appending it to the Hamiltonian to form the Lagrangian, and then proceeds in much the same way as in Chap. 3 dealing with mixed constraints. In the indirect method, the choice of controls, when a pure constraint is active, must be further limited by constraining approximately the value of the derivative of the active state constraint with respect to time. This derivative will involve time derivatives of the state variables, which can be written in terms of the control and state variables through the use of the state equations. Thus, the restrictions on the time derivatives of the pure state constraints are transformed in the form of mixed constraints, and these will be appended instead to the Hamiltonian to form the Lagrangian. Because the pure state constraints are adjoined in this indirect fashion, the corresponding Lagrange multipliers must satisfy some complementary slackness conditions in addition to those mentioned in Chap. 3.

With the formulation of the Lagrangian in each approach, we will write the respective maximum principle, where the choice of control will come from maximizing the Hamiltonian subject to both pure state constraints and mixed constraints. We will find, however, in contrast to Chap. 3, that in both approaches, the adjoint functions may be required to have jumps at those times where the pure state constraints become tight.

We begin with a simple example in Sect. 4.1 to motivate the necessity of possible jumps in the adjoint functions. Section 4.2 formulates the problem with pure state constraints along with the required assumptions. In Sect. 4.3, we use the direct method for stating the maximum principle necessary conditions for solving such problems. Sufficiency conditions are stated in Sect. 4.4. Section 4.5 is devoted to developing the maximum principle for the indirect method, which involves adjoining the first derivative of the pure state constraints to form the Lagrangian function and imposing some additional constraints on the Lagrange multipliers of the resulting formulation. Also, the adjoint variables and the Lagrange multipliers arising in this method will be related to those arising in the direct method. Finally, the current-value form of the maximum principle for the indirect method is described in Sect. 4.6.

## 4.1 Jumps in Marginal Valuations

In this section, we formulate an optimal control problem with a pure constraint, which can be solved merely by inspection and which exhibits a discontinuous marginal valuation of the state variable. Since the adjoint variables in Chaps. 2 and 3 provide these marginal valuations and since we would like this feature to continue, we must allow the adjoint variables to have jumps if the marginal valuations can be discontinuous. This will enable us to formulate a maximum principle in the next section, which is similar to (3.10) with the exception that the adjoint variables, and therefore also the Hamiltonian, may have possible jumps satisfying some jump conditions.

**Example 4.1** Consider the problem with a pure state constraint:

$$\max \left\{ J = \int_0^3 -u dt \right\} \tag{4.1}$$

subject to

$$\dot{x} = u, \quad x(0) = 0, \tag{4.2}$$

$$0 \leq u \leq 3, \tag{4.3}$$

$$x - 1 + (t-2)^2 \geq 0. \tag{4.4}$$

**Solution** From the objective function (4.1), one can see that it is good to have low values of $u$. If we use $u = 0$ to begin with, we see that $x(t) = 0$ as long as $u(t) = 0$. At $t = 1$, $x(1) = 0$ and the constraint (4.4) is satisfied with an equality. But continuing with $u(t) = 0$ beyond $t = 1$ is not feasible since $x(t) = 0$ would not satisfy the constraint (4.4) just after $t = 1$.

In Fig. 4.1, we see that the lowest possible feasible state trajectory from $t = 1$ to $t = 2$ satisfies the state constraint (4.4) with an equality. In order not to violate the constraint (4.4), its first time derivative $u(t) + 2(t-2)$ must be nonnegative. This gives us $u(t) = 2(2-t)$ to be the lowest feasible value for the control. This value will make the state $x(t)$ ride along the constraint boundary until $t = 2$, at which point $u(2) = 0$; see Fig. 4.1. Continuing with $u(t) = 2(2-t)$ beyond $t = 2$ will make $u(t)$ negative, and violate the lower bound in (4.3). It is easy to see, however, that $u(t) = 0$, $t \geq 2$, is the lowest feasible value, which can be followed all the way to the terminal time $t = 3$.

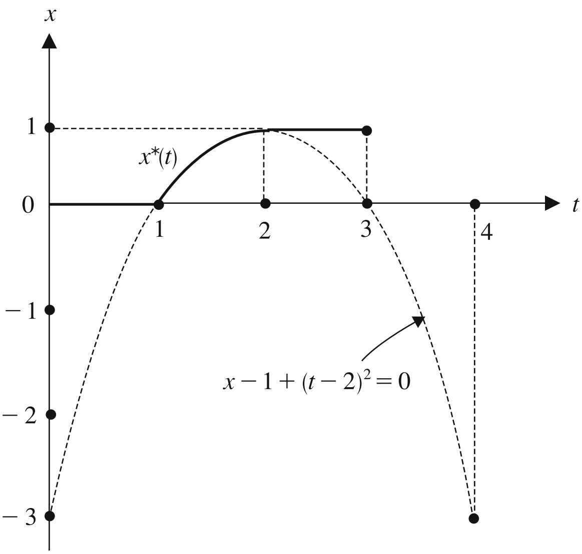


Figure 4.1: Feasible state space and optimal state trajectory for Examples 4.1 and 4.4

It can be seen from Fig. 4.1 that the bold trajectory is the lowest possible feasible state trajectory on the entire time interval [0,3]. Moreover, it is obvious that the lowest possible feasible control is used at any given $t \in [0,3]$, and therefore, the solution we have found is optimal. We can now restate the values of the state and control variables that we have obtained:

$$x^*(t) = \begin{cases} 0, & t \in [0,1), \\ 1-(t-2)^2, & t \in [1,2], \\ 1, & t \in (2,3], \end{cases} \quad u^*(t) = \begin{cases} 0, & t \in [0,1), \\ 2(2-t), & t \in [1,2], \\ 0, & t \in (2,3]. \end{cases} \tag{4.5}$$

Next we find the value function $V(x,t)$ for this problem. It is obvious that the feedback control $u^*(x,t) = 0$ is optimal at any point $(x,t)$ when $x \geq 1$ or when $(x,t)$ is on the right-hand side of the parabola in Fig. 4.1. Thus, $V(x,t) = 0$ on such points.

On the other hand, when $x \in [0,1]$ and it is on the left-hand side of the parabola, the optimal trajectory is very similar to the one shown in Fig. 4.1. Specifically, the control is zero until it hits the trajectory at time $\tau = 2 - \sqrt{1-x}$. Then, the control switches to $2(2-s)$ for $s \in (\tau, 2)$

to climb along the left-hand side of the parabola to reach its peak, and then switches back to zero on the time interval [2,3]. Thus, in this case,

$$\begin{aligned} V(x,t) &= -\int_t^\tau 0ds - \int_\tau^2 2(2-s)ds - \int_2^3 0ds \\ &= s^2 - 4s\Big]_{2-\sqrt{1-x}}^2 = (x-1). \end{aligned}$$

Thus, we have the value function

$$V(x,t) = \begin{cases} 0, & x \geq 1,\ t \in [0,3], \\ x-1, & x \geq 1-(t-2)^2,\ t \in [0,2), \\ 0, & 1-(t-2)^2 \leq x \leq 1,\ t \in [2,3]. \end{cases}$$

This gives us the marginal valuation along the optimal path $x^*(t)$ given in (4.5) as

$$V_x(x^*(t),t) = \begin{cases} 1, & t \in [0,2), \\ 0, & t \in [2,3]. \end{cases} \tag{4.6}$$

We can now see that this marginal valuation is discontinuous at $t = 2$, and it has a downward jump of size 1 at that time.

The maximum principle that we will state in Sect. 4.3 will have certain jump conditions in order to accommodate problems like Example 4.1. Indeed in Example 4.2, we will apply the maximum principle of Sect. 4.3 to the problem in Example 4.1, and see that the adjoint variable that represents the marginal valuation along the optimal path will have a jump consistent with (4.6).

In the next section, we state the general optimal control problem that is the subject of this chapter.

## 4.2 The Optimal Control Problem with Pure and Mixed Constraints

We will append to the problem (3.7) considered in Chap. 3, the pure state variable inequality constraint of type

$$h(x,t) \geq 0, \tag{4.7}$$

where we assume function $h : E^n \times E^1 \to E^p$ to be continuously differentiable in all its arguments. By the definition of function $h$, (4.7) represents a set of $p$ constraints $h_i(x,t) \geq 0$, $i = 1, 2, \ldots, p$. It is noted that the constraint $h_i \geq 0$ is called a constraint of $r$th order if the $r$th time derivative of $h_i$ is the first time a term in control $u$ appears in the expression by putting $f(x,u,t)$ for $\dot{x}$ after each differentiation. It is through this expression that the control acts to satisfy the constraint $h_i \geq 0$. The value of $r$ is referred to as the *order* of the constraint. Of course, if the constraint $h_i$ is of order $r$, then we would require $h_i$ to be $r$ times continuously differentiable.

Except for Exercise 4.12, in this book we will consider only first-order constraints, i.e., $r = 1$. For such constraints, the first-time derivative of $h$ has terms in $u$. Thus, we can define $h^1(x,u,t)$ as follows:

$$h^1 = \frac{dh}{dt} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t}. \tag{4.8}$$

In the important special case of the nonnegativity constraint

$$x(t) \geq 0, \quad t \in [0,T], \tag{4.9}$$

(4.8) is simply $h^1 = f$. For an upper bound constant $x(t) \leq M$, written as

$$M - x(t) \geq 0, \quad t \in [0,T], \tag{4.10}$$

(4.8) gives $h^1 = -f$. These will be of order one because the function $f(x,u,t)$ usually contains terms in $u$.

As in Chap. 3, the constraints (4.7) need also to satisfy a full-rank type constraint qualification before a maximum principle can be derived. With respect to the $i$th constraint $h_i(x,t) \geq 0$, an interval $(\theta_1, \theta_2) \subset [0,T]$ with $\theta_1 < \theta_2$ is called an *interior interval* if $h_i(x(t),t) > 0$ for all $t \in (\theta_1, \theta_2)$. If the optimal trajectory "hits the boundary," i.e., satisfies $h_i(x(t),t) = 0$ for $\tau_1 \leq t \leq \tau_2$ for some $i$, then $[\tau_1, \tau_2]$ is called a *boundary interval.* An instant $\tau_1$ is called an *entry time* if there is an interior interval ending at $t = \tau_1$ and a boundary interval starting at $\tau_1$. Correspondingly, $\tau_2$ is called an *exit time* if a boundary interval ends and an interior interval starts at $\tau_2$. If the trajectory just touches the boundary at time $\tau$, i.e., $h(x(\tau),\tau) = 0$ and if the trajectory is in the interior just before and just after $\tau$, then $\tau$ is called a *contact time.* Taken together, entry, exit, and contact times are called *junction times.*

In this book we shall not consider problems that require optimal state trajectories to have countably many junction times. In other words, we shall state the maximum principle necessary optimality conditions for state trajectories having only finitely many junction times. Also, all of the applications studied in this book exhibit optimal state trajectories containing finitely many junction times or no junction times.

Throughout the book, we will assume that the constraint qualification introduced in Sect. 3.1 as well as the following full-rank condition on any boundary interval $[\tau_1, \tau_2]$ hold:

$$\operatorname{rank}\begin{bmatrix} \partial h_1^1/\partial u \\ \partial h_2^1/\partial u \\ \vdots \\ \partial h_{\hat{p}}^1/\partial u \end{bmatrix} = \hat{p},$$

where for $t \in [\tau_1, \tau_2]$,

$$h_i(x^*(t), t) = 0,\ i = 1, 2, \ldots, \hat{p} \le p$$

and

$$h_i(x^*(t), t) > 0,\ i = \hat{p} + 1, \ldots, p.$$

Note that this full-rank condition on the constraints (4.7) is written when the order of each of the constraints in (4.7) is one. For the general case of higher-order constraints, see Hartl et al. (1995).

Let us recapitulate the optimal control problem for which we will state a direct maximum principle in the next section. The problem is

$$\begin{cases} \max\left\{J = \displaystyle\int_0^T F(x, u, t)dt + S[x(T), T]\right\}, \\ \text{subject to} \\ \dot{x} = f(x, u, t),\ x(0) = x_0, \\ g(x, u, t) \ge 0, \\ h(x, t) \ge 0, \\ a(x(T), T) \ge 0, \\ b(x(T), T) = 0. \end{cases} \tag{4.11}$$

Important special cases of the mixed constraint $g(x, u, t) \geq 0$ are $u_i \in [0, M]$ for $M > 0$ and $u_i(t) \in [0, x_i(t)]$, and those of the terminal constraints $a(x(T), T) \geq 0$ and $a(x(T), T) = 0$ are $x_i(T) \geq k$ and $x_i(T) = k$, respectively, where $k$ is a constant. Likewise, the special cases of the pure constraints $h(x, t) \geq 0$ are $x_i \geq 0$ and $x_i \leq M$, for which $h_{x_i} = +1$ and $h_{x_i} = -1$, respectively, and $h_t = 0$.

## 4.3 The Maximum Principle: Direct Method

For the problem (4.11), we will now state the direct maximum principle which includes the discussion above and the required jump conditions. For details, see Dubovitskii and Milyutin (1965), Feichtinger and Hartl (1986), Hartl et al. (1995), Boccia et al. (2016), and references therein. We will use superscript $d$ on various multipliers that arise in the direct method, to distinguish them from the corresponding multipliers (which are not superscripted) that arise in the indirect method, to be discussed in Sect. 4.5. Naturally, it will not be necessary to superscript the multipliers that are known to remain the same in both methods.

To formulate the maximum principle for the problem (4.11), we define the Hamiltonian function $H^d : E^n \times E^m \times E^1 \to E^1$ as

$$H^d = F(x, u, t) + \lambda^d f(x, u, t)$$

and the Lagrangian function $L^d : E^n \times E^m \times E^n \times E^q \times E^p \times E^1 \to E^1$ as

$$L^d(x, u, \lambda^d, \mu, \eta^d, t) = H^d(x, u, \lambda^d, t) + \mu g(x, u, t) + \eta^d h(x, t). \quad (4.12)$$

The maximum principle states the necessary conditions for $u^*$ (with the corresponding state trajectory $x^*$) to be optimal. The conditions are that there exist an adjoint function $\lambda^d$, which may be discontinuous at a time in a boundary interval or a contact time, multiplier functions $\mu, \alpha, \beta, \gamma^d, \eta^d$, and a jump parameter $\zeta^d(\tau)$, at each time $\tau$, where $\lambda^d$ is discontinuous, such that the following (4.13) holds:

$$
\begin{array}{|l|}
\hline
\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0, \text{ satisfying constraints} \\
g(x^*, u^*, t) \geq 0,\ h(x^*, t) \geq 0, \text{ and the terminal constraints} \\
a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0; \\
\dot{\lambda}^d = -L_x[x^*, u^*, \lambda^d, \mu, \eta^d, t] \\
\text{with the transversality conditions} \\
\lambda^d(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) \\
\qquad\qquad + \gamma^d h_x(x^*(T), T), \text{ and} \\
\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma^d \geq 0,\ \gamma^d h(x^*(T), T) = 0; \\
\text{the Hamiltonian maximizing condition} \\
H^d[x^*(t), u^*(t), \lambda^d(t), t] \geq H^d[x^*(t), u, \lambda^d(t), t] \\
\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying} \\
g[x^*(t), u, t] \geq 0; \\
\text{the jump conditions at any time } \tau, \\
\text{where } \lambda^d \text{ is discontinuous, are} \\
\lambda^d(\tau^-) = \lambda^d(\tau^+) + \zeta^d(\tau) h_x(x^*(\tau), \tau) \text{ and} \\
H^d[x^*(\tau), u^*(\tau^-), \lambda^d(\tau^-), \tau] = H^d[x^*(\tau), u^*(\tau^+), \lambda^d(\tau^+), \tau] \\
\qquad\qquad\qquad\qquad - \zeta^d(\tau) h_t(x^*(\tau), \tau); \\
\text{the Lagrange multipliers } \mu(t) \text{ are such that} \\
\partial L^d / \partial u|_{u=u^*(t)} = 0,\ dH^d/dt = dL^d/dt = \partial L^d / \partial t, \\
\text{and the complementary slackness conditions} \\
\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0, \\
\eta(t) \geq 0,\ \eta^d(t) h(x^*(t), t) = 0, \text{ and} \\
\zeta^d(\tau) \geq 0,\ \zeta^d(\tau) h(x^*(\tau), \tau) = 0 \text{ hold.} \\
\hline
\end{array}
\tag{4.13}
$$

As in the previous chapters, $\lambda^d(t)$ has the marginal value interpretation. Therefore, while it is not needed for the application of the maximum principle (4.13), we can trivially set

$$\lambda^d(T) = S_x(x^*(T), T). \tag{4.14}$$

If $T$ is also a decision variable constrained to lie in the interval $[T_1, T_2]$, $0 \le T_1 < T_2 < \infty$, then in addition to (4.13), if $T^*$ is the optimal terminal time, it must satisfy a condition similar to (3.15) and (3.81), i.e.,

$$\begin{aligned} &H^d[x^*(T^*), u^*(T^{*-}), \lambda^d(T^{*-}), T^*] + S_T[x^*(T^*), T^*] + \alpha a_T[x^*(T^*), T^*] \\ &+\beta b_T[x^*(T^*), T^*] + \gamma^d h_T[x^*(T^*), T^*] \begin{cases} \le 0 & \text{if } T^* = T_1, \\ = 0 & \text{if } T^* \in (T_1, T_2), \\ \ge 0 & \text{if } T^* = T_2. \end{cases} \end{aligned} \tag{4.15}$$

**Remark 4.1** In most practical examples, $\lambda^d$ and $H^d$ will only jump at junction times. However, in some cases a discontinuity may occur at a time in the interior of a boundary interval, e.g., when a mixed constraint becomes active at that time.

**Remark 4.2** It is known that the adjoint function $\lambda^d$ is continuous at a junction time $\tau$, i.e., $\zeta^d(\tau) = 0$, if (i) the entry or exit at time $\tau$ is non-tangential, i.e., if $h^1(x^*(\tau), u^*(\tau), \tau) \neq 0$, or (ii) if the control $u^*$ is continuous at $\tau$ and the

$$\text{rank} \begin{bmatrix} \partial g/\partial u & \text{diag}(g) & 0 \\ \partial h^1/\partial u & 0 & \text{diag}(h) \end{bmatrix} = m + p,$$

when evaluated at $x^*(\tau)$ and $u^*(\tau)$.

We will see that the jump conditions on the adjoint variables in (4.13) will give us precisely the jump in Example 4.2, where we will apply the direct maximum principle to the problem in Example 4.1. The jump condition on $H^d$ in (4.13) requires that the Hamiltonian should be continuous at $\tau$ if $h_t(x^*(\tau), \tau) = 0$. The continuity of the Hamiltonian (in case $h_t = 0$) makes intuitive sense when considered in the light of its interpretation given in Sect. 2.2.4.

This brief discussion of the jump conditions, limited here only to first-order pure state constraints, is far from complete, and a detailed discussion is beyond the scope of this book. An interested reader should consult the comprehensive survey by Hartl et al. (1995). For an example with a second-order state constraint, see Maurer (1977).

Needless to say, computational methods are required to solve problems with general inequality constraints in all but the simplest of the cases. The reader should consult the excellent book by Teo et al. (1991) and references therein for computational procedures and software; see also Polak et al. (1993), Bulirsch and Kraft (1994), Bryson (1998), and Pytlak and Vinter (1993, 1999). A MATLAB based software, used for solving finite and infinite horizon optimal control problems with pure state and mixed inequality constraints, is available at http://orcos.tuwien.ac.at/research/ocmat_software/.

**Example 4.2** Apply the direct maximum principle (4.13) to solve the problem in Example 4.1.

**Solution** Since we already have optimal $u^*$ and $x^*$ as obtained in (4.5), we can use these in (4.13) to obtain $\lambda^d, \mu_1, \mu_2, \gamma^d, \eta^d$, and $\zeta^d$. Thus,

$$H^d = -u + \lambda^d u, \tag{4.16}$$

$$L^d = H^d + \mu_1 u + \mu_2(3-u) + \eta^d[x - 1 + (t-2)^2], \tag{4.17}$$

$$L_u^d = -1 + \lambda^d + \mu_1 - \mu_2 = 0, \tag{4.18}$$

$$\dot{\lambda}^d = -L_x^d = -\eta^d,\ \lambda^d(3^-) = \gamma^d, \tag{4.19}$$

$$\gamma^d[x^*(3) - 1 + (3-2)^2] = 0, \tag{4.20}$$

$$\mu_1 \geq 0,\ \mu_1 u^* = 0,\ \mu_2 \geq 0,\ \mu_2(3-u^*) = 0, \tag{4.21}$$

$$\eta^d \geq 0,\ \eta^d[x^*(t) - 1 + (t-2)^2] = 0, \tag{4.22}$$

and if $\lambda^d$ is discontinuous for some $\tau \in [1,2]$, the boundary interval as seen from Fig. 4.1, then

$$\lambda^d(\tau^-) = \lambda^d(\tau^+) + \zeta^d(\tau),\ \zeta^d(\tau) \geq 0, \tag{4.23}$$

$$-u^*(\tau^-) + \lambda^d(\tau^-)u^*(\tau^-) = -u^*(\tau^+) + \lambda^d(\tau^+)u^*(\tau^+) - \zeta^d(\tau)2(\tau-2). \tag{4.24}$$

Since $\gamma^d = 0$ from (4.20), we have $\lambda^d(3-) = 0$ from (4.19). Also, we set $\lambda^d(3) = 0$ according to (4.14).

Interval (2,3]: We have $\eta^d = 0$ from (4.22), and thus $\dot{\lambda}^d = 0$ from (4.19), giving $\lambda^d = 0$. From (4.18) and (4.21), we have $\mu_1 = 1 > 0$ and $\mu_2 = 0$.

Interval [1,2]: We get $\mu_1 = \mu_2 = 0$ from $0 < u^* < 3$ and (4.21). Thus, (4.18) implies $\lambda^d = 1$ and (4.19) gives $\eta^d = -\dot{\lambda}^d = 0$. Thus $\lambda^d$ is discontinuous at the exit time $\tau = 2$, and we use (4.23) to see that the jump parameter $\zeta^d(2) = \lambda^d(2^-) - \lambda^d(2^+) = 1$. Furthermore, it is easy to check that (4.24) also holds at $\tau = 2$.

Interval [0,1): Clearly $\mu_2 = 0$ from (4.21). Also $u^* = 0$ would still be optimal if there were no lower bound constraint on $u$ in this interval. This means that the constraint $u \geq 0$ is not binding, giving us $\mu_1 = 0$. Then from (4.18), we have $\lambda^d = 1$. Finally, from (4.19), we have $\eta^d = -\dot{\lambda}^d = 0$.

We can now see that the adjoint variable

$$\lambda^d(t) = \begin{cases} 1, & t \in [0, 2), \\ 0, & t \in [2, 3], \end{cases} \tag{4.25}$$

is precisely the same as the marginal valuation $V_x(x^*(t), t)$ obtained in (4.6). We also see that $\lambda^d$ is continuous at time $t = 1$ where the entry to the constraint is non-tangential as stated in Remark 4.2.

## 4.4 Sufficiency Conditions: Direct Method

When first-order pure state constraints are present, sufficiency results are usually stated in terms of the maximum principle using the direct method described in Hartl et al. (1995).

We will now state the sufficiency result for the problem specified in (4.11). For this purpose, let us define the maximized Hamiltonian

$$H^{0d}(x, \lambda^d(t), t) = \max_{\{u | g(x,u,t) \geq 0\}} H^d(x, u, \lambda^d, t). \tag{4.26}$$

See Feichtinger and Hartl (1986) and Seierstad and Sydsæter (1987) for details.

**Theorem 4.1** *Let $(x^*, u^*, \lambda^d, \mu, \alpha, \beta, \gamma^d, \eta^d)$ and the jump parameters $\zeta^d(\tau)$ at each $\tau$, where $\lambda^d$ is discontinuous, satisfy the necessary conditions in (4.13). If $H^{0d}(x, \lambda^d(t), t)$ is concave in $x$ at each $t \in [0, T]$, $S$*

*in* (3.2) *is concave in* $x$, $g$ *in* (3.3) *is quasiconcave in* $(x, u)$, $h$ *in* (4.7) *and* $a$ *in* (3.4) *are quasiconcave in* $x$, *and* $b$ *in* (3.5) *is linear in* $x$, *then* $(x^*, u^*)$ *is optimal.*

We will illustrate an application of this theorem in Example 4.4, which shows that the solution obtained in Example 4.3 is optimal.

Theorem 4.1 is written for finite horizon problems. For infinite horizon problems, this theorem remains valid if the transversality condition on the adjoint variable in (4.29) is modified along the lines discussed in Sect. 3.6.

In concluding this section, we should note that the sufficiency conditions stated in Theorem 4.1 rely on the presence of appropriate concavity conditions. Sufficiency conditions can also be obtained without these concavity assumptions. These are called second-order conditions for a local maximum, which require the second variation on the linearized state equation to be negative definite. For further details on second-order sufficiency conditions, the reader is referred to Maurer (1981), Malanowski (1997), and references in Hartl et al. (1995).

## 4.5 The Maximum Principle: Indirect Method

The main idea underlying the indirect method is that when the pure state constraint (4.7), assumed to be of order one, becomes active, we must require its first derivative $h^1(x, u, t)$ in (4.8) to be nonnegative, i.e.,

$$h^1(x, u, t) \geq 0, \text{ whenever } h(x, t) = 0. \tag{4.27}$$

While this is a mixed constraint, it is different from those treated in Chap. 3 in the sense that it is imposed only when the constraint (4.8) is tight.

Since (4.27) is a mixed constraint, it is tempting to use the maximum principle (3.12) developed in Chap. 3. This can be done provided that we can find a way to impose (4.27) only when $h(x, t) = 0$. One way to accomplish this is to append (4.27) to the Hamiltonian when forming the Lagrangian, by using a multiplier $\eta \geq 0$, i.e., append $\eta h^1$, and require that $\eta h = 0$, which is equivalent to imposing $\eta_i h_i = 0,\ i = 1, 2, \ldots, p$. This means that when $h_i > 0$ for some $i$, we have $\eta_i = 0$ and it is then not a part of the Lagrangian.

Note that when we require $\eta h = 0$, we do not need to impose $\eta h^1 = 0$ as required for mixed constraints. This is because when $h_i > 0$ on an interval, then $\eta_i = 0$ and so $\eta_i h_i^1 = 0$ on that interval. On the other hand, when $h_i = 0$ on an interval, then it is because $h_i^1 = 0$, and thus, $\eta_i h_i^1 = 0$ on that interval. In either case, $\eta_i h_i^1 = 0$.

With these observations, we are ready to formulate the indirect maximum principle for the problem (4.11).

We form the Lagrangian as

$$L(x, u, \lambda, \mu, \eta, t) = H(x, u, \lambda, t) + \mu g(x, u, t) + \eta h^1(x, u, t), \quad (4.28)$$

where the Hamiltonian $H = F(x, u, t) + \lambda f(x, u, t)$ as defined in (3.8). We will now state the maximum principle which includes the discussion above and the required jump conditions.

The maximum principle states the necessary conditions for $u^*$ (with the state trajectory $x^*$) to be optimal. These conditions are that there exist an adjoint function $\lambda$, which may be discontinuous at each entry or contact time, multiplier functions $\mu, \alpha, \beta, \gamma, \eta$, and a jump parameter $\zeta(\tau)$ at each $\tau$, where $\lambda^d$ is discontinuous, such that (4.29) on the following page holds.

Once again, as before, we can set $\lambda(T) = S_x(x^*(T), T)$. If $T \in [T_1, T_2]$ is a decision variable, then (4.15) with $\lambda^d$ and $\gamma^d$ replaced by $\lambda$ and $\gamma$, respectively, must also hold.

In (4.29), we see that there are jump conditions on the adjoint variables and also the Hamiltonian in the indirect maximum principle. The remarks on the jump condition made in connection with the direct maximum principle (4.13) apply also to the jump conditions in (4.29). In (4.29), we also see a condition $\dot{\eta} \leq 0$, in addition to the complimentary conditions on $\eta$. The presence of this term will become clear after we relate this multiplier to those in the direct maximum principle, which we discuss next.

In various applications that are discussed in subsequent chapters of this book, we use the indirect maximum principle. Nevertheless, it is worthwhile to provide relationships between the multipliers of the two approaches, as these will be useful when checking for the sufficiency conditions of Theorem 4.1, developed in Sect. 4.4.

$$
\begin{aligned}
&\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0, \text{ satisfying constraints}\\
&g(x^*, u^*, t) \geq 0,\ h(x^*, t) \geq 0, \text{ and the terminal constraints}\\
&a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0;\\
&\dot{\lambda} = -L_x[x^*, u^*, \lambda, \mu, \eta, t] \text{ with the transversality conditions}\\
&\lambda(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T)\\
&\qquad\qquad +\gamma h_x(x^*(T), T), \text{ and}\\
&\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma \geq 0,\ \gamma h(x^*(T), T) = 0;\\
&\text{the Hamiltonian maximizing condition}\\
&H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t]\\
&\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying}\\
&g[x^*(t), u, t] \geq 0, \text{ and}\\
&h_i^1(x^*(t), u, t) \geq 0 \text{ whenever } h_i(x^*(t), t) = 0, i = 1, 2, \cdots, p;\\
&\text{the jump conditions at any entry/contact time } \tau,\\
&\text{where } \lambda \text{ is discontinuous, are}\\
&\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau) \text{ and}\\
&H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau]\\
&\qquad\qquad\qquad\qquad -\zeta(\tau) h_t(x^*(\tau), \tau);\\
&\text{the Lagrange multipliers } \mu(t) \text{ are such that}\\
&\partial L/\partial u|_{u=u^*(t)} = 0,\ dH/dt = dL/dt = \partial L/\partial t,\\
&\text{and the complementary slackness conditions}\\
&\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0,\\
&\eta(t) \geq 0,\ \eta(t) h(x^*(t), t) = 0, \text{ and}\\
&\zeta(\tau) \geq 0,\ \zeta(\tau) h(x^*(\tau), \tau) = 0 \text{ hold.}
\end{aligned}
\tag{4.29}
$$

We now obtain the multipliers of the direct maximum principle from those in the indirect maximum principle. Since the multipliers coincide in the interior, we let $[\tau_1, \tau_2]$ denote a boundary interval and $\tau$ a contact time. It is shown in Hartl et al. (1995) that

$$\eta^d(t) = -\dot{\eta}(t),\ t \in (\tau_1, \tau_2), \tag{4.30}$$

$$\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t),\ t \in (\tau_1, \tau_2), \tag{4.31}$$

Note that $\eta^d(t) \geq 0$ in (4.13). Thus, we have $\dot{\eta} \leq 0$, which we have already included in (4.29). The jump parameter at an entry time $\tau_1$, an exit time $\tau_2$, or a contact time $\tau$, respectively, satisfies

$$\zeta^d(\tau_1) = \zeta(\tau_1) - \eta(\tau_1^+),\ \zeta^d(\tau_2) = \eta(\tau_2^-),\ \zeta^d(\tau) = \zeta(\tau). \tag{4.32}$$

By comparing $\lambda^d(T^-)$ in (4.13) and $\lambda(T^-)$ in (4.29) and using (4.31), we have

$$\gamma^d = \gamma + \eta(T^-). \tag{4.33}$$

Going the other way, we have

$$\eta(t) = \int_t^{\tau_2} \eta^d(s)ds + \zeta^d(\tau_2),\ t \in (\tau_1, \tau_2),$$

$$\lambda(t) = \lambda^d(t) - \eta(t)h(x^*(t), t),\ t \in (\tau_1, \tau_2),$$

$$\zeta(\tau_1) = \zeta^d(\tau_1) + \eta(\tau_1^+),\ \zeta(\tau_2) = 0,\ \zeta(\tau) = \zeta^d(\tau),$$

$$\gamma = \gamma^d - \eta(T^-).$$

Finally, as we had mentioned earlier, the multipliers $\mu, \alpha$, and $\beta$ are the same in both methods.

**Remark 4.3** From (4.30), (4.32), and $\eta^d(t) \geq 0$ and $\zeta^d(\tau_1) \geq 0$ in (4.13), we can obtain the conditions

$$\dot{\eta}(t) \leq 0 \tag{4.34}$$

and

$$\zeta(\tau_1) \geq \eta(\tau_1^+) \text{ at each entry time } \tau_1, \tag{4.35}$$

which are useful to know about. Hartl et al. (1995) and Feichtinger and Hartl (1986) also add these conditions to the indirect maximum principle necessary conditions (4.29).

**Remark 4.4** In Exercise 4.12, we discuss the indirect method for higher-order constraints. For further details, see Pontryagin et al. (1962), Bryson and Ho (1975) and Hartl et al. (1995).

**Example 4.3** Consider the problem:

$$\max \left\{ J = \int_0^2 -x dt \right\}$$

subject to

$$\dot{x} = u, \ x(0) = 1, \tag{4.36}$$

$$u + 1 \geq 0, \ 1 - u \geq 0, \tag{4.37}$$

$$x \geq 0. \tag{4.38}$$

Note that this problem is the same as Example 2.3, except for the nonnegativity constraint (4.38).

**Solution** The Hamiltonian is

$$H = -x + \lambda u,$$

which implies the optimal control to have the form

$$u^*(x, \lambda) = \text{bang}[-1, 1; \lambda], \ \text{whenever } x > 0. \tag{4.39}$$

When $x = 0$, we impose $\dot{x} = u \geq 0$ in order to insure that (4.38) holds. Therefore, the optimal control on the state constraint boundary is

$$u^*(x, \lambda) = \text{bang}[0, 1; \lambda], \ \text{whenever } x = 0. \tag{4.40}$$

Now we form the Lagrangian

$$L = H + \mu_1(u + 1) + \mu_2(1 - u) + \eta u,$$

where $\mu_1, \mu_2$, and $\eta$ satisfy the complementary slackness conditions

$$\mu_1 \geq 0, \quad \mu_1(u + 1) = 0, \tag{4.41}$$

$$\mu_2 \geq 0, \quad \mu_2(1 - u) = 0, \tag{4.42}$$

$$\eta \geq 0, \quad \eta x = 0. \tag{4.43}$$

Furthermore, the optimal trajectory must satisfy

$$\frac{\partial L}{\partial u} = \lambda + \mu_1 - \mu_2 + \eta = 0. \tag{4.44}$$

From the Lagrangian we also get

$$\dot{\lambda} = -\frac{\partial L}{\partial x} = 1, \ \lambda(2^-) = \gamma \geq 0, \ \gamma x(2) = \ \lambda(2^-)x(2) = 0. \tag{4.45}$$

It is reasonable to guess that the optimal control $u^*$ will be the one that keeps $x^*$ as small as possible, subject to the state constraint (4.38). Thus,

$$u^*(t) = \begin{cases} -1, & t \in [0,1), \\ 0, & t \in [1,2]. \end{cases} \tag{4.46}$$

This gives

$$x^*(t) = \begin{cases} 1-t, & t \in [0,1), \\ 0, & t \in [1,2]. \end{cases}$$

To obtain $\lambda(t)$, let us first try $\lambda(2^-) = \gamma = 0$. Then, since $x^*(t)$ enters the boundary zero at $t = 1$, there are no jumps in the interval $(1,2]$, and the solution for $\lambda(t)$ is

$$\lambda(t) = t - 2, \ \ t \in (1,2). \tag{4.47}$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 0$ on $(1,2]$, we have $u^*(t) = 0$ by (4.40), as stipulated. Now let us see what must happen at $t = 1$. We know from (4.47) that $\lambda(1^+) = -1$. To obtain $\lambda(1^-)$, we see that $H(1^+) = -x^*(1^+) + \lambda(1^+)u^*(1^+) = 0$ and $H(1^-) = -x^*(1^-) + \lambda(1^-)u^*(1^-) = -\lambda(1^-)$. By equating $H(1^-)$ to $H(1^+)$ as required in (4.29), we obtain $\lambda(1^-) = 0$. Using now the jump condition on $\lambda(t)$ in (4.29), we get the value of the jump $\zeta(1) = \lambda(1^-) - \lambda(1^+) = 1 \geq 0$.

With $\lambda(1^-) = 0$, we can solve (4.45) to obtain

$$\lambda(t) = t - 1, \ \ t \in [0,1].$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 1-t > 0$ is positive on [0,1), we can use (4.39) to obtain $u^*(t) = -1$ for $0 \leq t < 1$, which is as stipulated in (4.46). In the time interval [0,1) by (4.42), $\mu_2 = 0$ since $u^* < 1$, and by (4.43), $\eta = 0$ because $x > 0$. Therefore, $\mu_1(t) = -\lambda(t) = 1 - t > 0$ for $0 \leq t < 1$, and this with $u^* = -1$ satisfies (4.41).

To complete the solution, we calculate the Lagrange multipliers in the interval [1,2]. Since $u^*(t) = 0$ on $t \in [1,2]$, we have $\mu_1(t) = \mu_2(t) = 0$. Then, from (4.44) we obtain $\eta(t) = -\lambda(t) = 2 - t \geq 0$ which, with

$x^*(t) = 0$ satisfies (4.43). Thus, our guess $\gamma = 0$ is correct, and we do not need to examine the possibility of $\gamma > 0$. The graphs of $x^*(t)$ and $\lambda(t)$ are shown in Fig. 4.2. In Exercise 4.1, you are asked to redo Example 4.3 by guessing that $\gamma > 0$ and see that it leads to a contradiction with a condition of the maximum principle.

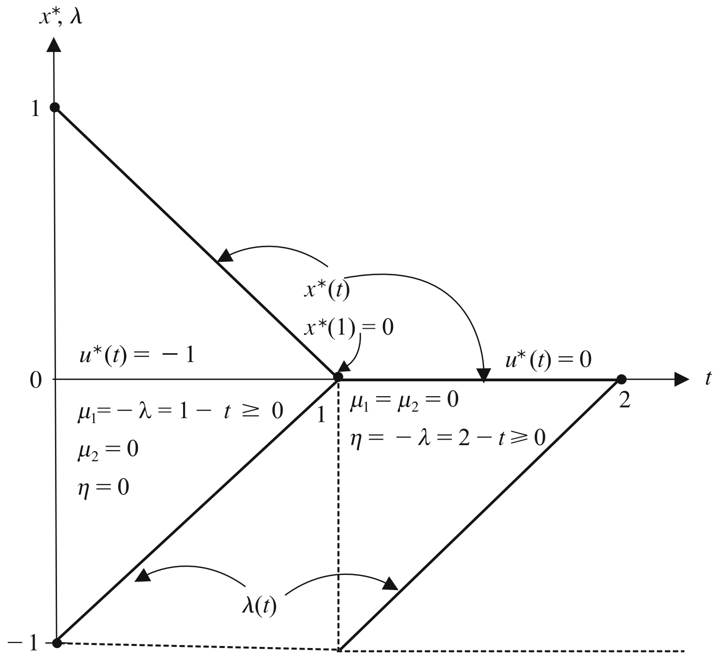


Figure 4.2: State and adjoint trajectories in Example 4.3

It should be obvious that if the terminal time were $T = 1.5$, the optimal control would be $u^*(t) = -1$, $t \in [0, 1)$ and $u^*(t) = 0$, $t \in [1, 1.5]$. You are asked in Exercise 4.10 to redo the above calculations in this case and show that one now needs to have $\gamma = 1/2$. In Exercise 4.3, you are asked to solve a similar problem with $F = -u$.

**Remark 4.5** Example 4.3 is a problem instance in which the state constraint is active at the terminal time. In instances where the initial state or the final state or both are on the constraint boundary, the maximum principle may *degenerate* in the sense that there is no nontrivial solution of the necessary conditions, i.e., $\lambda(t) \equiv 0$, $t \in [0, T]$, where $T$ is the terminal time. See Arutyunov and Aseev (1997) or Ferreira and Vinter (1994) for conditions that guarantee a nontrivial solution for the multipliers.

**Remark 4.6** It can easily be seen that Example 4.3 is a problem instance in which multipliers $\lambda$ and $\mu_1$ would not be unique if the jump condition on the Hamiltonian in (4.29) was not imposed. For references dealing with the issue of non-uniqueness of the multipliers and conditions under which the multipliers are unique, see Kurcyusz and Zowe (1979), Maurer (1977, 1979), Maurer and Wiegand (1992), and Shapiro (1997).

**Example 4.4** The purpose here is to show that the solution obtained in Example 4.3 satisfies the sufficiency conditions of Theorem 4.1. For this we first obtain the direct adjoint variable

$$\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t) = \begin{cases} t-1, & t \in [0,1), \\ 0, & t \in [1,2). \end{cases}$$

It is easy to see that

$$H(x, u, \lambda^d(t), t) = \begin{cases} -x + (t-1)u, & t \in [0,1), \\ -x, & t \in [1,2], \end{cases}$$

is linear and hence concave in $(x, u)$ at each $t \in [0, 2]$. Functions

$$g(x, u, t) = \begin{pmatrix} u+1 \\ 1-u \end{pmatrix}$$

and

$$h(x) = x$$

are linear and hence quasiconcave in $(x, u)$ and $x$, respectively. Functions $S \equiv 0$, $a \equiv 0$ and $b \equiv 0$ satisfy the conditions of Theorem 4.1 trivially. Thus, the solution obtained for Example 4.3 satisfies all conditions of Theorem 4.1, and is therefore optimal.

In Exercise 4.14, you are asked to use Theorem 4.1 to verify that the given solution there is optimal.

**Example 4.5** Consider Example 4.3 with $T = 3$ and the terminal state constraint

$$x(3) = 1.$$

**Solution** Clearly, the optimal control $u^*$ will be the one that keeps $x$ as small as possible, subject to the state constraint (4.38) and the boundary condition $x(0) = x(3) = 1$. Thus,

$$u^*(t) = \begin{cases} -1, & t \in [0,1), \\ 0, & t \in [1,2], \\ 1, & t \in (2,3], \end{cases} \quad x^*(t) = \begin{cases} 1-t, & t \in [0,1), \\ 0, & t \in [1,2], \\ t-2, & t \in (2,3]. \end{cases}$$

For brevity, we will not provide the same level of detailed explanation as we did in Example 4.3. Rather, we will only compute the adjoint function and the multipliers that satisfy the optimality conditions. These are

$$\lambda(t) = \begin{cases} t-1, & t \in [0,1], \\ t-2, & t \in (1,3), \end{cases} \tag{4.48}$$

$$\mu_1(t) = \mu_2(t) = 0,\ \eta(t) = -\lambda(t),\ t \in [1,2], \tag{4.49}$$

$$\gamma = 0, \quad \beta = \lambda(2^-) = 1, \tag{4.50}$$

and the jump $\zeta(1) = 1 \geq 0$ so that

$$\lambda(1^-) = \lambda(1^+) + \zeta(1) \text{ and } H(1^-) = H(1^+). \tag{4.51}$$

**Example 4.6** Introduce a discount rate $\rho > 0$ in Example 4.1 so that the objective function becomes

$$\max \left\{ J = \int_0^3 -e^{-\rho t} u dt \right\} \tag{4.52}$$

and re-solve using the indirect maximum principle (4.29).

**Solution** It is obvious that the optimal solution will remain the same as (4.5), shown also in Fig. 4.1.

With $u^*$ and $x^*$ as in (4.5), we must obtain $\lambda, \mu_1, \mu_2, \eta, \gamma$, and $\zeta$ so that the necessary optimality conditions (4.29) hold, i.e.,

$$H = -e^{-\rho t} u + \lambda u, \tag{4.53}$$

$$L = H + \mu_1 u + \mu_2 (3-u) + \eta[u + 2(t-2)], \tag{4.54}$$

$$L_u = -e^{-\rho t} + \lambda + \mu_1 - \mu_2 + \eta = 0, \tag{4.55}$$

$$\dot{\lambda} = -L_x = 0,\ \lambda(3^-) = 0, \tag{4.56}$$

$$\gamma[x^*(3) - 1 + (1-2)^2] = 0, \tag{4.57}$$

$$\mu_1 \geq 0,\ \mu_1 u = 0,\ \mu_2 \geq 0,\ \mu_2(3-u) = 0, \tag{4.58}$$

$$\eta \geq 0,\ \eta[x^*(t) - 1 + (t-2)^2] = 0, \tag{4.59}$$

and if $\lambda$ is discontinuous at the entry time $\tau = 1$, then

$$\lambda(1^-) = \lambda(1^+) + \zeta(1),\ \zeta(1) \geq 0, \tag{4.60}$$

$$-e^{-\rho}u^*(1^-) + \lambda(1^-)u^*(1^-) = -e^{-\rho}u^*(1^+) + \lambda(1^+) - \zeta(1)(-2). \tag{4.61}$$

From (4.60), we obtain $\lambda(1^-) = e^{-\rho}$. This with (4.56) gives

$$\lambda(t) = \begin{cases} e^{-\rho}, & 0 \leq t < 1, \\ 0, & 1 \leq t \leq 3, \end{cases}$$

as shown in Fig. 4.3,

$$\mu_1(t) = \begin{cases} e^{-\rho t} - e^{-\rho}, & 0 \leq t < 1, \\ 0, & 1 \leq t \leq 2, \\ e^{-\rho t}, & 2 < t \leq 3, \end{cases} \quad \mu_2(t) = 0,\ 0 \leq t \leq 3,$$

and

$$\eta(t) = \begin{cases} 0, & 0 \leq t < 1, \\ e^{-\rho t}, & 1 \leq t \leq 2, \\ 0, & 2 < t \leq 3, \end{cases}$$

which, along with $u^*$ and $x^*$, satisfy (4.29).

Note, furthermore, that $\lambda$ is continuous at the exit time $t = 2$. At the entry time $\tau_1 = 1$, $\zeta(1) = e^{-\rho} \geq \eta(1^+) = e^{-\rho}$, so that (4.35) also holds. Finally, $\gamma = \eta(3^-) = 0$.

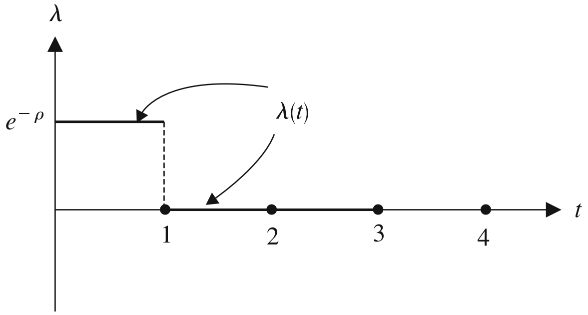


Figure 4.3: Adjoint trajectory for Example 4.4

## 4.6 Current-Value Maximum Principle: Indirect Method

Just as the necessary condition (3.42) represents the current-value formulation corresponding to (3.12), we can, when first-order pure state constraints are present, also state the current-value formulation of the necessary conditions (4.29). As in Sect. 3.3, with $F(x,u,t) = \phi(x,u)e^{-\rho t}$, $S(x,T) = \psi(x)e^{-\rho T}$, and $\rho > 0$, the objective function in the problem (4.11) is replaced by

$$\max \left\{ J = \int_0^T \phi(x,u)e^{-\rho t}dt + \psi[x(T)]e^{-\rho T} \right\}.$$

With the Hamiltonian $H$ as defined in (3.35), we can write the Lagrangian as

$$L[x,u,\lambda,\mu,\eta] := H + \mu g + \eta h^1 = \phi + \lambda f + \mu g + \eta h^1.$$

We can now state the current-value form of the maximum principle, giving the necessary conditions for $u^*$ (with the state trajectory $x^*$) to be optimal. These conditions are that there exist an adjoint function $\lambda$, which may be discontinuous at each entry or contact time, multiplier functions $\mu, \alpha, \beta, \gamma, \eta$, and a jump parameter $\zeta(\tau)$ at each $\tau$ where $\lambda^d$ is discontinuous, such that the following (4.62) holds:

$$
\boxed{
\begin{array}{l}
\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0, \text{ satisfying constraints} \\
g(x^*, u^*, t) \geq 0,\ h(x^*(t), t) \geq 0, \text{and the terminal constraints} \\
a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0; \\
\dot{\lambda} = \rho\lambda - L_x[x^*, u^*, \lambda, \mu, \eta, t] \\
\text{with the transversality conditions} \\
\lambda(T^-) = \psi_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) \\
\qquad\qquad +\gamma h_x(x^*(T), T), \text{ and} \\
\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma \geq 0,\ \gamma h(x^*(T), T) = 0; \\
\text{the Hamiltonian maximizing condition} \\
H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \\
\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying} \\
g[x^*(t), u, t] \geq 0, \text{ and} \\
h_i^1(x^*(t), u, t) \geq 0 \text{ whenever } h_i(x^*(t), t) = 0, i = 1, 2, \cdots, p; \\
\text{the jump conditions at any entry/contact time } \tau, \\
\text{where } \lambda \text{ is discontinuous, are} \\
\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau) \text{ and} \\
H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau] \\
\qquad\qquad\qquad\qquad -\zeta(\tau) h_t(x^*(\tau), \tau); \\
\text{the Lagrange multipliers } \mu(t) \text{ are such that} \\
\partial L/\partial u|_{u=u^*(t)} = 0,\ dH/dt = dL/dt = \partial L/\partial t + \rho\lambda f, \\
\text{and the complementary slackness conditions} \\
\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0, \\
\eta(t) \geq 0,\ \eta(t) h(x^*(t), t) = 0, \text{ and} \\
\zeta(\tau) \geq 0,\ \zeta(\tau) h(x^*(\tau), \tau) = 0 \text{ hold.}
\end{array}
}
\tag{4.62}
$$

If $T \in [T_1, T_2],\ 0 \le T_1 < T_2 < \infty$, is also a decision variable, then if $T^*$ is the optimal terminal time, then the optimal solution $x^*, u^*, T^*$ must satisfy (4.62) with $T$ replaced by $T^*$ and the condition

$$H[x^*(T^*), u^*(T^{*-}), \lambda^d(T^{*-}), T^*] - \rho\psi[x^*(T^*), T^*] + \alpha a_T[x^*(T^*), T^*]$$
$$+\beta b_T[x^*(T^*), T^*] + \gamma^d h_T[x^*(T^*), T^*] \begin{cases} \le 0 & \text{if } T^* = T_1, \\ = 0 & \text{if } T^* \in (T_1, T_2), \\ \ge 0 & \text{if } T^* = T_2. \end{cases} \quad (4.63)$$

Derivation of (4.63) starting from (4.15) is similar to that of (3.44) from (3.15).

**Remark 4.7** The current-value version of (4.34) in Remark 4.3 is $\dot{\eta}(t) \le \rho\eta(t)$ and (4.35).

The infinite horizon problem with pure and mixed constraints can be stated as (3.97) with an additional constraint (4.7). As in Sect. 3.6, the conditions in (4.62) except the transversality condition on the adjoint variable are still necessary for optimality. As for the sufficiency conditions, an analogue of Theorem 4.1 holds, subject to the discussion on infinite horizon transversality conditions in Sect. 3.6.

We conclude this chapter with the following cautionary remark.

**Remark 4.8** While various subsets of conditions specified in the maximum principles (4.13), (4.29), or (4.62) have been proved in the literature, proofs of the entire results are still not available. For this reason, Hartl (1995) call (4.13), (4.29), or (4.62) as *informal theorems*. Seierstad and Sydsæter (1987) call them *almost necessary conditions* since, very rarely, problems arise where the optimal solution requires more complicated multipliers and adjoint variables than those specified in this chapter.

## Exercises for Chapter 4

**E 4.1** Rework Example 4.3 by guessing that $\gamma > 0$, and show that it leads to a contradiction with a condition of the maximum principle.

**E 4.2** Rework Example 4.3 with terminal time $T = 1/2$.

**E 4.3** Change the objective function of Example 4.3 as follows:

$$\max \left\{ J = \int_0^2 (-u) dt \right\}.$$

Re-solve and show that the solution is not unique.

**E 4.4** Specialize the maximum principle (4.29) for the nonnegativity state constraint of the form

$$x(t) \geq 0 \text{ for all } t \text{ satisfying } 0 \leq t \leq T,$$

in place of $h(x, t) \geq 0$ in (4.7).

**E 4.5** Consider the problem:

$$\max \left\{ J = \int_0^T (-x) dt \right\}$$

subject to

$$\dot{x} = -u - 1, \; x(0) = 1,$$
$$x(t) \geq 0, \; 0 \leq u(t) \leq 1.$$

Show that

(a) If $T = 1$, there is exactly one feasible and optimal solution.
(b) If $T > 1$, then there is no feasible solution.
(c) If $0 < T < 1$, then there is a unique optimal solution.
(d) If the control constraint is $0 \leq u(t) \leq K$, there is a unique optimal solution for every $K \geq 1$ and $T = 1/2$.
(e) The value of the objective in (d) increases as $K$ increases.
(f) If the control constraint in (d) is $u(t) \geq 0$, then the optimal control is an impulse control defined by the limit of the solution in (e).

**E 4.6** Impose the constraint $x \geq 0$ on Exercise 3.16(b) to obtain the problem:

$$\max \left\{ J = \int_0^4 (-x) dt \right\}$$

subject to

$$\dot{x} = u, \;\; x(0) = 1, \;\; x(4) = 1,$$
$$u + 1 \geq 0, \;\; 1 - u \geq 0,$$
$$x \geq 0.$$

Find the optimal trajectories of the control variable, the state variable, and other multipliers. Also, graph these trajectories.

**E 4.7** Transform the problem (4.11) with the pure constraint of type (4.7) to a problem with the nonnegativity constraint of type (4.9).

Hint: Define $y = h(x,t)$ as an additional state variable. Recall that we have assumed (4.7) to be a first-order constraint.

**E 4.8** Consider a two-reservoir system such as that shown in Fig. 4.4, where $x_i(t)$ is the volume of water in reservoir $i$ and $u_i(t)$ is the rate of discharge from reservoir $i$ at time $t$. Thus,

$$\dot{x}_1(t) = -u_1(t), \qquad x_1(0) = 4,$$
$$\dot{x}_2(t) = u_1(t) - u_2(t), \quad x_2(0) = 4.$$

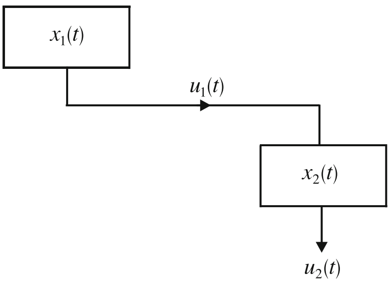


Figure 4.4: Two-reservoir system of Exercise 4.8

Solve the problem of maximizing

$$J = \int_0^{10} [(10 - t)u_1(t) + tu_2(t)]dt$$

subject to the above state equations and the constraints

$$0 \leq u_i(t) \leq 1, \;\; x_i(t) \geq 0 \text{ for all } t \in [0, 10].$$

Also compute the optimal value of the objective function.

Hint: Guess the optimal solution and verify it by using the Lagrangian form of the maximum principle.

**E 4.9** *An Inventory Control Problem.* Solve

$$\max_P \int_0^T -\left(hI + \frac{P^2}{2}\right) dt$$

subject to

$$\dot{I} = P - S, \; I(0) = I_0 > \frac{S^2}{2h},$$

and the control and the pure state inequality constraints

$$P \geq 0 \text{ and } I \geq 0,$$

respectively. Assume that $S > 0$ and $h > 0$ are constants and $T$ is sufficiently large. Note that $I$ represents inventory, $P$ represents production rate, and $S$ represents demand. The constraints on $P$ and $I$ mean that production must be nonnegative and backlogs are not allowed, respectively.

Hint: By $T$ being sufficiently large, we mean $T > I_0/S + S/(2h)$.

**E 4.10** Redo Example 4.3 with $T = 1.5$.

**E 4.11** Redo Example 4.6 using the current-value maximum principle (4.62) in Sect. 4.6.

**E 4.12** For this exercise only, assume that $h(x,t) \geq 0$ in (4.7) is a second-order constraint, i.e., $r = 2$. Transform the problem to one with nonnegativity constraints. Use the result in Exercise 4.4 to derive a maximum principle for problems with second-order constraints.

Hint: As in Exercise 4.7, define $y = h$. In addition, define yet another state variable $z = \dot{y} = dh/dt$. Note further that this procedure can be generalized to handle problems with $r$th-order constraints for any positive integer $r$.

**E 4.13** Re-solve Example 4.6 when $\rho < 0$.

**E 4.14** Consider the following problem:

$$\min \left\{ J = \int_0^5 u dt \right\}$$

subject to the state equation

$$\dot{x} = u - x, \; x(0) = 1,$$

and the control and state constraints

$$0 \le u \le 1, \; x(t) \ge 0.7 - 0.2t.$$

Use the sufficiency conditions in Theorem 4.1 to verify that the optimal control for the problem is

$$u^*(t) = \begin{cases} 0, & 0 \le t \le \theta, \\ 0.5 - 0.2t, & \theta < t \le 2.5, \\ 0, & 2.5 < t \le 5, \end{cases}$$

where $\theta \approx 0.51626$. Sketch the optimal state trajectory $x^*(t)$ for the problem.

**E 4.15** In Example 4.6, let $t^{\pm}(x) = 2 \pm \sqrt{1-x}$. Show that the value function

$$V(x,t) = \begin{cases} -\frac{2e^{-2\rho}+2(\rho\sqrt{1-x}-1)e^{-\rho(2-\sqrt{1-x})}}{\rho^2}, & \text{for } x < 1, 0 \le t \le t^-(x), \\ 0, & \text{for } x \ge 1 \text{ or } t^+(x) \le t \le 3. \end{cases}$$

Note that $V(x,t)$ is not defined for $x < 1, \; t^-(x) < t \le 3$. Show furthermore that for the given initial condition $x(0) = 0$, the marginal valuation is

$$V_x(x^*(t),t) = \lambda^d(t) = \lambda(t) + \eta(t) = \begin{cases} e^{-\rho}, & \text{for } t \in [0,1), \\ e^{-\rho t}, & \text{for } t \in [1,2], \\ 0, & \text{for } t \in (2,3]. \end{cases}$$

In this case, it is interesting to note that the marginal valuation is discontinuous at the constraint exit time $t = 2$.

**E 4.16** Show in Example 4.3 that the value function

$$V(x,t) = \begin{cases} -x^2/2, & \text{for } x \le 2-t,\ 0 \le t \le 2, \\ -2x+2-2t+xt+t^2/2, & \text{for } x > 2-t,\ 0 \le t \le 2. \end{cases}$$

Then verify that for the given initial condition $x(0) = 1$,

$$V_x(x^*(t),t) = \lambda^d(t) = \lambda(t) + \eta(t) = \begin{cases} t-1, & \text{for } t \in [0,1), \\ 0, & \text{for } t \in [1,2]. \end{cases}$$

**E 4.17** Rework Example 4.5 by using the direct maximum principle (4.13).

**E 4.18** Solve the linear inventory control problem of minimizing

$$\int_0^T (cP(T) + hI(t))dt$$

subject to

$$\dot{I}(t) = P(t) - S, \quad I(0) = 1,$$
$$P(t) \ge 0 \text{ and } I(t) \ge 0, \quad t \in [0,T],$$

where $P(t)$ denotes the production rate and $I(t)$ is the inventory level at time $t$ and where $c, h$ and $S$ are positive constants and the given terminal time $T > \sqrt{2S}$.

**E 4.19** A machine with quality $x(t) \ge 0$ produces goods with $ax(t)$ dollars per unit time at time $t$. The quality deteriorates at the rate $\delta$, but the decay can be slowed by a preventive maintenance $u(t)$ as follows:

$$\dot{x} = u - \delta x,\ x(0) = x_0 > 0.$$

Obtain the optimal maintenance rate $u(t),\ 0 \le t \le T$, so as to maximize

$$\int_0^T (ax - u)dt$$

subject to $u \in [0, \bar{u}]$ and $x \le \bar{x}$, where $\bar{u} > \delta\bar{x},\ a > \delta$, and $\bar{x} > x_0$.

Hint: Solve first the problem without the state constraint $x \le \bar{x}$. You will need to treat two cases: $\delta T \le \ln a - \ln(a-\delta)$ and $\delta T > \ln a - \ln(a-\delta)$.

**E 4.20** Maximize

$$J = \int_0^3 (u - x)dt$$

subject to

$$\dot{x} = 1 - u,\ x(0) = 2,$$

$$0 \le u \le 3,\ x + u \le 4,\ x \ge 0.$$

**E 4.21** Maximize

$$J = \int_0^2 (1 - x)dt$$

subject to

$$\dot{x} = u,\ x(0) = 1,$$

$$-1 \le u \le 1,\ x \ge 0.$$

**E 4.22** Maximize

$$J = \int_0^3 (4 - t)u dt$$

subject to

$$\dot{x} = u,\ x(0) = 0,\ x(3) = 3,$$

$$0 \le u \le 2,\ 1 + t - x \ge 0.$$

**E 4.23** Maximize

$$J = -\int_0^4 e^{-t}(u - 1)^2 dt$$

subject to

$$\dot{x} = u, \quad x(0) = 0,$$

$$x \le 2 + e^{-3}.$$

**E 4.24** Solve the following problem:

$$\max\left\{J=\int_0^2 (2u-x)dt\right\}$$

$$\dot{x}=-u,\quad x(0)=e,$$

$$-3\le u\le 3,\quad x-u\ge 0,\quad x\ge t.$$

**E 4.25** Solve the following problem:

$$\max\left\{J=\int_0^3 -2x_1 dt\right\}$$

$$\dot{x}_1=x_2,\quad x_1(0)=2,$$

$$\dot{x}_2=u,\quad x_2(0)=0,$$

$$x_1\ge 0.$$

**E 4.26** Re-solve Example 4.6 with the control constraint (4.3) replaced by $0\le u\le 1$.

**E 4.27** Solve explicitly the following problem:

$$\max\left\{J=-\int_0^2 x(t)dt\right\}$$

subject to

$$\dot{x}(t)=u(t),\ x(0)=1,$$

$$-a\ \le\ u(t)\le b,\ a>1/2,\ b>0,$$

$$x(t)\ \ge\ t-2.$$

Obtain $x^*(t),\ u^*(t)$ and all the required multipliers.

**E 4.28** Minimize

$$\int_0^T \frac{1}{2}(x^2+c^2u^2)dt$$

subject to

$$\dot{x}=u,\ \ x(0)=x_0>0,\ \ x(T)=0,$$

$$h_1(x,t)=x-a_1+b_1t\ge 0,$$

$$h_2(x,t)=a_2-b_2t-x\ge 0,$$

where $a_i,b_i>0, a_2>x_0>a_1$, and $a_2/b_2>a_1/b_1$; see Fig. 4.5. The optional path must begin at $x_0$ on the $x$-axis, stay in the shaded area, and end on the $t$-axis.

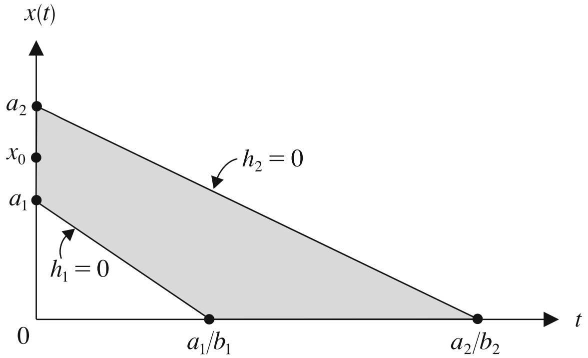


Figure 4.5: Feasible space for Exercise 4.28

(a) First, assume that the problem parameters are such that the optimal solution $x^*(t)$ satisfies $h_1(x^*(t), t) > 0$ for $t \in [0, T]$. Show that

$$x^*(t) = k_1 e^{t/c} + k_2 e^{-t/c},$$

where $k_1$ and $k_2$ are the constants to be determined. Write down the two conditions that would determine the constants. Also, illustrate graphically the optimal state trajectory.

(b) How would your solution change if the problem parameters do not satisfy the condition in (a)? Characterize and graphically illustrate the optimal state trajectory.

**E 4.29** With $a > 0,\ b > 0$, and $\dot{\gamma}(t)/\gamma(t) = -\rho(t) < 0$,

$$\max_{u,T} \left\{ J = \int_0^T \frac{a}{b}(1 - e^{-bu(t)})\gamma(t)dt \right\}$$

subject to

$$\dot{x} = -u, \quad x(0) = x_0 > 0 \text{ given},$$

and the constraint

$$x(t) \geq 0.$$

Obtain the expressions satisfied by the optimal terminal time $T^*$, the optimal control $u^*(t),\ 0 \leq t \leq T^*$, and the optimal state trajectory $x^*(t),\ 0 \leq t \leq T^*$. Furthermore, obtain them explicitly in the special case when $\rho(t) = \rho > 0$, a constant positive discount rate.

**E 4.30** Set $\rho = 0$ in the solution of Example 4.6 and obtain $\lambda, \gamma, \eta, \zeta(1)$ for the undiscounted problem. Then use the transformation formulas (4.30)–(4.33) on these and the fact that $\zeta(2) = 0$ to obtain $\lambda^d, \gamma^d, \eta^d$, and $\zeta^d(1)$ and $\zeta^d(2)$, and show that they are the same as those obtained in Example 4.2 along with $\zeta^d(1) = 0$, which holds trivially.

**E 4.31** Consider a finite-time economy in which production can be used for consumption as well as investment, but production also pollutes. The state equations for the capital stock $K$ and stock of pollution $W$ are

$$\dot{K} = suK,\ K(0) = K_0,$$

$$\dot{W} = uK - \delta W,\ W(0) = W_0,$$

where a fraction $s$ of the production output $uK$ is invested, with $u$ denoting the capacity utilization rate. The control constraints are

$$0 \le s \le 1,\ 0 \le u \le 1,$$

and the state constraint

$$W \le \bar{W}$$

implies that the pollution stock cannot exceed the upper bound $\bar{W}$.

The aim of the economy is to choose $s$ and $u$ so as to maximize the consumption utility

$$\int_0^T (1-s)uK dt.$$

Assume that $W_0 < \bar{W},\ T > 1$ and $W_0 - K_0/\delta)e^{-\delta T} + K_0/\delta < \bar{W}$, which means that even with $s(t) \equiv 0$, the pollution stock never reaches $\bar{W}$ even with $u(t) \equiv 1$.