Suresh P. Sethi

Optimal Control Theory

Applications to Management Science and Economics

Third Edition

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Suresh P. Sethi
Jindal School of Management, SM30
University of Texas at Dallas
Richardson, TX, USA

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This book is dedicated to the memory of my parents

Manak Bai and Gulab Chand Sethi

Preface to Third Edition

The third edition of this book will not see my co-author Gerald L. Thompson, who very sadly passed away on November 9, 2009. Gerry and I wrote the first edition of the 1981 book sitting practically side by side, and I learned a great deal about book writing in the process. He was also my PhD supervisor and mentor and he is greatly missed.

After having used the second edition of the book in the classroom for many years, the third edition arrives with new material and many improvements. Examples and exercises related to the interpretation of the adjoint variables and Lagrange multipliers are inserted in Chaps. 2–4. Direct maximum principle is now discussed in detail in Chap. 4 along with the existing indirect maximum principle from the second edition. Chattering or relaxed controls leading to pulsing advertising policies are introduced in Chap. 7. An application to information systems involving chattering controls is added as an exercise.

The objective function in Sect. 11.1.3 is changed to the more popular objective of maximizing the total discounted society's utility of consumption. Further discussion leading to obtaining a saddle-point path on the phase diagram leading to the long-run stationary equilibrium is provided in Sect. 11.2. For this purpose, a global saddle-point theorem is stated in Appendix D.7. Also inserted in Appendix D.8 is a discussion of the Sethi-Skiba points which lead to nonunique stable equilibria. Finally, a new Sect. 11.4 contains an adverse selection model with continuum of the agent types in a principal-agent framework, which requires an application of the maximum principle.

Chapter 12 of the second edition is removed except for the material on differential games and the distributed parameter maximum principle. The differential game material joins new topics of stochastic Nash differential games and Stackelberg differential games via their applications to marketing to form a new Chap. 13 titled Differential Games. As a result, Chap. 13 of the second edition becomes Chap. 12. The material on the distributed parameter maximum principle is now Appendix D.9.

The exposition is revised in some places for better reading. New exercises are added and the list of references is updated. Needless to say, the errors in the second edition are corrected, and the notation is made consistent.

Thanks are due to Huseyin Cavusoglu, Andrei Dmitruk, Gustav Feichtinger, Richard Hartl, Yonghua Ji, Subodha Kumar, Sirong Lao, Helmut Maurer, Ernst Presman, Anyan Qi, Andrea Seidl, Atle Seierstad, Xi Shan, Lingling Shi, Xiahong Yue, and the students in my Optimal Control Theory and Applications course over the years for their suggestions for improvement. Special thanks go to Qi (Annabelle) Feng for her dedication in updating and correcting the forthcoming solution manual that went with the first edition. I cannot thank Barbara Gordon and Lindsay Wilson enough for their assistance in the preparation of the text, solution manual, and presentation materials. In addition, the meticulous copy editing of the entire book by Lindsay Wilson is much appreciated. Anshuman Chutani, Pooja Kamble, and Shivani Thakkar are also thanked for their assistance in drawing some of the figures in the book.

Richardson, TX, USA — Suresh P. Sethi
June 2018

Preface to Second Edition

The first edition of this book, which provided an introduction to optimal control theory and its applications to management science to many students in management, industrial engineering, operations research and economics, went out of print a number of years ago. Over the years we have received feedback concerning its contents from a number of instructors who taught it, and students who studied from it. We have also kept up with new results in the area as they were published in the literature. For this reason we felt that now was a good time to come out with a new edition. While some of the basic material remains, we have made several big changes and many small changes which we feel will make the use of the book easier.

The most visible change is that the book is written in Latex and the figures are drawn in CorelDRAW, in contrast to the typewritten text and hand-drawn figures of the first edition. We have also included some problems along with their numerical solutions obtained using Excel.

The most important change is the division of the material in the old Chap. 3, into Chaps. 3 and 4 in the new edition. Chapter 3 now contains models having mixed (control and state) constraints, current value formulations, terminal conditions and model types, while Chap. 4 covers the more difficult topic of pure state constraints, together with mixed constraints. Each of these chapters contain new results that were not available when the first edition was published.

The second most important change is the expansion of the material in the old Sect. 12.4 on stochastic optimal control theory and its becoming the new Chap. 13. The new Chap. 12 now contains the following advanced topics on optimal control theory: differential games, distributed parameter systems, and impulse control. The new Chap. 13 provides a brief introduction to stochastic optimal control problems. It contains formulations of simple stochastic models in production, marketing and finance, and their solutions. We deleted the old Chap. 11 of the first edition on computational methods, since there are a number of excellent references now available on this topic. Some of these references are listed in Sect. 4.2 of Chap. 4 and Sect. 8.3 of Chap. 8.

The emphasis of this book is not on mathematical rigor, but rather on developing models of realistic situations faced in business and management. For that reason we have given, in Chaps. 2 and 8, proofs of the continuous and discrete maximum principles by using dynamic programming and Kuhn-Tucker theory, respectively. More general maximum principles are stated without proofs in Chaps. 3, 4 and 12.

One of the fascinating features of optimal control theory is its extraordinarily wide range of possible applications. We have covered some of these as follows: Chap. 5 covers finance; Chap. 6 considers production and inventory problems; Chap. 7 covers marketing problems; Chap. 9 treats machine maintenance and replacement; Chap. 10 deals with problems of optimal consumption of natural resources (renewable or exhaustible); and Chap. 11 discusses a number of applications of control theory to economics. The contents of Chaps. 12 and 13 have been described earlier.

Finally, four appendices cover either elementary material, such as the theory of differential equations, or very advanced material, whose inclusion in the main text would interrupt its continuity. At the end of the book is an extensive but not exhaustive bibliography of relevant material on optimal control theory including surveys of material devoted to specific applications.

We are deeply indebted to many people for their part in making this edition possible. Onur Arugaslan, Gustav Feichtinger, Neil Geismar, Richard Hartl, Steffen Jørgensen, Subodha Kumar, Helmut Maurer, Gerhard Sorger, and Denny Yeh made helpful comments and suggestions about the first edition or preliminary chapters of this revision. Many students who used the first edition, or preliminary chapters of this revision, also made suggestions for improvements. We would like to express our gratitude to all of them for their help. In addition we express our appreciation to Eleanor Balocik, Frank (Youhua) Chen, Feng Cheng, Howard Chow, Barbara Gordon, Jiong Jiang, Kuntal Kotecha, Ming Tam, and Srinivasa Yarrakonda for their typing of the various drafts of the manuscript. They were advised by Dirk Beyer, Feng Cheng, Subodha Kumar, Young Ryu, Chelliah Sriskandarajah, Wulin Suo, Houmin Yan, Hanqin Zhang, and Qing Zhang on the technical problems of using LATEX.

We also thank our wives and children—Andrea, Chantal, Anjuli, Dorothea, Allison, Emily, and Abigail—for their encouragement and understanding during the time-consuming task of preparing this revision.

Finally, while we regret that lack of time and pressure of other duties prevented us from bringing out a second edition soon after the first edition went out of print, we sincerely hope that the wait has been worthwhile. In spite of the numerous applications of optimal control theory which already have been made to areas of management science and economics, we continue to believe there is much more that remains to be done. We hope the present revision will rekindle interest in furthering such applications, and will enhance the continued development in the field.

Richardson, TX, USA — Suresh P. Sethi
Pittsburgh, PA, USA — Gerald L. Thompson
January 2000

Preface to First Edition

The purpose of this book is to exposit, as simply as possible, some recent results obtained by a number of researchers in the application of optimal control theory to management science. We believe that these results are very important and deserve to be widely known by management scientists, mathematicians, engineers, economists, and others. Because the mathematical background required to use this book is two or three semesters of calculus plus some differential equations and linear algebra, the book can easily be used to teach a course in the junior or senior undergraduate years or in the early years of graduate work. For this purpose, we have included numerous worked-out examples in the text, as well as a fairly large number of exercises at the end of each chapter. Answers to selected exercises are included in the back of the book. A solutions manual containing completely worked-out solutions to all of the 205 exercises is also available to instructors.

The emphasis of the book is not on mathematical rigor, but on modeling realistic situations faced in business and management. For that reason, we have given in Chaps. 2 and 7 only heuristic proofs of the continuous and discrete maximum principles, respectively. In Chap. 3 we have summarized, as succinctly as we can, the most important model types and terminal conditions that have been used to model management problems. We found it convenient to put a summary of almost all the important management science models on two pages: see Tables 3.1 and 3.3.

One of the fascinating features of optimal control theory is the extraordinarily wide range of its possible applications. We have tried to cover a wide variety of applications as follows: Chap. 4 covers finance; Chap. 5 considers production and inventory; Chap. 6 covers marketing; Chap. 8 treats machine maintenance and replacement; Chap. 9 deals with problems of optimal consumption of natural resources (renewable or exhaustible); and Chap. 10 discusses several economic applications.

In Chap. 11 we treat some computational algorithms for solving optimal control problems. This is a very large and important area that needs more development.

Chapter 12 treats several more advanced topics of optimal control: differential games, distributed parameter systems, optimal filtering, stochastic optimal control, and impulsive control. We believe that some of these models are capable of wider applications and further theoretical development.

Finally, four appendixes cover either elementary material, such as differential equations, or advanced material, whose inclusion in the main text would spoil its continuity. Also at the end of the book is a bibliography of works actually cited in the text. While it is extensive, it is by no means an exhaustive bibliography of management science applications of optimal control theory. Several surveys of such applications, which contain many other important references, are cited.

We have benefited greatly during the writing of this book by having discussions with and obtaining suggestions from various colleagues and students. Our special thanks go to Gustav Feichtinger for his careful reading and suggestions for improvement of the entire book. Carl Norström contributed two examples to Chaps. 4 and 5 and made many suggestions for improvement. Jim Bookbinder used the manuscript for a course at the University of Toronto, and Tom Morton suggested some improvements for Chap. 5. The book has also benefited greatly from various coauthors with whom we have done research over the years. Both of us also have received numerous suggestions for improvements from the students in our applied control theory courses taught during the past several years. We would like to express our gratitude to all these people for their help.

The book has gone through several drafts, and we are greatly indebted to Eleanor Balocik and Rosilita Jones for their patience and careful typing.

Although the applications of optimal control theory to management science are recent and many fascinating applications have already been made, we believe that much remains to be done. We hope that this book will contribute to the popularity of the area and will enhance future developments.

Toronto, ON, Canada — Suresh P. Sethi
Pittsburgh, PA, USA — Gerald L. Thompson
August 1981

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Chapter 1

What Is Optimal Control Theory?

Many management science applications involve the control of dynamic systems, i.e., systems that evolve over time. They are called *continuous-time systems* or *discrete-time systems* depending on whether time varies continuously or discretely. We will deal with both kinds of systems in this book, although the main emphasis will be on continuous-time systems.

Optimal control theory is a branch of mathematics developed to find optimal ways to control a dynamic system. The purpose of this book is to give an elementary introduction to the mathematical theory, and then apply it to a wide variety of different situations arising in management science. We have deliberately kept the level of mathematics as simple as possible in order to make the book accessible to a large audience. The only mathematical requirements for this book are elementary calculus, including partial differentiation, some knowledge of vectors and matrices, and elementary ordinary and partial differential equations. The last topic is briefly covered in Appendix A. Chapter 12 on stochastic optimal control also requires some concepts in stochastic calculus, which are introduced at the beginning of that chapter.

The principle management science applications discussed in this book come from the following areas: finance, economics, production and inventory, marketing, maintenance and replacement, and the consumption of natural resources. In each major area we have formulated one or more simple models followed by a more complicated model. The reader may

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wish at first to cover only the simpler models in each area to get an idea of what could be accomplished with optimal control theory. Later, the reader may wish to go into more depth in one or more of the applied areas.

Examples are worked out in most of the chapters to facilitate the exposition. At the end of each chapter, we have listed exercises that the reader should solve for deeper understanding of the material presented in the chapter. Hints are supplied with some of the exercises. Answers to selected exercises are given in Appendix E.

1.1 Basic Concepts and Definitions

We will use the word *system* as a primitive term in this book. The only property that we require of a system is that it is capable of existing in various *states*. Let the (real) variable $x(t)$ be the *state variable* of the system at time $t \in [0, T]$, where $T > 0$ is a specified time horizon for the system under consideration. For example, $x(t)$ could measure the inventory level at time t, the amount of advertising goodwill at time t, or the amount of unconsumed wealth or natural resources at time t.

We assume that there is a way of controlling the state of the system. Let the (real) variable $u(t)$ be the *control variable* of the system at time t. For example, $u(t)$ could be the production rate at time t, the advertising rate at time t, etc.

Given the values of the state variable $x(t)$ and the control variable $u(t)$ at time t, the *state equation*, a differential equation,

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \tag{1.1}$$

specifies the instantaneous rate of change in the state variable, where $\dot{x}(t)$ is a commonly used notation for $dx(t)/dt$, f is a given function of x, u, and t, and x_0 is the initial value of the state variable. If we know the initial value x_0 and the *control trajectory*, i.e., the values of $u(t)$ over the whole time interval $0 \le t \le T$, then we can integrate (1.1) to get the *state trajectory*, i.e., the values of $x(t)$ over the same time interval. We want to choose the control trajectory so that the state and control trajectories maximize the *objective functional*, or simply the *objective function*,

$$J = \int_0^T F(x(t), u(t), t)dt + S[x(T), T]. \tag{1.2}$$

In (1.2), F is a given function of x, u, and t, which could measure the benefit minus the cost of advertising, the utility of consumption, the negative of the cost of inventory and production, etc. Also in (1.2), the function S gives the *salvage value* of the ending state $x(T)$ at time T. The salvage value is needed so that the solution will make "good sense" at the end of the horizon.

Usually the control variable $u(t)$ will be constrained. We indicate this as

$$u(t) \in \Omega(t), \quad t \in [0, T], \tag{1.3}$$

where $\Omega(t)$ is the set of feasible values for the control variable at time t.

Optimal control problems involving (1.1), (1.2), and (1.3) will be treated in Chap. 2.

In Chap. 3, we will replace (1.3) by inequality constraints involving control variables. In addition, we will allow these constraints to depend on state variables. These are called mixed inequality constraints and written as

$$g(x(t), u(t), t) \geq 0, \quad t \in [0, T], \tag{1.4}$$

where g is a given function of u, t, and possibly x.

In addition, there may be constraints involving only state variables, but not control variables. These are written as

$$h(x(t), t) \geq 0, \quad t \in [0, T], \tag{1.5}$$

where h is a given function of x and t. Such constraints are the most difficult to deal with, and are known as pure state inequality constraints. Problems involving (1.1), (1.2), (1.4), and (1.5) will be treated in Chap. 4.

Finally, we note that all of the imposed constraints limit the values that the terminal state $x(T)$ may take. We denote this by saying

$$x(T) \in X, \tag{1.6}$$

where X is called the *reachable set* of the state variable at time T. Note that X depends on the initial value x_0. Here X is the set of possible terminal values that can be reached when $x(t)$ and $u(t)$ obey imposed constraints.

Although the above description of the control problem may seem abstract, you will find that in each specific application, the variables and parameters will have specific meanings that make them easy to understand and remember. The examples that follow will illustrate this point.

1.2 Formulation of Simple Control Models

We now formulate three simple models chosen from the areas of production, advertising, and economics. Our only objective here is to identify and interpret in these models each of the variables and functions described in the previous section. The solutions for each of these models will be given in detail in later chapters.

Example 1.1 *A Production-Inventory Model.* The various quantities that define this model are summarized in Table 1.1 for easy comparison with the other models that follow.

Table 1.1: The production-inventory model of Example 1.1

State variable	$I(t)$ = Inventory level
Control variable	$P(t)$ = Production rate
State equation	$\dot{I}(t) = P(t) - S(t),\ I(0) = I_0$
Objective function	Maximize $\left\{J = \int_0^T -[h(I(t)) + c(P(t))]dt\right\}$
State constraint	$I(t) \geq 0$
Control constraints	$0 \leq P_{\min} \leq P(t) \leq P_{\max}$
Terminal condition	$I(T) \geq I_{\min}$
Exogenous functions	$S(t)$ = Demand rate
	$h(I)$ = Inventory holding cost
	$c(P)$ = Production cost
Parameters	T = Terminal time
	$I_{\min}$ = Minimum ending inventory
	$P_{\min}$ = Minimum possible production rate
	$P_{\max}$ = Maximum possible production rate
	I_0 = Initial inventory level

We consider the production and inventory storage of a given good, such as steel, in order to meet an exogenous demand. The state variable $I(t)$ measures the number of tons of steel that we have on hand at time $t \in [0, T]$. There is an exogenous demand rate $S(t)$ tons of steel per day at time $t \in [0, T]$, and we must choose the production rate $P(t)$ tons of steel per day at time $t \in [0, T]$. Given the initial inventory of I_0 tons of steel on hand at $t = 0$, the state equation

$$\dot{I}(t) = P(t) - S(t)$$

describes how the steel inventory changes over time. Since $h(I)$ is the cost of holding inventory I in dollars per day, and $c(P)$ is the cost of producing steel at rate P, also in dollars per day, the objective function is to maximize the negative of the sum of the total holding and production costs over the period of T days. Of course, maximizing the negative sum is the same as minimizing the sum of holding and production costs. The state variable constraint, $I(t) \geq 0$, is imposed so that the demand is satisfied for all t. In other words, *backlogging* of demand is not permitted. (An alternative formulation is to make $h(I)$ become very large when I becomes negative, i.e., to impose a *stockout* penalty cost.) The control constraints keep the production rate $P(t)$ between a specified lower bound $P_{\min}$ and a specified upper bound $P_{\max}$. Finally, the terminal constraint $I(T) \geq I_{\min}$ is imposed so that the terminal inventory is at least $I_{\min}$.

The statement of the problem is lengthy because of the number of variables, functions, and parameters which are involved. However, with the production and inventory interpretations as given, it is not difficult to see the reasons for each condition. In Chap. 6, various versions of this model will be solved in detail. In Sect. 12.2, we will deal with a stochastic version of this model.

Example 1.2 *An Advertising Model.* The various quantities that define this model are summarized in Table 1.2.

We consider a special case of the Nerlove-Arrow advertising model which will be discussed in detail in Chap. 7. The problem is to determine the rate at which to advertise a product at each time t. Here the state variable is *advertising goodwill*, $G(t)$, which measures how well the product is known at time t. We assume that there is a *forgetting coefficient* δ, which measures the rate at which customers tend to forget the product.

To counteract forgetting, advertising is carried out at a rate measured by the control variable $u(t)$. Hence, the state equation is

$$\dot{G}(t) = u(t) - \delta G(t),$$

with $G(0) = G_0 > 0$ specifying the initial goodwill for the product.

Table 1.2: The advertising model of Example 1.2

State variable	$G(t) =$ Advertising goodwill
Control variable	$u(t) =$ Advertising rate
State equation	$\dot{G}(t) = u(t) - \delta G(t), G(0) = G_0$
Objective function	Maximize $\left\{ J = \int_0^\infty [\pi(G(t)) - u(t)]e^{-\rho t} dt \right\}$
State constraint	$\cdots$
Control constraints	$0 \leq u(t) \leq Q$
Terminal condition	$\cdots$
Exogenous function	$\pi(G) =$ Gross profit rate
Parameters	$\delta =$ Goodwill decay constant
	$\rho =$ Discount rate
	$Q =$ Upper bound on advertising rate
	$G_0 =$ Initial goodwill level

The objective function J requires special discussion. Note that the integral defining J is from time $t = 0$ to time $t = \infty$; we will later call a problem having an upper time limit of ∞, an *infinite horizon problem*. Because of this upper limit, the integrand of the objective function includes the discount factor $e^{-\rho t}$, where $\rho > 0$ is the (constant) discount rate. Without this discount factor, the integral would (in most cases) diverge to infinity. Hence, we will see that such a discount factor is an essential part of infinite horizon models. The rest of the integrand in the objective function consists of the gross profit rate $\pi(G(t))$, which results from the goodwill level $G(t)$ at time t less the cost of advertising assumed to be proportional to $u(t)$ (proportionality factor = 1); thus $\pi(G(t)) - u(t)$ is the net profit rate at time t. Also $[\pi(G(t)) - u(t)]e^{-\rho t}$ is the net profit rate at time t discounted to time 0, i.e., the present value of the time t profit rate. Hence, J can be interpreted as the total value of discounted future profits, and is the quantity we are trying to maximize.

There are control constraints $0 \leq u(t) \leq Q$, where Q is the upper bound on the advertising rate. However, there is no state constraint. It can be seen from the state equation and the control constraints that the goodwill $G(t)$ in fact never becomes negative.

You will find it instructive to compare this model with the previous one and note the similarities and differences between the two.

Example 1.3 *A Consumption Model.* Rich Rentier plans to retire at age 65 with a lump sum pension of W_0 dollars. Rich estimates his remaining life span to be T years. He wants to consume his wealth during these T retirement years, beginning at the age of 65, and leave a bequest to his heirs in a way that will maximize his total utility of consumption and bequest.

Since he does not want to take investment risks, Rich plans to put his money into a savings account that pays interest at a continuously compounded rate of r. In order to formulate Rich's optimization problem, let $t = 0$ denote the time when he turns 65 so that his retirement period can be denoted by the interval $[0, T]$. If we let the state variable $W(t)$ denote Rich's wealth and the control variable $C(t) \geq 0$ denote his rate of consumption at time $t \in [0, T]$, it is easy to see that the state equation is

$$\dot{W}(t) = rW(t) - C(t),$$

with the initial condition $W(0) = W_0 > 0$. It is reasonable to require that $W(t) \geq 0$ and $C(t) \geq 0, \;\; t \in [0, T]$. Letting $U(C)$ be the utility function of consumption C and $B(W)$ be the bequest function of leaving a bequest of amount W at time T, we see that the problem can be stated as an optimal control problem with the variables, equations, and constraints shown in Table 1.3.

Note that the objective function has two parts: first the integral of the discounted utility of consumption from time 0 to time T with ρ as the discount rate; and second the bequest function $e^{-\rho T}B(W)$, which measures Rich's discounted utility of leaving an estate W to his heirs at time T. If he has no heirs and does not care about charity, then $B(W) = 0$. However, if he has heirs or a favorite charity to whom he wishes to leave money, then $B(W)$ measures the strength of his desire to leave an estate of amount W. The nonnegativity constraints on state and control variables are obviously natural requirements that must be imposed.

You will be asked to solve this problem in Exercise 2.1 after you have learned the maximum principle in the next chapter. Moreover, a stochastic extension of the consumption problem, known as a consumption/investment problem, will be discussed in Sect. 12.4.

Table 1.3: The consumption model of Example 1.3

State variable	$W(t)$ = Wealth
Control variable	$C(t)$ = Consumption rate
State equation	$\dot{W}(t) = rW(t) - C(t),\ W(0) = W_0$
Objective function	$\text{Max}\left\{J = \int_0^T U(C(t))e^{-\rho t}dt + B(W(T))e^{-\rho T}\right\}$
State constraint	$W(t) \geq 0$
Control constraint	$C(t) \geq 0$
Terminal condition	$\cdots$
Exogenous	$U(C)$ = Utility of consumption
Functions	$B(W)$ = Bequest function
Parameters	T = Terminal time
	W_0 = Initial wealth
	ρ = Discount rate
	r = Interest rate

1.3 History of Optimal Control Theory

Optimal control theory is an extension of the calculus of variations (see Appendix B), so we discuss the history of the latter first.

The creation of the calculus of variations occurred almost immediately after the formalization of calculus by Newton and Leibniz in the seventeenth century. An important problem in calculus is to find an argument of a function at which the function takes on its maximum or minimum. The extension of this problem posed in the calculus of variations is to find a function which maximizes or minimizes the value of an integral or functional of that function. As might be expected, the extremum problem in the calculus of variations is much harder than the extremum problem in differential calculus. Euler and Lagrange are generally considered to be the founders of the calculus of variations. Newton, Legendre, and the Bernoulli brothers also contributed much to the early development of the field.

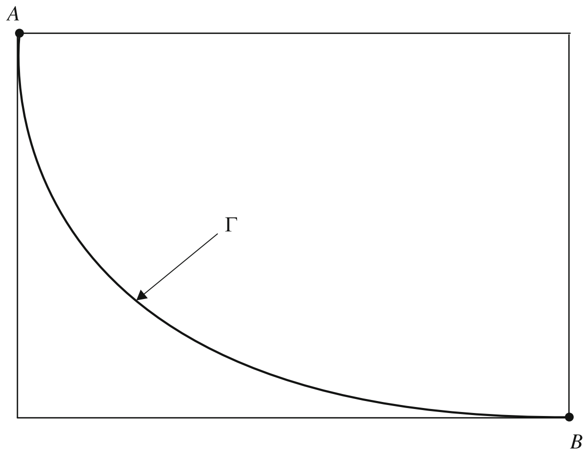

Figure 1.1: The Brachistochrone problem

A celebrated problem first solved using the calculus of variations was the *path of least time* or the *Brachistochrone problem*. The problem is illustrated in Fig. 1.1. It involves finding the shape of a curve Γ connecting the two points A and B in the vertical plane with the property that a bead sliding along the curve under the influence of gravity will move from A to B in the shortest possible time. The problem was posed by Johann Bernoulli in 1696, and it played an important part in the development of calculus of variations. It was solved by Johann Bernoulli, Jakob Bernoulli, Newton, Leibnitz, and L'Hôpital. In Sect. B.4, we provide a solution to the Brachistochrone problem by using what is known as the *Euler-Lagrange equation*, stated in Sect. B.2, and show that the shape of the solution curve is represented by a *cycloid*.

In the nineteenth and early twentieth centuries, many mathematicians contributed to the calculus of variations; these include Hamilton, Jacobi, Bolza, Weierstrass, Carathéodory, and Bliss.

Converting calculus of variations problems into control theory problems requires one more conceptual step—the addition of control variables to the state equations. Isaacs (1965) made such an extension in two-person pursuit-evasion games in the period 1948–1955. Bellman (1957) made a similar extension with the idea of dynamic programming.

Modern control theory began with the publication (in Russian in 1961 and English in 1962) of the book, *The Mathematical Theory of Optimal Processes*, by Pontryagin et al. (1962). Well-known American mathematicians associated with the maximum principle include Valentine, McShane, Hestenes, Berkovitz, and Neustadt. The importance of the book by Pontryagin et al. lies not only in a rigorous formulation of a calculus of variations problem with constrained control variables, but also in the proof of the maximum principle for optimal control problems. See Pesch and Bulirsch (1994) and Pesch and Plail (2009) for historical perspectives on the topics of the calculus of variations, dynamic programming, and optimal control.

The maximum principle permits the *decoupling* of the dynamic problem over time, using what are known as *adjoint variables* or *shadow prices*, into a series of problems, each of which holds at a single instant of time. The optimal solution of the instantaneous problems can be shown to give the optimal solution to the overall problem.

In this book we will be concerned principally with the application of the maximum principle in its various forms to find the solutions of a wide variety of applied problems in management science and economics. It is hoped that the reader, after reading some of these problems and their solutions, will appreciate, as we do, the importance of the maximum principle.

Some important books and surveys of the applications of the maximum principle to management science and economics are Connors and Teichroew (1967), Arrow and Kurz (1970), Hadley and Kemp (1971), Bensoussan et al. (1974), Stöppler (1975), Clark (1976), Sethi (1977a, 1978a), Tapiero (1977, 1988), Wickwire (1977), Bookbinder and Sethi (1980), Lesourne and Leban (1982), Tu (1984), Feichtinger and Hartl (1986), Carlson and Haurie (1987b), Seierstad and Sydsæter (1987), Erickson (2003), Léonard and Long (1992), Kamien and Schwartz (1992), Van Hilten et al. (1993), Feichtinger et al. (1994a), Maimon et al. (1998), Dockner et al. (2000), Caputo (2005), Grass et al. (2008), and Bensoussan (2011). Nevertheless, we have included in our bibliography many works of interest.

1.4 Notation and Concepts Used

In order to make the book readable, we will adopt the following notation which will hold throughout the book. In addition, we will define some important concepts that are required, including those of concave, convex and affine functions, and saddle points.

We use the symbol "=" to mean "is equal to" or "is defined to be equal to" or "is identically equal to" depending on the context. The symbol ":=" means "is defined to be equal to," the symbol "≡" means "is identically equal to," and the symbol "≈" means "is approximately equal to." The double arrow "⇒" means "implies," "∀" means "for all," and "∈" means "is a member of." The symbol □ indicates the end of a proof.

Let y be an n-component column vector and z be an m-component row vector, i.e.,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = (y_1, \ldots, y_n)^T \text{ and } z = (z_1, \ldots, z_m),$$

where the superscript ^{T} on a vector (or, a matrix) denotes the transpose of the vector (or, the matrix). At times, when convenient and not confusing, we will use the superscript $'$ for the transpose operation. If y and

z are functions of time t, a scalar, then the time derivatives $\dot{y} := dy/dt$ and $\dot{z} := dz/dt$ are defined as

$$\dot{y} = \frac{dy}{dt} = (\dot{y}_1, \cdots, \dot{y}_n)^T \text{ and } \dot{z} = \frac{dz}{dt} = (\dot{z}_1, \ldots, \dot{z}_m),$$

where $\dot{y}_i$ and $\dot{z}_j$ denote the time derivatives dy_i/dt and dz_j/dt, respectively.

When $n = m$, we can define the inner product

$$zy = \Sigma_{i=1}^n z_i y_i. \tag{1.7}$$

More generally, if

$$A = \{a_{ij}\} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix}$$

is an $m \times k$ matrix and $B = \{b_{ij}\}$ is a $k \times n$ matrix, we define the matrix product $C = \{c_{ij}\} = AB$, which is an $m \times n$ matrix with components

$$c_{ij} = \Sigma_{r=1}^k a_{ir} b_{rj}. \tag{1.8}$$

Let E^k denote the k-dimensional Euclidean space. Its elements are k-component vectors, which may be either row or column vectors, depending on the context. Thus in (1.7), $y \in E^n$ is a column vector and $z \in E^m$ is a row vector.

Next, in Sects. 1.4.1–1.4.4, we provide the notation for multivariate differentiation. Needless to say, the functions introduced are assumed to be appropriately differentiable for their derivatives being defined.

1.4.1 Differentiating Vectors and Matrices with Respect To Scalars

Let $f : E^1 \to E^k$ be a k-dimensional function of a scalar variable t. If f is a row vector, then we define

$$\frac{df}{dt} = f_t = (f_{1t}, f_{2t}, \cdots, f_{kt}), \text{ a row vector.}$$

We will also use the notation $f' = (f_1', f_2', \cdots, f_k')$ and $f'(t)$ in place of f_t.

If f is a column vector, then

$$\frac{df}{dt} = f_t = \begin{bmatrix} f_{1t} \\ f_{2t} \\ \vdots \\ f_{kt} \end{bmatrix} = (f_{1t}, f_{2t}, \cdots, f_{kt})^T, \text{a column vector.}$$

Once again, $f(t)$ may also be written as f' or $f'(t)$.

A similar rule applies if a matrix function is differentiated with respect to a scalar.

Example 1.4 Let $f(t) = \begin{bmatrix} t^2 & 2t+3 \\ e^{3t} & 1/t \end{bmatrix}$. Find f_t.

Solution $f_t = \begin{bmatrix} 2t & 2 \\ 3e^{3t} & -1/t^2 \end{bmatrix}$.

1.4.2 Differentiating Scalars with Respect to Vectors

If $F(y, z)$ is a scalar function defined on $E^n \times E^m$ with y an n-dimensional column vector and z an m-dimensional row vector, then the *gradients* F_y and F_z are defined, respectively, as

$$F_y = (F_{y_1}, \cdots, F_{y_n}), \text{ a row vector,} \tag{1.9}$$

and

$$F_z = (F_{z_1}, \cdots, F_{z_m}), \text{ a row vector,} \tag{1.10}$$

where F_{y_i} and F_{z_j} denote the partial derivatives with respect to the subscripted variables.

Thus, we always define the gradient with respect to a row or column vector as a row vector. Alternatively, F_y and F_z are also denoted as $\nabla_y F$ and $\nabla_z F$, respectively. In this notation, if F is a function of y only or z only, then the subscript can be dropped and the gradient of F can be written simply as ∇F.

Example 1.5 Let $F(y,z) = {y_1}^2 y_3 z_2 + 3y_2 \ln z_1 + y_1 y_2$, where $y = (y_1, y_2, y_3)^T$ and $z = (z_1, z_2)$. Obtain F_y and F_z.

Solution $F_y = (F_{y_1}, F_{y_2}, F_{y_3}) = (2y_1 y_3 z_2 + y_2, 3\ln z_1 + y_1, {y_1}^2 z_2)$ and $F_z = (F_{z_1}, F_{z_2}) = (3y_2/z_1, {y_1}^2 y_3)$.

1.4.3 Differentiating Vectors with Respect to Vectors

If $f : E^n \times E^m \to E^k$ is a k-dimensional vector function, f either row or column, i.e.,

$$f = (f_1, \cdots, f_k) \text{ or } f = (f_1, \cdots, f_k)^T,$$

where each component $f_i = f_i(y,z)$ depends on the column vector $y \in E^n$ and the row vector $z \in E^m$, then f_z will denote the $k \times m$ matrix

$$f_z = \begin{bmatrix} \partial f_1/\partial z_1, & \partial f_1/\partial z_2, & \cdots & \partial f_1/\partial z_m \\ \partial f_2/\partial z_1, & \partial f_2/\partial z_2, & \cdots & \partial f_2/\partial z_m \\ \vdots & \vdots & \cdots & \vdots \\ \partial f_k/\partial z_1, & \partial f_k/\partial z_2, & \cdots & \partial f_k/\partial z_m \end{bmatrix} = \{\partial f_i/\partial z_j\}, \qquad (1.11)$$

and f_y will denote the $k \times n$ matrix

$$f_y = \begin{bmatrix} \partial f_1/\partial y_1 & \partial f_1/\partial y_2 & \cdots & \partial f_1/\partial y_n \\ \partial f_2/\partial y_1 & \partial f_2/\partial y_2 & \cdots & \partial f_2/\partial y_n \\ \vdots & \vdots & \cdots & \vdots \\ \partial f_k/\partial y_1 & \partial f_k/\partial y_2 & \cdots & \partial f_k/\partial y_n \end{bmatrix} = \{\partial f_i/\partial y_j\}. \qquad (1.12)$$

Matrices f_z and f_y are known as *Jacobian* matrices. It should be emphasized that the rule of defining a Jacobian does not depend on the row or column nature of the function or its arguments. Thus,

$$f_z = (f^T)_z = f_{z^T} = (f^T)_{z^T}.$$

Example 1.6 Let $f : E^3 \times E^2 \to E^3$ be defined by $f(y,z) = ({y_1}^2 y_3 z_2 + 3y_2 \ln z_1, z_1 {z_2}^2 y_3, z_1 y_1 + z_2 y_2)^T$ with $y = (y_1, y_2, y_3)^T$ and $z = (z_1, z_2)$. Obtain f_z and f_y.

Solution.

$$f_z = \begin{bmatrix} 3y_2/z_1 & {y_1}^2 y_3 \\ {z_2}^2 y_3 & 2z_1 z_2 y_3 \\ y_1 & y_2 \end{bmatrix},$$

$$f_y = \begin{bmatrix} 2y_1 y_3 z_2 & 3\ln z_1 & {y_1}^2 z_2 \\ 0 & 0 & z_1 {z_2}^2 \\ z_1 & z_2 & 0 \end{bmatrix}.$$

Applying the rule (1.11) to F_y in (1.9), we obtain $F_{yz} = (F_y)_z$ to be the $n \times m$ matrix

$$F_{yz} = \begin{bmatrix} F_{y_1 z_1} & F_{y_1 z_2} & \cdots & F_{y_1 z_m} \\ F_{y_2 z_1} & F_{y_2 z_2} & \cdots & F_{y_2 z_m} \\ \vdots & \vdots & \cdots & \vdots \\ F_{y_n z_1} & F_{y_n z_2} & \cdots & F_{y_n z_m} \end{bmatrix} = \left\{ \frac{\partial^2 F}{\partial y_i \partial z_j} \right\}. \tag{1.13}$$

Applying the rule (1.12) to F_z in (1.10), we obtain $F_{zy} = (F_z)_y$ to be the $m \times n$ matrix

$$F_{zy} = \begin{bmatrix} F_{z_1 y_1} & F_{z_1 y_2} & \cdots & F_{z_1 y_n} \\ F_{z_2 y_1} & F_{z_2 y_2} & \cdots & F_{z_2 y_n} \\ \vdots & \vdots & \cdots & \vdots \\ F_{z_m y_1} & F_{z_m y_2} & \cdots & F_{z_m y_n} \end{bmatrix} = \left\{ \frac{\partial^2 F}{\partial z_i \partial y_j} \right\}. \tag{1.14}$$

Note that if $F(y, z)$ is twice continuously differentiable, then we also have $F_{zy} = (F_{yz})^T$.

Example 1.7 Obtain F_{yz} and F_{zy} for $F(y, z)$ specified in Example 1.5. Since the given $F(y, z)$ is twice continuously differentiable, check also that $F_{zy} = (F_{yz})^T$.

Solution. Applying rule (1.11) to F_y obtained in Example 1.5 and rule (1.12) to F_z obtained in Example 1.5, we have, respectively,

$$F_{yz} = \begin{bmatrix} 0 & 2y_1y_3 \\ 3/z_1 & 0 \\ 0 & {y_1}^2 \end{bmatrix} \text{ and } F_{zy} = \begin{bmatrix} 0 & 3/z_1 & 0 \\ 2y_1y_3 & 0 & {y_1}^2 \end{bmatrix}.$$

Also, it is easily seen from these matrices that $F_{zy} = (F_{yz})^T$.

1.4.4 Product Rule for Differentiation

Let g be an n-component row vector function and f be an n-component column vector function of an n-component vector x. Then in Exercise 1.9, you are asked to show that

$$(gf)_x = gf_x + f^T g_x = gf_x + f^T(g^T)_x. \tag{1.15}$$

In Exercise 1.10, you are asked to show further that with $g = F_x$, where $x \in E^n$ and the function $F : E^n \to E^1$ is twice continuously differentiable so that $F_{xx} = (F_{xx})^T$, called the Hessian, then

$$(gf)_x=(F_x f)_x=F_x f_x + f^T F_{xx} = F_x f_x + (F_{xx} f)^T. \tag{1.16}$$

The latter result will be used in Chap. 2 for the derivation of (2.25).

Many mathematical expressions in this book will be vector equations or inequalities involving vectors and vector functions. Since scalars are a special case of vectors, these expressions hold just as well for scalar equations or inequalities involving scalars and scalar functions. In fact, it may be a good idea to read them as scalar expressions on the first reading. Then in the second and further readings, the extension to vector form will be easier.

1.4.5 Miscellany

The *norm* of an m-component row or column vector z is defined to be

$$\| z \| = \sqrt{z_1^2 + \cdots + z_m^2}. \tag{1.17}$$

The norm of a vector is commonly used to define a *neighborhood* N_{z_0} of a point, e.g.,

$$N_{z_0} = \{z | \; \| z - z_0 \| < \varepsilon\}, \tag{1.18}$$

where $\varepsilon > 0$ is a small positive real number.

We will occasionally make use of the so-called "little-o" notation $o(z)$. A function $F(z) : E^m \to E^1$ is said to be of the order $o(z)$, if

$$\lim_{\|z\|\to 0} \frac{F(z)}{\| z \|} = 0.$$

The most common use of this notation will be to collect higher order terms in a series expansion.

In the continuous-time models discussed in this book, we generally will use $x(t)$ to denote the state (column) vector, $u(t)$ to denote the control (column) vector, and $\lambda(t)$ to denote the adjoint (row) vector. Whenever there is no possibility of confusion, we will suppress the time indicator (t) from these vectors and write them as x, u, and λ, respectively. When talking about *optimal* state and control vectors, we put an asterisk "*" as a superscript, i.e., as x^* and u^*, respectively, whereas u will refer to an admissible control with x as the corresponding state. No asterisk, however, needs to be put on the adjoint vector λ as it is only defined along an optimal path.

Thus, the values of the control, state and adjoint variables at time t along an optimal path will be written as $u^*(t), x^*(t)$, and $\lambda(t)$. When the control is expressed in terms of the state, it is called a *feedback control.* With an abuse of notation, we will express it as $u(x)$, or $u(x,t)$ if an explicit time dependence is required. Likewise, the optimal feedback control will be denoted as $u^*(x)$ or $u^*(x,t)$.

We also use the simplified notation $x'(t)$ to mean $(x(t))'$, the transpose of $x(t)$. Likewise, for a matrix $A(t)$, we use $A'(t)$ to mean $(A(t))'$ or the transpose of $A(t)$, and $A^{-1}(t)$ to mean $(A(t))^{-1}$ or the inverse of $A(t)$, when the inverse exists.

The *norm* of an m-dimensional row or column vector function $z(t)$, $t \in [0,T]$, is defined to be

$$\| z \| = \left[\Sigma_{j=1}^m \int_0^T z_j^2(\tau) d\tau \right]^{\frac{1}{2}}. \tag{1.19}$$

In Chap. 4 and some other chapters, we will encounter functions of time with jumps. For such functions, it is useful to have the concepts of *left* and *right limits.* With $\varepsilon > 0$, these are defined, respectively, for a function $x(t)$ as

$$x(T^-) = \lim_{\tau \uparrow T} x(\tau) = \lim_{\varepsilon \to 0} x(T-\varepsilon) \text{ and } x(T^+) = \lim_{\tau \downarrow T} x(\tau) = \lim_{\varepsilon \to 0} x(T+\varepsilon). \tag{1.20}$$

These limits are illustrated for a function $x(t)$ graphed in Fig. 1.2. Here,

$$x(0) = 1,\ x(0^+) = 2,$$

$$x(1^-) = 3,\ x(1^+) = x(1) = 4,$$

$$x(2^-) = 3,\ x(2) = 2,\ x(2^+) = 1,$$

$$x(3^-) = 2,\ x(3) = 3.$$

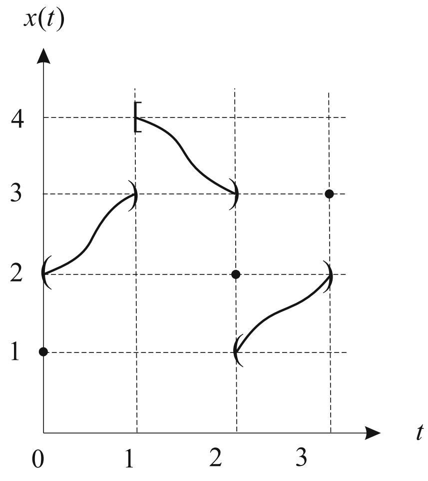

Figure 1.2: Illustration of left and right limits

In the discrete-time models introduced in Chap. 8 and applied in Chap. 9, we use x^k, u^k, and λ^k to denote state, control, and adjoint vectors, respectively, at time k, $k = 0, 1, 2, \ldots, T$. We also denote the difference operator by

$$\Delta x^k := x^{k+1} - x^k.$$

As in the continuous-time case, the optimal values of the state variable x^k and the control variable u^k will have an asterisk as a superscript; thus, x^{k*} and u^{k*} denote the corresponding quantities along an optimal path. Once again, the adjoint variable λ^k along an optimal path will not have an asterisk.

In order to specify the optimal control for linear control problems, we will introduce a special notation, called the *bang function*, as

$$\text{bang}[b_1, b_2; W] = \begin{cases} b_1 & \text{if} \quad W < 0, \\ \text{arbitrary} & \text{if} \quad W = 0, \\ b_2 & \text{if} \quad W > 0. \end{cases} \tag{1.21}$$

In order to specify the optimal control for linear-quadratic problems, we define another special function, called the *sat function*, as

$$\text{sat}[y_1, y_2; W] = \begin{cases} y_1 & \text{if} \quad W < y_1, \\ W & \text{if} \quad y_1 \le W \le y_2, \\ y_2 & \text{if} \quad W > y_2. \end{cases} \tag{1.22}$$

The word "sat" is short for the word "saturation." The latter name comes from an electrical engineering application to *saturated amplifiers.*

In several applications to be discussed, we will need the concept of impulse control, which is sometimes needed in cases when an unbounded control can be applied for a very short time. An example is the advertising model in Table 1.2 when $Q = \infty$. We apply unbounded control for a short time in order to cause a jump discontinuity in the state variable. For the example in Table 1.2, this might mean an intense advertising campaign (a media blitz) in order to increase advertising goodwill by a finite amount in a very short time. The impulse function defined below is required to evaluate the integral in the objective function, which measures the cost of the intense advertising campaign.

Suppose we want to apply an impulse control at time t to change the state variable from $x(t) = x_1$ to the value x_2 "immediately" after t, i.e., $x(t^+) = x_2$. To compute its contribution to the objective function (1.2), we use the following procedure: given $\varepsilon > 0$ and a constant control $u(\varepsilon)$, integrate (1.1) from t to $t + \varepsilon$ with $x(t) = x_1$ and choose $u(\varepsilon)$ so that $x(t + \varepsilon) = x_2$; this gives the trajectory $x(\tau; \varepsilon, u(\varepsilon))$ for $\tau \in [t, t + \varepsilon]$. We can now compute

$$\text{imp}(x_1, x_2; t) = \lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} F(x, u, \tau) d\tau. \tag{1.23}$$

If the impulse is applied only at time t, then we can calculate (1.2) as

$$J = \int_0^t F(x,u,\tau)d\tau + \text{imp}(x_1, x_2; t) + \int_t^T F(x,u,\tau)d\tau + S[x(T),T]. \tag{1.24}$$

If there are several instants at which impulses are applied, then this procedure is easily extended. Examples of the use of (1.24) occur in Chaps. 5 and 6. We frequently omit t in (1.23) when the impulse function is independent of t.

1.4.6 Convex Set and Convex Hull

A set $D \subset E^n$ is a *convex set* if for each pair of points $y, z \in D$, the entire line segment joining these two points is also in D, i.e.,

$$py + (1-p)z \in D, \text{ for each } p \in [0,1].$$

Given $x^i \in E^n, i = 1, 2, \ldots, l$, we define $y \in E^n$ to be a *convex combination* of $x^i \in E^n$, if there exists $p_i \geq 0$ such that

$$\sum_{i=1}^{l} p_i = 1 \text{ and } y = \sum_{i=1}^{l} p_i x^i.$$

The *convex hull* of a set $D \subset E^n$ is

$$\text{co}D := \left\{ \sum_{i=1}^{l} p_i x^i : \sum_{i=1}^{l} p_i = 1,\ p_i \geq 0,\ x^i \in D,\ i = 1, 2, \ldots, l \right\}.$$

In other words, co$D$ is the set of all convex combinations of points in D.

1.4.7 Concave and Convex Functions

A real-valued function ψ defined on a convex set $D \subset E^n$, i.e., $\psi : D \to E^1$, is *concave*, if for each pair of points $y, z \in D$ and for all $p \in [0,1]$,

$$\psi(py + (1-p)z) \geq p\psi(y) + (1-p)\psi(z).$$

If the inequalities in the above definition are strict for all $y, z \in D$ with $y \neq z$, and $0 < p < 1$, then ψ is called a *strictly concave function.*

In the single dimensional case of $n = 1$, there is an enlightening geometrical interpretation. Namely, $\psi(x)$ defined on an interval $D = [a, b]$ is concave if, for each pair of points on the graph of $\psi(x)$, the line segment joining these two points lies entirely below or on the graph of $\psi(x)$; see Fig. 1.3.

Reverting back to the n-dimensional case, if ψ is a differentiable function on a convex set $D \subset E^n$, then it is *concave*, if for each pair of points $y, z \in D$,

$$\psi(z) \leq \psi(y) + \psi_x(y)(z - y),$$

where we understand y and z to be column vectors. Furthermore, if the function ψ is twice differentiable, then it is *concave*, if at each point in D, the $n \times n$ symmetric matrix ψ_{xx} is negative semidefinite, i.e., all of its eigenvalues are non-positive.

Finally, if ψ is a concave function, then the negative of the function ψ, i.e., $-\psi : D \to E^1$, is a *convex function.*

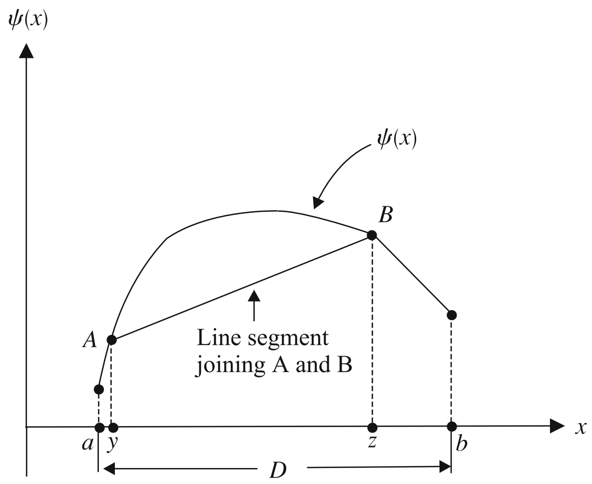

Figure 1.3: A concave function

1.4.8 Affine Function and Homogeneous Function of Degree k

A function $\psi : E^n \to E^1$ is said to be *affine*, if the function $\psi(x) - \psi(0)$ is linear. Thus, ψ can be represented as $\psi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n a_i x_i + b$, where $a_i,\ i = 1, 2, \ldots, n$, and b are scalar constants.

A function $\psi : E^n \to E^1$ is said to be *homogeneous of degree k*, if $\psi(bx) = b^k \psi(x)$, where $b > 0$ is a scalar constant.

In economics, we often assume that a firm's production function is homogeneous of degree 1, i.e., if all inputs are multiplied by b, then output is multiplied by b. Such a production function is said to exhibit the property of *constant return to scale*. A linear function $\psi(x) = ax = \sum_{i=1}^n a_i x_i$ is a simple example of a homogeneous function of degree 1. Other examples are $\psi(x) = \min\{x_i,\ i = 1, 2, \ldots, n\}$ and $\psi(x) = a(\Pi_{i=1}^n {x_i}^{\alpha_i})^{1/\sum_{i=1}^n \alpha_i}$ with $a > 0$ and $\alpha_i > 0,\ i = 1, 2, \ldots, n$. An important special case of the last example, known as the Cobb-Douglas production function, is $\psi(K, L) = aK^{\alpha_1} L^{\alpha_2}$ with $\alpha_1 + \alpha_2 = 1$, where K and L are factors of production called capital and labor, respectively, and a denotes the total factor productivity.

1.4.9 Saddle Point

An important concept in two-person zero-sum games is that of a saddle point. Let $\psi(x, y)$, a real-valued function defined on the space $E^n \times E^m$, i.e., $\psi : E^n \times E^m \to E^1$, be the payoff of player 1 and $-\psi(x, y)$ be the payoff of player 2, when they make decisions x and y, respectively, in a zero-sum game. A point $(\hat{x}, \hat{y}) \in E^n \times E^m$ is called a *saddle point* of $\psi(x, y)$ or of the game, if

$$\psi(\hat{x}, y) \geq \psi(\hat{x}, \hat{y}) \geq \psi(x, \hat{y}) \text{ for all } x \in E^n \text{ and } y \in E^m.$$

Note that a saddle point may not exist, and even if it exists, it may not be unique. Note also that

$$\psi(\hat{x}, \hat{y}) = \max_x \psi(x, \hat{y}) = \min_y \psi(\hat{x}, y).$$

Intuitively, this could produce a picture like a horse saddle as shown in Fig. 1.4, hence the name saddle point for a point like $(\hat{x}, \hat{y})$. This concept will be used in Sect. 13.1.

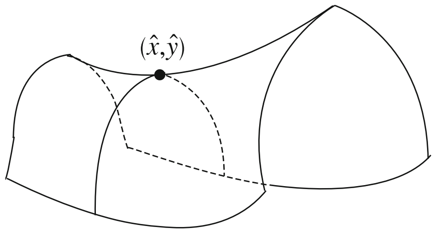

Figure 1.4: An illustration of a saddle point

1.4.10 Linear Independence and Rank of a Matrix

A set of vectors $a_1, a_2, \ldots, a_m$ in E^n, $m \leq n$, is said to be *linearly dependent* if there exist scalars p_i not all zero such that

$$\sum_{i=1}^{m} p_i a_i = 0. \tag{1.25}$$

If (1.25) holds only when $p_1 = p_2 = \cdots = p_m = 0$, then the vectors are said to be *linearly independent*. In particular, if one of the vectors in the set $\{a_1, a_2, \ldots, a_m\}$ is a *null vector*, then the set is linearly dependent.

The *rank* of an $m \times n$ matrix A, written $\text{rank}(A)$, is the maximum number of linearly independent rows or, equivalently, the maximum number of linearly independent columns of A. An $m \times n$ matrix is of *full rank* if

$$\text{rank}(A) = \min\{m, n\}.$$

1.5 Plan of the Book

The book has thirteen chapters and five appendices: A, B, C, D, and E, covering a variety of topics which are listed in the table of contents and explained in the prefaces.

In any given chapter, say Chap. 7, sections are numbered consecutively as 7.1, 7.2, 7.3, etc. Subsections are numbered consecutively within each section, i.e., 7.2.1, 7.2.2, 7.2.3, etc. Mathematical expressions are numbered consecutively by chapter as (7.1), (7.2), (7.3), etc. Theorems are also numbered consecutively by chapter as Theorem 7.1, Theorem 7.2, Theorem 7.3, etc. Similarly, definitions, examples, exercises, figures, propositions, remarks, and tables are numbered consecutively by chapter. These elements will be referenced throughout the book by use of their designated numbers. The same scheme is used in the appendices, thus, sections in Appendix B, for example, are numbered as B.1, B.2, B.3, etc.

Exercises for Chapter 1

E 1.1 In Example 1.1, let the functions and parameters of the production- inventory model be given by:

$$h(I) = 10I,\ c(P) = 20P,\ T = 10,\ I_0 = 1,000$$

$$P_{\min} = 600,\ P_{\max} = 1200,\ I_{\min} = 800,\ S(t) = 900 + 10t.$$

(a) Set $P(t) = 1000$ for $0 \leq t \leq 10$. Determine whether this control is feasible; if it is feasible, compute the value J of the objective function.

(b) If $P(t) = 800$, show that the terminal constraint is violated and hence the control is infeasible.

(c) If $P(t) = P_{\min}$ for $0 \leq t \leq 6$ and $P(t) = P_{\max}$ for $6 < t \leq 10$, show that the control is infeasible because the state constraint is violated.

E 1.2 In Example 1.1, suppose there is a cost associated with changing the rate of production. One way to formulate this problem is to let the control variable $u(t)$ denote the rate of change of the production rate $P(t)$, having a cost cu^2 associated with such changes, where $c > 0$. Formulate the new problem.

Hint: Let $P(t)$ be an additional state variable.

E 1.3 For the advertising model in Example 1.2, let $\pi(G) = 2\sqrt{G}$, $\delta = 0.05$, $\rho = 0.2$, $Q = 2$, and $G_0 = 16$. Set $u(t) = 0.8$ for $t \geq 0$, and show that $G(t)$ is constant for all t. Compute the value J of the objective function.

E 1.4 In Example 1.2, suppose G measures the number of people who know about the product. Hence, if A is the total population, then $A - G$ is the number of people who do not know about the product. If $u(t)$ measures the advertising rate at time t, assume that $u(A - G)$ is the corresponding rate of increase of G due to this advertising. Formulate the new model.

E 1.5 Rich Rentier in Example 1.3 has initial wealth $W_0 = \$1,000,000$. Assume $B = 0, \rho = 0.1, r = 0.15$, and assume that Rich expects to live for exactly 20 years.

(a) What is the maximum constant consumption level that Rich can afford during his remaining life?

(b) If Rich's utility function is $U(C) = \ln C$, what is the present value of the total utility in part (a)?

(c) Suppose Rich sets aside \$100,000 to start the Rentier Foundation. What is the maximum constant grant level that the foundation can support if it is to last forever?

E 1.6 Suppose Rich in Exercise 1.5 takes on a part-time job, which yields an income of $y(t)$ at time t. Assume $y(t) = 10,000e^{-0.05t}$ and that he has a bequest function $B(W) = 0.5 \ln W$.

(a) Reformulate this new optimal control problem.

(b) If Rich (no longer a *rentier*) consumes at the constant rate found in Exercise 1.5(a), find his terminal wealth and his new total utility.

E 1.7 Consider the following educational policy question. Let $S(t)$ denote the total number of scientists at time t, and let δ be the retirement rate of scientists. Let $E(t)$ be the number of teaching scientists and $R(t)$ be the number of research scientists, so that $S(t) = E(t) + R(t)$. Assume $\gamma E(t)$ is the number of newly graduated scientists at time t, of which the policy allocates $u\gamma E(t)$ to the pool of teachers, where $0 \le u \le 1$. The remaining graduates are added to the pool of researchers. The government has a target of maximizing the function $\alpha E(T) + \beta R(T)$ at a given future time T, where α and β are positive constants. Formulate the optimal control problem for the government.

E 1.8 For $F(x, y)$ defined in Example 1.5, obtain the matrices F_{xx} and F_{yy}.

E 1.9 Let $x \in E^m$, g be an n-component row vector function of x, and f be an n-component column vector function of x. Use the ordinary product rule of calculus for functions of scalars to derive the formula

$$(gf)_x = gf_x + f^T(g^T)_x = gf_x + f^T g_x.$$

E 1.10 Let F be a scalar function of $x \in E^n$ and f as defined in Exercise 1.9. Assume F to be twice continuously differentiable. Show that

$$(F_x f)_x = F_x f_x + f^T F_{xx} = F_x f_x + f^T (F_{xx})^T = F_x f_x + (F_{xx} f)^T.$$

Hint: Set the gradient $F_x = g$, a row vector, and then use Exercise 1.9 to derive the first equality. Note in connection with the second equality that the function F being twice continuously differentiable implies that $F_{xx} = (F_{xx})^T$.

E 1.11 For F_y obtained in Example 1.5 and f defined in Example 1.6, obtain $(F_y f)_y$ and verify the relation shown in Exercise 1.10.

E 1.12 Use the bang function defined in (1.21) to sketch the optimal control

$$u^*(t) = \text{bang}[-1, 1; W(t)] \text{ for } 0 \leq t \leq 5,$$

when

(a) $W(t) = t - 2$

(b) $W(t) = t^2 - 4t + 3$

(c) $W(t) = \sin \pi t$.

E 1.13 Use the sat function defined in (1.22) to sketch the optimal control

$$u^*(t) = \text{sat}[2, 3; W(t)] \text{ for } 0 \leq t \leq 5,$$

when

(a) $W(t) = 4 - t$

(b) $W(t) = 2 + t^2$

(c) $W(t) = 4 - 4e^{-t}$.

E 1.14 Evaluate the function $\text{imp}(G_1, G_2; t)$ for the advertising model of Table 1.2 when $G_2 > G_1$, $Q = \infty$, and $\pi(G) = pG$, where p is a constant.

Chapter 2

The Maximum Principle: Continuous Time

The main purpose of this chapter is to introduce the maximum principle as a necessary condition that must be satisfied by any optimal control for the basic problem specified in Sect. 2.1. Although vector notation is used, the reader can consider the problem as one with only a single state variable and a single control variable on the first reading. In Sect. 2.2, the method of dynamic programming is used to derive the maximum principle. We use this method because of the simplicity and familiarity of the dynamic programming concept. The derivation also yields significant economic interpretations. In Appendix C, the maximum principle is also derived by using a more general method similar to that of Pontryagin et al. (1962), but with certain simplifications. In Sect. 2.3, we apply the maximum principle to solve a number of simple, but illustrative, examples. In Sect. 2.4, the maximum principle is shown to be sufficient for optimal control under an appropriate concavity condition, which holds in many management science applications. Finally, Sect. 2.5 illustrates the use of Excel spreadsheet software to solve an optimal control problem.

2.1 Statement of the Problem

Optimal control theory deals with the problem of optimizing dynamic systems. The problem must be well posed before any solution can be attempted. This requires a clear mathematical description of the system

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to be optimized, the constraints imposed on the system, and the objective function to be maximized (or minimized).

2.1.1 The Mathematical Model

An important part of any control problem is the process of modeling the dynamic system under consideration, be it physical, business, or otherwise. The aim is to arrive at a mathematical description which is simple enough to deal with, and realistic enough to be able to predict the response of the system to any given input. Our model is restricted to systems that can be characterized by a set of ordinary differential equations (or, ordinary difference equations in the discrete-time case treated in Chap. 8). Thus, given the initial state x_0 of the system and control history $u(t),\ t \in [0,T]$, of the process, the evolution of the system may be described by the first-order differential equation, known also as the *state equation*,

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \tag{2.1}$$

where the vector of *state variables*, $x(t) \in E^n$, the vector of *control variables*, $u(t) \in E^m$, and $f : E^n \times E^m \times E^1 \to E^n$. Furthermore, the function f is assumed to be continuously differentiable. Here we assume x to be a column vector and f to be a column vector of functions. The path $x(t),\ t \in [0,T]$, is called a *state trajectory* and $u(t),\ t \in [0,T]$, is called a *control trajectory* or simply, a *control*. The terms *vector of state variables*, *state vector*, and *state* will be used interchangeably; similarly for the terms *vector of control variables*, *control vector*, and *control*. As mentioned earlier, when no confusion arises, we will usually suppress the time notation (t); thus, e.g., $x(t)$ will be written simply as x. Furthermore, it should be inferred from the context whether x denotes the state at time t or the entire state trajectory. A similar statement holds for u.

2.1.2 Constraints

In this chapter, we are concerned with problems of types (1.4) and (1.5) that do not have state constraints. Such constraints are considered in Chaps. 3 and 4, as indicated in Sect. 1.1. We do impose constraints of type (1.3) on the control variables. We define an *admissible control* to be a control trajectory $u(t)$, $t \in [0,T]$, which is piecewise continuous and satisfies, in addition,

$$u(t) \in \Omega(t) \subset E^m, \quad t \in [0,T]. \tag{2.2}$$

Usually the set $\Omega(t)$ is determined by physical or economic constraints on the values of the control variables at time t.

2.1.3 The Objective Function

An objective function is a quantitative measure of the performance of the system over time. An *optimal control* is defined to be an admissible control which maximizes the objective function. In business or economic problems, a typical objective function gives some appropriate measure of quantities such as profit or sales. If the aim is to minimize cost, then the objective function to be maximized is the negative of cost. Mathematically, we let

$$J = \int_0^T F(x(t), u(t), t)dt + S(x(T), T) \tag{2.3}$$

denote the objective function, where the functions $F : E^n \times E^m \times E^1 \rightarrow E^1$ and $S : E^n \times E^1 \rightarrow E^1$ are assumed for our purposes to be continuously differentiable. In a typical business application, $F(x, u, t)$ could be the *instantaneous profit rate* and $S(x, T)$ could be the *salvage value* of having x as the system state at the *terminal time* T.

2.1.4 The Optimal Control Problem

Given the preceding definitions we can state the optimal control problem, which we will be concerned with in this chapter. The problem is to find an admissible control u^*, which maximizes the objective function (2.3) subject to the state equation (2.1) and the control constraints (2.2). We now restate the optimal control problem as:

$$\begin{cases} \max\limits_{u(t)\in\Omega(t)} \left\{ J = \displaystyle\int_0^T F(x, u, t)dt + S(x(T), T) \right\} \\ \text{subject to} \\ \dot{x} = f(x, u, t), \; x(0) = x_0. \end{cases} \tag{2.4}$$

The control u^* is called an *optimal control* and x^*, determined by means of the state equation with $u = u^*$, is called the *optimal trajectory* or an *optimal path.* The optimal value $J(u^*)$ of the objective function will be denoted as J^*, and occasionally as $J^*_{(x_0)}$ when we need to emphasize its dependence on the initial state x_0.

The optimal control problem (2.4) is said to be in *Bolza form* because of the form of the objective function in (2.3). It is said to be in *Lagrange form* when $S \equiv 0$. We say the problem is in *Mayer form* when $F \equiv 0$. Furthermore, it is in *linear Mayer form* when $F \equiv 0$ and S is linear, i.e.,

$$\begin{cases} \max_{u(t)\in\Omega(t)} \{J = cx(T)\} \\ \text{subject to} \\ \dot{x} = f(x,u,t),\ x(0) = x_0, \end{cases} \tag{2.5}$$

where $c = (c_1, c_2, \cdots, c_n)$ is an n-dimensional row vector of given constants. In the next paragraph and in Exercise 2.5, it will be demonstrated that all of these forms can be converted into the linear Mayer form.

To show that the Bolza form can be reduced to the linear Mayer form, we define a new state vector $y = (y_1, y_2, \ldots, y_{n+1})$, having $n+1$ components defined as follows: $y_i = x_i$ for $i = 1, \ldots, n$ and y_{n+1} defined by the solution of the equation

$$\dot{y}_{n+1} = F(x,u,t) + \frac{\partial S(x,t)}{\partial x} f(x,u,t) + \frac{\partial S(x,t)}{\partial t}, \tag{2.6}$$

with $y_{n+1}(0) = S(x_0, 0)$. By writing $f(x,u,t)$ as $f(y,u,t)$, with a slight abuse of notation, and by denoting the right-hand side of (2.6) as $f_{n+1}(y,u,t)$, we can write the new state equation in the vector form as

$$\dot{y} = \begin{pmatrix} \dot{x} \\ \\ \dot{y}_{n+1} \end{pmatrix} = \begin{pmatrix} f(y,u,t) \\ \\ f_{n+1}(y,u,t) \end{pmatrix},\ y(0) = \begin{pmatrix} x_0 \\ \\ S(x_0,0) \end{pmatrix}. \tag{2.7}$$

We also put $c = (0, \cdots, 0, 1)$, where c has $n+1$ components with the first n terms all 0. If we integrate (2.6) from 0 to T, we see that

$$y_{n+1}(T) - y_{n+1}(0) = \int_0^T F(x,u,t)dt + S(x(T),T) - S(x_0,0).$$

In view of setting the initial condition as $y_{n+1}(0) = S(x_0, 0)$, the problem in (2.4) can be expressed as that of maximizing

$$J = \int_0^T F(x,u,t)dt + S(x(T),T) = y_{n+1}(T) = cy(T) \tag{2.8}$$

over $u(t) \in \Omega(t)$, subject to (2.7). Of course, the price paid for going from Bolza to linear Mayer form is an additional state variable and its associated differential equation (2.6). Also, for the function f_{n+1} to be continuously differentiable, in keeping with the assumptions made in Sect. 2.1.1, we need to assume that the salvage value function $S(x,t)$ is twice continuously differentiable.

Exercise 2.5 presents the task of showing in a similar way that the Lagrange and Mayer forms can also be reduced to the linear Mayer form.

Example 2.1 Convert the following single-state problem in Bolza form to its linear Mayer form:

$$\max \left\{ J = \int_0^T \left(x - \frac{u^2}{2} \right) dt + \frac{1}{4} [x(T)]^2 \right\}$$

subject to

$$\dot{x} = u, \quad x(0) = x_0.$$

Solution. We use (2.6) to introduce the additional state variable y_2 as follows:

$$\dot{y}_2 = x - \frac{u^2}{2} + \frac{1}{2} x u, \quad y_2(0) = \frac{1}{4} x_0^2.$$

Then,

$$\begin{aligned}
y_2(T) &= y_2(0) + \int_0^T \left(x - \frac{u^2}{2} + \frac{1}{2} x u \right) dt \\
&= \int_0^T \left(x - \frac{u^2}{2} \right) dt + \int_0^T \left(\frac{1}{2} x \dot{x} \right) dt + y_2(0) \\
&= \int_0^T \left(x - \frac{u^2}{2} \right) dt + \int_0^T d \left(\frac{1}{4} x^2 \right) \\
&= \int_0^T \left(x - \frac{u^2}{2} \right) dt + \frac{1}{4} [x(T)]^2 - \frac{1}{4} x_0^2 + y_2(0) \\
&= \int_0^T \left(x - \frac{u^2}{2} \right) dt + \frac{1}{4} x(T)^2 \\
&= J.
\end{aligned}$$

Thus, the linear Mayer form version with the two-dimensional state $y = (x, y_2)$ can be stated as

$$\max \{J = y_2(T)\}$$

subject to

$$\begin{aligned} \dot{x} &= u, \quad x(0) = x_0, \\ \dot{y}_2 &= x - \frac{u^2}{2} + \frac{1}{2}xu, \quad y_2(0) = \frac{1}{4}x_0^2. \end{aligned}$$

In Sect. 2.2, we derive necessary conditions for optimal control in the form of the maximum principle, and in Sect. 2.4 we derive sufficient conditions. In these derivations, we shall assume the existence of an optimal control, while providing references where needed, as the topic of existence is beyond the scope of this book. In any particular application, however, the existence of a solution will be demonstrated by actually finding a solution that satisfies both the necessary and the sufficient conditions for optimality. We thus avoid the necessity of having to prove general existence theorems, which require advanced and difficult mathematics. Nevertheless, interested readers can consult Hartl et al. (1995) and Seierstad and Sydsæter (1987) for brief discussions of existence results and references therein including Cesari (1983).

2.2 Dynamic Programming and the Maximum Principle

We will now derive the maximum principle by using a dynamic programming approach. The proof is intuitive in nature and is not intended to be mathematically rigorous. For more rigorous derivations, we refer the reader to Appendix C, Berkovitz (1961), Pontryagin et al. (1962), Halkin (1967), Boltyanskii (1971), Hartberger (1973), Bryant and Mayne (1974), Leitmann (1981), and Seierstad and Sydsæter (1987). Additional references can be found in the survey by Hartl et al. (1995). For discussions of maximum principles for more general optimal control problems, including those with nondifferentiable functions, see Clarke (1983, 1989).

2.2.1 The Hamilton-Jacobi-Bellman Equation

Suppose $V(x,t) : E^n \times E^1 \to E^1$ is a function whose value is the maximum value of the objective function of the control problem for the system, given that we start at time t in state x. That is,

$$V(x,t) = \max_{u(s)\in\Omega(s)} \left[\int_t^T F(x(s),u(s),s)ds + S(x(T),T) \right], \tag{2.9}$$

where for $s \geq t$,

$$\frac{dx(s)}{ds} = f(x(s),u(s),s),\ x(t) = x.$$

We initially assume that the *value function* $V(x,t)$ exists for all x and t in the relevant ranges. Later we will make additional assumptions about the function $V(x,t)$.

Bellman (1957) in his book on dynamic programming states the *principle of optimality* as follows:

> An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the outcome resulting from the initial decision.

Intuitively this principle is obvious, for if we were to start in state x at time t and did not follow an optimal path from then on, there would then exist (by assumption) a better path from t to T, hence, we could improve the proposed solution by following this better path. We will use the principle of optimality to derive conditions on the value function $V(x,t)$.

Figure 2.1 is a schematic picture of the optimal path $x^*(t)$ in the state-time space, and two nearby points (x,t) and $(x+\delta x, t+\delta t)$, where δt is a small increment of time and $x+\delta x = x(t+\delta t)$. The value function changes from $V(x,t)$ to $V(x+\delta x, t+\delta t)$ between these two points. By the principle of optimality, the change in the objective function is made up of two parts: first, the incremental change in J from t to $t+\delta t$, which is given by the integral of $F(x,u,t)$ from t to $t+\delta t$; second, the value function $V(x+\delta x, t+\delta t)$ at time $t+\delta t$. The control actions $u(\tau)$ should be chosen to lie in $\Omega(\tau)$, $\tau \in [t, t+\delta t]$, and to maximize the sum of these two terms. In equation form this is

$$V(x,t) = \max_{\substack{u(\tau)\in\Omega(\tau)\\ \tau\in[t,t+\delta t]}} \left\{ \int_t^{t+\delta t} F[x(\tau),u(\tau),\tau]d\tau + V[x(t+\delta t),t+\delta t] \right\}, \tag{2.10}$$

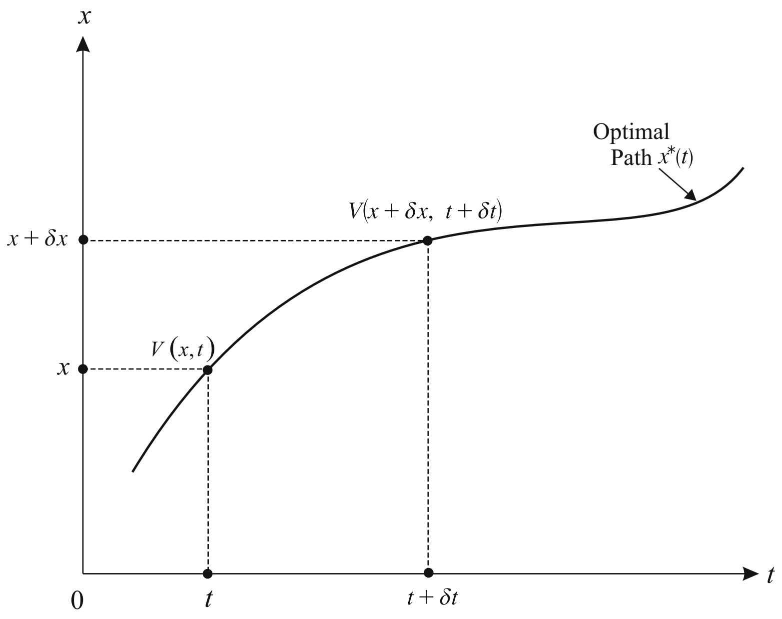

Figure 2.1: An optimal path in the state-time space

where δt represents a small increment in t. It is instructive to compare this equation to definition (2.9).

Since F is a continuous function, the integral in (2.10) is approximately $F(x,u,t)\delta t$ so we can rewrite (2.10) as

$$V(x,t) = \max_{u \in \Omega(t)} \{F(x,u,t)\delta t + V[x(t+\delta t), t+\delta t]\} + o(\delta t), \quad (2.11)$$

where $o(\delta t)$ denotes a collection of higher-order terms in δt. (By definition given in Sect. 1.4.4, $o(\delta t)$ is a function such that $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$.)

We now make an assumption that we will return to again later. We assume that the value function V is a continuously differentiable function of its arguments. This allows us to use the Taylor series expansion of V with respect to δt and obtain

$$V[x(t+\delta t), t+\delta t] = V(x,t) + [V_x(x,t)\dot{x} + V_t(x,t)]\delta t + o(\delta t), \quad (2.12)$$

where V_x and V_t are partial derivatives of $V(x,t)$ with respect to x and t, respectively.

Substituting for $\dot{x}$ from (2.1) in the above equation and then using it in (2.11), we obtain

$$V(x,t) = \max_{u\in\Omega(t)} \{F(x,u,t)\delta t + V(x,t) + V_x(x,t)f(x,u,t)\delta t + V_t(x,t)\delta t\} + o(\delta t). \tag{2.13}$$

Canceling $V(x,t)$ on both sides and then dividing by δt we get

$$0 = \max_{u\in\Omega(t)} \{F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)\} + \frac{o(\delta t)}{\delta t}. \tag{2.14}$$

Now we let $\delta t \to 0$ and obtain the following equation

$$0 = \max_{u\in\Omega(t)} \{F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)\}, \tag{2.15}$$

for which the boundary condition is

$$V(x,T) = S(x,T). \tag{2.16}$$

This boundary condition follows from the fact that the value function at $t = T$ is simply the salvage value function.

The components of the vector $V_x(x,t)$ can be interpreted as the marginal contributions of the state variables x to the value function or the maximized objective function (2.9). We denote the marginal return vector (along the optimal path $x^*(t)$) by the *adjoint* (row) vector $\lambda(t) \in E^n$, i.e.,

$$\lambda(t) = V_x(x^*(t),t) := V_x(x,t) \mid_{x=x^*(t)}. \tag{2.17}$$

From the preceding remark, we can interpret $\lambda(t)$ as the per unit change in the objective function value for a small change in $x^*(t)$ at time t. In other words, $\lambda(t)$ is the highest hypothetical unit price which a rational decision maker would be willing to pay for an infinitesimal addition to $x^*(t)$. See Sect. 2.2.4 for further discussion.

Next we introduce a function $H: E^n \times E^m \times E^n \times E^1 \to E^1$ called the *Hamiltonian*

$$H(x,u,\lambda,t) = F(x,u,t) + \lambda f(x,u,t). \tag{2.18}$$

We can then rewrite Eq. (2.15) as the equation

$$\max_{u\in\Omega(t)} [H(x,u,V_x,t) + V_t] = 0, \tag{2.19}$$

called the *Hamilton-Jacobi-Bellman equation* or, simply, the HJB equation to be satisfied along an optimal path. Note that it is possible to take V_t out of the maximizing operation since it does not depend on u.

The Hamiltonian maximizing condition of the maximum principle can be obtained from (2.19) and (2.17) by observing that, if $x^*(t)$ and $u^*(t)$ are optimal values of the state and control variables and $\lambda(t)$ is the corresponding value of the adjoint variable at time t, then the optimal control $u^*(t)$ must satisfy (2.19), i.e., for all $u \in \Omega(t)$,

$$\begin{aligned} H[x^*(t), u^*(t), \lambda(t), t] + V_t(x^*(t), t) \quad \geq \quad & H[x^*(t), u, \lambda(t), t] \\ & + V_t(x^*(t), t). \end{aligned} \tag{2.20}$$

Canceling the term V_t on both sides, we obtain the Hamiltonian maximizing condition

$$H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \tag{2.21}$$

for all $u \in \Omega(t)$.

In order to complete the statement of the maximum principle, we must still obtain the adjoint equation.

Remark 2.1 We use u^* and x^* for optimal control and state to distinguish them from an admissible control u and the corresponding state x, respectively. However, since the adjoint variable λ is defined only along the optimal path, there is no need for such a distinction, and therefore we do not use the superscript * on λ.

2.2.2 Derivation of the Adjoint Equation

The derivation of the adjoint equation proceeds from the HJB equation (2.19), and is similar to those in Fel'dbaum (1965) and Kirk (1970). Note that, given the optimal path x^*, the optimal control u^* maximizes the left-hand side of (2.19), and its maximum value is zero. We now consider small perturbations of the values of the state variables in a neighborhood of the optimal path x^*. Thus, let

$$x(t) = x^*(t) + \delta x(t), \tag{2.22}$$

where $\| \delta x(t) \| < \varepsilon$ for a small positive ε.

We now consider a 'fixed' time instant t. We can then write (2.19) as

$$\begin{aligned} 0 &= H[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_t(x^*(t), t) \\ &\geq H[x(t), u^*(t), V_x(x(t), t), t] + V_t(x(t), t). \end{aligned} \tag{2.23}$$

To explain, we note from (2.19) that the left-hand side of $\geq$ in (2.23) equals zero. The right-hand side can attain the value zero only if $u^*(t)$ is also an optimal control for $x(t)$. In general, for $x(t) \neq x^*(t)$, this will not be so. From this observation, it follows that the expression on the right-hand side of (2.23) attains its maximum (of zero) at $x(t) = x^*(t)$. Furthermore, $x(t)$ is not explicitly constrained. In other words, $x^*(t)$ is an unconstrained local maximum of the right-hand side of (2.23), so that the derivative of this expression with respect to x must vanish at $x^*(t)$, i.e.,

$$H_x[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_{tx}(x^*(t), t) = 0, \tag{2.24}$$

provided the derivative exists, and for which, we must further assume that V is a twice continuously differentiable function of its arguments. With $H = F + V_x f$ from (2.17) and (2.18), we obtain

$$H_x = F_x + V_x f_x + f^T V_{xx} = F_x + V_x f_x + (V_{xx} f)^T$$

by using $g = V_x$ in the identity (1.15). Substituting this in (2.24) and recognizing the fact that $V_{xx} = (V_{xx})^T$, we obtain

$$F_x + V_x f_x + f^T V_{xx} + V_{tx} = F_x + V_x f_x + (V_{xx} f)^T + V_{tx} = 0, \tag{2.25}$$

where the superscript ^{T} denotes the transpose operation. See (1.16) or Exercise 1.10 for further explanation.

The derivation of the necessary condition (2.25) is the crux of the reasoning in the derivation of the adjoint equation. It is easy to obtain the so-called adjoint equation from it. We begin by taking the time derivative of $V_x(x, t)$. Thus,

$$\begin{aligned} \frac{dV_x}{dt} &= \left(\frac{dV_{x_1}}{dt}, \frac{dV_{x_2}}{dt}, \cdots, \frac{dV_{x_n}}{dt} \right) \\ &= (V_{x_1 x}\dot{x} + V_{x_1 t}, V_{x_2 x}\dot{x} + V_{x_2 t}, \cdots, V_{x_n x}\dot{x} + V_{x_n t}) \\ &= (\textstyle\sum_{i=1}^n V_{x_1 x_i}\dot{x}_i, \sum_{i=1}^n V_{x_2 x_i}\dot{x}_i, \cdots, \sum_{i=1}^n V_{x_n x_i}\dot{x}_i) \; + \; (V_x)_t \\ &= (V_{xx}\dot{x})^T + V_{xt} \\ &= (V_{xx} f)^T + V_{tx}. \end{aligned} \tag{2.26}$$

Note in the above that

$$V_{x_i x} = (V_{x_i x_1}, V_{x_i x_2}, \cdots, V_{x_i x_n})$$

and

$$V_{xx}\dot{x} = \begin{pmatrix} V_{x_1x_1} & V_{x_1x_2} & \cdots & V_{x_1x_n} \\ V_{x_2x_1} & V_{x_2x_2} & \cdots & V_{x_2x_n} \\ \vdots & \vdots & \cdots & \vdots \\ V_{x_nx_1} & V_{x_nx_2} & \cdots & V_{x_nx_n} \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}. \tag{2.27}$$

Since the terms on the right-hand side of (2.26) are the same as the last two terms in (2.25), we see that (2.26) becomes

$$\frac{dV_x}{dt} = -F_x - V_x f_x. \tag{2.28}$$

Because λ was defined in (2.17) to be V_x, we can rewrite (2.28) as

$$\dot{\lambda} = -F_x - \lambda f_x.$$

To see that the right-hand side of this equation can be written simply as $-H_x$, we need to go back to the definition of H in (2.18) and recognize that when taking the partial derivative of H with respect to x, the adjoint variables λ are considered to be independent of x. We note further that along the optimal path, λ is a function of t only. Thus,

$$\dot{\lambda} = -H_x. \tag{2.29}$$

Also, from the definition of λ in (2.17) and the boundary condition (2.16), we have the *terminal boundary condition*, which is also called the *transversality condition*:

$$\lambda(T) = \frac{\partial S(x,T)}{\partial x} \mid_{x=x^*(T)} = S_x(x^*(T),T). \tag{2.30}$$

The *adjoint equation* (2.29) together with its boundary condition (2.30) determine the adjoint variables.

This completes our derivation of the maximum principle using dynamic programming. We can now summarize the main results in the following section.

2.2.3 The Maximum Principle

The necessary conditions for $u^*(t),\ t \in [0, T]$, to be an optimal control are:

$$\begin{cases} \dot{x}^* = f(x^*, u^*, t), x^*(0) = x_0, \\ \dot{\lambda} = -H_x[x^*, u^*, \lambda, t], \quad \lambda(T) = S_x(x^*(T), T), \\ H[x^*, u^*, \lambda, t] \geq H[x^*, u, \lambda, t], \forall u \in \Omega(t), t \in [0, T]. \end{cases} \tag{2.31}$$

It should be emphasized that the state and the adjoint arguments of the Hamiltonian are $x^*(t)$ and $\lambda(t)$ on both sides of the Hamiltonian maximizing condition in (2.31), respectively. Furthermore, $u^*(t)$ must provide a *global* maximum of the Hamiltonian $H[x^*(t), u, \lambda(t), t]$ over $u \in \Omega(t)$. For this reason the necessary conditions in (2.31) are called the *maximum principle*.

Note that in order to apply the maximum principle, we must simultaneously solve two sets of differential equations with u^* obtained from the Hamiltonian maximizing condition in (2.31). With the control variable u^* so obtained, the state equation for x^* is given with the initial value x_0, and the adjoint equation for λ is specified with a condition on the terminal value $\lambda(T)$. Such a system of equations, where initial values of some variables and final values of other variables are specified, is called a *two-point boundary value problem* (TPBVP). The general solution of such problems can be very difficult; see Bryson and Ho (1975), Roberts and Shipman (1972), and Feichtinger and Hartl (1986). However, there are certain special cases which are easy. One such is the case in which the adjoint equation is independent of the state and the control variables; here we can solve the adjoint equation first, then get the optimal control u^*, and then solve for x^*.

Note also that if we can solve the Hamiltonian maximizing condition for an optimal control function in closed form $u^*(x, \lambda, t)$ so that

$$u^*(t) = u^*[x^*(t), \lambda(t), t],$$

then we can substitute this into the state and adjoint equations to get the TPBVP just in terms of a set of differential equations, i.e.,

$$\begin{cases} \dot{x}^* = f(x^*, u^*(x^*, \lambda, t), t), \quad x^*(0) = x_0, \\ \dot{\lambda} = -H_x(x^*, u^*(x^*, \lambda, t), \lambda, t), \quad \lambda(T) = S_x(x^*(T), T). \end{cases} \tag{2.32}$$

We should note that we are making a slight abuse of notation here by using $u^*(x, \lambda, t)$ to denote the optimal control function and $u^*(t)$ as the optimal control at time t. Thus, depending on the context, when we use u^* without any argument, it may mean the optimal control function $u^*(x, \lambda, t)$, or the optimal control at time t, or the entire optimal control trajectory $\{u^*(t),\ t \in [0, T]\}$.

In Sect. 2.5, we derive the TPBVP for a specific example, and solve its discrete version by using Excel. In subsequent chapters we will solve many TPBVPs of varying degrees of difficulty.

One final remark should be made. Because an integral is unaffected by values of the integrand at a finite set of points, some of the arguments made in this chapter may not hold at a finite set of points. This does not affect the validity of the results.

In the next section, we give economic interpretations of the maximum principle, and in Sect. 2.3, we solve five simple examples by using the maximum principle.

2.2.4 Economic Interpretations of the Maximum Principle

Recall from Sect. 2.1.3 that the objective function (2.3) is

$$J = \int_0^T F(x, u, t)dt + S(x(T), T),$$

where F is considered to be the instantaneous profit rate measured in dollars per unit of time, and $S(x, T)$ is the salvage value, in dollars, of the system at time T when the terminal state is x. For purposes of discussion it will be convenient to consider the system as a firm and the state $x(t)$ as the stock of capital at time t.

In (2.17), we interpreted $\lambda(t)$ to be the per unit change in the value function $V(x, t)$ for small changes in capital stock x. In other words, $\lambda(t)$ is the marginal value per unit of capital at time t, and it is also referred to as the *price* or *shadow price* of a unit of capital at time t. In particular, the value of $\lambda(0)$ is the marginal rate of change of the maximum value of J (the objective function) with respect to the change in the initial capital stock, x_0.

Remark 2.2 As mentioned in Appendix C, where we prove a maximum principle without any smoothness assumption on the value function, there arise cases in which the value function may not be differentiable with respect to the state variables. In such cases, when $V_x(x^*(t), t)$ does not exist, then (2.17) has no meaning. See Bettiol and Vinter (2010), Yong and Zhou (1999), and Cernea and Frankowska (2005) for interpretations of the adjoint variables or extensions of (2.17) in such cases.

Next we interpret the Hamiltonian function in (2.18). Multiplying (2.18) formally by dt and using the state equation (2.1) gives

$$Hdt = Fdt + \lambda f dt = Fdt + \lambda \dot{x} dt = Fdt + \lambda dx.$$

The first term $F(x, u, t)dt$ represents the *direct contribution* to J in dollars from time t to $t + dt$, if the firm is in state x (i.e., it has a capital stock of x), and we apply control u in the interval $[t, t + dt]$. The differential $dx = f(x, u, t)dt$ represents the change in capital stock from time t to $t + dt$, when the firm is in state x and control u is applied. Therefore, the second term λdx represents the value in dollars of the incremental capital stock dx, and hence can be considered as the *indirect contribution* to J in dollars. Thus, Hdt can be interpreted as the *total contribution* to J from time t to $t + dt$ when $x(t) = x$ and $u(t) = u$ in the interval $[t, t + dt]$.

With this interpretation, it is easy to see why the Hamiltonian must be maximized at each instant of time t. If we were just to maximize F at each instant t, we would not be maximizing J, because we would ignore the effect of the control in changing the capital stock, which gives rise to indirect contributions to J. The maximum principle derives the adjoint variable $\lambda(t)$, the price of capital at time t, in such a way that $\lambda(t)dx$ is the correct valuation of the indirect contribution to J from time t to $t + dt$. As a consequence, the Hamiltonian maximizing problem can be treated as a static problem at each instant t. In other words, the maximum principle *decouples* the dynamic maximization problem (2.4) in the interval $[0, T]$ into a set of static maximization problems associated with instants t in $[0, T]$. Thus, the Hamiltonian can be interpreted as a surrogate profit rate to be maximized at each instant of time t.

The value of λ to be used in the maximum principle is given by (2.29) and (2.30), i.e.,

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x} - \lambda \frac{\partial f}{\partial x},\ \lambda(T) = S_x(x(T), T).$$

Rewriting the first equation as

$$-d\lambda = H_x dt = F_x dt + \lambda f_x dt,$$

we can observe that along the optimal path, $-d\lambda$, the negative of the increase or, in other words, the decrease in the price of capital from t to $t + dt$, which can be considered as the *marginal cost of holding that capital*, equals the *marginal revenue* $H_x dt$ *of investing the capital.* In turn the marginal revenue $H_x dt$ consists of the sum of the direct marginal contribution $F_x dt$ and the indirect marginal contribution $\lambda f_x dt$. Thus, the adjoint equation becomes the equilibrium relation—*marginal cost equals marginal revenue*, which is a familiar concept in the economics literature; see, e.g., Cohen and Cyert (1965, p. 189) or Takayama (1974, p. 712).

Further insight can be obtained by integrating the above adjoint equation from t to T as follows:

$$\begin{aligned} \lambda(t) &= \lambda(T) + \int_t^T H_x(x(\tau), u(\tau), \lambda(\tau), \tau) d\tau \\ &= S_x(x(T), T) + \int_t^T H_x d\tau. \end{aligned}$$

Note that the price $\lambda(T)$ of a unit of capital at time T is its marginal salvage value $S_x(x(T), T)$. In the special case when $S \equiv 0$, we have $\lambda(T) = 0$, as clearly no value can be derived or lost from an infinitesimal increase in $x(T)$. The price $\lambda(t)$ of a unit of capital at time t is the sum of its terminal price $\lambda(T)$ plus the integral of the marginal surrogate profit rate H_x from t to T.

The above interpretations show that the adjoint variables behave in much the same way as the *dual variables* in linear (and nonlinear) programming, with the differences being that here the adjoint variables are time dependent and satisfy derived differential equations. These connections will become clearer in Chap. 8, which addresses the discrete maximum principle.

2.3 Simple Examples

In order to absorb the maximum principle, the reader should study very carefully the examples in this section, all of which are problems having only one state and one control variable. Some or all of the exercises at the end of the chapter should also be worked.

In the following examples and others in this book, we will at times omit the superscript $*$ on the optimal values of the state variables as long as no confusion arises from doing so.

Example 2.2 Consider the problem:

$$\max \left\{ J = \int_0^1 -x dt \right\} \tag{2.33}$$

subject to the state equation

$$\dot{x} = u, \; x(0) = 1 \tag{2.34}$$

and the control constraint

$$u \in \Omega = [-1, 1]. \tag{2.35}$$

Note that $T = 1$, $F = -x, S = 0$, and $f = u$. Because $F = -x$, we can interpret the problem as one of minimizing the (signed) area under the curve $x(t)$ for $0 \leq t \leq 1$.

Solution First, we form the Hamiltonian

$$H = -x + \lambda u \tag{2.36}$$

and note that, because the Hamiltonian is linear in u, the form of the optimal control, i.e., the one that would maximize the Hamiltonian, is

$$u^*(t) = \begin{cases} 1 & \text{if} \quad \lambda(t) > 0, \\ \text{arbitrary} & \text{if} \quad \lambda(t) = 0, \\ -1 & \text{if} \quad \lambda(t) < 0, \end{cases} \tag{2.37}$$

or referring to the notation in Sect. 1.4,

$$u^*(t) = \text{bang}[-1, 1; \lambda(t)]. \tag{2.38}$$

To find λ, we write the adjoint equation

$$\dot{\lambda} = -H_x = 1, \; \lambda(1) = S_x(x(T), T) = 0. \tag{2.39}$$

Because this equation does not involve x and u, we can easily solve it as

$$\lambda(t) = t - 1. \tag{2.40}$$

It follows that $\lambda(t) = t - 1 < 0$ for $t \in [0, 1)$ and so $u^*(1) = -1, \; t \in [0, 1)$. Since $\lambda(1) = 0$, for simplicity we can also set $u^*(1) = -1$ at the single point $t = 1$. We can then specify the optimal control to be

$$u^*(t) = -1 \text{ for all } t \in [0, 1].$$

Substituting this into the state equation (2.34) we have

$$\dot{x} = -1,\ x(0) = 1, \tag{2.41}$$

whose solution is

$$x^*(t) = 1 - t \text{ for } t \in [0, 1]. \tag{2.42}$$

The graphs of the optimal state and adjoint trajectories appear in Fig. 2.2. Note that the optimal value of the objective function is $J^* = -1/2$.

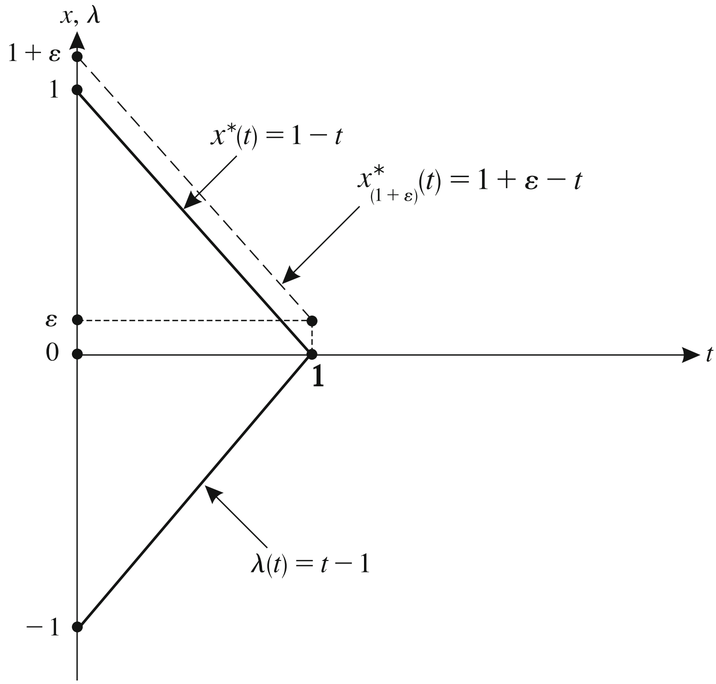

Figure 2.2: Optimal state and adjoint trajectories for Example 2.2

In Sect. 2.2.4, we stated that the adjoint variable $\lambda(t)$ gives the marginal value per unit increment in the state variable $x(t)$ at time t. Let us illustrate this claim at time $t = 0$ with the help of Example 2.2. Note from (2.40) that $\lambda(0) = -1$. Thus, if we increase the initial value $x(0)$ from 1, by a small amount ε, to a new value $1 + \varepsilon$, where ε may be positive or negative, then we expect the optimal value of the objective function to change from $J^* = -1/2$ to

$$J^*_{(1+\varepsilon)} = -1/2 + \lambda(0)\varepsilon + o(\varepsilon) = -1/2 - \varepsilon + o(\varepsilon),$$

where we use the subscript $(1+\varepsilon)$ to distinguish the new value from J^* as well as to emphasize its dependence on the new initial condition $x(0)=1+\varepsilon$. To verify this, we first observe that $u^*(t)=-1,\ t\in[0,1]$, remains optimal in this example for the new initial condition. Then from (2.41) with $x(0)=1+\varepsilon$, we can obtain the new optimal state trajectory, shown by the dotted line in Fig. 2.2 as

$$x^*_{(1+\varepsilon)}(t)=1+\varepsilon-t,\ t\in[0,1],$$

where the notation $x^*_{(y)}(t)$ indicates the dependence of the optimal trajectory on the initial value $x(0)=y$. Substituting this for x in (2.33) and integrating, we get the new objective function value to be $-1/2-\varepsilon$. Since 0 is of the order $o(\varepsilon)$, our claim has been illustrated.

We should note that in general it may be necessary to perform separate calculations for positive and negative ε. It is easy to see, however, that this is not the case in this example.

Example 2.3 Let us solve the same problem as in Example 2.2 over the interval $[0,2]$ so that the objective is:

$$\max\left\{J=\int_0^2 -x dt\right\}. \tag{2.43}$$

The dynamics and constraints are (2.34) and (2.35), respectively, as before. Here we want to minimize the *signed* area between the horizontal axis and the trajectory of $x(t)$ for $0\le t\le 2$.

Solution As before, the Hamiltonian is defined by (2.36) and the optimal control is as in (2.38). The adjoint equation

$$\dot{\lambda}=1,\ \lambda(2)=0 \tag{2.44}$$

is the same as (2.39) except that now $T=2$ instead of $T=1$. The solution of (2.44) is easily found to be

$$\lambda(t)=t-2,\quad t\in[0,2]. \tag{2.45}$$

The graph of $\lambda(t)$ is shown in Fig. 2.3.

With $\lambda(t)$ as in (2.45), we can determine $u^*(t)=-1$ throughout. Thus, the state equation is the same as (2.41). Its solution is given by (2.42) for $t\in[0,2]$. The optimal value of the objective function is $J^*=0$. The graph of $x^*(t)$ is also sketched in Fig. 2.3.

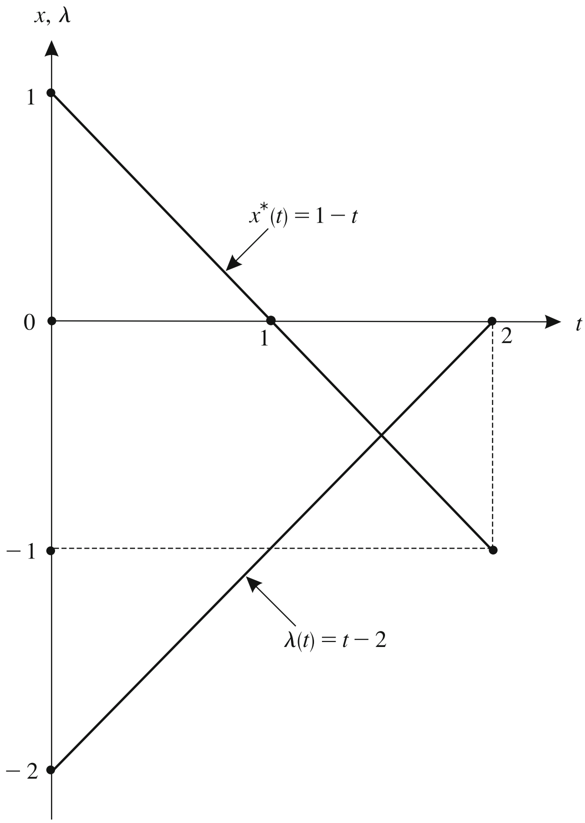

Figure 2.3: Optimal state and adjoint trajectories for Example 2.3

Example 2.4 The next example is:

$$\max\left\{J = \int_0^1 -\frac{1}{2}x^2 dt\right\} \tag{2.46}$$

subject to the same constraints as in Example 2.2, namely,

$$\dot{x} = u,\ x(0) = 1,\ u \in \Omega = [-1, 1]. \tag{2.47}$$

Here $F = -(1/2)x^2$ so that the interpretation of the objective function (2.46) is that we are trying to find the trajectory $x(t)$ in order that the area under the curve $(1/2)x^2$ is minimized.

Solution The Hamiltonian is

$$H = -\frac{1}{2}x^2 + \lambda u. \tag{2.48}$$

The control function $u^*(x, \lambda)$ that maximizes the Hamiltonian in this case depends only on λ, and it has the form

$$u^*(x, \lambda) = \text{bang}[-1, 1; \lambda]. \tag{2.49}$$

Then, the optimal control at time t can be expressed as $u^*(t) = \text{bang}[-1, 1, \lambda(t)]$.

The adjoint equation is

$$\dot{\lambda} = -H_x = x, \ \lambda(1) = 0. \tag{2.50}$$

Here the adjoint equation involves x, so we cannot solve it directly. Because the state equation (2.47) involves u, which depends on λ, we also cannot integrate it independently without knowing λ.

A way out of this dilemma is to use some intuition. Since we want to minimize the area under $(1/2)x^2$ and since $x(0) = 1$, it is clear that we want x to decrease as quickly as possible. Let us therefore temporarily *assume* that λ is nonpositive in the interval $[0, 1]$ so that from (2.49) we have $u = -1$ throughout the interval. (In Exercise 2.8, you will be asked to show that this assumption is correct.) With this assumption, we can solve (2.47) as

$$x(t) = 1 - t. \tag{2.51}$$

Substituting this into (2.50) gives

$$\dot{\lambda} = 1 - t.$$

Integrating both sides of this equation from t to 1 gives

$$\int_t^1 \dot{\lambda}(\tau)d\tau = \int_t^1 (1 - \tau)d\tau,$$

or

$$\lambda(1) - \lambda(t) = (\tau - \frac{1}{2}\tau^2) \mid_t^1 ,$$

which, using $\lambda(1) = 0$, yields

$$\lambda(t) = -\frac{1}{2}t^2 + t - \frac{1}{2}. \tag{2.52}$$

The reader may now verify that $\lambda(t)$ is nonpositive in the interval $[0, 1]$, verifying our original assumption. Hence, (2.51) and (2.52) satisfy the necessary conditions. In Exercise 2.26, you will be asked to show that they satisfy sufficient conditions derived in Sect. 2.4 as well, so that they are indeed optimal. Thus, $x^*(t) = 1 - t$, and using this in (2.46), we can get $J^* = -1/6$. Figure 2.4 shows the graphs of the optimal state and adjoint trajectories.

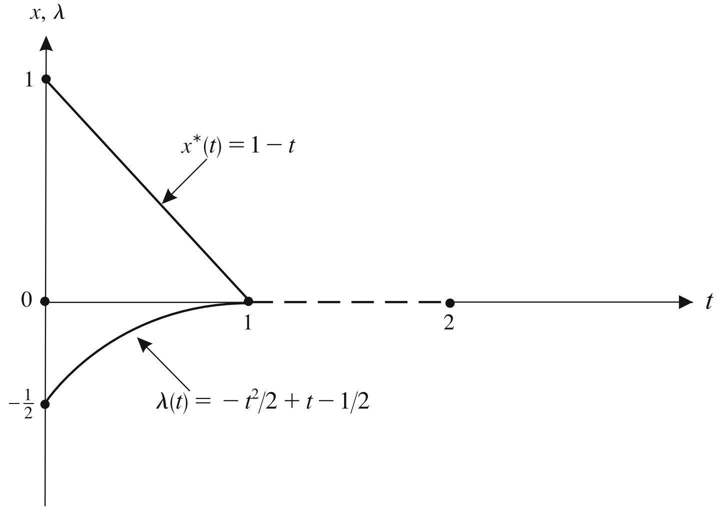

Figure 2.4: Optimal trajectories for Examples 2.4 and 2.5

Example 2.5 Let us rework Example 2.4 with $T = 2$, i.e., with the objective function:

$$\max \left\{ J = \int_0^2 -\frac{1}{2}x^2 dt \right\} \tag{2.53}$$

subject to the constraints (2.47).

Solution The Hamiltonian is still as in (2.48) and the form of the optimal policy remains as in (2.49). The adjoint equation is

$$\dot{\lambda} = x, \ \lambda(2) = 0,$$

which is the same as (2.50) except $T = 2$ instead of $T = 1$. Let us try to extend the solution of the previous example from $T = 1$ to $T = 2$. Thus, we keep $\lambda(t)$ as in (2.52) for $t \in [0, 1]$ with $\lambda(1) = 0$. If we recall from the definition of the bang function that bang $[-1, 1; 0]$ is not defined, it allows us to choose u in (2.49) arbitrarily when $\lambda = 0$. This is an instance of *singular control*, so let us see if we can *maintain* the singular control by choosing u appropriately. To do this we choose $u = 0$ when $\lambda = 0$. Since $\lambda(1) = 0$ we set $u(1) = 0$ so that from (2.47), we have $\dot{x}(1) = 0$. Now note that if we set $u(t) = 0$ for $t > 1$, then by integrating equations (2.47) and (2.50) forward from $t = 1$ to $t = 2$, we see that $x(t) = 0$ and $\lambda(t) = 0$ for $1 < t \leq 2$; in other words, $u(t) = 0$ maintains singular control in the interval. Intuitively, this is the correct answer since once we get $x = 0$, we should keep it at 0 in order to maximize the objective function J in (2.53). We will later give further discussion of *singular control* and state an additional necessary condition in Sect. D.6 for such cases; see also Bell and Jacobson (1975). In Fig. 2.4, we can get the singular solution by extending the graphs shown to the right (as shown by thick dotted line), making $x^*(t) = 0,\ \lambda(t) = 0$, and $u^*(t) = 0$ for $1 < t \leq 2$.

With the trajectory $x^*(t),\ 0 \leq t \leq 2$, thus obtained, we can use (2.53) to compute the optimal value of the objective function as

$$J^* = \int_0^1 -(1/2)(1-t)^2 dt + \int_1^2 -(1/2)(0)dt = -1/6.$$

Now suppose that the initial $x(0)$ is perturbed by a small amount ε to $x(0) = 1 + \varepsilon$, where ε may be positive or negative. According to the marginal value interpretation of $\lambda(0)$, whose value is $-1/2$ in this example, we can estimate the change in the objective function to be $\lambda(0)\varepsilon + o(\varepsilon) = -\varepsilon/2 + o(\varepsilon)$.

Next we calculate directly the impact of the perturbation in the initial value. For this we must obtain new control and state trajectories. These are clearly

$$u^*_{(1+\varepsilon)}(t) = \begin{cases} -1, & t \in [0, 1+\varepsilon], \\ 0, & t \in (1+\varepsilon, 2], \end{cases}$$

and

$$x^*_{(1+\varepsilon)}(t) = \begin{cases} 1+\varepsilon-t, & t \in [0, 1+\varepsilon], \\ 0, & t \in (1+\varepsilon, 2], \end{cases}$$

where we have used the subscript $(1+\varepsilon)$ to distinguish these from the original trajectories as well as to indicate their dependence on the initial value $x(0) = 1+\varepsilon$. We can then obtain the corresponding optimal value of the objective function as

$$\begin{aligned} J^*_{(1+\varepsilon)} &= \int_0^{1+\varepsilon} -(1/2)(1+\varepsilon-t)^2 dt = -1/6 - \varepsilon/2 - \varepsilon^2/2 - \varepsilon^3/6 \\ &= -1/6 + \lambda(0)\varepsilon + o(\varepsilon), \end{aligned}$$

where $o(\varepsilon) = -\varepsilon^2/2 - \varepsilon^3/6$.

In this example and Example 2.2, we have, by direct calculation, demonstrated the significance of $\lambda(0)$ as the marginal value of the change in the initial state. This could have also been accomplished by obtaining the value function $V(x,t)$ for $x(t)=x,\ t\in[0,2]$, and then showing that $\lambda(0)=V_x(1,0)$. This, of course, is the relationship (2.17) at $x(0)=x=1$ and $t=0$.

Keep in mind, however, that deriving $V(x,t)$ is more than just finding the solution of the problem, which we have already found by using the maximum principle. $V(x,t)$ also yields additional insights into the problem. In order to completely specify $V(x,t)$ for all $x\in E^1$ and all $t\in[0,2]$, we need to deal with a number of cases. Here, we will carry out the details only in the case of any $t\in[0,2]$ and $0\le x\le 2-t$, and leave the listing of the other cases and the required calculations as Exercise 2.13.

We know from (2.9) that we need to solve the optimal control problem for any given $t\in[0,2]$ with $0\le x\le 2-t$. However, from our earlier analysis of this example, it is clear that the optimal control

$$u^*_{(x,t)}(s)=\begin{cases}-1, & s\in[t,t+x],\\ 0, & s\in(t+x,2],\end{cases}$$

and the corresponding

$$x^*_{(x,t)}(s)=\begin{cases}x-(s-t), & s\in[t,t+x],\\ 0, & s\in(t+x,2],\end{cases}$$

where we use the subscript to show the dependence of the control and state trajectories of a problem beginning at time t with the state $x(t)=x$. Thus,

$$V(x,t)=\int_t^{t+x}-\frac{1}{2}[x^*_{(x,t)}(s)]^2ds=-\frac{1}{2}\int_t^{t+x}(x-s+t)^2ds.$$

While this expression can be easily integrated to obtain an explicit solution for $V(x,t)$, we do not need to do this for our immediate purpose at hand, which is to obtain $V_x(x,t)$. Differentiating the right-hand side with respect to x, we obtain

$$V_x(x,t)=-\frac{1}{2}\int_t^{x+t}2(x-s+t)ds.$$

Furthermore, since

$$x^*(t) = \begin{cases} 1 - t, & t \in [0,1], \\ 0, & t \in (1,2], \end{cases}$$

we obtain

$$V_x(x^*(t),t) = \begin{cases} -\frac{1}{2}\int_t^1 2(x-s+t)ds = -\frac{1}{2}t^2 + t - \frac{1}{2}, & t \in [0,1], \\ 0, & t \in (1,2], \end{cases}$$

which equals $\lambda(t)$ obtained as the adjoint variable in Example 2.5. Note that for $t \in [0,1]$, $\lambda(t)$ in Example 2.5 is the same as that in Example 2.4 obtained in (2.52).

Example 2.6 This example is slightly more complicated and the optimal control is not bang-bang. The problem is:

$$\max \left\{ J = \int_0^2 (2x - 3u - u^2)dt \right\} \tag{2.54}$$

subject to

$$\dot{x} = x + u,\ x(0) = 5 \tag{2.55}$$

and the control constraint

$$u \in \Omega = [0,2]. \tag{2.56}$$

Solution Here $T = 2$, $F = 2x - 3u - u^2$, $S = 0$, and $f = x + u$. The Hamiltonian is

$$\begin{aligned} H &= (2x - 3u - u^2) + \lambda(x+u) \\ &= (2+\lambda)x - (u^2 + 3u - \lambda u). \end{aligned} \tag{2.57}$$

Let us find the optimal control policy by differentiating (2.57) with respect to u. Thus,

$$\frac{\partial H}{\partial u} = -2u - 3 + \lambda = 0,$$

so that the form of the optimal control is

$$u^*(t) = \frac{\lambda(t) - 3}{2}, \tag{2.58}$$

provided this expression stays within the interval $\Omega = [0, 2]$. Note that the second derivative of H with respect to u is $\partial^2 H/\partial u^2 = -2 < 0$, so that (2.58) satisfies the second-order condition for the maximum of a function.

We next derive the adjoint equation as

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2 - \lambda,\ \lambda(2) = 0. \tag{2.59}$$

Referring to Appendix A.1, we can use the integrating factor e^t to obtain

$$e^t(d\lambda + \lambda dt) = d(e^t\lambda) = -2e^t dt.$$

We then integrate it on both sides from t to 2 and use the terminal condition $\lambda(2) = 0$ to obtain the solution of the adjoint equation (2.59) as

$$\lambda(t) = 2(e^{2-t} - 1).$$

If we substitute this into (2.58) and impose the control constraint (2.56), we see that the optimal control is

$$u^*(t) = \begin{cases} 2 & \text{if } \ e^{2-t} - 2.5 > 2, \\ e^{2-t} - 2.5 & \text{if } \ 0 \le e^{2-t} - 2.5 \le 2, \\ 0 & \text{if } \ e^{2-t} - 2.5 < 0, \end{cases} \tag{2.60}$$

or referring to the notation defined in (1.22),

$$u^*(t) = \text{sat}[0, 2; e^{2-t} - 2.5].$$

The graph of $u^*(t)$ appears in Fig. 2.5. In the figure, t_1 satisfies $e^{2-t_1} - 2.5 = 2$, i.e., $t_1 = 2 - \ln 4.5 \approx 0.496$, while t_2 satisfies $e^{2-t_2} - 2.5 = 0$, which gives $t_2 = 2 - \ln 2.5 \approx 1.08$.

In Exercise 2.2 you will be asked to compute the optimal state trajectory $x^*(t)$ corresponding to $u^*(t)$ shown in Fig. 2.5 by piecing together the solutions of three separate differential equations obtained from (2.55) and (2.60).

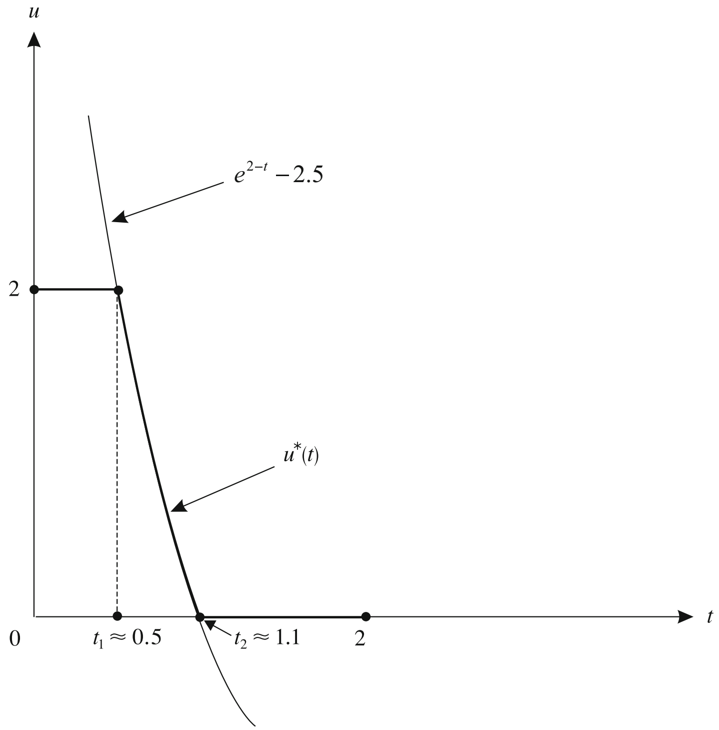

Figure 2.5: Optimal control for Example 2.6

2.4 Sufficiency Conditions

So far, we have shown the necessity of the maximum principle conditions for optimality. Next we prove a theorem that gives qualifications under which the maximum principle conditions are also sufficient for optimality. This theorem is important from our point of view since the models derived from many management science applications will satisfy conditions required for the sufficiency result. As remarked earlier, our technique for proving existence will be to display for any given model, a solution that satisfies both necessary and sufficient conditions. A good reference for sufficiency conditions is Seierstad and Sydsæter (1987).

We first define a function $H^0 : E^n \times E^m \times E^1 \to E^1$ called the *derived Hamiltonian* as follows:

$$H^0(x, \lambda, t) = \max_{u \in \Omega(t)} H(x, u, \lambda, t). \tag{2.61}$$

We assume that by this equation a function $u^*(x,\lambda,t)$ is implicitly and uniquely defined. Given these assumptions we have by definition,

$$H^0(x,\lambda,t) = H(x,u^*,\lambda,t). \tag{2.62}$$

For our proof of the sufficiency of the maximum principle, we also need the derivative $H^0_x(x,\lambda,t)$, which by use of the Envelope Theorem can be given as

$$H^0_x(x,\lambda,t) = H_x(x,u^*,\lambda,t) := H_x(x,u,\lambda,t)|_{u=u^*}. \tag{2.63}$$

To see this in the case when $u^*(x,\lambda,t)$ is differentiable in x, let us differentiate (2.62) with respect to x:

$$H^0_x(x,\lambda,t) = H_x(x,u^*,\lambda,t) + H_u(x,u^*,\lambda,t)\frac{\partial u^*}{\partial x}. \tag{2.64}$$

To obtain (2.63) from (2.64), we need to show that the second term on the right-hand side of (2.64) vanishes, i.e.,

$$H_u(x,u^*,\lambda,t)\frac{\partial u^*}{\partial x} = 0 \tag{2.65}$$

for each x. There are two cases to consider. If u^* is in the interior of $\Omega(t)$, then it satisfies the first-order condition $H_u(x,u^*,\lambda,t) = 0$, thereby implying (2.65). Otherwise, u^* is on the boundary of $\Omega(t)$. Then, for each i,j, either $H_{u_i} = 0$ or $\partial u_i^*/\partial x_j = 0$ or both. Once again, (2.65) holds. Exercise 2.25 gives a specific instance of this case.

Remark 2.3 We have shown the result in (2.63) in cases when u^* is a differentiable function of x. The result holds more generally, provided that $\Omega(t)$ is appropriately qualified; see Derzko et al. (1984). Such results are known as *Envelope Theorems*, and are used often in economics.

Theorem 2.1 (Sufficiency Conditions). *Let $u^*(t)$, and the corresponding $x^*(t)$ and $\lambda(t)$ satisfy the maximum principle necessary condition (2.31) for all $t \in [0,T]$. Then, u^* is an optimal control if $H^0(x,\lambda(t),t)$ is concave in x for each t and $S(x,T)$ is concave in x.*

Proof. The proof is a minor extension of the arguments in Arrow and Kurz (1970). By definition

$$H[x(t),u(t),\lambda(t),t] \le H^0[x(t),\lambda(t),t]. \tag{2.66}$$

Since H^0 is differentiable and concave, we can use the applicable definition of concavity given in Sect. 1.4 to obtain

$$H^0[x(t),\lambda(t),t] \le H^0[x^*(t),\lambda(t),t] + H_x^0[x^*(t),\lambda(t),t][x(t)-x^*(t)]. \tag{2.67}$$

Using (2.66), (2.62), and (2.63) in (2.67), we obtain

$$\begin{aligned} H[x(t),u(t),\lambda(t),t] \;\le\; & H[x^*(t),u^*(t),\lambda(t),t] \\ & +H_x[x^*(t),u^*(t),\lambda(t),t][x(t)-x^*(t)]. \end{aligned} \tag{2.68}$$

By definition of H in (2.18) and the adjoint equation of (2.31)

$$\begin{aligned} F[x(t),u(t),t]+\lambda(t)f[x(t),u(t),t] \le{} & F[x^*(t),u^*(t),t] \\ & +\lambda(t)f[x^*(t),u^*(t),t]-\dot{\lambda}(t)[x(t)-x^*(t)]. \end{aligned} \tag{2.69}$$

Using the state equation in (2.31), transposing, and regrouping,

$$\begin{aligned} F[x^*(t),u^*(t),t]-F[x(t),u(t),t] \;\ge\; & \dot{\lambda}(t)[x(t)-x^*(t)] \\ & +\lambda(t)[\dot{x}(t)-\dot{x}^*(t)]. \end{aligned} \tag{2.70}$$

Furthermore, since $S(x,T)$ is a differential and concave function in its first argument, we have

$$S(x(T),T) \le S(x^*(T),T) + S_x(x^*(T),T)[x(T)-x^*(T)] \tag{2.71}$$

or,

$$S(x^*(T),T) - S(x(T),T) \ge S_x(x^*(T),T)[x(T)-x^*(T)]. \tag{2.72}$$

Integrating both sides of (2.70) from 0 to T and adding (2.72), we have

$$\begin{aligned} & \left[\int_0^T F(x^*(t),u^*(t),t)dt + S(x^*(T),T)\right] \\ & \quad -\left[\int_0^T F(x(t),u(t),t)dt + S(x(T),T)\right] \\ & \quad \ge [\lambda(T)-S_x(x^*(T),T)][x(T)-x^*(T)] - \lambda(0)[x(0)-x^*(0)] \end{aligned}$$

or,

$$J(u^*) - J(u) \qquad (2.73)$$
$$\geq [\lambda(T) - S_x(x^*(T), T)][x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)],$$

where $J(u)$ is the value of the objective function associated with a control u. Since $x^*(0) = x(0) = x_0$, the initial condition, and since $\lambda(T) = S_x(x^*(T), T)$ from the terminal adjoint condition in (2.31), we have

$$J(u^*) \geq J(u). \qquad (2.74)$$

Thus, u^* is an optimal control. This completes the proof. □

Because $\lambda(t)$ is not known *a priori*, it is usual to test H^0 for a stronger assumption, i.e., to check for the concavity of the function $H^0(x, \lambda, t)$ in x for any λ and t. Sometimes the stronger condition given in Exercise 2.27 can be used.

Mangasarian (1966) gives a sufficient condition in which the concavity of $H^0(x, \lambda(t), t)$ in Theorem 2.1 is replaced by a stronger condition requiring the Hamiltonian $H(x, u, \lambda(t), t)$ to be jointly concave in (x, u).

Example 2.7 Let us show that the problems in Examples 2.2 and 2.3 satisfy the sufficient conditions. We have from (2.36) and (2.61),

$$H^0 = -x + \lambda u^*,$$

where u^* is given by (2.37). Since u^* is a function of λ only, $H^0(x, \lambda, t)$ is certainly concave in x for any t and λ (and in particular for $\lambda(t)$ supplied by the maximum principle). Since $S(x, T) = 0$, the sufficient conditions hold.

Finally, it is important to mention that thus far in this chapter, we have considered problems in which the terminal values of the state variables are not constrained. Such problems are called *free-end-point problems*. The problems at the other extreme, where the terminal values of the state variables are completely specified, are termed *fixed-end-point problems*. Then, there are problems in between these two extremes. While a detailed discussion of terminal conditions on state variables appears in Sect. 3.4 of the next chapter, it is instructive here to briefly indicate how the maximum principle needs to be modified in the case of fixed-end-point problems. Suppose $x(T)$ is completely specified, i.e.,

$x(T) = k \in E^n$, where k is a vector of constants. Observe then that the first term on the right-hand side of inequality (2.73) vanishes regardless of the value of $\lambda(T)$, since $x(T) - x^*(T) = k - k = 0$ in this case. This means that the sufficiency result would go through for any value of $\lambda(T)$. Not surprisingly, therefore, the transversality condition (2.30) in the fixed-end-point case changes to

$$\lambda(T) = \beta, \tag{2.75}$$

where $\beta \in E^n$ is a vector of constants to be determined.

Indeed, one can show that (2.75) is also the necessary transversality condition for fixed point problems. With this observation, the maximum principle for fixed-end-point problems can be obtained by modifying (2.31) as follows: adding $x(T) = k$ and removing $\lambda(T) = S_x(x^*(T), T)$. Likewise, the resulting TPBVP (2.32) can be modified correspondingly; it will have initial and final values on the state variables, whereas both initial and terminal values for the adjoint variables are unspecified, i.e., $\lambda(0)$ and $\lambda(T)$ are constants to be determined.

In Exercises 2.28 and 2.19, you are asked to solve the fixed-end-point problems given there.

2.5 Solving a TPBVP by Using Excel

A number of examples and exercises found throughout this book involve finding a numerical solution to a two-point boundary value problem (TPBVP). In this section we will show how the GOAL SEEK function in Excel can be used for this purpose. We will solve the following example.

Example 2.8 Consider the problem:

$$\max \left\{ J = \int_0^1 -\frac{1}{2}(x^2 + u^2)dt \right\}$$

subject to

$$\dot{x} = -x^3 + u, \; x(0) = 5. \tag{2.76}$$

Solution We form the Hamiltonian

$$H = -\frac{1}{2}(x^2 + u^2) + \lambda(-x^3 + u),$$

where the adjoint variable λ satisfies the equation

$$\dot{\lambda} = x + 3x^2\lambda,\ \lambda(1) = 0. \tag{2.77}$$

Since u is unconstrained, we set $H_u = 0$ to obtain $u^* = \lambda$. With this, the state equation (2.76) becomes

$$\dot{x} = -x^3 + \lambda,\ x(0) = 5. \tag{2.78}$$

Thus, the TPBVP is given by the system of equations (2.77) and (2.78).

A simple method to solve the TPBVP uses what is known as the *shooting method*, explained in the flowchart in Fig. 2.6.

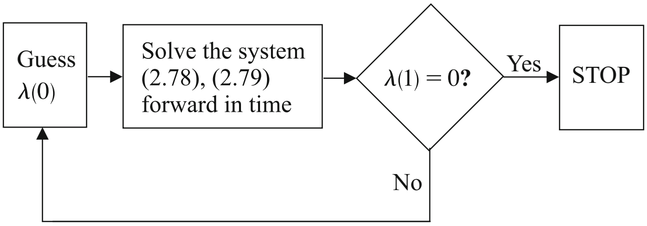

Figure 2.6: The flowchart for Example 2.8

We will use Excel functions to implement the shooting method. For this we discretize (2.77) and (2.78) by replacing dx/dt and $d\lambda/dt$ by

$$\frac{\triangle x}{\triangle t} = \frac{x(t+\triangle t) - x(t)}{\triangle t} \text{ and } \frac{\triangle \lambda}{\triangle t} = \frac{\lambda(t+\triangle t) - \lambda(t)}{\triangle t},$$

respectively. Substitution of $\triangle x/\triangle t$ for $\dot{x}$ in (2.78) and $\triangle\lambda/\triangle t$ for $\dot{\lambda}$ in (2.77) gives the discrete version of the TPBVP:

$$x(t+\triangle t) = x(t) + [-x(t)^3 + \lambda(t)] \triangle t,\ x(0) = 5, \tag{2.79}$$

$$\lambda(t+\triangle t) = \lambda(t) + [x(t) + 3x(t)^2\lambda(t)] \triangle t,\ \lambda(1) = 0. \tag{2.80}$$

In order to solve these equations, open an empty spreadsheet, choose the unit of time to be $\triangle t = 0.01$, make a guess for the initial value $\lambda(0)$ to be, say -0.2, and make the entries in the cells of the spreadsheet as specified below:

> Enter -0.2 in cell A1.
> Enter 5 in cell B1.
> Enter = A1 + (B1 + 3 ∗ (B1ˆ2)∗ A1)∗ 0.01 in cell A2.
> Enter = B1 + (-B1ˆ 3 + A1) ∗ 0.01 in cell B2.

Here we have entered the right-hand side of the difference equation (2.80) for $t = 0$ in cell A2 and the right-hand side of the difference equation (2.79) for $t = 0$ in cell B2. Note that $\lambda(0) = -0.2$ shown as the entry -0.2 in cell A1 is merely a guess. The correct value will be determined by the use of the GOAL SEEK function.

Next highlight cells A2 and B2 and drag the combination down to row 101 of the spreadsheet. Using EDIT in the menu bar, select FILL DOWN. Thus, Excel will solve Eqs. (2.80) and (2.79) from $t = 0$ to $t = 1$ in steps of $\triangle t = 0.01$, and that solution will appear as entries in columns A and B of the spreadsheet, respectively. In other words, the guessed solution for $\lambda(t)$ will appear in cells A1 to A101 and the corresponding solution for $x(t)$ will appear in cells B1 to B101. To find the correct value for $\lambda(0)$, use the GOAL SEEK function under TOOLS in the menu bar and make the following entries:

> Set cell: A101.
> To value: 0.
> By changing cell: A1.

It finds the correct initial value for the adjoint variable as $\lambda(0) = -0.10437$, which should appear in cell A1, and the correct ending value of the state variable as $x(1) = 0.62395$, which should appear in cell B101. You will notice that the entry in cell A101 may not be exactly zero as instructed, although it will be very close to it. In our example, it is -0.0007. By using the CHART function, the graphs of $x^*(t)$ and $\lambda(t)$ can be printed out by Excel as shown in Fig. 2.7.

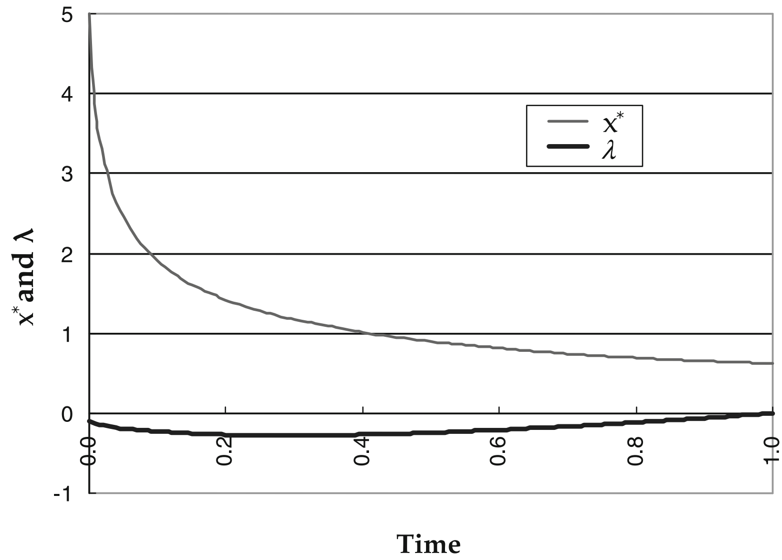

Figure 2.7: Solution of TPBVP by excel

As we discuss more complex problems involving control and state inequality constraints in Chaps. 3 and 4, we will realize that the shooting method is no longer adequate to solve such problems. However, there is a large amount of literature devoted to computational methods for solving optimal control problems. While a detailed treatment of this topic is beyond the scope of this book, we suggest some references as well as a software in Chap. 4, Sect. 4.3.

Exercises for Chapter 2

E 2.1 Perform the following:

(a) In Example 2.2, show $J^* = -1/2$.

(b) In Example 2.3, show $J^* = 0$.

(c) In Example 2.4, show $J^* = -1/6$.

(d) In Example 2.5, show $J^* = -1/6$.

E 2.2 Complete Example 2.6 by writing the optimal $x^*(t)$ in the form of integrals over the three intervals $(0, t_1), (t_1, t_2)$, and $(t_2, 2)$ shown in Fig. 2.5.

Hint: It is not necessary to actually carry out the numerical evaluation of these integrals unless you are ambitious.

E 2.3 Find the optimal solution for Example 2.1 with $x_0 = 0$ and $T = 1$.

E 2.4 Rework Example 2.6 with $F = 2x - 3u$.

E 2.5 Show that both the Lagrange and Mayer forms of the optimal control problem can be reduced to the linear Mayer form (2.5).

E 2.6 Show that the optimal control obtained from the application of the maximum principle satisfies the principle of optimality: if $u^*(t)$ is an optimal control and $x^*(t)$ is the corresponding optimal path for $0 \le t \le T$ with $x(0) = x_0$, then verify the above proposition by showing that $u^*(t)$ for $\tau \le t \le T$ satisfies the maximum principle for the problem beginning at time τ with the initial condition $x(\tau) = x^*(\tau)$.

E 2.7 Provide an alternative derivation of the adjoint equation in Sect. 2.2.2 by starting with a restatement of the Eq. (2.19) as $-V_t = H^0$ and differentiating it with respect to x.

E 2.8 In Example 2.4, show that in view of (2.47) any $\lambda(t), t \in [0, 1]$, that satisfies (2.50) must be nonnegative.

E 2.9 The system defined in (2.4) is termed *autonomous* if F, f, S and Ω are not explicit functions of time t. In this case, show that the Hamiltonian is constant along the optimal path, i.e., show that $dH/dt = 0$. Furthermore, verify this result in Example 2.4 by a direct substitution for x and λ from (2.51) and (2.52), respectively, into H given in (2.48).

E 2.10 In Example 2.4, verify by direct calculation that with a new initial value $x(0) = 1+\varepsilon$ with ε small, the new optimal objective function value will be

$$J^*_{(1+\varepsilon)} = -1/6 + \lambda(0)\varepsilon + o(\varepsilon) = -1/6 - \varepsilon/2 - \varepsilon^2/2.$$

E 2.11 In Example 2.6, verify by direct calculation that with a new initial $x(0) = 5+\varepsilon$ with ε small, the objective function value will change by

$$\lambda(0)\varepsilon + o(\varepsilon) = 2(e^2 - 1)\varepsilon + o(\varepsilon).$$

E 2.12 Obtain the value function $V(x,t)$ explicitly in Example 2.4 and verify the relation $V_x(x^*(t),t) = \lambda(t)$ for the example by showing that $V_x(1-t,t) = -(1/2)t^2 + t - 1/2$.

E 2.13 Obtain the value function $V(x,t)$ explicitly in Example 2.5 for every $x \in E^1$ and $t \in [0,2]$.

Hint: You need to deal with the following cases for $t \in [0,2]$:

(i) $0 \leq x \leq 2-t$,
(ii) $x > 2-t$,
(iii) $t-2 \leq x < 0$, and
(iv) $x < t-2$.

E 2.14 Obtain $V(x,t)$ in Example 2.6 for small positive and negative x for $t \in [t_2, 2]$. Then, show that $V_x(x,t) = 2(e^{2-t}-1),\ t \in [t_2,2]$, is the same as $\lambda(t),\ t \in [t_2,2]$ obtained in Example 2.6.

E 2.15 Solve the problem:

$$\max\left\{J = \int_0^T (x - \frac{u^2}{2})dt\right\}$$

subject to

$$\dot{x} = u,\ x(0) = x_0,$$

$$u \in [0,1],$$

for optimal control and optimal state trajectory. Verify that your solution is optimal by using the maximum principle sufficiency condition.

E 2.16 Solve completely the problem:

$$\max\left\{\int_0^1 (x+u)dt\right\}$$

$$\dot{x} = 1 - u^2,\ \ x(0) = 1;$$

that is, find $x^*(t), u^*(t)$ and $\lambda(t), 0 \leq t \leq 1$.

E 2.17 Use the maximum principle to solve the following problem given in the Mayer form:

$$\max[8x_1(18) + 4x_2(18)]$$

subject to

$$\dot{x}_1 = x_1 + x_2 + u,\ x_1(0) = 15,$$

$$\dot{x}_2 = 2x_1 - u,\ x_2(0) = 20,$$

and the control constraint

$$0 \le u \le 1.$$

Hint: Use the method in Appendix A to solve the simultaneous differential equations.

E 2.18 In Fig. 2.8, a water reservoir being used for the purpose of fire-fighting is leaking, and its water height $x(t)$ is governed by

$$\dot{x} = -0.1x + u, x(0) = 10,$$

where $u(t)$ denotes the net inflow at time t and $0 \le u \le 3$.

Note that $x(t)$ also represents the water pressure in appropriate units. Since high water pressure is useful for fire-fighting, the objective function in (a) below involves keeping the average pressure high, while that in (b) involves building up a high pressure at $T = 100$. Furthermore, we do not need to impose the state constraints $0 \le x(t) \le 50$, as these will always be satisfied for every feasible control $u(t),\ 0 \le t \le 100$.

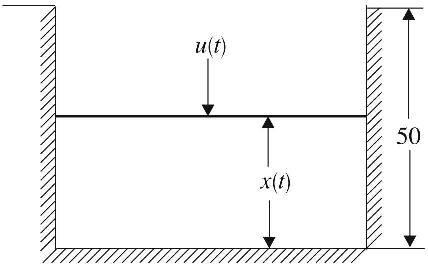

Figure 2.8: Water reservoir of Exercise 2.18

(a) Find the optimal control which maximizes

$$J = \int_0^{100} x dt.$$

Find the maximum level reached.

(b) Replace the objective function in (a) by

$$J = 5x(100),$$

and re-solve the problem.

(c) Redo the problem with $J = \int_0^{100} (x - 5u)dt$.

E 2.19 Consider the following fixed-end-point problem:

$$\max_u \left\{ J = -\int_0^T (g(x) + cu^2)dt \right\}$$

subject to

$$\dot{x} = f(x) + b(x)u, \; x(0) = x_0, \; x(T) = 0,$$

where functions $g \geq 0$, f, and b are assumed to be continuously differentiable. Derive the two-point boundary value problem (TPBVP) satisfied by the optimal state and control trajectories.

E 2.20 *A Machine Maintenance Problem.* Consider the machine state dynamics

$$\dot{x} = -\delta x + u, \; x(0) = x_0 > 0,$$

where $\delta > 0$ is the rate of deterioration of the machine state and u is the rate of machine maintenance. Find the optimal maintenance rate:

$$\max \left\{ J = \int_0^T e^{-\rho t}(\pi x - \frac{u^2}{2})dt + e^{-\rho T} Sx(T) \right\},$$

where $\pi > 0$ with πx representing the profit rate when the machine state is x, $u^2/2$ is the cost of maintaining the machine at rate u, $\rho > 0$ is the discount rate, T is the time horizon, and $S > 0$ is the salvage value of the machine for each unit of the machine state at time T. Furthermore, show that the optimal maintenance rate decreases, increases, or remains constant over time depending on whether the difference $S - \pi/(\rho + \delta)$ is negative, positive, or zero, respectively.

E 2.21 Transform the machine maintenance problem of Exercise 2.20 into Mayer Form. Then solve it to obtain the optimal maintenance rate.

E 2.22 *Regional Allocation of Investment.* Let K_i, $i = 1, 2$, denote the capital stock in Region i. Let b_i be the productivity of capital and s_i be the marginal propensity to save in Region i. Since the investment funds for the two regions come from the savings in the whole economy, we have

$$\dot{K}_1 + \dot{K}_2 = b_1 s_1 K_1 + b_2 s_2 K_2 = g_1 K_1 + g_2 K_2,$$

where $g_i = b_i s_i$. Let u denote the control variable representing the fraction of investment allocated to Region 1 with the remainder going to Region 2. Clearly,

$$0 \le u \le 1, \tag{2.81}$$

and

$$\dot{K}_1 = u(g_1 K_1 + g_2 K_2), K_1(0) = a_1 > 0, \tag{2.82}$$

$$\dot{K}_2 = (1-u)(g_1 K_1 + g_2 K_2), K_2(0) = a_2 > 0. \tag{2.83}$$

The optimal control problem is to maximize the productivity of the whole economy at time T. Thus, the objective is:

$$\max\{J = b_1 K_1(T) + b_2 K_2(T)\}$$

subject to (2.81), (2.82), and (2.83).

(a) Use the maximum principle to derive the form of the optimal policy.

(b) Assume $b_2 > b_1$. Show that $u^*(t) = 0$ for $t \in [\hat{t}, T]$, where $\hat{t}$ is a switching point and $0 \le \hat{t} < T$.

(c) If you are ambitious, find the $\hat{t}$ of part (b).

E 2.23 *Investment Allocation.* Let K denote the capital stock and λK its output rate with $\lambda > 0$. For simplicity in notation, we set the productivity factor $\lambda = 1$. Let u denote the invested fraction of the output. Then, uK is the investment rate and $(1-u)K$ is the consumption rate. Let us assume an exponential utility $1 - e^{-C}$ of consumption C. Solve the resulting optimal control problem:

$$\max\left\{J = \int_0^T [1 - e^{-(1-u(t))K(t)}]dt\right\}$$

subject to

$$\dot{K}(t) = u(t)K(t),\ K(0) = K_0,\ K(T) \text{ free},\ 0 \le u(t) \le 1,\ 0 \le t \le T.$$

Assume $T > 1$ and $0 < K_0 < 1 - e^{1-T}$. Obtain explicitly the optimal investment allocation $u^*(t)$, optimal capital $K^*(t)$, and the adjoint variable $\lambda(t),\ 0 \le t \le T$.

E 2.24 The rate at which a new product can be sold at any time t is $f(p(t))g(Q(t))$ where p is the price and Q is *cumulative* sales. We assume $f'(p) < 0$; sales vary inversely with price. Also $g'(Q) \gtreqless 0$ for $Q \lesseqgtr Q_1$, respectively, where $Q_1 > 0$ is a constant known as the saturation level. For a given price, current sales grow with past sales in the early stages as people learn about the good from past purchasers. But as cumulative sales increase, there is a decline in the number of people who have not yet purchased the good. Eventually the sales rate for any given price falls, as the market becomes saturated. The unit production cost c may be constant or may decline with cumulative sales if the firm learns how to produce less expensively with experience: $c = c(Q),\ c'(Q) \leq 0$. Formulate and solve the optimal control problem in order to characterize the price policy $p(t),\ 0 \leq t \leq T$, that maximizes profits from this new "fad" over a fixed horizon T. Specifically, show that in marketing a new product, its optimal price rises while the market expands to its saturation level and falls as the market matures beyond the saturation level.

E 2.25 Suppose $H(x, u, \lambda, t) = \lambda ux - \frac{1}{2}u^2$ and $\Omega(t) = [0, 1]$ for all t.

(a) Show that the form of the optimal control is given by the function

$$u^*(x, \lambda) = \text{sat}[0, 1; \lambda x] = \begin{cases} \lambda x & \text{if} \quad 0 \leq \lambda x \leq 1, \\ 1 & \text{if} \quad \lambda x > 1, \\ 0 & \text{if} \quad \lambda x < 0. \end{cases}$$

(b) Verify that (2.63) holds for all values of x and λ.

E 2.26 Show that the derived Hamiltonians H^0 found in Examples 2.4 and 2.6 satisfy the concavity condition required for the sufficiency result in Sect. 2.4.

E 2.27 If F and f are concave in x and u and if $\lambda(t) \geq 0$, then show that the derived Hamiltonian H^0 is concave in x. Note that the concavity of F and f are easier to check than the concavity of H^0 as required in Theorem 2.1 on sufficiency conditions.

E 2.28 A simple controlled dynamical system is modeled by the scalar equation

$$\dot{x} = x + u.$$

The fixed-end-point optimal control problem consists in steering $x(t)$ from an initial state $x(0) = x_0$ to the target $x(1) = 0$, such that

$$J(u) = \frac{1}{4}\int_0^1 u^4 dt$$

is minimized. Use the maximum principle to show that the optimal control is given by

$$u^*(t) = \frac{4x_0}{3}(e^{-4/3} - 1)^{-1}e^{-t/3}.$$

E 2.29 Perform the following:

(a) Solve the optimal consumption problem of Example 1.3 with $U(C) = \ln C$ and $B = 0$.

Hint: Since $C(t) \geq 0$, we can replace the state constraint $W(t) \geq 0,\ t \in [0, T]$, by the terminal condition $W(T) = 0$, and then use the transversality condition given in (2.75).

(b) Find the rate of change of optimal consumption over time and conclude that consumption remains constant when $r = \rho$, increases when $r > \rho$, and decreases when $r < \rho$.

E 2.30 Perform the following:

(a) Formulate the TPBVP (2.32) and its discrete version for the problem in Example 2.8, but with a new initial condition $x(0) = 1$.

(b) Solve the discrete version of the TPBVP by using Excel.

E 2.31 Solve explicitly

$$\max\left\{J = -\int_0^2 x(t)dt\right\}$$

subject to

$$\dot{x}(t) = u(t),\ x(0) = 1,\ x(2) \ = \ 0,$$
$$-a \ \leq \ u(t) \leq b,\ a > 1/2,\ b > 0.$$

Obtain optimal $x^*(t),\ u^*(t)$, and all required multipliers.

Chapter 3

The Maximum Principle: Mixed Inequality Constraints

The problems to which the maximum principle derived in the previous chapter was applicable had constraints involving only the control variables. We will see that in many applied models it is necessary to impose constraints involving both control and state variables. Inequality constraints involving control and possibly state variables are called *mixed inequality constraints.*

In the solution spaces of problems with mixed constraints, there may be regions in which one or more of the constraints is tight. When this happens, the system must be controlled in such a way that the tight constraints are not violated. As a result, the maximum principle of Chap. 2 must be revised so that the Hamiltonian is maximized subject to the constraints. This is done by appending the Hamiltonian with the mixed constraints and the associated Lagrange multipliers to form a Lagrangian, and then setting the derivatives of the resulting Lagrangian with respect to the control variables to zero.

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In Sect. 3.1, a Lagrangian form of the maximum principle is discussed for models in which there are some constraints that involve only control variables, and others that involve both state and control variables simultaneously. Problems having pure state variable inequality constraints, i.e., those involving state variables but no control variables, are more difficult and will be dealt with in Chap. 4.

In Sect. 3.2, we state conditions under which the Lagrangian maximum principle is also sufficient for optimality.

Economists frequently analyze optimal control problems involving a discount rate. By combining the discount factor with the adjoint variables and the Lagrange multipliers and making suitable changes in the definitions of the Hamiltonian and Lagrangian functions, it is possible to derive the *current-value formulation* of the maximum principle as described in Sect. 3.3.

It is often the case in finite horizon problems that some restrictions are imposed on the state variables at the end of the horizon. In Sect. 3.4, we discuss the transversality conditions to be satisfied by the adjoint variable in special cases of interest. Section 3.5 is devoted to the study of free terminal time problems where the terminal time itself is a decision variable to be determined. Models with infinite horizons and their stationary equilibrium solutions are covered in Sect. 3.6.

Section 3.7 presents a classification of a number of the most important and commonly used kinds of optimal control models, together with a brief description of the forms of their optimal solutions. The reader may wish to refer to this section from time to time while working through later chapters in the book.

3.1 A Maximum Principle for Problems with Mixed Inequality Constraints

We will state the maximum principle for optimal control problems with mixed inequality constraints without proof. For further details see Pontryagin et al. (1962), Hestenes (1966), Arrow and Kurz (1970), Hadley and Kemp (1971), Bensoussan et al. (1974), Feichtinger and Hartl (1986), Seierstad and Sydsæter (1987), and Grass et al. (2008).

Let the system under consideration be described by the following vector differential equation

$$\dot{x} = f(x, u, t),\ x(0) = x_0 \tag{3.1}$$

given the initial conditions x_0 and a control trajectory $u(t)$, $t \in [0,T]$, $T > 0$, where T can be the terminal time to be optimally determined or given as a fixed positive number. Note that in the above equation, $x(t) \in E^n$ and $u(t) \in E^m$, and the function $f : E^n \times E^m \times E^1 \to E^n$ is assumed to be continuously differentiable.

Let us consider the following objective:

$$\max \left\{ J = \int_0^T F(x,u,t)dt + S[x(T),T] \right\}, \tag{3.2}$$

where $F : E^n \times E^m \times E^1 \to E^1$ and $S : E^n \times E^1 \to E^1$ are continuously differentiable functions and where T denotes the terminal time. Depending on the situation being modeled, the terminal time T may be given or to be determined. In the case when T is given, the function $S(x(T),T)$ should be viewed as merely a function of the terminal state, and can be revised as $S(x(T))$.

Next we impose constraints on state and control variables. Specifically, for each $t \in [0,T]$, $x(t)$ and $u(t)$ must satisfy

$$g(x,u,t) \geq 0, \ t \in [0,T], \tag{3.3}$$

where g: $E^n \times E^m \times E^1 \to E^q$ is continuously differentiable in all its arguments and *must* contain terms in u. An important special case is that of controls having an upper bound that depends on the current state, i.e., $u(t) \leq M(x(t)), \ t \in [0,T]$, which can be written as $M(x) - u \geq 0$. Inequality constraints without terms in u will be introduced later in Chap. 4.

It is important to note that the mixed constraints (3.3) allow for inequality constraints of the type $g(u,t) \geq 0$ as special cases. Thus, the control constraints of the form $u(t) \in \Omega(t)$ treated in Chap. 2 can be subsumed in (3.3), provided that they can be expressed in terms of a finite number of inequality constraints of the form $g(u,t) \geq 0$. In most problems that are of interest to us, this will indeed be the case. Thus, from here on, we will formulate control constraints either directly as inequality constraints and include them as parts of (3.3), or as $u(t) \in \Omega(t)$, which can be easily converted into a set of inequality constraints to be included as parts of (3.3).

Finally, the terminal state is constrained by the following inequality and equality constraints:

$$a(x(T),T) \geq 0, \tag{3.4}$$

$$b(x(T),T) = 0, \tag{3.5}$$

where $a: E^n \times E^1 \to E^{l_a}$ and $b: E^n \times E^1 \to E^{l_b}$ are continuously differentiable in all their arguments. Clearly, a and b are not functions of T, if T is a given fixed number. In the specific cases when T is given, the terminal state constraints will be written as $a(x(T)) \geq 0$ and $b(x(T)) = 0$. Important special cases of (3.4) are $x(T) \geq k$.

We can now define a control $u(t),\ t \in [0,T]$, or simple u, to be admissible if it is piecewise continuous and if, together with its corresponding state trajectory $x(t),\ t \in [0,T]$, it satisfies the constraints (3.3), (3.4), and (3.5).

At times we may find terminal inequality constraints given as

$$x(T) \in Y(T) \subset X(T), \tag{3.6}$$

where $Y(T)$ is a convex set and $X(T)$ is the set of all feasible terminal states, also called the *reachable set* from the initial state x_0, i.e.,

$$X(T) = \{x(T) \mid x(T) \text{ obtained by an admissible control } u \text{ and (3.1)}\}.$$

Remark 3.1 The feasible set defined by (3.4) and (3.5) need not be convex. Thus, if the convex set $Y(T)$ can be expressed by a finite number of inequalities $a(x(T),T) \geq 0$ and equalities $b(x(T),T) = 0$, then (3.6) becomes a special case of (3.4) and (3.5). In general, (3.6) is not a special case of (3.4) and (3.5), since it may not be possible to define a given $Y(T)$ by a finite number of inequalities and equalities.

In this book, we will only deal with problems in which the following *full-rank conditions* hold. That is,

$$\text{rank}[\partial g/\partial u,\ \text{diag}(g)] = q$$

holds for all arguments $x(t),\ u(t),\ t$, that could arise along an optimal solution, and

$$\text{rank}\begin{bmatrix} \partial a/\partial x & \text{diag}(a) \\ \partial b/\partial x & 0 \end{bmatrix} = l_a + l_b$$

hold for all possible values of $x(T)$ and T. The first of these conditions means that the gradients with respect to u of all active constraints in (3.3) must be linearly independent. Similarly, the second condition means that the gradients with respect to x of the equality constraints (3.5) and of the active inequality constraints in (3.4) must be linearly independent. These conditions are also referred to as the *constraint qualifications*. In cases when these do not hold, see Seierstad and Sydsæter (1987) for details on weaker constraint qualifications.

Before proceeding further, let us recapitulate the optimal control problem under consideration in this chapter:

$$\begin{cases} \max\left\{J=\int_0^T F(x,u,t)dt+S[x(T),T]\right\}, \\ \text{subject to} \\ \dot{x}=f(x,u,t),\ x(0)=x_0, \\ g(x,u,t)\geq 0, \\ a(x(T),T)\geq 0, \\ b(x(T),T)=0. \end{cases} \tag{3.7}$$

To state the maximum principle we define the Hamiltonian function $H:\ E^n\times E^m\times E^n\times E^1\to E^1$ as

$$H(x,u,\lambda,t):=F(x,u,t)+\lambda f(x,u,t), \tag{3.8}$$

where $\lambda\in E^n$ (a row vector). We also define the Lagrangian function $L:\ E^n\times E^m\times E^n\times E^q\times E^1\to E^1$ as

$$L(x,u,\lambda,\mu,t):=H(x,u,\lambda,t)+\mu g(x,u,t), \tag{3.9}$$

where $\mu\in E^q$ is a row vector, whose components are called Lagrange multipliers. These Lagrange multipliers satisfy the complementary slackness conditions

$$\mu\geq 0,\ \mu g(x,u,t)=0,$$

which, in view of (3.3), can be expressed equivalently as

$$\mu_i\geq 0,\ \mu_i g_i(x,u,t)=0,\quad i=1,2,\ldots,q.$$

The adjoint vector satisfies the differential equation

$$\dot{\lambda}=-L_x(x,u,\lambda,\mu,t) \tag{3.10}$$

with the terminal condition

$$\begin{cases} la(T) = S_x(x(T),T) + \alpha a_x(x(T),T) + \beta b_x(x(T),T), \\ \alpha \geq 0, \;\; \alpha a(x(T),T) = 0, \end{cases} \tag{3.11}$$

where $\alpha \in E^{l_a}$ and $\beta \in E^{l_b}$ are constant vectors.

The maximum principle states that the necessary conditions for u^*, with the corresponding state trajectory x^*, to be an optimal control are that there should exist continuous and piecewise continuously differentiable functions λ, piecewise continuous functions μ, and constants α and β such that (3.12) holds, i.e.,

$$\boxed{\begin{array}{l} \dot{x}^* = f(x^*,u^*,t),\; x^*(0) = x_0, \\ \text{satisfying the terminal constraints} \\ a(x^*(T),T) \geq 0 \text{ and } b(x^*(T),T) = 0, \\ \dot{\lambda} = -L_x(x^*,u^*,\lambda,\mu,t) \\ \text{with the terminal conditions} \\ \lambda(T) = S_x(x^*(T),T) + \alpha a_x(x^*(T),T) + \beta b_x(x^*(T),T), \\ \alpha \geq 0,\; \alpha a(x^*(T),T) = 0, \\ \text{the Hamiltonian maximizing condition} \\ H[x^*(t),u^*(t),\lambda(t),t] \geq H[x^*(t),u,\lambda(t),t] \\ \text{at each } t \in [0,T] \text{ for all } u \text{ satisfying} \\ g[x^*(t),u,t] \geq 0, \\ \text{and the Lagrange multipliers } \mu(t) \text{ are such that} \\ \dfrac{\partial L}{\partial u}|_{u=u^*(t)} = \left(\dfrac{\partial H}{\partial u} + \mu \dfrac{\partial g}{\partial u}\right)|_{u=u^*(t)} = 0 \\ \text{and the complementary slackness conditions} \\ \mu(t) \geq 0,\; \mu(t) g(x^*,u^*,t) = 0 \text{ hold.} \end{array}} \tag{3.12}$$

In the case of the terminal constraint (3.6), note that the terminal conditions on the state and the adjoint variables in (3.12) will be replaced, respectively, by

$$x^*(T) \in Y(T) \subset X(T) \tag{3.13}$$

and

$$[\lambda(T) - S_x(x^*(T), T)][y - x^*(T)] \geq 0, \quad \forall y \in Y(T). \tag{3.14}$$

In Exercise 3.5, you are asked to derive (3.14) from (3.12) in the one dimensional case when $Y(T) = Y = [\underline{x}, \bar{x}]$ for each $T > 0$, where $\underline{x}$ and $\bar{x}$ are two constants such that $\bar{x} > \underline{x}$.

In the case when the terminal time $T \geq 0$ in the problem (3.10) is also a decision variable, there is an additional necessary transversality condition for T^* to be optimal, namely,

$$\begin{aligned} H[x^*(T^*), u^*(T^*), \lambda(T^*), T^*] + S_T[x^*(T^*), T^*] \\ +\alpha a_T[x^*(T^*), T^*] + \beta_T[x^*(T^*), T^*] = 0, \end{aligned} \tag{3.15}$$

provided T^* is an interior solution, i.e., $T^* \in (0, \infty)$. In other words, optimal T^* and $x^*(t)$, $u^*(t)$, $t \in [0, T^*]$, must satisfy (3.12) with T replaced by T^* and (3.15). This condition will be further discussed and illustrated with examples in Sect. 3.5. The discussion will also include the case when T is restricted to lie in the interval $[T_1, T_2], T_2 > T_1 \geq 0$.

We will now illustrate the use of the maximum principle (3.12) by solving a simple example.

Example 3.1 Consider the problem:

$$\max \left\{ J = \int_0^1 u dt \right\}$$

subject to

$$\dot{x} = u, \; x(0) = 1, \tag{3.16}$$

$$u \geq 0, \; x - u \geq 0. \tag{3.17}$$

Note that constraints (3.17) are of the mixed type (3.3). They can also be rewritten as $0 \leq u \leq x$.

Solution The Hamiltonian is

$$H = u + \lambda u = (1 + \lambda)u,$$

so that the optimal control has the form

$$u^*(x, \lambda) = \text{bang}[0, x; 1 + \lambda]. \tag{3.18}$$

To get the adjoint equation and the multipliers associated with constraints (3.17), we form the Lagrangian:

$$L = H + \mu_1 u + \mu_2(x - u) = \mu_2 x + (1 + \lambda + \mu_1 - \mu_2)u.$$

From this we get the adjoint equation

$$\dot{\lambda} = -\frac{\partial L}{\partial x} = -\mu_2, \; \lambda(1) = 0. \tag{3.19}$$

Also note that the optimal control must satisfy

$$\frac{\partial L}{\partial u} = 1 + \lambda + \mu_1 - \mu_2 = 0, \tag{3.20}$$

and μ_1 and μ_2 must satisfy the complementary slackness conditions

$$\mu_1 \geq 0, \; \mu_1 u = 0, \tag{3.21}$$
$$\mu_2 \geq 0, \; \mu_2(x - u) = 0. \tag{3.22}$$

It is reasonable in this simple problem to guess that $u^*(t) = x(t)$ is an optimal control for all $t \in [0, 1]$. We now show that this control satisfies all the conditions of the Lagrangian form of the maximum principle.

Since $x(0) = 1$, the control $u^* = x$ gives $x = e^t$ as the solution of (3.16). Because $x = e^t > 0$, it follows that $u^* = x > 0$. Thus, $\mu_1 = 0$ from (3.21).

From (3.20) we then have

$$\mu_2 = 1 + \lambda.$$

Substituting this into (3.19) and solving gives

$$1 + \lambda(t) = e^{1-t}. \tag{3.23}$$

Since the right-hand side of (3.23) is always positive, $u^* = x$ satisfies (3.18). Notice that $\mu_2 = e^{1-t} \geq 0$ and $x - u^* = 0$, so (3.22) holds.

Using $u^* = x$ in (3.16), we can obtain the optimal state trajectory $x^*(t) = e^t$. Thus, the optimal value of the objective function is

$$J^* = \int_0^1 e^t dt = (e - 1).$$

Let us now examine the consequence of changing the constraint $x - u \geq 0$ on control u to $x - u \geq -\varepsilon$, which gives $u \leq x + \varepsilon$ for a small ε. In this case, it is clear that the optimal control $u^* = x + \varepsilon$, which we can use in (3.16) to obtain $x^*(t) = e^t(1+\varepsilon) - \varepsilon$. The optimal value of the objective function changes to

$$\int_0^1 u(t)dt = \int_0^1 e^t(1+\varepsilon)dt = (e-1)(1+\varepsilon).$$

This means that J^* increases by $(e-1)\varepsilon$, which in this case equals $\varepsilon \int_0^1 \mu_2(t)dt = \varepsilon \int_0^1 e^{1-t}dt$, as stipulated in Remark 3.8.

We conclude Sect. 3.1 with the following remarks.

Remark 3.2 Strictly speaking, we should have $H = \lambda_0 F + \lambda f$ in (3.8) with $(\lambda_0, \lambda(t)) \neq (0, 0)$ for all $t \in [0, T]$. However, when $\lambda_0 = 0$, the conditions in the maximum principle do not change if we replace F by any other function. Therefore, the problems where the maximum principle holds only with $\lambda_0 = 0$ are termed *abnormal.* Such problems may arise when there are terminal state constraints such as (3.4) and (3.5) or pure state constraints treated in Chap. 4. In this book, as is standard in the economics literature dealing with optimal control theory, we will set $\lambda_0 = 1$. This is because the problems that are of interest to us will be *normal.* For examples of abnormal problems and further discussion on this issue, see Seierstad and Sydsæter (1987).

Remark 3.3 The function defined in (3.9) is not a Lagrangian function in the sense of the continuous-time counterpart of the Lagrangian function defined in (8.45) in Chap. 8. However, it can be viewed, roughly speaking, as a Lagrangian function associated with the problem of maximizing the Hamiltonian (3.8) subject to the constraints (3.3) along the optimal path. As in this book, some people refer to (3.9) as a Lagrangian function, while others call it an *extended Pontryagin function.*

Remark 3.4 It should be pointed out that if the set Y in (3.6) consists of a single point $Y = \{k\}$, making the problem a fixed-end-point problem, then the transversality condition reduces to simply $\lambda(T)$ to equal a constant to be determined, since $x^*(T) = k$. In this case the salvage function S becomes a constant, and can therefore be disregarded. When $Y = X$, the terminal condition in (3.12) reduces to (2.30). Further discussion of the terminal conditions can be found in Sect. 3.4 along with a summary in Table 3.1.

Remark 3.5 As in Chap. 2, it can be shown that $\lambda_i(t),\ i = 1, 2, ..., n$, is interpreted as the marginal value of an increment in the state variable x_i at time t. Specifically, the relation (2.17) holds so long as the value function $V(x, t)$, defined in (2.10), is continuously differentiable in x_i; see Seierstad and Sydsæter (1987).

Remark 3.6 The Lagrange multiplier $\alpha_i,\ i = 1, 2, \ldots, n$ represents the shadow price associated with the terminal state constraint $a_i(x(T), T) \geq 0$. Thus, if we change this constraint to $a_i(x(T), T) \geq \varepsilon$ for a small ε, then the change in the objective function will be $-\varepsilon\alpha_i + o(\varepsilon)$. A similar interpretation holds for the multiplier β; see Sect. 3.4 for further discussion. This will be illustrated in Example 3.4 and Exercise 3.17.

Remark 3.7 In the case when the terminal constraint (3.4) or (3.5) is binding, the transversality condition $\lambda(T)$ in (3.12) should be viewed as the left-hand limit, $\lim_{t\uparrow T} \lambda(t)$, sometimes written as $\lambda(T^-)$, and then we would express $\lambda(T) = S_x(x^*(T), T)$. However, the standard practice for problems treated in Chaps. 2 and 3 is to use the notation that we have used. Nevertheless, care should be exercised in distinguishing the marginal value of the state at time T given by $S_x(x^*(T), T)$ and the shadow prices for the terminal constraints (3.4) and (3.5) given by α and β, respectively. See Sect. 3.4 and Example 3.4 for further elaboration.

Remark 3.8 It is also possible to provide marginal value interpretations to Lagrange multipliers $\mu_i,\ i = 1, 2, \ldots, m$. If we change the constraint $g_i(x, u, t) \geq 0$ to $g_i(x, u, t) \geq \varepsilon$ for a small ε, then we expect the change in the optimal value of the objective function to be $-\varepsilon \int_0^T \mu_i(t)dt + o(\varepsilon)$; see Peterson (1973, 1974) or Malanowski (1984). If $\varepsilon < 0$, then the constraint is being relaxed, and $\int_0^T \mu_i(t)dt \geq 0$ provides the marginal value of relaxing the constraint. We will illustrate this concept with the help of Example 3.1.

Remark 3.9 In the case when the problem (3.7) is changed by interchanging $x(T)$ and $x(0)$ so that the initial condition $x(0) = x_0$ is replaced by $x(T) = x_T$, and $S(x(T), T)$, $a(x(T), T)$ and $b(x(T), T)$ are replaced by $S(x(0))$, $a(x(0))$ and $b(x(0))$, respectively, then in the maximum principle (3.12), we need to replace initial condition $x^*(0) = x_0$ by $x^*(T) = x_T$ and the terminal condition on the adjoint variable λ by the initial condition $\lambda(0) = S_x(x^*(0)) + \alpha a_x(x^*(0)) + \beta b_x(x^*(0))$ with $\alpha \geq 0$ and $\alpha a(x^*(0)) = 0$.

3.2 Sufficiency Conditions

In this section we will state, without proof, a number of sufficiency results. These results require the concepts of concave and quasiconcave functions.

Recall from Sect. 1.4 that with $D \subset E^n$, a convex set, a function $\psi : D \to E^1$ is *concave*, if for all $y, z \in D$ and for all $p \in [0, 1]$,

$$\psi(py + (1-p)z) \geq p\psi(y) + (1-p)\psi(z). \tag{3.24}$$

The function ψ is *quasiconcave* if (3.24) is relaxed to

$$\psi(py + (1-p)z) \geq \min\{\psi(y), \psi(z)\}, \tag{3.25}$$

and ψ is *strictly concave* if $y \neq z$ and $p \in (0, 1)$ and (3.24) holds with a strict inequality. Furthermore, ψ is *convex*, *quasiconvex*, or *strictly convex* if $-\psi$ is concave, quasiconcave, or strictly concave, respectively. Note that linearity implies both concavity and convexity, and concavity implies quasiconcavity. For further details on the properties of such functions, see Mangasarian (1969).

We can now state a sufficiency result concerning the problem with mixed constraints stated in (3.7). For this purpose, let us define the maximized Hamiltonian

$$H^0(x, \lambda, t) = \max_{\{u|g(x,u,t)\geq 0\}} H(x, u, \lambda, t). \tag{3.26}$$

Theorem 3.1 *Let $(x^*, u^*, \lambda, \mu, \alpha, \beta)$ satisfy the necessary conditions in (3.12). If $H^0(x, \lambda(t), t)$ is concave in x at each $t \in [0, T]$, S in (3.2) is concave in x, g in (3.3) is quasiconcave in (x, u), a in (3.4) is quasiconcave in x, and b in (3.5) is linear in x, then (x^*, u^*) is optimal.*

The result is a straightforward extension of Theorem 2.1. See, e.g., Seierstad and Sydsæter (1977, 1987) and Feichtinger and Hartl (1986).

In Exercise 3.7 you are asked to check these sufficiency conditions for Example 3.1.

3.3 Current-Value Formulation

In most management science and economics problems, the objective function is usually formulated in terms of money or utility. These quantities have time value, and therefore the future streams of money or utility are discounted. The discounted objective function can be written as a special case of (3.2) by assuming that the time dependence of the relevant functions comes only through the discount factor. Thus,

$$F(x,u,t) = \phi(x,u)e^{-\rho t} \text{ and } S(x,T) = \psi(x)e^{-\rho T}, \tag{3.27}$$

where we assume the discount rate $\rho > 0$. We should also mention that if $F(x,u,t) = \phi(x,u,t)e^{-\rho t}$ and $S(x,T) = \psi(x,T)e^{-\rho T}$, then there is no advantage of developing a current-value version of the maximum principle, and it is recommended that the present-value formulation be used in this case.

Now, the objective in problem (3.7) can be written as:

$$\max \left\{ J = \int_0^T \phi(x,u)e^{-\rho t}dt + \psi[x(T)]e^{-\rho T} \right\}. \tag{3.28}$$

For this problem, the Hamiltonian, which we shall now refer to as the present-value Hamiltonian, H^{pv}, is

$$H^{pv} := e^{-\rho t}\phi(x,u) + \lambda^{pv} f(x,u,t) \tag{3.29}$$

and the present-value Lagrangian is

$$L^{pv} := H^{pv} + \mu^{pv} g(x,u,t) \tag{3.30}$$

with the present-value adjoint variables λ^{pv} and present-value multipliers α^{pv} and β^{pv} satisfying

$$\dot{\lambda}^{pv} = -L_x^{pv}, \tag{3.31}$$

$$\begin{aligned} \lambda^{pv}(T) &= S_x[x(T),T] + \alpha^{pv} a_x(x(T),T) + \beta^{pv} b_x(x(T),T) \\ &= e^{-\rho T}\psi_x[x(T)] + \alpha^{pv} a_x(x(T),T) + \beta^{pv} b_x(x(T),T), \end{aligned} \tag{3.32}$$

$$\alpha^{pv} \geq 0, \quad \alpha^{pv} a(x(T),T) = 0, \tag{3.33}$$

and μ^{pv} satisfying

$$\mu^{pv} \geq 0, \quad \mu^{pv} g = 0. \tag{3.34}$$

We use superscript ^{pv} in this section to distinguish these from the current-value functions defined as follows. Elsewhere, we do not need to make the distinction explicitly since we will either be using the present-value definitions or the current-value definitions of these functions. The reader will always be able to tell what is meant from the context.

We now define the current-value Hamiltonian

$$H[x, u, \lambda, t] := \phi(x, u) + \lambda f(x, u, t) \tag{3.35}$$

and the current-value Lagrangian

$$L[x, u, \lambda, \mu, t] := H + \mu g(x, u, t). \tag{3.36}$$

To see why we can do this, we note that if we define

$$\lambda := e^{\rho t}\lambda^{pv} \text{ and } \mu := e^{\rho t}\mu^{pv}, \tag{3.37}$$

we can rewrite (3.29) and (3.30) as

$$H = e^{\rho t}H^{pv} \text{ and } L = e^{\rho t}L^{pv}. \tag{3.38}$$

Since $e^{\rho t} > 0$, maximizing H^{pv} with respect to u at time t is equivalent to maximizing the current-value Hamiltonian H with respect to u at time t. Furthermore, from (3.37),

$$\dot{\lambda} = \rho e^{\rho t}\lambda^{pv} + e^{\rho t}\dot{\lambda}^{pv}. \tag{3.39}$$

The first term on the right-hand side of (3.39) is simply $\rho\lambda$ using the definition in (3.37). To simplify the second term we use the differential equation (3.31) for λ^{pv} and the fact that $L_x = e^{\rho t}L_x^{pv}$ from (3.38). Thus,

$$\dot{\lambda} = \rho\lambda - L_x,$$

$$\lambda(T) = \psi_x[x(T)] + \alpha a_x(x(T), T) + \beta b_x(x(T), T), \tag{3.40}$$

where the terminal condition for $\lambda(T)$ follows immediately from the terminal condition for $\lambda^{pv}(T)$ in (3.32), the definition (3.38),

$$\alpha = e^{\rho t}\alpha^{pv} \quad \text{and} \quad \beta = e^{\rho t}\beta^{pv}. \tag{3.41}$$

The complementary slackness conditions satisfied by the current-value Lagrange multipliers μ and α are

$$\mu \geq 0,\ \mu g = 0,\ \ \alpha \geq 0,\ \text{and } \alpha a = 0$$

on account of (3.33), (3.34), (3.37), and (3.41).

We will now state the maximum principle in terms of the current-value functions. It states that the necessary conditions for u^*, with the corresponding state trajectory x^*, to be an optimal control are that there exist λ and μ such that the conditions (3.42) hold, i.e.,

$$\boxed{\begin{array}{l}
\dot{x}^* = f(x^*, u^*, t),\\
a(x^*(T), T) \geq 0,\ b(x^*(T), T) = 0,\\
\dot{\lambda} = \rho\lambda - L_x[x^*, u^*, \lambda, \mu, t], \text{ with the terminal conditions}\\
\lambda(T) = \psi_x(x^*(T)) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T),\\
\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\\
\text{and the Hamiltonian maximizing condition}\\
H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t]\\
\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying}\\
g[x^*(t), u, t] \geq 0,\\
\text{and the Lagrange multipliers } \mu(t) \text{ are such that}\\
\frac{\partial L}{\partial u}|_{u=u^*(t)} = 0, \text{ and the complementary slackness}\\
\text{conditions } \mu(t) \geq 0 \text{ and } \mu(t)g(x^*, u^*, t) = 0 \text{ hold.}
\end{array}} \tag{3.42}$$

As in Sect. 3.1, when the terminal constraint is given by (3.6) instead of (3.4) and (3.5), we need to replace the terminal condition on the state and the adjoint variables, respectively, by (3.13) and

$$[\lambda(T) - \psi_x(x^*(T))][y - x^*(T)] \geq 0,\ \forall y \in Y(T). \tag{3.43}$$

See also Remark 3.4, which applies here as well.

If $T \geq 0$ is also a decision variable and if T^* is the optimal terminal time, then the optimal solution x^*, u^*, and T^* must satisfy (3.42) with T replaced by T^* along with

$$\begin{aligned} &H[x^*(T^*), u^*(T^*), \lambda(T^*), T^*] - \rho\psi[x^*(T^*)]\\ &+\alpha a_T[x^*(T^*), T^*] + \beta_T[x^*(T^*), T^*] = 0. \end{aligned} \tag{3.44}$$

You are asked in Exercise 3.8 to show that (3.44) is the current-value version of (3.15) under the relation (3.27). Furthermore, show how (3.44) should be modified if $S(x,T) = \psi(x,T)e^{-\rho T}$ in (3.27).

As for the sufficiency conditions for the current-value formulation, one can simply use Theorem 3.1 as if it were stated for the current-value formulation.

Example 3.2 We illustrate an application of the current-value maximum principle by solving the consumption problem of Example 1.3 with $U(C) = \ln C$ and $W(T) = 0$. Thus, we solve

$$\max_{C(t)\geq 0} \left\{ J = \int_0^T e^{-\rho t} \ln C(t)dt + B(0)e^{-\rho T} \right\}$$

subject to the wealth dynamics

$$\dot{W} = rW - C,\ W(0) = W_0,\ W(T) = 0,$$

where $W_0 > 0$. As hinted in Exercise 2.29(a), we do not need to impose the pure state constraint $W(t) \geq 0,\ t \in [0,T]$, in view of $C(t) \geq 0,\ t \in [0,T]$, and $W(T) = 0$. Also, the salvage function reduces to $B(0)$, which is a constant; see Remark 3.4.

Solution In Exercise 2.29(a) we used the standard Hamiltonian formulation to solve the problem. We now demonstrate the use of the current-value Hamiltonian formulation:

$$H = \ln C + \lambda(rW - C), \tag{3.45}$$

with the adjoint equation

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial W} = (\rho - r)\lambda,\ \lambda(T) = \beta, \tag{3.46}$$

where β is some constant to be determined. The solution of (3.46) is

$$\lambda(t) = \beta e^{(\rho - r)(t-T)}. \tag{3.47}$$

To find the optimal control, we maximize H by differentiating (3.45) with respect to C and setting the result to zero:

$$\frac{\partial H}{\partial C} = \frac{1}{C} - \lambda = 0,$$

which implies

$$C^*(t) = \frac{1}{\lambda(t)} = \frac{1}{\beta} e^{(\rho - r)(T-t)}. \tag{3.48}$$

Using this consumption level in the wealth dynamics gives

$$\dot{W}(t) = rW(t) - \frac{1}{\beta} e^{(\rho-r)(T-t)},\ W(0) = W_0,$$

which can be solved as

$$W^*(t) = e^{rt} \left[W_0 - \frac{e^{(\rho-r)T}(1 - e^{-\rho t})}{\rho\beta} \right]. \tag{3.49}$$

Setting $W^*(T) = 0$ gives $\beta = e^{(\rho-r)T}(1 - e^{-\rho T})/\rho W_0$. Therefore, the optimal consumption rate and wealth at time t are

$$C^*(t) = \frac{\rho W_0 e^{(r-\rho)t}}{1 - e^{-\rho T}},\ W^*(t) = e^{rt} W_0 \left[\frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}} \right]. \tag{3.50}$$

The optimal value of the objective function is

$$J^* = \frac{1 - e^{-\rho T}}{\rho} \left[\ln \frac{\rho W_0}{1 - e^{-\rho T}} \right] + \frac{r - \rho}{\rho} \left[\frac{1}{\rho} - e^{-\rho T} \left(T + \frac{1}{\rho} \right) \right] + B(0) e^{-\rho T}. \tag{3.51}$$

The interpretation of the current-value functions are that these functions reflect the values at time t in terms of the current (or, time-t) dollars. The standard functions, on the other hand, reflect the values at time t in terms of time-zero dollars. For example, the standard adjoint variable $\lambda^{pv}(t)$ can be interpreted as the marginal value per unit increase in the state at time t, in the same units as that of the objective function (3.28), i.e., in terms of time-zero dollars; see Sect. 2.2.4. On the other hand, $\lambda(t) = e^{\rho t}\lambda^{pv}(t)$ is obviously the same value expressed in terms of current (or, time-t) dollars.

For the consumption problem of Example 3.2, note that the current-value adjoint function

$$\lambda(t) = e^{(\rho - r)t}(1 - e^{-\rho T})/\rho W_0. \tag{3.52}$$

This gives the marginal value per unit increase in wealth at time t in time-t dollars. In Exercise 2.29(a), the standard adjoint variable was $\lambda^{pv}(t) = e^{-rt}(1 - e^{-\rho T})/\rho W_0$, which can be written as $\lambda^{pv}(t) = e^{-\rho t}\lambda(t)$.

Thus, it is clear that $\lambda^{pv}(t)$ expresses the same marginal value in time-zero dollars. In particular,

$$dJ^*/dW_0 = (1 - e^{-\rho T})/\rho W_0 = \lambda(0) = \lambda^{pv}(0)$$

gives the marginal value per unit increase in the initial wealth W_0.

In Exercise 3.11, you are asked to formulate and solve a consumption problem of an economy. The problem is a linear version of the famous Ramsey model; see Ramsey (1928) and Feichtinger and Hartl (1986, p. 201).

Before concluding this section on the current-value formulation, let us also provide the current-value version of the HJB equation (2.15) or (2.19) along with the terminal condition (2.16). As in (2.9), we now define the value function for the problem (3.7), with its objective function replaced by (3.28), as follows:

$$\begin{aligned} V(x,t) = \quad & \max_{\{u|g(x,u,t)\geq 0\}} \left[\int_t^T \phi(x(s),u(s))ds + e^{-\rho(T-t)}\psi(x(T)) \right] \\ & \text{if } x(T) \text{ satisfies } a(x(T),T) \geq 0 \text{ and } b(x(T),T) = 0, \\ & \text{and } V(x,t) = -\infty, \text{ otherwise.} \end{aligned} \tag{3.53}$$

Then proceeding as in Sect. 2.1.1, we have

$$V(x,t) = \max_{\substack{\{u(\tau)|g(x(\tau),u(\tau),\tau)\geq 0\} \\ \tau\in[t,\, t+\delta t]}} \left\{ \phi[x(\tau),u(\tau)]d\tau + e^{-\rho\delta dt} V[x(t+\delta t), t+\delta t] \right\}. \tag{3.54}$$

Noting that $e^{-\rho\delta t} = 1 - \rho\delta t + 0(\delta t)$ and continuing on as in Sect. 2.1.1, we can obtain the current-value version of (2.15) and (2.19) as

$$\begin{aligned} \rho V(x,t) &= \max_{\{u|g(x,u,t)\geq 0\}} \{\phi(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)\} \\ &= \max_{\{u|g(x,u,t)\geq 0\}} \{H(x,u,V_x,t) + V_t\} = 0, \end{aligned} \tag{3.55}$$

where H is defined as in (3.35).

Finally, we can write the terminal condition as

$$V(x,T) = \begin{cases} \psi(x), \text{ if } a(x,T) \geq 0 \text{ and } b(x,T) = 0, \\ -\infty, \text{ otherwise.} \end{cases} \tag{3.56}$$

3.4 Transversality Conditions: Special Cases

Terminal conditions on the adjoint variables, also known as *transversality conditions*, are extremely important in optimal control theory. Because the salvage value function $\psi(x)$ is known, we know the marginal value per unit change in the state at terminal time T. Since $\lambda(T)$ must be equal to this marginal value, it provides us with the boundary conditions for the differential equations for the adjoint variables. We will now derive the terminal or transversality conditions for the current-value adjoint variables for some important special cases of the general problem treated in Sect. 3.3. We also summarize these conditions in Table 3.1.

Case 1: *Free-end point.* In this case, we do not put any constraints on the terminal state $x(T)$. Thus,

$$x(T) \in X(T).$$

From the terminal conditions in (3.42), it is obvious that for the free-end-point problem, i.e., when $Y(T) = X(T)$,

$$\lambda(T) = \psi_x[x^*(T)]. \tag{3.57}$$

This includes the condition $\lambda(T) = 0$ in the special case of $\psi(x) \equiv 0$; see Example 3.1, specifically (3.19). These conditions are repeated in Table 3.1, Row 1.

The economic interpretation of $\lambda(T)$ is that it equals the marginal value of a unit increment in the terminal state evaluated at its optimal value $x^*(T)$.

Case 2: *Fixed-end point.* In this case, which is the other extreme from the free-end-point case, the terminal constraint is

$$b(x(T), T) = x(T) - k = 0,$$

and the terminal conditions in (3.42) do not provide any information for $\lambda(T)$. However, as mentioned in Remark 3.4 and recalled subsequently in connection with (3.42), $\lambda(T)$ will be some constant β, which will be determined by solving the boundary value problem, where the system of differential equations consists of the state equations with both initial and terminal conditions and the adjoint equations with no boundary conditions. This condition is repeated in Table 3.1, Row 2. Example 3.2 solved in the previous section illustrates this case.

The economic interpretation of $\lambda(T) = \beta$ is as follows. The constant β times ε, i.e., $\beta\varepsilon$, provides the value that could be lost if the fixed-end point were specified to be $k + \varepsilon$ instead of k; see Exercise 3.12.

Case 3: *Lower bound.* Here we restrict the ending value of the state variable to be bounded from below, namely,

$$a(x(T), T) = x(T) - k \geq 0,$$

where $k \in X$. In this case, the terminal conditions in (3.42) reduce to

$$\lambda(T) \geq \psi_x[x^*(T)] \tag{3.58}$$

and

$$\{\lambda(T) - \psi_x[x^*(T)]\}\{x^*(T) - k\} = 0, \tag{3.59}$$

with the recognition that the shadow price of the inequality constraint (3.4) is

$$\alpha = \lambda(T) - \psi_x[x^*(T)] \geq 0. \tag{3.60}$$

For $\psi(x) \equiv 0$, these terminal conditions can be written as

$$\lambda(T) \geq 0 \text{ and } \lambda(T)[x^*(T) - k] = 0. \tag{3.61}$$

These conditions are repeated in Table 3.1, Row 3.

Case 4: *Upper bound.* Similarly, when the ending value of the state variable is bounded from above, i.e., when the terminal constraint is

$$k - x(T) \geq 0,$$

the conditions for this opposite case are

$$\lambda(T) \leq \psi_x[x^*(T)] \tag{3.62}$$

and (3.59). These are repeated in Table 3.1, Row 4. Furthermore, (3.62) can be related to the condition on $\lambda(T)$ in (3.42) by setting

$$\alpha = \psi_x[x^*(T)] - \lambda(T) \geq 0. \tag{3.63}$$

Case 5: *A general case.* A general ending condition is

$$x(T) \in Y(T) \subset X(T),$$

which is already stated in (3.6). The transversality conditions are specified in (3.43) and repeated in Table 3.1, Row 5.

An important situation which gives rise to a one-sided constraint occurs when there is an *isoperimetric* or *budget constraint* of the form

$$\int_0^T l(x,u,t)dt \leq K, \tag{3.64}$$

where $l : E^n \times E^m \times E^1 \rightarrow E^1$ is assumed to be nonnegative, bounded, and continuously differentiable, and K is a positive constant representing the amount of a budgeted resource. To see how this constraint can be converted into a lower bound constraint, we define an additional state variable x_{n+1} by the state equation

$$\dot{x}_{n+1} = -l(x,u,t),\ x_{n+1}(0) = K,\ x_{n+1}(T) \geq 0. \tag{3.65}$$

We employ the index $n+1$ simply because we already have n state variables $x = (x_1, x_2, \ldots, x_n)$. Also Eq. (3.65) becomes an additional equation which is added to the original system.

In Exercise 3.13 you will be asked to rework the leaky reservoir problem of Exercise 2.18 with an additional isoperimetric constraint on the total amount of water available. Later in Chap. 7, you'll be asked to solve Exercises 7.10–7.12 involving budgets for advertising expenditures.

In Table 3.1, we have summarized all the terminal or transversality conditions discussed previously. In Sect. 3.7 we discuss model types. We will see that, given the initial state x_0, we can completely specify a control model by selecting a model type and a transversality condition. In what follows, we solve two examples with lower bounds on the terminal state illustrating the use of transversality conditions (3.61), also stated in Table 3.1, Row 3. Example 3.3 is a variation of the consumption problem in Example 3.2. It illustrates the use of the transversality conditions (3.61).

Example 3.3 Let us modify the objective function of the consumption problem (Example 3.2) to take into account the salvage (bequest) value of terminal wealth. This is the utility to the individual of leaving an estate to his heirs upon death. Let us now assume that T denotes the time of the individual's death and $BW(T)$, where B is a positive constant, denotes his utility of leaving wealth $W(T)$ to his heirs upon death. Then, the problem is:

$$\max_{C(t)\geq 0}\left\{J=\int_0^T e^{-\rho t}\ln C(t)dt+e^{-\rho T}BW(T)\right\} \tag{3.66}$$

Table 3.1: Summary of the transversality conditions

	Constraint on $x(T)$	Description	$\lambda(T)$	$\lambda(T)$ when $\psi\equiv 0$
1	$x(T)\in Y(T)=X(T)$	Free-end point	$\lambda(T)=\psi_x[x^*(T)]$	$\lambda(T)=0$
2	$x(T)=k\in X(T)$, i.e., $Y(T)=\{k\}$	Fixed-end point	$\lambda(T)=\beta$, a constant to be determined	$\lambda(T)=\beta$, a constant to be determined
3	$x(T)\in X(T)\cap[k,\infty)$, i.e., $Y(T)=\{x\mid x\geq k\}$	lower bound $x(T)\geq k$	$\lambda(T)\geq\psi_x[x^*(T)]$ and $\{\lambda(T)-\psi_x[x^*(T)]\}\{x^*(T)-k\}=0$	$\lambda(T)\geq 0$ and $\lambda(T)[x^*(T)-k]=0$
4	$x(T)\in X(T)\cap(-\infty,k]$, i.e., $Y(T)=\{x\mid x\leq k\}$	upper bound $x(T)\leq k$	$\lambda(T)\leq\psi_x[x^*(T)]$ and $\{\lambda(T)-\psi_x[x^*(T)]\}\{k-x^*(T)\}=0$	$\lambda(T)\leq 0$ and $\lambda(T)[k-x^*(T)]=0$
5	$x(T)\in Y(T)\subset X(T)$	General constraints	$\{\lambda(T)-\psi_x[x^*(T)]\}\{y-x^*(T)\}\geq 0$ $\forall y\in Y(T)$	$\lambda(T)[y-x^*(T)]\geq 0$ $\forall y\in Y(T)$

Note 1. In Table 3.1, $x(T)$ denotes the (column) vector of n state variables and $\lambda(T)$ denotes the (row) vector of n adjoint variables at the terminal time T; $X(T)\subset E^n$ denotes the reachable set of terminal states obtained by using all possible admissible controls; and $\psi: E^n\to E^1$ denotes the salvage value function

Note 2. Table 3.1 will provide transversality conditions for the standard Hamiltonian formulation if we replace ψ with S, and reinterpret λ as being the standard adjoint variable everywhere in the table. Also (3.15) is the standard form of (3.44)

subject to the wealth equation

$$\dot{W}=rW-C,\ W(0)=W_0,\ W(T)\geq 0. \tag{3.67}$$

Solution The Hamiltonian for the problem is given in (3.45), and the adjoint equation is given in (3.46) except that the transversality conditions are from Table 3.1, Row 3:

$$\lambda(T)\geq B,\ [\lambda(T)-B]W^*(T)=0. \tag{3.68}$$

In Example 3.2, the value of β, the terminal value of the adjoint variable, was

$$\beta=\frac{1-e^{-rT}}{rW_0}.$$

We now have two cases: (i) $\beta\geq B$ and (ii) $\beta<B$.

In case (i), the solution of the problem is the same as that of Example 3.2, because by setting $\lambda(T) = \beta$ and recalling that $W^*(T) = 0$ in that example, it follows that (3.68) holds.

In case (ii), we set $\lambda(T) = B$. Then, by using B in place of β in (3.47)–(3.49), we get $\lambda(t) = Be^{(\rho-r)(t-T)}$, $C^*(t) = (1/B)e^{(\rho-r)(T-t)}$, and

$$W^*(t) = e^{rt}\left[W_0 - \frac{e^{(\rho-r)T}(1-e^{-\rho t})}{\rho B}\right]. \tag{3.69}$$

Since $\beta < B$, we can see from (3.49) and (3.69) that the wealth level in case (ii) is larger than that in case (i) at $t \in (0, T]$. Furthermore, the amount of bequest is

$$W^*(T) = W_0 e^{rT} - \frac{e^{\rho T} - 1}{\rho B} > 0.$$

Note that (3.68) holds for case (ii). Also, if we had used (3.42) instead of Table 3.1, Row 3, we would have $\lambda(T) = B + \alpha,\ \alpha \geq 0,\ \alpha W^*(T) = 0$, equivalently, in place of (3.68). It is easy to see that $\alpha = \beta - B$ in case (i) and $\alpha = 0$ in case (ii).

Example 3.4 Consider the problem:

$$\max\left\{J = \int_0^2 -x dt\right\}$$

subject to

$$\dot{x} = u,\ x(0) = 1,\ x(2) \geq 0, \tag{3.70}$$

$$-1 \leq u \leq 1. \tag{3.71}$$

Solution The Hamiltonian is

$$H = -x + \lambda u.$$

Here, we do not need to introduce the Lagrange multipliers for the control constraints (3.71), since we can easily deduce that the Hamiltonian maximizing control has the form

$$u^* = \text{bang}[-1, 1; \lambda]. \tag{3.72}$$

The adjoint equation is

$$\dot{\lambda} = 1 \tag{3.73}$$

with the transversality conditions

$$\lambda(2) \geq 0 \text{ and } \lambda(2)x(2) = 0, \tag{3.74}$$

obtained from (3.61) or from Table 3.1, Row 3. Since $\lambda(t)$ is monotonically increasing, the control (3.72) can switch at most once, and it can only switch from $u^* = -1$ to $u^* = 1$. Let the switching time be $t^* \leq 2$. Then the optimal control is

$$u^*(t) = \begin{cases} -1 & \text{for } 0 \leq t \leq t^*, \\ +1 & \text{for } t^* < t \leq 2. \end{cases} \tag{3.75}$$

Since the control switches at t^*, $\lambda(t^*)$ must be 0. Solving (3.73) gives

$$\lambda(t) = t - t^*.$$

There are two cases: (i) $t^* < 2$ and (ii) $t^* = 2$. We analyze case (i) first. Here $\lambda(2) = 2 - t^* > 0$; therefore from (3.74), $x(2) = 0$. Solving for $x(t)$ with $u^*(t)$ given in (3.75), we obtain

$$x(t) = \begin{cases} 1 - t & \text{for } 0 \leq t \leq t^*, \\ (t - t^*) + x(t^*) = t + 1 - 2t^* & \text{for } t^* < t \leq 2. \end{cases}$$

Therefore, setting $x(2) = 0$ gives

$$x(2) = 3 - 2t^* = 0,$$

which makes $t^* = 3/2$. Since this satisfies $t^* < 2$, we do not have to deal with case (ii), and we have

$$x^*(t) = \begin{cases} 1 - t & \text{for } 0 \leq t \leq 3/2, \\ t - 2 & \text{for } 3/2 < t \leq 2 \end{cases} \quad \text{and } \lambda(t) = t - \frac{3}{2}.$$

Figure 3.1 shows the optimal state and adjoint trajectories. Using the optimal state trajectory in the objective function, we can obtain its optimal value $J^* = -1/4$.

In Exercise 3.15, you are asked to consider case (ii) by setting $t^* = 2$, and show that the maximum principle will not be satisfied in this case.

Finally, we can verify the marginal value interpretation of the adjoint variable as indicated in Remark 3.5. For this, we first note that the feasible region for the problem is given by $x \geq t-2,\ t \in [0,2]$. To obtain the value function $V(x,t)$, we can easily obtain the optimal solution in the interval $[t,2]$ for the problem beginning with $x(t)=x$. We use the notation introduced in Example 2.5 to specify the optimal solution as

$$u^*_{(x,t)}(s) = \begin{cases} -1, & s \in [t,\ \frac{1}{2}(x+t)+1), \\ 1, & s \in [\frac{1}{2}(x+t)+1, 2], \end{cases}$$

and

$$x^*_{(x,t)}(s) = \begin{cases} x+t-s, & s \in [t,\ \frac{1}{2}(x+t)+1), \\ s-2, & s \in [\frac{1}{2}(x+t)+1, 2]. \end{cases}$$

Then for $x \geq t-2$,

$$\begin{aligned} V(x,t) &= \textstyle\int_t^2 -x^*_{(x,t)}(s)ds \\ &= -\textstyle\int_t^{(1/2)(x+t)+1}(x+t-s)ds - \int_{(1/2)(x+t)+1}^2 (s-2)ds \\ &= (1/4)t^2 - (1/4)x^2 + (1/2)t(x-2) - (x-1). \end{aligned} \tag{3.76}$$

For $x < t-2$, there is no feasible solution, and we therefore set $V(x,t) = -\infty$.

We can now verify that for $0 \leq t \leq 3/2$, the value function $V(x,t)$ is continuously differentiable at $x = x^*(t) = 1-t$, and

$$\begin{aligned} V_x(x^*(t),t) &= -(1/2)x^*(t) + (1/2)t - 1 \\ &= -(1/2)(1-t) + (1/2)t - 1 \\ &= t - 3/2 \\ &= \lambda(t). \end{aligned}$$

What happens when $t \in (3/2, 2]$? Clearly, for $x \geq x^*(t) = t-2$, we may still use (3.76) to obtain the right-hand derivative $V_x^+(x^*(t),t) = -(1/2)x^*(t) + (1/2)t - 1 = -(1/2)(t-2) + (1/2)t - 1 = 0$. However, for $x < x^*(t)$, we have $x < t-2$ for which there is no feasible solution, and we set the left-hand derivative $V_x^-(x^*(t),t) = -\infty$. Thus, the value function $V(x,t)$ is not differentiable at $x^*(t)$, and since $V_x(x^*(t),t)$ does not exist for $t \in (3/2, 2]$, (2.17) has no meaning; see Remark 2.2.

It is possible, however, to provide an economic meaning for $\lambda(2)$. In Exercise 3.17, you are asked to rework Example 3.4 with the terminal condition $x(2) \geq 0$ replaced by $x(2) \geq \varepsilon$, where ε is small. Furthermore,

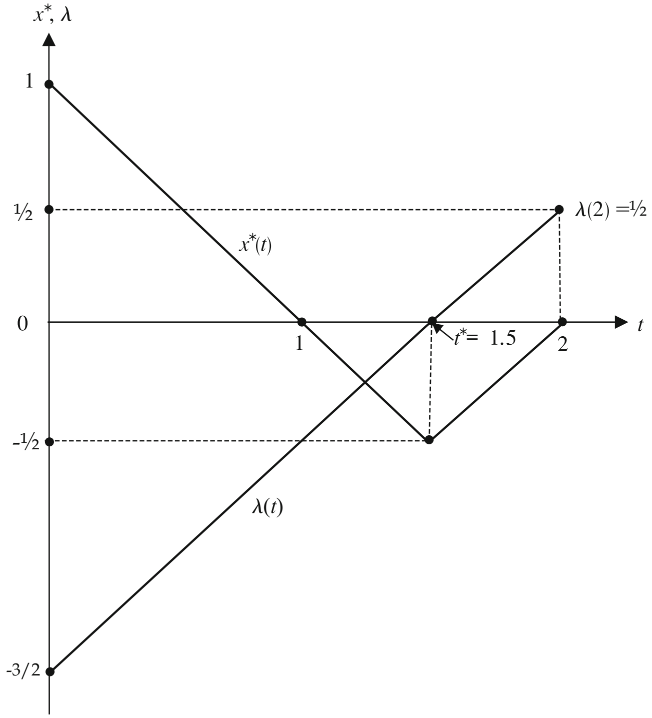

Figure 3.1: State and adjoint trajectories in Example 3.4

the solution will illustrate that $\alpha = \lambda(2) - 0 = 1/2$, obtained by using (3.60), represents the shadow price of the constraint as indicated in Remark 3.7.

3.5 Free Terminal Time Problems

In some cases, the terminal time is not given but needs to be determined as an additional decision. Here, a necessary condition for a terminal time to be optimal in the present-value and current-value formulations are given in (3.15) and (3.44), respectively. In this section, we elaborate further on these conditions as well as solve two free terminal time examples: Examples 3.5 and 3.6.

Let us begin with a special case of the condition (3.15) for the simple problem (2.4) when $T \geq 0$ is a decision variable. When compared with the problem (3.7), the simple problem is without the mixed constraints and constraints at the terminal time T. Thus the transversality condition (3.15) reduces to

$$H[x^*(T^*), u^*(T^*), \lambda(T^*), T^*] + S_T[x^*(T^*), T^*] = 0. \tag{3.77}$$

This condition along with the Maximum Principle (2.31) with T replaced by T^* give us the necessary conditions for the optimality of T^* and $u^*(t),\ t \in [0, T^*]$ for the simple problem (2.4) when $T \geq 0$ is also a decision variable.

An intuitively appealing way to check if the optimal $T^* \in (0, \infty)$ must satisfy (3.77) is to solve the problem (2.4) with the terminal time T^* with $u^*(t), t \in [0, T^*]$ as the optimal control trajectory, and then show that the first-order condition for T^* to maximize the objective function in a neighborhood $(T^* - \delta, T^* + \delta)$ of T^* with $\delta > 0$ leads to (3.77). For this, let us set $u^*(t) = u^*(T^*),\ t \in [T^*, T^* + \delta)$, so that we have a control $u^*(t)$ that is feasible for (2.4) for any $T \in (T^* - \delta, T^* + \delta)$, as well as continuous at T^*. Let $x^*(t),\ t \in [0, T^* + \delta]$ be the corresponding state trajectory. With these we can obtain the corresponding objective function value

$$J(T) = \int_0^T F(x^*(t), u^*(t), t)dt + S(x^*(T), T),\ T \in (T^* - \delta, T^* + \delta), \tag{3.78}$$

which, in particular, represents the optimal value of the objective function for the problem (2.4) when $T = T^*$. Furthermore, since $u^*(t)$ is continuous at T^*, $x^*(t)$ is continuously differentiable there, and so is $J(T)$. In this case, since T^* is optimal, it must satisfy

$$J'(T^*) := \frac{dJ(T)}{dT}|_{T=T^*} = 0. \tag{3.79}$$

Otherwise, we would have either $J'(T^*) > 0$ or $J'(T^*) < 0$. The former situation would allow us to find a $T \in (T^*, T^* + \delta)$ for which $J(T) > J(T^*)$, and T^* could not be optimal since the choice of an optimal control for (2.4) defined on the interval $[0, T]$ would only improve the value of the objective function. Likewise, the later situation would allow us to find a $T \in (T^* - \delta, T^*)$ for which $J(T) > J(T^*)$. By taking the derivative of (3.78), we can write (3.79) as

$$F(x^*(T^*), u^*(T^*), T^*) + S_x[x^*(T^*), T^*]\dot{x}^*(T^*) + S_T[x^*(T^*), T^*] = 0. \tag{3.80}$$

Furthermore, using the definition of the Hamiltonian in (2.18) and the state equation and the transversality condition in (2.31), we can easily see that (3.80) can be written as (3.77).

Remark 3.10 An intuitive way to obtain optimal T^* is to first solve the problem (2.4) with a given terminal time T and obtain the optimal value of the objective function $J^*(T)$, and then maximize $J^*(T)$ over T. Hartl and Sethi (1983) show that the first-order condition for maximizing $J^*(T)$, namely, $dJ^*(T)/dT = 0$ can also be used to derive the transversality condition (3.77).

If T is restricted to lie in the interval $[T_1, T_2]$, where $T_2 > T_1 \geq 0$, then (3.77) is still valid provided $T^* \in (T_1, T_2)$. As is standard, if $T^* = T_1$, then the $=$ sign in (3.77) is replaced by $\leq$, and if $T^* = T_2$, then the $=$ sign in (3.77) is replaced by $\geq$. In other words, if we must have $T^* \in [T_1, T_2]$, then we can replace (3.77) by

$$H[x^*(T^*), u^*(T^*), \lambda(T^*), T^*] + S_T[x^*(T^*), T^*] \begin{cases} \leq 0 & \text{if } T^* = T_1, \\ = 0 & \text{if } T^* \in (T_1, T_2), \\ \geq 0 & \text{if } T^* = T_2. \end{cases} \tag{3.81}$$

Similarly, we can also obtain the corresponding versions of (3.15) and (3.44) for the problem (3.7) and its current value version (specified in Sect. 3.3), respectively.

We shall now illustrate (3.77) and (3.81) by solving Examples 3.5 and 3.6. To illustrate the idea in Remark 3.10, you are asked in Exercise 3.6 to solve Example 3.5 by using $dJ^*(T)/dt = 0$ to obtain the optimal T^*.

Example 3.5 Consider the problem:

$$\max_{u,T} \left\{ J = \int_0^T (x - u)dt + x(T) \right\} \tag{3.82}$$

subject to

$$\dot{x} = -2 + 0.5u, \; x(0) = 17.5, \tag{3.83}$$
$$u \in [0, 1], \; T \geq 0.$$

Solution The Hamiltonian is

$$H = x - u + \lambda(-2 + 0.5u),$$

where $\dot{\lambda} = -1,\ \lambda(T) = 1$, which gives

$$\lambda(t) = 1 + (T - t).$$

Then, the optimal control is given by

$$u^*(t) = \text{bang}[0, 1; 0.5(T - 1 - t)]. \tag{3.84}$$

In other words, $u^*(t) = 1$ for $0 \le t \le T - 1$ and $u^*(t) = 0$ for $T - 1 < t \le T$.

Since we must also determine the optimal terminal time T^*, it must satisfy (3.77), which, in view of the fact that $u^*(T^*) = 0$ from (3.84), reduces to

$$x^*(T^*) - 2 = 0. \tag{3.85}$$

By substituting $u^*(t)$ in (3.83) and integrating, we obtain

$$x^*(t) = \begin{cases} 17.5 - 1.5t, & 0 \le t \le T - 1, \\ 17 + 0.5T - 2t, & T - 1 < t \le T. \end{cases} \tag{3.86}$$

We can now apply (3.85) to obtain

$$x^*(T^*) - 2 = 17 - 1.5T^* - 2 = 0,$$

which gives $T^* = 10$. Thus, the optimal solution of the problem is given by $T^* = 10$ and

$$u^*(t) = \text{bang}[0, 1; 0.5(9 - t)].$$

Note that if we had restricted T to be in the interval $[T_1, T_2] = [2, 8]$, we would have $T^* = 8,\ u^*(t) = \text{bang}[0, 1; 0.5(7 - t)]$, and $x^*(8) - 2 = 5 - 2 = 3 \ge 0$, which would satisfy (3.81) at $T^* = T_2 = 8$. On the other hand, if T were restricted in the interval $[T_1, T_2] = [11, 15]$, then $T^* = 11,\ u^*(t) = \text{bang}[0, 1; 0.5(10 - t)]$, and $x^*(11) - 2 = 0.5 - 2 = -1.5 \le 0$ would satisfy (3.81) at $T^* = T_1 = 11$.

Next, we will apply the maximum principle to solve a well known *time-optimal control problem*. It is one of the problems used by Pontryagin et al. (1962) to illustrate the applications of the maximum principle.

The problem also elucidates a specific instance of the synthesis of optimal controls.

By the *synthesis of optimal controls*, we mean the procedure of "patching" together various forms of the optimal controls obtained from the Hamiltonian maximizing condition. A simple example of the synthesis occurs in Example 2.5, where $u^* = 1$ when $\lambda > 0$, $u^* = -1$ when $\lambda < 0$, and the control is singular when $\lambda = 0$. An optimal trajectory starting at the given initial state variables is synthesized from these. In Example 2.5, this synthesized solution is $u^* = -1$ for $0 \leq t < 1$ and $u^* = 0$ for $1 \leq t \leq 2$. Our next example requires a synthesis procedure which is more complex. In Chap. 5, both the cash management and equity financing models require such synthesis procedures.

Example 3.6 *A Time-Optimal Control Problem.* Consider a subway train of mass m moving horizontally along a smooth linear track with negligible friction. Let $x(t)$ denote the position of the train, measured in miles from the origin called the main station, along the track at time t, measured in minutes. Then the equation of the train's motion is governed by Newton's Second Law of Motion, which states that force equals mass times acceleration. In mathematical terms, the equation of the motion is the second-order differential equation

$$m\frac{d^2x(t)}{dt^2} = m\ddot{x}(t) = u(t),$$

where $u(t)$ denotes the external force applied to the train at time t and $\ddot{x}(t)$ represents the acceleration in miles per minute per minute, or miles/minute^2. This equation, along with

$$x(0) = x_0 \text{ and } \dot{x}(0) = y_0,$$

respectively, as the initial position of the train and its initial velocity in miles per minute, characterizes its motion completely.

For convenience in further exposition, we may assume $m = 1$ so that the equation of motion can be written as

$$\ddot{x} = u. \tag{3.87}$$

Then, the force u can be expressed simply as acceleration or deceleration (i.e., negative acceleration) depending on whether u is positive or negative, respectively.

In order to develop the time-optimal control problem under consideration, we transform (3.87) into a system of two first-order differential equations (see Appendix A)

$$\begin{cases} \dot{x} = y, \quad x(0) = x_0, \\ \dot{y} = u, \quad y(0) = y_0, \end{cases} \tag{3.88}$$

where $y(t)$ denotes the velocity of the train in miles/minute at time t.

Assume further that, for the comfort of the passengers, the maximum acceleration and deceleration are required to be at most 1 mile/minute^2. Thus, the control variable constraint is

$$u \in \Omega = [-1, 1]. \tag{3.89}$$

The problem is to find a control satisfying (3.89) such that the train stops at the main station located at $x = 0$ in a minimum possible time T. Of course, for the train to come to rest at $x = 0$ at time T, we must have $x(T) = 0$ and $y(T) = 0$. We have thus defined the following fixed-end-point optimal control problem:

$$\begin{cases} \max \left\{ J = \int_0^T -1 dt \right\} \\ \text{subject to} \\ \dot{x} = y,\ x(0) = x_0,\ x(T) = 0, \\ \dot{y} = u,\ y(0) = y_0,\ y(T) = 0, \\ \text{and the control constraint} \\ u \in \Omega = [-1, 1]. \end{cases} \tag{3.90}$$

Note that (3.90) is a fixed-end-point problem with unspecified terminal time. For this problem to be nontrivial, we must not have $x_0 = y_0 = 0$, i.e., we must have either $x_0 \neq 0$ or $y_0 \neq 0$ or both are nonzero.

Solution Here we have only control constraints of the type treated in Chap. 2, and so we can use the maximum principle (2.31). The standard Hamiltonian function is

$$H = -1 + \lambda_1 y + \lambda_2 u,$$

where the adjoint variables λ_1 and λ_2 satisfy

$$\dot{\lambda}_1 = 0,\ \lambda_1(T) = \beta_1 \text{ and } \dot{\lambda}_2 = -\lambda_1,\ \lambda_2(T) = \beta_2,$$

and β_1 and β_2 are constants to be determined in the case of a fixed-end-point problem; see Table 3.1, Row 2. We can integrate these equations and write the solution in the form

$$\lambda_1 = \beta_1 \text{ and } \lambda_2 = \beta_2 + \beta_1(T-t),$$

where β_1 and β_2 are constants to be determined from the maximum principle (2.31), condition (3.15), and the specified initial and terminal values of the state variables. The Hamiltonian maximizing condition yields the form of the optimal control to be

$$u^*(t) = \text{bang}\{-1, 1;\ \beta_2 + \beta_1(T-t)\}. \tag{3.91}$$

As for the minimum time T^*, it is clearly zero if the train is initially at rest at the main station, i.e., $(x_0, y_0) = 0$. In this case, the problem is trivial, $u^*(0) = 0$, and there is nothing further to solve. Otherwise, at least one of x_0 or y_0 is not zero, in which case the minimum time $T^* > 0$ and the transversality condition (3.15) applies. Since $y(T) = 0$ and $S \equiv 0$, we have

$$H + S_T|_{T=T^*} = \lambda_2(T^*)u^*(T^*) - 1 = \beta_2 u^*(T^*) - 1 = 0,$$

which together with the bang-bang control policy (3.91) implies either

$$\lambda_2(T^*) = \beta_2 = -1 \text{ and } u^*(T^*) = -1,$$

or

$$\lambda_2(T^*) = \beta_2 = +1 \text{ and } u^*(T^*) = +1.$$

Since the switching function $\beta_2 + \beta_1(T^* - t)$ is a linear function of the time remaining, it can change sign at most once. Therefore, we have two cases: (i) $u^*(\tau) = -1$ in the interval $t \le \tau \le T^*$ for some $t \ge 0$; (ii) $u^*(\tau) = +1$ in the interval $t \le \tau \le T^*$ for some $t \ge 0$. We can integrate (3.88) in each of these cases as shown in Table 3.2. Also in the table we have the curves Γ^- and Γ^+, which are obtained by eliminating t from the expressions for x and y in each case. The parabolic curves Γ^- and Γ^+ are called *switching curves* and are shown in Fig. 3.2.

It should be noted parenthetically that Fig. 3.2 is different from the figures we have seen thus far, where the abscissa represented the time

Table 3.2: State trajectories and switching curves

(i) $u^*(\tau) = -1$ for $(t \le \tau \le T^*)$	(ii) $u^*(\tau) = +1$ for $(t \le \tau \le T^*)$
$y(t) = T^* - t$	$y(t) = t - T^*$
$x(t) = -(T^* - t)^2/2$	$x(t) = (t - T^*)^2/2$
$\Gamma^- : x = -y^2/2$ for $y \ge 0$	$\Gamma^+ : x = y^2/2$ for $y \le 0$

dimension. In Fig. 3.2, the abscissa represents the train's location and the ordinate represents the train's velocity. Thus, the point (x_0, y_0) represents the vector of the train's initial position and initial velocity. A trajectory of the train over time can be represented by a curve in this figure. For example, the bold-faced trajectory beginning at (x_0, y_0) represents a train that is moving in the positive direction and it is slowing down. It passes through the main station located at the origin and comes to a momentary rest at the point that is $\sqrt{y_0^2 + 2x_0}$ miles to the right of the main station. At this location, the train reverses its direction and speeds up to reach the location x_* and attain the velocity of y_*. At this point, it slows down gradually until it comes to rest at the main station. In the ensuing discussion we will show that this trajectory is in fact the minimal time trajectory beginning at the location x_0 at a velocity of y_0. We will furthermore obtain the control representing the optimal acceleration and deceleration along the way. Finally, we will obtain the various instants of interest, which are implicit in the depiction of the trajectory in Fig. 3.2.

We can put Γ^+ and Γ^- into a single switching curve Γ as

$$y = \Gamma(x) = \begin{cases} \Gamma^+(x) = -\sqrt{2x}, & x \ge 0, \\ \Gamma^-(x) = +\sqrt{-2x}, & x < 0. \end{cases} \tag{3.92}$$

If the initial state $(x_0, y_0) \neq 0$, lies on the switching curve, then we have $u^* = +1$ (resp., $u^* = -1$) if $x_0 > 0$ (resp., $x_0 < 0$); i.e., if (x_0, y_0) lies on Γ^+ (resp., Γ^-). In the common parlance, this means that we apply the brakes to bring the train to a full stop at the main station. If the initial state (x_0, y_0) is not on the switching curve, then we choose, between $u^* = 1$ and $u^* = -1$, that which moves the system toward the switching

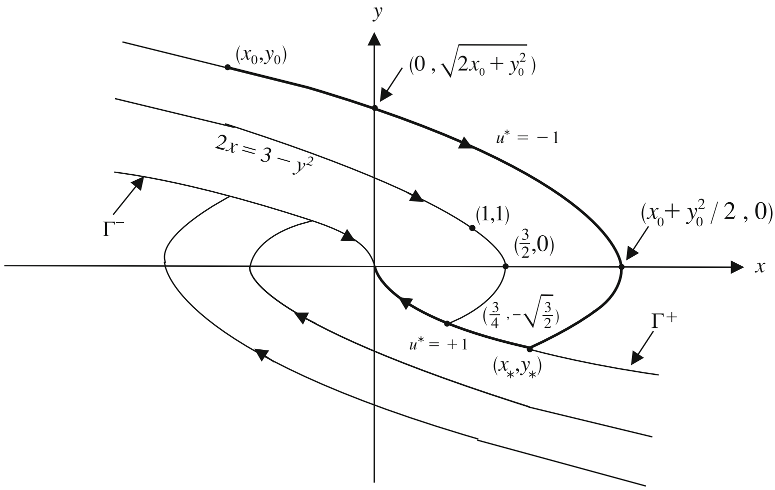

Figure 3.2: Minimum time optimal response for Example 3.6

curve. By inspection, it is obvious that above the switching curve we must choose $u^* = -1$ and below we must choose $u^* = +1$.

The other curves in Fig. 3.2 are solutions of the differential equations starting from initial points (x_0, y_0). If (x_0, y_0) lies above the switching curve Γ as shown in Fig. 3.2, we use $u^* = -1$ to compute the curve as follows:

$$\dot{x} = y, \; x(0) = x_0,$$
$$\dot{y} = -1, \; y(0) = y_0.$$

Integrating these equations gives

$$y = -t + y_0,$$
$$x = -\frac{t^2}{2} + y_0 t + x_0.$$

Elimination of t between these two gives

$$x = \frac{y_0^2 - y^2}{2} + x_0. \tag{3.93}$$

This is the equation of the parabola in Fig. 3.2 through (x_0, y_0). The point of intersection of the curve (3.93) with the switching curve Γ^+ is obtained by solving (3.93) and the equation for Γ^+, namely $2x = y^2$, simultaneously, which gives

$$x_* = \frac{y_0^2 + 2x_0}{4}, \; y_* = -\sqrt{(y_0^2 + 2x_0)/2}, \tag{3.94}$$

where the minus sign in the expression for y_* in (3.94) was chosen since the intersection occurs when y_* is negative. The time t_* that it takes to reach the switching curve, called the *switching time*, given that we start above it, is

$$t_* = y_0 - y_* = y_0 + \sqrt{(y_0^2 + 2x_0)/2}. \tag{3.95}$$

To find the minimum total time to go from the starting point (x_0, y_0) to the origin (0,0), we substitute t_* into the equation for Γ^+ in Column (ii) of Table 3.2; this gives

$$T^* = t_* - y_* = y_0 + \sqrt{2(y_0^2 + 2x_0)}. \tag{3.96}$$

Here t_* is the time to get to the switching curve and $-y_*$ is the time spent along the switching curve.

Note that the parabola (3.93) intersects the y-axis at the point $(0, +\sqrt{2x_0 + y_0^2})$ and the x-axis at the point $(x_0 + y_0^2/2, 0)$. This means that for the initial position (x_0, y_0) depicted in Fig. 3.2, the train first passes the main station at the velocity of $+\sqrt{2x_0 + y_0^2}$ and comes to a momentary stop at the distance of $(x_0 + y_0^2/2)$ to the right of the main station. There it reverses its direction, comes to within the distance of x_* from the main station, switches then to $u^* = +1$, which slows it to a complete stop at the main station at time T^* given by (3.96).

As a numerical example, start at the point (x_0, y_0) =(1,1). Then, the equation of the parabola (3.93) is

$$2x = 3 - y^2.$$

The switching point given by (3.94) is $(3/4, -\sqrt{3/2})$. Finally from (3.95), the switching time is $t_* = 1 + \sqrt{3/2}$ min. Substituting into (3.96), we find the minimum time to stop is $T^* = 1 + \sqrt{6}$ min.

To complete the solution of this example let us evaluate β_1 and β_2, which are needed to obtain λ_1 and λ_2. Since (1,1) is above the switching curve, the approach to the main station is on the curve Γ^+, and therefore, $u^*(T^*) = 1$ and $\beta_2 = 1$. To compute β_1, we observe that $\lambda_2(t_*) = \beta_2 + \beta_1(T^* - t_*) = 0$ so that $\beta_1 = -\beta_2/(T^* - t_*) = -1/\sqrt{3/2} = -\sqrt{2/3}$. Finally, we obtain $x_* = 3/4$ and $y_* = -\sqrt{3/2}$ from (3.94).

Let us now describe the optimal solution from $(1, 1)$ in the common parlance. The position $(1, 1)$ means the train is 1 mile to the right of the main station, moving away from it at the speed of 1 mile per minute. The control $u^* = -1$ means that the brakes are applied to slow the train down. This action brings the train to a momentary stop at a distance of $\sqrt{3}$ miles to the right of the main station. Moreover, the continuation of control $u^* = -1$ means the train reverses its direction at that point and starts speeding toward the station. When it comes to within 3/4 miles to the right of the main station at time $t_* = 1 + \sqrt{3/2}$, its velocity of $-\sqrt{3/2}$ or the speed of $\sqrt{3/2}$ miles per minute toward the station is too fast to come to a rest at the main station without application of the brakes. So the control is switched to $u^* = +1$ at time t_*, which means the brakes are applied at that time. This action brings the train to a complete stop at the main station at the time of $T^* = 1 + \sqrt{6}$ min after the train left its initial position $(1, 1)$.

In Exercises 3.19–3.22, you are asked to work other examples with different starting points *above, below,* and *on* the switching curve. Note that $t_* = 0$ by definition, if the starting point is on the switching curve.

3.6 Infinite Horizon and Stationarity

Thus far, we have studied problems whose horizon is finite or whose horizon length is a decision variable to be determined. In this section, we briefly discuss the case of $T = \infty$ in the problem (3.7), called the *infinite horizon case.* This case is especially important in many economics and management science problems. Our treatment of this case is largely heuristic, since a general theory of the necessary optimality conditions is not available. Nevertheless, we can rely upon an infinite-horizon extension of the sufficiency optimality conditions stated in Theorem 3.1.

When we put $T = \infty$ in (3.7) along with $\rho > 0$, we will generally get a nonstationary infinite horizon problem in the sense that the various functions involved depend explicitly on the time variable t. Such problems are extremely hard to solve. So, in this section we will devote our attention to only stationary infinite horizon problems, which do not depend explicitly on time t. Furthermore, it is reasonable in most cases to assume $\sigma(x) \equiv 0$ in infinite horizon problems. Moreover, in most economics and management science problems, the terminal constraints, if any, require the state variables to be nonnegative. Thus, to begin with, we consider the problem:

$$\begin{cases} \max \left\{ J = \int_0^\infty \phi(x,u)e^{-\rho t}dt \right\}, \\ \text{subject to} \\ \dot{x} = f(x,u), \; x(0) = x_0, \\ g(x,u) \geq 0. \end{cases} \tag{3.97}$$

This *stationarity* assumption means that the state equations, the current-value adjoint equations, and the current-value Hamiltonian in (3.35) are all explicitly independent of time t.

Remark 3.11 The concept of *stationarity* introduced here is different from the concept of *autonomous systems* introduced in Exercise 2.9. This is because, in the presence of discounting in (3.28), the stationarity assumption (3.97) does not give us an autonomous system as defined there. See Exercise 3.42 for further comparison between the two concepts.

When it comes to the transversality conditions in the infinite horizon case, the situation is somewhat more complicated. Even the economic argument for the finite horizon case fails to extend here because we do not have a meaningful analogue of the salvage value function. Moreover, in the free-end-point case with no salvage value, the standard maximum principle (2.31) gives $\lambda^{pv}(T) = 0$, which can no longer be necessary in general for $T = \infty$, as confirmed by a simple counter-example in Exercise 3.37. As a matter of fact, we have no general results giving conditions under which the limit of the finite horizon transversality conditions are necessary. What is true is that the maximum principle (3.42) holds except for the transversality condition on $\lambda(T)$.

When it comes to the sufficiency of the limiting transversality conditions obtained by letting $T \to \infty$ in Theorem 3.1, the situation is much better. As a matter of fact, we can see from the inequality (2.73) with $S(x) \equiv 0$ that all we need is

$$\lim_{T\to\infty} \lambda^{pv}(T)[x(T) - x^*(T)] = \lim_{T\to\infty} e^{-\rho T}\lambda(T)[x(T) - x^*(T)] \geq 0 \tag{3.98}$$

for Theorem 2.1, and therefore Theorem 3.1, to hold. See Seierstad and Sydsæter (1987) and Feichtinger and Hartl (1986) for further details.

In the important free-end-point case (3.97), since $x(T)$ is arbitrary, (3.98) will imply

$$\lim_{T\to\infty} \lambda^{pv}(T) = \lim_{T\to\infty} e^{-\rho T}\lambda(T) = 0. \tag{3.99}$$

While not a necessary condition as indicated earlier, it is interesting to note that (3.99) is the limiting version of the condition in Table 3.1, Row 1.

Another important case is that of nonnegativity constraints

$$\lim_{T\to\infty} x(T) \geq 0. \tag{3.100}$$

Then, it is clear that the transversality conditions

$$\lim_{T\to\infty} e^{-\rho T}\lambda(T) \geq 0 \text{ and } \lim_{T\to\infty} e^{-\rho T}\lambda(T)x^*(T) = 0, \tag{3.101}$$

imply (3.98). Note that these are also analogous to Table 3.1, Row 3.

We leave it as Exercise 3.38 for you to show that the limiting version of the condition in the rightmost column of Rows 2, 3, and 4 in Table 3.1 imply (3.98). This would mean that Theorem 3.1 provides sufficient optimality conditions for the problem (3.97), except in the free-end-point case, i.e., when the terminal constraints $a(x(T)) \geq 0$ and $b(x(T)) = 0$ are not present. Moreover, in the free-end-point case, we can use (3.98), or even (3.99) with some qualifications, as discussed earlier.

Example 3.7 Let us return to Example 3.3 and now assume that we have a perpetual charitable trust with initial fund W_0, which wants to maximize its total discounted utility of charities $C(t)$ over time, subject to the terminal condition

$$\lim_{T\to\infty} W(T) \geq 0. \tag{3.102}$$

For convenience we restate the problem:

$$\max_{C(t)\geq 0} \left\{ J = \int_0^\infty e^{-\rho t} \ln C(t) dt \right\}$$

subject to

$$\dot{W} = rW - C, \; W(0) = W_0 > 0, \tag{3.103}$$

and (3.102).

Solution We already know from Example 3.3 with $B = 0$ that we are in case (i), and the optimal solution is given by (3.50) in Example 3.2. It seems reasonable to explore whether or not we can obtain an optimal solution for our infinite horizon problem by letting $T \to \infty$ in (3.50). Furthermore, since the limiting version of the maximum principle (3.42) is sufficient for optimality in this case, all we need to do is to check if the limiting solution satisfies the condition

$$\lim_{T\to\infty} e^{-\rho T}\lambda(T) \geq 0 \text{ and } \lim_{T\to\infty} e^{-\rho T}\lambda(T)W^*(T) = 0. \tag{3.104}$$

With $T \to \infty$ in (3.50) and (3.52), we have

$$W^*(t) = e^{(r-\rho)t}W_0,\ C^*(t) = \rho W^*(t),\ \lambda(t) = 1/\rho W^*(t). \tag{3.105}$$

Since $\lambda(t) \geq 0$ and $\lambda(t)W^*(t) = 1/\rho$, it is clear that (3.104) holds. Thus, (3.105) gives the optimal solution. Using this solution in the objective function, we obtain

$$J^* = \frac{1}{\rho}\ln \rho W_0 + \frac{r-\rho}{\rho^2}, \tag{3.106}$$

which we can verify to be the same as (3.51) as $T \to \infty$.

It is interesting to observe from (3.105) that the optimal consumption is increasing, constant, or decreasing if r is greater than, equal to, or less than ρ, respectively. Moreover, if $\rho = r$, then $W^*(t) = W_0, C^*(t) = rW_0$, and $\lambda(t) = 1/rW_0$, which means that it is optimal to consume just the interest earned on the invested wealth—no more, no less—and, therefore, none of the initial wealth is ever consumed!

In the case of stationary systems, considerable attention is focused on equilibrium where all motion ceases, i.e., the values of x and λ for which $\dot{x} = 0$ and $\dot{\lambda} = 0$. The notion is that of optimal *long-run stationary equilibrium*; see Arrow and Kurz (1970, Chapter 2) and Carlson and Haurie (1987a, 1996). If an equilibrium exists, then it is defined by the quadruple $\{\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}\}$ satisfying

$$\boxed{\begin{array}{l} f(\bar{x},\bar{u}) = 0, \\ \rho\bar{\lambda} = L_x[\bar{x},\bar{u},\bar{\lambda},\bar{\mu}], \\ \bar{\mu} \geq 0,\ \bar{\mu}g(\bar{x},\bar{u}) = 0, \\ \text{and} \\ H(\bar{x},\bar{u},\bar{\lambda}) \geq H(\bar{x},u,\bar{\lambda}) \\ \text{for all } u \text{ satisfying} \\ g(\bar{x},u) \geq 0. \end{array}} \tag{3.107}$$

Clearly, if the initial condition $x_0 = \bar{x}$, the optimal control is $u^*(t) = \bar{u}$ for all t. If $x_0 \neq \bar{x}$, the optimal solution will have a transient phase. Moreover, depending on the problem, the equilibrium may be attained in a finite time or an approach to it may be asymptotic.

If the nonnegativity constraint (3.100) is added to problem (3.97), then we may include the requirement $\bar{\lambda} \geq 0$ and $\bar{\lambda}\bar{x} = 0$ in (3.107).

If the constraint involving g is not imposed in (3.97), $\bar{\mu}$ may be dropped from the quadruple. In this case, the long-run stationary equilibrium is defined by the triple $\{\bar{x},\ \bar{u},\ \bar{\lambda}\}$ satisfying

$$f(\bar{x},\ \bar{u}) = 0,\ \rho\bar{\lambda} = H_x(\bar{x},\ \bar{u},\ \bar{\lambda}),\ \text{and}\ H_u(\bar{x},\ \bar{u},\ \bar{\lambda}) = 0. \tag{3.108}$$

Also known in this case is that the optimal value of the objective function can be expressed as

$$J^* = H(x_0, u^*(0), \lambda(0))/\rho. \tag{3.109}$$

You are asked to prove this relation in Exercise 3.40. That it holds in Example 3.7 is quite clear when we use (3.105) in (3.109) and see that we get (3.106).

Also, we see from Example 3.7 that when we let $t \to \infty$ in (3.105), we formally obtain

$$(\bar{W},\bar{C},\bar{\lambda}) = \begin{cases} (0,0,\infty) & \text{if} \quad \rho > r, \\ (W_0,\rho W_0,1/\rho W_0) & \text{if} \quad \rho = r, \\ (\infty,\infty,0) & \text{if} \quad \rho < r. \end{cases} \tag{3.110}$$

This is precisely the long-run stationary equilibrium that we will obtain if we apply (3.108) along with $\bar{\lambda} \geq 0$ and $\bar{\lambda}\bar{W} = 0$ directly to the optimal control problem in Example 3.7. This verification is left as Exercise 3.41.

Example 3.8 For another application of (3.108), let us return to Example 3.7 and now assume that the wealth W is invested in a productive activity resulting in an output rate $\ln W$, and that the horizon $T = \infty$. Since $\ln W$ is only defined for $W > 0$, we do not need to impose the terminal constraint (3.102) here.

Thus, the problem is

$$\max_{C(t)\geq 0} \left\{ J = \int_0^\infty e^{-\rho t} \ln C(t) dt \right\}$$

subject to

$$\dot{W} = \ln W - C, \; W(0) = W_0 > 0, \tag{3.111}$$

and one task is to find the long-run stationary equilibrium for it. Note that since the horizon is infinite, it is usual to assume no salvage value and no terminal conditions on the state.

Solution By (3.108) we set

$$\ln \bar{W} - \bar{C} = 0, \; \rho = 1/\bar{W}, \; 1/\bar{C} - \bar{\lambda} = 0,$$

which gives the equilibrium $\{\bar{W}, \bar{C}, \bar{\lambda}\} = \{1/\rho, -\ln\rho, -1/\ln\rho\}$. Since, $0 < \rho < 1$, we have $\bar{C} > 0$, which satisfies the requirement that the consumption be nonnegative. Also, the equilibrium wealth $\bar{W} > 0$.

It is important to note that the optimal *long-run stationary equilibrium* (which is also called the *turnpike*) is not the same as the optimal steady-state *among* the set of all possible steady-states. The latter concept is termed the *Golden Rule* or *Golden Path* in economics, and a procedure to obtain it is described below. However, the two concepts are identical if the discount rate $\rho = 0$; see Exercise 3.43.

The Golden Path is obtained by setting $\dot{x} = f(x, u) = 0$, which provides the feedback control $u(x)$ that would keep $x(t) = x$ over time. Then, substitute $u(x)$ in the integrand $\phi(x, u)$ of (3.28) to obtain $\phi(x, u(x))$. The value of x that maximizes $\phi(x, u(x))$ yields the Golden Path. Of course, all of the constraints imposed on the problem have to be respected when obtaining the Golden Path.

In some cases, there may be more than one equilibria defined by (3.107). If so, the equilibrium that is attained may depend on the initial starting point. Moreover, from some special starting points, the system may have an option to go to two or more different equilibria. Such points are called the Sethi-Skiba points; see Appendix D.8.

For multidimensional systems consisting of two or more states, optimal trajectories may exhibit more complex behaviors. Of particular importance is the concept of limit cycles. If the optimal trajectory of a dynamical system tends to spiral in toward a closed loop in the state space, then that closed loop is called a *limit cycle*. For more on this topic, refer to Vidyasagar (2002) and Grass et al. (2008).

3.7 Model Types

Optimal control theory has been used to solve problems occurring in engineering, economics, management science, and other fields. In each field of application, certain general kinds of models which we will call *model types* are likely to occur, and each such model requires a specialized form of the maximum principle. In Chap. 2 we derived, in considerable detail, a simple form of the continuous-time maximum principle. However, to continue to provide such details for each different version of the maximum principle needed in later chapters of this book would be both repetitive and lengthy.

The purpose of this section is to avoid the latter by listing most of the different management science model types that we will use in later chapters. For each model type, we will give a brief description of the corresponding objective function, state equations, control and state inequality constraints, terminal conditions, adjoint equations, and the form of the optimal control policy. We will also indicate where each of these model types is applied in later chapters.

The reader may wish to skim this section on first reading to get an idea of what it contains, work a few of the exercises, and go on to the various functional areas discussed in later chapters. Then, when specific model types are encountered, the reader may return to read the relevant parts of this section in more detail.

We are now able to state the general forms of all the models (with one or two exceptions) that we will use to analyze the applications discussed in the rest of the book. Some other model types will be explained in later chapters.

In Table 3.3 we have listed six different combinations of ϕ and f functions. If we specify the initial value x_0 of the state variable x and the constraints on the control and state variables, we can get a completely specified optimal control model by selecting one of the model types in Table 3.3 together with one of the terminal conditions given in Table 3.1.

The reader will see numerous examples of the uses of Tables 3.1 and 3.3 when we construct optimal control models of various applied situations in later chapters. To help in understanding these, we will give a brief mathematical discussion of the six model types in Table 3.3, with an indication of where each model type will be used later in the book.

In Model Type (a) of Table 3.3 we see that both ϕ and f are linear functions of their arguments. Hence it is called the *linear-linear case*. The Hamiltonian is

$$\begin{aligned} H &= Cx + Du + \lambda(Ax + Bu + d) \\ &= Cx + \lambda Ax + \lambda d + (D + \lambda B)u. \end{aligned} \tag{3.112}$$

From (3.112) it is obvious that the optimal policy is bang-bang with the switching function $(D + \lambda B)$. Since the adjoint equation is independent of both control and state variables, it can be solved completely without resorting to two-point boundary value methods. Examples of (a) occur in the cash balance problem of Sect. 5.1.1 and the maintenance and replacement model of Sect. 9.1.1.

Model Type (b) of Table 3.3 is the same as Model Type (a) except that the function $C(x)$ is nonlinear. Thus, the term C_x appears in the adjoint equation, and two-point boundary value methods are needed to solve the problem. Here, there is a possibility of singular control, and a specific example is the Nerlove-Arrow model in Sect. 7.1.1.

Model Type (c) of Table 3.3 has linear functions in the state equation and quadratic functions in the objective function. Therefore, it is sometimes called the *linear-quadratic case*. In this case, the optimal control can be expressed in a form in which the state variables enter linearly. Such a form is known as the *linear decision rule*; see (D.36) in Appendix D. A specific example of this case occurs in the production-inventory example of Sect. 6.1.1.

Model Type (d) is a more general version of Model Type (b) in which the state equation is nonlinear in x. Here again, there is a possibility of singular control. The wheat trading model of Sect. 6.2.1 illustrates this model type. The solution of a special case of the model in Sect. 6.2.3 exhibits the occurrence of a singular control.

Table 3.3: Objective, state, and adjoint equations for various model types

	Objective function integrand $\phi =$	State equation $\dot{x} = f =$	Current-value adjoint equation $\dot{\lambda} =$	Form of optimal control policy
(a)	$Cx + Du$	$Ax + Bu + d$	$\lambda(\rho - A) - C$	Bang-bang
(b)	$C(x) + Du$	$Ax + Bu + d$	$\lambda(\rho - A) - C_x$	Bang-bang+Singular
(c)	$x^T Cx + u^T Du$	$Ax + Bu + d$	$\lambda(\rho - A) - 2x^T C$	Linear decision rule
(d)	$C(x) + Du$	$A(x) + Bu + d$	$\lambda(\rho - A_x) - C_x$	Bang-bang+Singular
(e)	$c(x) + q(u)$	$(ax + d)b(u) + e(x)$	$\lambda(\rho - ab(u) - e_x) - c_x$	Interior or boundary
(f)	$c(x)q(u)$	$(ax + d)b(u) + e(x)$	$\lambda(\rho - ab(u) - e_x) - c_x q(u)$	Interior or boundary

Note. The current-value Hamiltonian is often used when $\rho > 0$ is the discount rate; the standard formulation is identical to the current-value formulation when $\rho = 0$. In Table 3.3, capital letters indicate vector functions and small letters indicate scalar functions or vectors. A function followed by an argument in parentheses indicates a nonlinear function; when it is followed by an argument without parenthesis, it indicates a linear function. Thus, $A(x)$ and $e(x)$ are nonlinear vector and scalar functions, while Ax and ax are linear. The function d is always to be interpreted as an exogenous function of time only

In Model Types (e) and (f), the functions are scalar functions, and there is only one state equation, so λ is also a scalar function. In these cases, the Hamiltonian function is nonlinear in u. If it is concave in u, then the optimal control is usually obtained by setting $H_u = 0$. If it is convex, then the optimal control is the same as in Model Type (b).

Several examples of Model Type (e) occur in this book: the optimal financing model in Sect. 5.2.1, the Vidale-Wolfe advertising model in Sect. 7.2.1, the nonlinear extension of the maintenance and replacement model in Sect. 9.1.4, the forestry model in Sect. 10.2.1, the exhaustible resource model in Sect. 10.3.1, and all of the models in Chap. 11. Model Type (f) examples are: The Kamien-Schwartz model in Sect. 9.2.1 and the sole-owner fishery resource model in Sect. 10.1.

Although the general forms of the model are specified in Tables 3.1 and 3.3, there are a number of additional modeling *tricks* that are useful, which will be employed later. We collect these as a series of remarks below.

Remark 3.12 We sometimes need to use the absolute value function $|u|$ of a control variable u in forming the functions ϕ or f. For example, in the simple cash balance model of Sect. 5.1, $u < 0$ represents buying and $u > 0$ represents selling; in either case there is a transaction cost which can be represented as $c|u|$. In order to handle this, we define new control variables u_1 and u_2 satisfying the following relations:

$$u := u_1 - u_2,\ u_1 \geq 0,\ u_2 \geq 0, \tag{3.113}$$

$$u_1 u_2 = 0. \tag{3.114}$$

Thus, we represent u as the difference of two nonnegative variables, u_1 and u_2, together with the quadratic constraint (3.114). We can then write

$$|u| = u_1 + u_2, \tag{3.115}$$

which expresses the nonlinear function $|u|$ as a linear function with the constraint (3.114).

We now observe that we need not impose (3.114) explicitly, provided there are costs associated with the controls u_1 and u_2, since in the presence of these costs no optimal policy would ever choose to make both of them *simultaneously positive*. This is indeed the case in the cash balance problem of Sect. 5.1, where the associated transaction costs prevent us from simultaneously buying and selling the same security.

Thus, by doubling the number of variables and adding inequality constraints, we are able to represent $|u|$ as a linear function in the model.

Remark 3.13 Tables 3.1 and 3.3 are constructed for continuous-time models. Exactly the same kinds of models can be developed in the discrete-time case; see Chap. 8.

Remark 3.14 Consider Model Types (a) and (b) when the control variable constraints are defined by linear inequalities of the form

$$g(u,t) = g(t)u \geq 0. \tag{3.116}$$

Then, the problem of maximizing the Hamiltonian function becomes:

$$\begin{cases} \max(D + \lambda B)u \\ \text{subject to} \\ g(t)u \geq 0. \end{cases} \tag{3.117}$$

This is clearly a linear programming problem for each given instant of time t, since the Hamiltonian function is linear in u.

Further in Model Type (a), the adjoint equation does not contain terms in x and u, so we can solve it for $\lambda(t)$, and hence the objective function of (3.117) varies parametrically with $\lambda(t)$. In this case we can use parametric linear programming techniques to solve the problem over time. Since the optimal solution to the linear program always occurs at an extreme point of the convex set defined by $g(t)u \geq 0$, it follows that as $\lambda(t)$ changes, the optimal solution to (3.117) will "bang" from one extreme point of the feasible set to another. This is called a *generalized bang-bang optimal policy*. Such a policy occurs, e.g., in the optimal financing model treated in Sect. 5.2; see Table 5.1, Row 5.

In Model Type (b), the adjoint equation contains terms in x, so we cannot solve for the trajectory of $\lambda(t)$ without knowing the trajectory of $x(t)$. It is still true that (3.117) is a linear program for any given t, but the parametric linear programming techniques will not usually work. Instead, some type of iterative procedure is needed in general; see Bryson and Ho (1975).

Remark 3.15 The salvage value part $S[x(T), T]$ of the objective function is relevant in the optimization context in the following two cases:

Case (i) T is free and part of the problem is to determine the optimal terminal time; see, e.g., Sect. 9.1.

Case (ii) T is fixed and the problem is that of maximizing the objective function involving the salvage value of the ending state $x(T)$, which in this case can be written simply as $S[x(T)]$.

For the fixed-end-point problem and for the infinite horizon problem, it does not usually make much sense to define a salvage value function.

Remark 3.16 One important model type that we did not include in Table 3.3 is the *impulse control* model of Bensoussan and Lions (1975). In this model, an infinite control is instantaneously exerted on a state variable in order to cause a finite jump in its value. This model is particularly appropriate for the instantaneous reordering of inventory as required in lot-size models; see Bensoussan et al. (1974). Further discussion of impulse control is given in Sect. D.9.

Exercises for Chapter 3

E 3.1 Consider the constraint set

$$\Omega = \{(u_1, u_2)|0 \leq u_1 \leq x,\ -1 \leq u_2 \leq u_1\}.$$

Write these in the form shown in (3.3).

E 3.2 Find the reachable set X, defined in Sect. 3.1, if x and u satisfy

$$\dot{x} = u - 1,\ x_0 = 5,\ -1 \leq u \leq 1,$$

and $T = 3$.

E 3.3 Assume the constraint (3.3) to be of the form $g(u,t) \geq 0$, i.e., g does not contain x explicitly, and assume $x(T)$ is free. Apply the Lagrangian form of the maximum principle and derive the Hamiltonian form (2.31) with

$$\Omega(t) = \{u | g(u,t) \geq 0\}.$$

Assume $g(u,t)$ to be of the form $\alpha \leq u \leq \beta$.

E 3.4 Use the Lagrangian form of the maximum principle to obtain the optimal control for the following problem:

$$\max\{J = x_1(2)\}$$

subject to

$$\dot{x}_1(t) = u_1 - u_2, \quad x_1(0) = 2,$$

$$\dot{x}_2(t) = u_2, \qquad x_2(0) = 1,$$

and the constraints

$$u_1(t) \geq u_2(t),\ 0 \leq u_1(t) \leq x_2(t),\ 0 \leq u_2(t) \leq 2,\ 0 \leq t \leq 2.$$

An interpretation of this problem is that $x_1(t)$ is the stock of steel at time t and $x_2(t)$ is the total capacity of the steel mill at time t. Production of steel at rate u_1, which is bounded by the current steel mill capacity, can be split into u_2 and $u_1 - u_2$, where u_2 goes into increasing the steel mill capacity and $u_1 - u_2$ adds to the stock of steel. The objective is to build as large a stockpile of steel as possible by time $T = 2$. With this interpretation, we clearly need to have $x_1(t) \geq 0$ and $x_2(t) \geq 0$. However, it is easily seen that these constraints are automatically satisfied for every feasible solution of the problem. You may find it interesting to show why this is true. (It is possible to make the problem more interesting by assuming an exogenous demand d for steel so that $\dot{x}_1 = u_1 - u_2 - d$.)

E 3.5 Specialize the terminal condition (3.13) in the one-dimensional case (i.e., $n = 1$) with $Y(T) = Y = [\underline{x}, \bar{x}]$ for each $T > 0$, where $\underline{x}$ and $\bar{x}$ are two constants satisfying $\bar{x} > \underline{x}$. Use (3.12) to derive (3.14).

E 3.6 Obtain the optimal value $J^*(T)$ of the objective function for Example 3.5 for a given terminal time T, and then maximize it with respect to T by using the conditions $dJ^*(T)/dT = 0$. Show that you get the same optimal T^* as the one obtained for Example 3.5 by using (3.77).

E 3.7 Check that the solution of Example 3.1 satisfies the sufficiency conditions in Theorem 3.1.

E 3.8 Starting from (3.15), obtain the current-value version (3.44) for the problem defined by (3.27) and (3.28). Show further that if we were to require the function ψ to also depend on T, i.e. if $S(x, T) = \psi(x, T)e^{-\rho T}$ then the left-hand side of condition (3.44) would be modified to $H[x^*(T^*), u^*(T^*), \lambda(T^*), T^*] + \psi_T[x^*(T^*), T^*] - \rho\psi[x^*(T^*), T^*]$.

E 3.9 Develop the current-value formulation of Sect. 3.3 for a time-varying nonnegative discount rate $\rho(t)$, by replacing the factors $e^{-\rho t}$ and $e^{-\rho T}$ in (3.28), respectively, by

$$\alpha(t) = e^{-\int_0^t \rho(s)ds} \text{ and } \alpha(T) = e^{-\int_0^T \rho(s)ds}.$$

E 3.10 Begin with (3.54) and perform the steps leading to (3.55).

E 3.11 *Optimal Consumption of An Initial Investment Over a Finite Horizon.* Begin with an initial investment of x_0. Assets $x(t)$ at time t earn at the rate of r per dollar per unit time. A portion of the earnings is consumed at a rate of $c(t)$ per unit time at time t, while the remainder is invested. Neither a negative consumption rate nor a consumption rate exceeding the earnings is allowed. Assets depreciate at the constant rate δ. Assume $r > \delta + \rho$, where ρ is the discount rate applied on consumption. Find the optimal consumption rate over a finite horizon T such that the *present value* of the consumption stream over the finite horizon is maximized. Assume that T is sufficiently large. Let us note that the optimal capital accumulation model treated in Sect. 11.1.1 represents a generalization of this problem.

E 3.12 Show that if we require $W(T) = \varepsilon > 0$, ε small, instead of $W(T) = 0$ in Example 3.2, then the optimal value of the objective function will decrease by an amount $\beta\varepsilon = \varepsilon(1 - e^{rT})/rW_0 + o(\varepsilon)$.

E 3.13 Recall Exercise 2.18 of the leaky reservoir in Chap. 2. In this problem there was no explicit constraint on the total amount of water available. Suppose we impose the following isoperimetric constraint on that problem:

$$\int_0^{100} u dt = K,$$

where $K > 0$ is the total amount of water which must be used. Assume also that the reservoir has infinite capacity. Re-solve this problem for various values of K and the objective functions in parts (a) and (b) of Exercise 2.18.

E 3.14 From the transversality conditions for the general terminal constraints in Row 5 of Table 3.1, derive the transversality conditions in Row 1 for the free-end-point case, in Row 2 for the fixed-end-point case, and in Rows 3 and 4 for the one-sided constraint cases. Assume $\psi(x) = 0$, i.e., there is no salvage value and $X = E^1$ for simplicity.

E 3.15 For solving Example 3.3, consider case (ii) by starting with $t^* = 2$, and show that the maximum principle will not be satisfied in this case.

E 3.16 Rework Example 3.4 with $T = 4$ and the following different terminal conditions:

(a) $x(4)$ unconstrained,

(b) $x(4) = 1$,

(c) $x(4) \leq 1$,

(d) $x(4) \geq 1$.

E 3.17 Rework Example 3.4 with the terminal condition (3.70) replaced by $x(2) \geq \varepsilon$, where ε is small. Verify that the change in the optimal value of the objective function is $-\varepsilon/2 \approx -\alpha\varepsilon + o(\varepsilon)$, as stipulated in Remark 3.6.

E 3.18 Introduce a terminal value in Example 3.4 as follows:

$$\max \left\{ J = \int_0^2 (-x)dt + Bx(2) \right\}$$

subject to

$$\dot{x} = u, \; x(0) = 1,$$

$$x(2) \geq 0, \text{ i.e., } Y = [0, \infty) \text{ in Table 3.1, Row 3,}$$

$$-1 \leq u \leq 1.$$

Note that for $B = 0$, the problem is the same as Example 3.4. Solve this problem for $B = 1/2$, 1, 3/2, 2, 3. Conclude that for $B \geq 2$, the solution for the state variable does not change.

E 3.19 In Example 3.6, determine the optimal control and the corresponding state trajectory starting at the point (-4,6), which lies *above* the switching curve.

E 3.20 Carry out the synthesis of the optimal control for Example 3.6 when the starting point (x_0, y_0) lies *below* the switching curve.

E 3.21 Use the results of Exercise 3.20 to find the optimal control and the corresponding trajectory starting at the point $(-1, -1)$.

E 3.22 Find the optimal control, the minimum time, and the corresponding trajectory for Example 3.6 starting at the point $(-2, 2)$, which lies *on* the switching curve.

E 3.23 What is the shortest time in which a passenger can be transported in a ballistic missile from Los Angeles to New York? Assume that a missile with the ultimate mechanical and thermodynamical properties is available, but that the passenger imposes the restraint that the maximum acceleration or deceleration is 100 ft/s^2. The missile starts from rest in Los Angeles and stops in New York. Assume that the path is a straight line of length 2400 miles and ignore the rotation and curvature of the earth.

E 3.24 In the time-optimal control problem (3.90), replace the state equations by

$$\dot{x} = ay,\ x(0) = x_0 \geq 0,\ x(T) = \bar{x} > x_0,$$

$$\dot{y} = u,\ \ y(0) = y_0 \geq 0,\ y(T) = 0,$$

and the control constraint by

$$u \in \Omega = [U_{\min}, U_{\max}].$$

Assume $a > 0$ and $U_{\max} > 0 > U_{\min}$. Observe here that $x(t)$ could be interpreted as the cumulative value of gold mined by a gold-producing country and $y(t)$ could be interpreted as the total value of gold-mining machinery employed by the country at time $t \geq 0$. The required machinery is to be imported. Because of some inertia in the world market for the machinery, the country cannot control $y(t)$ directly, but is able to control its rate of change $\dot{y}(t)$. Thus $u(t)$ represents at time t, the import rate of the machinery when positive and the export rate when negative. The terminal value $\bar{x}$ represents the required amount of gold to be produced in a minimum possible time. Obtain the optimal solution.

E 3.25 Solve the following minimum weighted energy and time problem:

$$\max_{u,T}\left\{J=\int_0^T -(\frac{1}{2})(u^2+1)dt\right\}$$

subject to

$$\dot{x}=u,\ x(0)=5,\ x(T)=0,$$

and the control constraint

$$|u|\le 2.$$

Hint. Use (3.77) to determine T^*, the optimal value of T.

E 3.26 Rework Exercise 3.25 with the new integrand $F = -(1/2)(u^2+16)$ in the objective function.

Hint: Note that use of (3.77) gives an infeasible u. This means that we should look for a boundary solution for u. To obtain this, calculate $J^*(T)$ as defined in Exercise 3.6, and then choose T to maximize it. In doing so, take care to see that $x(T)=0$, and the control constraint is satisfied.

E 3.27 Exercise 3.26 becomes a minimum energy problem if we set $F=-u^2/2$. Show that the Hamiltonian maximizing condition of the maximum principle implies $u^*=k$, where k is a constant. Note that the application of (3.77) implies that $k=0$, which gives $x(t)=5$ for all $t\ge 0$ so that the terminal condition $x(T)=0$ cannot be satisfied.

To see that there exists no optimal control in this situation, let $k<0$ and compute J^*. It is now possible to see that $\lim_{k\to 0} J^* = 0$. This means that we can make the objective function value as close to zero as we wish, but not equal to zero. Note that in this case there are no feasible solutions satisfying the necessary conditions so we cannot check the sufficiency conditions; see the last paragraph of Sect. 2.1.4.

E 3.28 Show that every feasible control of the problem

$$\max_{T,u}\left\{J=\int_0^T -udt\right\}$$

subject to

$$\dot{x}=u,\ x(0)=x_0,\ x(T)=0,$$

$$|u|\le q,\ \text{where } q>0,$$

is an optimal control.

E 3.29 Let $x_0 > 0$ be the initial velocity of a rocket. Let u be the amount of acceleration (or deceleration) caused by applying a force which consumes fuel at the rate $|u|$. We want to bring the rocket to rest using minimum total amount of fuel. Hence, we have the following optimal control problem:

$$\max_{T,u} \left\{ J = \int_0^T -|u|dt \right\}$$

subject to

$$\dot{x} = u, \; x(0) = x_0, \; x(T) = 0,$$

$$-1 \leq u \leq +1.$$

Hint: Use (3.113)–(3.115) to deal with $|u|$. Show that for $x_0 > 0$, say $x_0 = 5$, every feasible control is optimal.

E 3.30 Analyze Exercise 3.29 with the state equation

$$\dot{x} = -ax + u,$$

where $a > 0$. Show that no optimal control exists for the problem.

E 3.31 By using the maximum principle, show that the problem

$$\begin{cases} \max \int_0^1 x dt \\ \text{subject to} \\ \dot{x} = x + u, \; x(0) = 0, \\ 1 - u \geq 0, \; 1 + u \geq 0, \; 2 - x - u \geq 0, \end{cases}$$

has the optimal control

$$u^*(t) = \begin{cases} 1, & t \in [0, \ln 2], \\ 1 + 2\ln 2 - 2t, & t \in (\ln 2, 1]. \end{cases}$$

Also, provide the values of the state variable, the adjoint variable, and the Lagrange multipliers along the optimal path.

E 3.32 If, in Exercise 3.31, we perturb the constraint $2 - x - u \geq 0$ by $2 - x - u \geq \varepsilon$, where ε is small, then show that the change in value of the objective function equals

$$\varepsilon \int_0^1 \mu_3 dt + o(\varepsilon),$$

where μ_3 is the Lagrange multiplier associated with the constraint $2-x-u \geq 0$ in Exercise 3.31. Moreover, if $\varepsilon < 0$, implying that we are relaxing the constraint, then verify that the change in the objective function is positive.

E 3.33 Obtain the value function $V(x,t)$ explicitly in Exercise 3.31 for every $x \in E^1$ and $t \in [0,1]$. Furthermore, verify that $\lambda(t) = V_x(x^*(t),t), \quad t \in [0,1]$, where $\lambda(t)$ is the adjoint variable obtained in the solution of Exercise 3.31.

E 3.34 Solve the problem:

$$\max_{u,T} \left\{ J = \int_0^T [-2 + (1 - u(t))x(t)]dt \right\}$$

subject to

$$\dot{x} = u, \; x(0) = 0, \; x(T) \geq 1,$$
$$u \in [0,1],$$
$$T \in [1,8].$$

Hint: First, show that $u^* = \text{bang}[0,1; \lambda - x]$ and that control can switch at most once from 1 to 0. Then, let $t^*(T)$ denote that switching time, if any, for a given $T \in [1,8]$. Consider three cases: (i) $T = 1$, (ii) $1 < T < 8$, and (iii) $T = 8$. Note that $\lambda(t^*(T)) - x(t^*(T)) = 0$. Use (3.15) in case (ii). Find the optimal solution in each of the three cases. The best of these solutions will be the solution of the problem.

E 3.35 Consider the problem:

$$\max_{u,T} \left\{ J = \int_0^T [-3 - u(t) + x(t)]dt \right\}$$

subject to

$$\dot{x} = u, \; x(0) = 0, \; x(T) \geq 1,$$

$$u \in [0, 1],$$
$$T \in [1, 4 + 2\sqrt{2}].$$

The problem has two different optimal solutions with different values for optimal T^*. Find both of these solutions.

E 3.36 Perform the following:

(a) Find the optimal consumption rate $C^*(t),\ t \in [0, T]$, in the problem:

$$\max \left\{ J = \int_0^T e^{-\rho t} \ln C(t) dt \right\}$$

subject to

$$\dot{W}(t) = -C(t), W(0) = W_0,$$

where T is given and $\rho > 0$.

(b) Assume that T is not given in (a), and is to be chosen optimally. Show for this free terminal time version that the optimal T^* decreases as the discount rate ρ increases.

Hint: It is possible to obtain $dT^*/d\rho$ by implicit differentiation.

E 3.37 An example, which illustrates that

$$\lim_{t \to \infty} \lambda(t) = 0$$

is not a necessary transversality condition in general, is:

$$\max \left\{ J = \int_0^\infty (1 - x) u dt \right\}$$

such that

$$\dot{x} = (1 - x)u,\ x(0) = 0,$$
$$0 \le u \le 1.$$

Show this by finding an optimal control.

E 3.38 Show that the limiting conditions in the rightmost column of Rows 2, 3, and 4 in Table 3.1 imply (3.98) when $T \to \infty$.

E 3.39 Consider the regulator problem defined by the scalar equation

$$\dot{x} = u,\ x(0) = x_0,$$

with the objective function

$$J = -\int_0^\infty \left(\frac{x^4}{4} + \frac{u^2}{2}\right) dt.$$

(a) Show that the long-term stationary equilibrium $(\bar{x}, \bar{u}, \bar{\lambda}) = (0, 0, 0)$, and conclude that in feedback form $u^*(x) = \bar{u} = 0$ when $x = \bar{x} = 0$.

(b) By using the maximum principle and the relation $\dot{u}^* = \frac{du^*(x)}{dx}\dot{x}$, derive a differential equation for the optimal feedback control $u^*(x)$ and solve it with the boundary condition $u^*(0) = 0$ to obtain

$$u^*(x) = \begin{cases} -x^2/\sqrt{2}, & x > 0, \\ 0, & x = 0, \\ +x^2/\sqrt{2}, & x < 0. \end{cases}$$

(c) Solve for $x^*(t)$ and $\lambda(t)$ and show that $\lim_{t\to\infty} x^*(t) = 0$ and that the limiting condition (3.99), i.e., $\lim_{t\to\infty} \lambda(t) = 0$, holds for this problem.

E 3.40 Show that for the problem (3.97) without the constraint $g(x, u) \geq 0$, the optimal value of the objective function

$$J^* = H(x_0, u^*(0), \lambda(0))/\rho.$$

See Grass et al. (2008).

E 3.41 Apply (3.108), along with the requirement $\bar{\lambda} \geq 0$ and $\bar{\lambda}\bar{W} = 0$ in view of the constraint (3.102), to Example 3.7 to verify that the long-run stationary equilibrium is as shown in (3.110).

E 3.42 For a stationary system as defined in Sect. 3.6, show that

$$\frac{dH}{dt} = \rho\lambda f(x^*(t), u^*(t))$$

and

$$\frac{dH^{pv}}{dt} = -\rho e^{-\rho t}\phi(x^*(t), u^*(t))$$

along the optimal path. Also, contrast these results with that of Exercise 2.9.

E 3.43 Consider the inventory problem:

$$\max \left\{ J = \int_0^\infty -e^{-\rho t}[(I - I_1)^2 + (P - P_1)^2]dt \right\}$$

subject to

$$\dot{I} = P - S,\ I(0) = I_0,$$

where I denotes inventory level, P denotes production rate, and S denotes a given constant demand rate.

(a) Find the optimal long-run stationary equilibrium, i.e., the turnpike defined in (3.107).

(b) Find the Golden Rule by setting $\dot{I} = 0$ in the state equation, solve for P, and substitute it into the integrand of the objective function. Then, maximize the *integrand* with respect to I.

(c) Verify that the Golden Rule inventory level obtained in (b) is the same as the turnpike inventory level found in (a) when $\rho = 0$.

Chapter 4

The Maximum Principle: Pure State and Mixed Inequality Constraints

In Chap. 2 we addressed optimal control problems having constraints only on control variables. We extended the discussion in Chap. 3 to treat mixed constraints that may involve state variables in addition to control variables.

Often in management science and economics problems there are nonnegativity constraints on state variables, such as inventory levels or wealth. These constraints do not include control variables. Also, there may be more general inequality constraints only on state variables, which include constraints that require certain state variables to remain nonnegative. Such constraints are known as *pure state variable inequality constraints* or, simply, *pure state constraints*.

Pure state constraints are more difficult to deal with than mixed constraints. We can intuitively appreciate this fact by keeping in mind that only control variables are under the direct influence of the decision maker. This enables the decision maker, when a mixed constraint becomes tight, to choose from the controls that would keep it tight for as long as required for optimality. Whereas with pure state constraints, the situation is different and more complicated. That is, when a constraint becomes tight, it does not provide any direct information to the decision maker on how to choose values for the control variables so as not to

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violate the constraint. Hence, considerable changes in the controls may be required to keep the constraint tight if needed for optimality.

This chapter considers pure state constraints together with mixed constraints. In the literature there are two ways of handling pure state constraints: *direct* and *indirect*. The direct method associates a multiplier with each constraint for appending it to the Hamiltonian to form the Lagrangian, and then proceeds in much the same way as in Chap. 3 dealing with mixed constraints. In the indirect method, the choice of controls, when a pure constraint is active, must be further limited by constraining approximately the value of the derivative of the active state constraint with respect to time. This derivative will involve time derivatives of the state variables, which can be written in terms of the control and state variables through the use of the state equations. Thus, the restrictions on the time derivatives of the pure state constraints are transformed in the form of mixed constraints, and these will be appended instead to the Hamiltonian to form the Lagrangian. Because the pure state constraints are adjoined in this indirect fashion, the corresponding Lagrange multipliers must satisfy some complementary slackness conditions in addition to those mentioned in Chap. 3.

With the formulation of the Lagrangian in each approach, we will write the respective maximum principle, where the choice of control will come from maximizing the Hamiltonian subject to both pure state constraints and mixed constraints. We will find, however, in contrast to Chap. 3, that in both approaches, the adjoint functions may be required to have jumps at those times where the pure state constraints become tight.

We begin with a simple example in Sect. 4.1 to motivate the necessity of possible jumps in the adjoint functions. Section 4.2 formulates the problem with pure state constraints along with the required assumptions. In Sect. 4.3, we use the direct method for stating the maximum principle necessary conditions for solving such problems. Sufficiency conditions are stated in Sect. 4.4. Section 4.5 is devoted to developing the maximum principle for the indirect method, which involves adjoining the first derivative of the pure state constraints to form the Lagrangian function and imposing some additional constraints on the Lagrange multipliers of the resulting formulation. Also, the adjoint variables and the Lagrange multipliers arising in this method will be related to those arising in the direct method. Finally, the current-value form of the maximum principle for the indirect method is described in Sect. 4.6.

4.1 Jumps in Marginal Valuations

In this section, we formulate an optimal control problem with a pure constraint, which can be solved merely by inspection and which exhibits a discontinuous marginal valuation of the state variable. Since the adjoint variables in Chaps. 2 and 3 provide these marginal valuations and since we would like this feature to continue, we must allow the adjoint variables to have jumps if the marginal valuations can be discontinuous. This will enable us to formulate a maximum principle in the next section, which is similar to (3.10) with the exception that the adjoint variables, and therefore also the Hamiltonian, may have possible jumps satisfying some jump conditions.

Example 4.1 Consider the problem with a pure state constraint:

$$\max\left\{J=\int_0^3 -udt\right\} \tag{4.1}$$

subject to

$$\dot{x}=u,\ \ x(0)=0, \tag{4.2}$$

$$0\le u\le 3, \tag{4.3}$$

$$x-1+(t-2)^2\ge 0. \tag{4.4}$$

Solution From the objective function (4.1), one can see that it is good to have low values of u. If we use $u=0$ to begin with, we see that $x(t)=0$ as long as $u(t)=0$. At $t=1$, $x(1)=0$ and the constraint (4.4) is satisfied with an equality. But continuing with $u(t)=0$ beyond $t=1$ is not feasible since $x(t)=0$ would not satisfy the constraint (4.4) just after $t=1$.

In Fig. 4.1, we see that the lowest possible feasible state trajectory from $t=1$ to $t=2$ satisfies the state constraint (4.4) with an equality. In order not to violate the constraint (4.4), its first time derivative $u(t)+2(t-2)$ must be nonnegative. This gives us $u(t)=2(2-t)$ to be the lowest feasible value for the control. This value will make the state $x(t)$ ride along the constraint boundary until $t=2$, at which point $u(2)=0$; see Fig. 4.1. Continuing with $u(t)=2(2-t)$ beyond $t=2$ will make $u(t)$ negative, and violate the lower bound in (4.3). It is easy to see, however, that $u(t)=0$, $t\ge 2$, is the lowest feasible value, which can be followed all the way to the terminal time $t=3$.

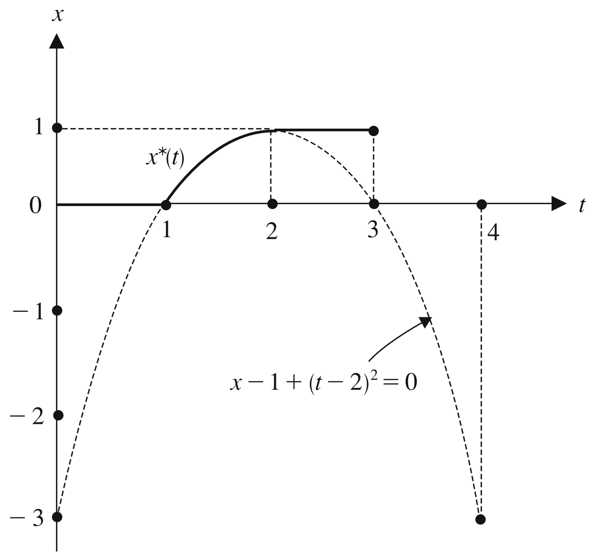

Figure 4.1: Feasible state space and optimal state trajectory for Examples 4.1 and 4.4

It can be seen from Fig. 4.1 that the bold trajectory is the lowest possible feasible state trajectory on the entire time interval [0,3]. Moreover, it is obvious that the lowest possible feasible control is used at any given $t \in [0, 3]$, and therefore, the solution we have found is optimal. We can now restate the values of the state and control variables that we have obtained:

$$x^*(t) = \begin{cases} 0, & t \in [0,1), \\ 1-(t-2)^2, & t \in [1,2], \\ 1, & t \in (2,3], \end{cases} \quad u^*(t) = \begin{cases} 0, & t \in [0,1), \\ 2(2-t), & t \in [1,2], \\ 0, & t \in (2,3]. \end{cases} \tag{4.5}$$

Next we find the value function $V(x, t)$ for this problem. It is obvious that the feedback control $u^*(x, t) = 0$ is optimal at any point (x, t) when $x \geq 1$ or when (x, t) is on the right-hand side of the parabola in Fig. 4.1. Thus, $V(x, t) = 0$ on such points.

On the other hand, when $x \in [0, 1]$ and it is on the left-hand side of the parabola, the optimal trajectory is very similar to the one shown in Fig. 4.1. Specifically, the control is zero until it hits the trajectory at time $\tau = 2 - \sqrt{1-x}$. Then, the control switches to $2(2 - s)$ for $s \in (\tau, 2)$

to climb along the left-hand side of the parabola to reach its peak, and then switches back to zero on the time interval [2,3]. Thus, in this case,

$$\begin{aligned} V(x,t) &= -\int_t^{\tau} 0ds - \int_{\tau}^{2} 2(2-s)ds - \int_2^3 0ds \\ &= s^2 - 4s\Big]_{2-\sqrt{1-x}}^{2} = (x-1). \end{aligned}$$

Thus, we have the value function

$$V(x,t) = \begin{cases} 0, & x \geq 1,\ t \in [0,3], \\ x-1, & x \geq 1-(t-2)^2,\ t \in [0,2), \\ 0, & 1-(t-2)^2 \leq x \leq 1,\ t \in [2,3]. \end{cases}$$

This gives us the marginal valuation along the optimal path $x^*(t)$ given in (4.5) as

$$V_x(x^*(t),t) = \begin{cases} 1, & t \in [0,2), \\ 0, & t \in [2,3]. \end{cases} \tag{4.6}$$

We can now see that this marginal valuation is discontinuous at $t = 2$, and it has a downward jump of size 1 at that time.

The maximum principle that we will state in Sect. 4.3 will have certain jump conditions in order to accommodate problems like Example 4.1. Indeed in Example 4.2, we will apply the maximum principle of Sect. 4.3 to the problem in Example 4.1, and see that the adjoint variable that represents the marginal valuation along the optimal path will have a jump consistent with (4.6).

In the next section, we state the general optimal control problem that is the subject of this chapter.

4.2 The Optimal Control Problem with Pure and Mixed Constraints

We will append to the problem (3.7) considered in Chap. 3, the pure state variable inequality constraint of type

$$h(x,t) \geq 0, \tag{4.7}$$

where we assume function $h : E^n \times E^1 \to E^p$ to be continuously differentiable in all its arguments. By the definition of function h, (4.7) represents a set of p constraints $h_i(x,t) \geq 0$, $i = 1, 2, \ldots, p$. It is noted that the constraint $h_i \geq 0$ is called a constraint of rth order if the rth time derivative of h_i is the first time a term in control u appears in the expression by putting $f(x,u,t)$ for $\dot{x}$ after each differentiation. It is through this expression that the control acts to satisfy the constraint $h_i \geq 0$. The value of r is referred to as the *order* of the constraint. Of course, if the constraint h_i is of order r, then we would require h_i to be r times continuously differentiable.

Except for Exercise 4.12, in this book we will consider only first-order constraints, i.e., $r = 1$. For such constraints, the first-time derivative of h has terms in u. Thus, we can define $h^1(x,u,t)$ as follows:

$$h^1 = \frac{dh}{dt} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t}. \tag{4.8}$$

In the important special case of the nonnegativity constraint

$$x(t) \geq 0, \quad t \in [0,T], \tag{4.9}$$

(4.8) is simply $h^1 = f$. For an upper bound constant $x(t) \leq M$, written as

$$M - x(t) \geq 0, \quad t \in [0,T], \tag{4.10}$$

(4.8) gives $h^1 = -f$. These will be of order one because the function $f(x,u,t)$ usually contains terms in u.

As in Chap. 3, the constraints (4.7) need also to satisfy a full-rank type constraint qualification before a maximum principle can be derived. With respect to the ith constraint $h_i(x,t) \geq 0$, an interval $(\theta_1, \theta_2) \subset [0,T]$ with $\theta_1 < \theta_2$ is called an *interior interval* if $h_i(x(t),t) > 0$ for all $t \in (\theta_1, \theta_2)$. If the optimal trajectory "hits the boundary," i.e., satisfies $h_i(x(t),t) = 0$ for $\tau_1 \leq t \leq \tau_2$ for some i, then $[\tau_1, \tau_2]$ is called a *boundary interval.* An instant τ_1 is called an *entry time* if there is an interior interval ending at $t = \tau_1$ and a boundary interval starting at τ_1. Correspondingly, τ_2 is called an *exit time* if a boundary interval ends and an interior interval starts at τ_2. If the trajectory just touches the boundary at time τ, i.e., $h(x(\tau),\tau) = 0$ and if the trajectory is in the interior just before and just after τ, then τ is called a *contact time.* Taken together, entry, exit, and contact times are called *junction times.*

In this book we shall not consider problems that require optimal state trajectories to have countably many junction times. In other words, we shall state the maximum principle necessary optimality conditions for state trajectories having only finitely many junction times. Also, all of the applications studied in this book exhibit optimal state trajectories containing finitely many junction times or no junction times.

Throughout the book, we will assume that the constraint qualification introduced in Sect. 3.1 as well as the following full-rank condition on any boundary interval $[\tau_1, \tau_2]$ hold:

$$\text{rank}\begin{bmatrix} \partial h_1^1/\partial u \\ \partial h_2^1/\partial u \\ \vdots \\ \partial h_{\hat{p}}^1/\partial u \end{bmatrix} = \hat{p},$$

where for $t \in [\tau_1, \tau_2]$,

$$h_i(x^*(t), t) = 0,\ i = 1, 2, \ldots, \hat{p} \leq p$$

and

$$h_i(x^*(t), t) > 0,\ i = \hat{p} + 1, \ldots, p.$$

Note that this full-rank condition on the constraints (4.7) is written when the order of each of the constraints in (4.7) is one. For the general case of higher-order constraints, see Hartl et al. (1995).

Let us recapitulate the optimal control problem for which we will state a direct maximum principle in the next section. The problem is

$$\left\{\begin{array}{l} \max\left\{J = \int_0^T F(x, u, t)dt + S[x(T), T]\right\}, \\ \text{subject to} \\ \dot{x} = f(x, u, t),\ x(0) = x_0, \\ g(x, u, t) \geq 0, \\ h(x, t) \geq 0, \\ a(x(T), T) \geq 0, \\ b(x(T), T) = 0. \end{array}\right. \tag{4.11}$$

Important special cases of the mixed constraint $g(x, u, t) \geq 0$ are $u_i \in [0, M]$ for $M > 0$ and $u_i(t) \in [0, x_i(t)]$, and those of the terminal constraints $a(x(T), T) \geq 0$ and $a(x(T), T) = 0$ are $x_i(T) \geq k$ and $x_i(T) = k$, respectively, where k is a constant. Likewise, the special cases of the pure constraints $h(x, t) \geq 0$ are $x_i \geq 0$ and $x_i \leq M$, for which $h_{x_i} = +1$ and $h_{x_i} = -1$, respectively, and $h_t = 0$.

4.3 The Maximum Principle: Direct Method

For the problem (4.11), we will now state the direct maximum principle which includes the discussion above and the required jump conditions. For details, see Dubovitskii and Milyutin (1965), Feichtinger and Hartl (1986), Hartl et al. (1995), Boccia et al. (2016), and references therein. We will use superscript d on various multipliers that arise in the direct method, to distinguish them from the corresponding multipliers (which are not superscripted) that arise in the indirect method, to be discussed in Sect. 4.5. Naturally, it will not be necessary to superscript the multipliers that are known to remain the same in both methods.

To formulate the maximum principle for the problem (4.11), we define the Hamiltonian function $H^d : E^n \times E^m \times E^1 \to E^1$ as

$$H^d = F(x, u, t) + \lambda^d f(x, u, t)$$

and the Lagrangian function $L^d : E^n \times E^m \times E^n \times E^q \times E^p \times E^1 \to E^1$ as

$$L^d(x, u, \lambda^d, \mu, \eta^d, t) = H^d(x, u, \lambda^d, t) + \mu g(x, u, t) + \eta^d h(x, t). \quad (4.12)$$

The maximum principle states the necessary conditions for u^* (with the corresponding state trajectory x^*) to be optimal. The conditions are that there exist an adjoint function λ^d, which may be discontinuous at a time in a boundary interval or a contact time, multiplier functions $\mu, \alpha, \beta, \gamma^d, \eta^d$, and a jump parameter $\zeta^d(\tau)$, at each time τ, where λ^d is discontinuous, such that the following (4.13) holds:

$$
\left\{
\begin{aligned}
&\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0, \text{ satisfying constraints} \\
&g(x^*, u^*, t) \geq 0,\ h(x^*, t) \geq 0, \text{ and the terminal constraints} \\
&a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0; \\
&\dot{\lambda}^d = -L_x[x^*, u^*, \lambda^d, \mu, \eta^d, t] \\
&\text{with the transversality conditions} \\
&\lambda^d(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) \\
&\qquad\qquad +\gamma^d h_x(x^*(T), T), \text{ and} \\
&\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma^d \geq 0,\ \gamma^d h(x^*(T), T) = 0; \\
&\text{the Hamiltonian maximizing condition} \\
&H^d[x^*(t), u^*(t), \lambda^d(t), t] \geq H^d[x^*(t), u, \lambda^d(t), t] \\
&\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying} \\
&g[x^*(t), u, t] \geq 0; \\
&\text{the jump conditions at any time } \tau, \\
&\text{where } \lambda^d \text{ is discontinuous, are} \\
&\lambda^d(\tau^-) = \lambda^d(\tau^+) + \zeta^d(\tau) h_x(x^*(\tau), \tau) \text{ and} \\
&H^d[x^*(\tau), u^*(\tau^-), \lambda^d(\tau^-), \tau] = H^d[x^*(\tau), u^*(\tau^+), \lambda^d(\tau^+), \tau] \\
&\qquad\qquad\qquad\qquad -\zeta^d(\tau) h_t(x^*(\tau), \tau); \\
&\text{the Lagrange multipliers } \mu(t) \text{ are such that} \\
&\partial L^d / \partial u|_{u=u^*(t)} = 0,\ dH^d/dt = dL^d/dt = \partial L^d / \partial t, \\
&\text{and the complementary slackness conditions} \\
&\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0, \\
&\eta(t) \geq 0,\ \eta^d(t) h(x^*(t), t) = 0, \text{ and} \\
&\zeta^d(\tau) \geq 0,\ \zeta^d(\tau) h(x^*(\tau), \tau) = 0 \text{ hold.}
\end{aligned}
\right. \tag{4.13}
$$

As in the previous chapters, $\lambda^d(t)$ has the marginal value interpretation. Therefore, while it is not needed for the application of the maximum principle (4.13), we can trivially set

$$\lambda^d(T) = S_x(x^*(T), T). \tag{4.14}$$

If T is also a decision variable constrained to lie in the interval $[T_1, T_2]$, $0 \le T_1 < T_2 < \infty$, then in addition to (4.13), if T^* is the optimal terminal time, it must satisfy a condition similar to (3.15) and (3.81), i.e.,

$$\begin{aligned} &H^d[x^*(T^*), u^*(T^{*-}), \lambda^d(T^{*-}), T^*] + S_T[x^*(T^*), T^*] + \alpha a_T[x^*(T^*), T^*] \\ &+\beta b_T[x^*(T^*), T^*] + \gamma^d h_T[x^*(T^*), T^*] \begin{cases} \le 0 & \text{if } T^* = T_1, \\ = 0 & \text{if } T^* \in (T_1, T_2), \\ \ge 0 & \text{if } T^* = T_2. \end{cases} \end{aligned} \tag{4.15}$$

Remark 4.1 In most practical examples, λ^d and H^d will only jump at junction times. However, in some cases a discontinuity may occur at a time in the interior of a boundary interval, e.g., when a mixed constraint becomes active at that time.

Remark 4.2 It is known that the adjoint function λ^d is continuous at a junction time τ, i.e., $\zeta^d(\tau) = 0$, if (i) the entry or exit at time τ is non-tangential, i.e., if $h^1(x^*(\tau), u^*(\tau), \tau) \neq 0$, or (ii) if the control u^* is continuous at τ and the

$$\text{rank} \begin{bmatrix} \partial g/\partial u & \text{diag}(g) & 0 \\ \partial h^1/\partial u & 0 & \text{diag}(h) \end{bmatrix} = m + p,$$

when evaluated at $x^*(\tau)$ and $u^*(\tau)$.

We will see that the jump conditions on the adjoint variables in (4.13) will give us precisely the jump in Example 4.2, where we will apply the direct maximum principle to the problem in Example 4.1. The jump condition on H^d in (4.13) requires that the Hamiltonian should be continuous at τ if $h_t(x^*(\tau), \tau) = 0$. The continuity of the Hamiltonian (in case $h_t = 0$) makes intuitive sense when considered in the light of its interpretation given in Sect. 2.2.4.

This brief discussion of the jump conditions, limited here only to first-order pure state constraints, is far from complete, and a detailed discussion is beyond the scope of this book. An interested reader should consult the comprehensive survey by Hartl et al. (1995). For an example with a second-order state constraint, see Maurer (1977).

Needless to say, computational methods are required to solve problems with general inequality constraints in all but the simplest of the cases. The reader should consult the excellent book by Teo et al. (1991) and references therein for computational procedures and software; see also Polak et al. (1993), Bulirsch and Kraft (1994), Bryson (1998), and Pytlak and Vinter (1993, 1999). A MATLAB based software, used for solving finite and infinite horizon optimal control problems with pure state and mixed inequality constraints, is available at http://orcos.tuwien.ac.at/research/ocmat_software/.

Example 4.2 Apply the direct maximum principle (4.13) to solve the problem in Example 4.1.

Solution Since we already have optimal u^* and x^* as obtained in (4.5), we can use these in (4.13) to obtain $\lambda^d, \mu_1, \mu_2, \gamma^d, \eta^d$, and ζ^d. Thus,

$$H^d = -u + \lambda^d u, \tag{4.16}$$

$$L^d = H^d + \mu_1 u + \mu_2(3-u) + \eta^d[x - 1 + (t-2)^2], \tag{4.17}$$

$$L_u^d = -1 + \lambda^d + \mu_1 - \mu_2 = 0, \tag{4.18}$$

$$\dot{\lambda}^d = -L_x^d = -\eta^d,\ \lambda^d(3^-) = \gamma^d, \tag{4.19}$$

$$\gamma^d[x^*(3) - 1 + (3-2)^2] = 0, \tag{4.20}$$

$$\mu_1 \geq 0,\ \mu_1 u^* = 0,\ \mu_2 \geq 0,\ \mu_2(3-u^*) = 0, \tag{4.21}$$

$$\eta^d \geq 0,\ \eta^d[x^*(t) - 1 + (t-2)^2] = 0, \tag{4.22}$$

and if λ^d is discontinuous for some $\tau \in [1,2]$, the boundary interval as seen from Fig. 4.1, then

$$\lambda^d(\tau^-) = \lambda^d(\tau^+) + \zeta^d(\tau),\ \zeta^d(\tau) \geq 0, \tag{4.23}$$

$$-u^*(\tau^-) + \lambda^d(\tau^-)u^*(\tau^-) = -u^*(\tau^+) + \lambda^d(\tau^+)u^*(\tau^+) - \zeta^d(\tau)2(\tau - 2). \tag{4.24}$$

Since $\gamma^d = 0$ from (4.20), we have $\lambda^d(3-) = 0$ from (4.19). Also, we set $\lambda^d(3) = 0$ according to (4.14).

Interval (2,3]: We have $\eta^d = 0$ from (4.22), and thus $\dot{\lambda}^d = 0$ from (4.19), giving $\lambda^d = 0$. From (4.18) and (4.21), we have $\mu_1 = 1 > 0$ and $\mu_2 = 0$.

Interval [1,2]: We get $\mu_1 = \mu_2 = 0$ from $0 < u^* < 3$ and (4.21). Thus, (4.18) implies $\lambda^d = 1$ and (4.19) gives $\eta^d = -\dot{\lambda}^d = 0$. Thus λ^d is discontinuous at the exit time $\tau = 2$, and we use (4.23) to see that the jump parameter $\zeta^d(2) = \lambda^d(2^-) - \lambda^d(2^+) = 1$. Furthermore, it is easy to check that (4.24) also holds at $\tau = 2$.

Interval [0,1): Clearly $\mu_2 = 0$ from (4.21). Also $u^* = 0$ would still be optimal if there were no lower bound constraint on u in this interval. This means that the constraint $u \geq 0$ is not binding, giving us $\mu_1 = 0$. Then from (4.18), we have $\lambda^d = 1$. Finally, from (4.19), we have $\eta^d = -\dot{\lambda}^d = 0$.

We can now see that the adjoint variable

$$\lambda^d(t) = \begin{cases} 1, & t \in [0, 2), \\ 0, & t \in [2, 3], \end{cases} \tag{4.25}$$

is precisely the same as the marginal valuation $V_x(x^*(t), t)$ obtained in (4.6). We also see that λ^d is continuous at time $t = 1$ where the entry to the constraint is non-tangential as stated in Remark 4.2.

4.4 Sufficiency Conditions: Direct Method

When first-order pure state constraints are present, sufficiency results are usually stated in terms of the maximum principle using the direct method described in Hartl et al. (1995).

We will now state the sufficiency result for the problem specified in (4.11). For this purpose, let us define the maximized Hamiltonian

$$H^{0d}(x, \lambda^d(t), t) = \max_{\{u|g(x,u,t)\geq 0\}} H^d(x, u, \lambda^d, t). \tag{4.26}$$

See Feichtinger and Hartl (1986) and Seierstad and Sydsæter (1987) for details.

Theorem 4.1 *Let $(x^*, u^*, \lambda^d, \mu, \alpha, \beta, \gamma^d, \eta^d)$ and the jump parameters $\zeta^d(\tau)$ at each τ, where λ^d is discontinuous, satisfy the necessary conditions in (4.13). If $H^{0d}(x, \lambda^d(t), t)$ is concave in x at each $t \in [0, T]$, S*

in (3.2) *is concave in* x, g *in* (3.3) *is quasiconcave in* (x, u), h *in* (4.7) *and* a *in* (3.4) *are quasiconcave in* x, *and* b *in* (3.5) *is linear in* x, *then* (x^*, u^*) *is optimal.*

We will illustrate an application of this theorem in Example 4.4, which shows that the solution obtained in Example 4.3 is optimal.

Theorem 4.1 is written for finite horizon problems. For infinite horizon problems, this theorem remains valid if the transversality condition on the adjoint variable in (4.29) is modified along the lines discussed in Sect. 3.6.

In concluding this section, we should note that the sufficiency conditions stated in Theorem 4.1 rely on the presence of appropriate concavity conditions. Sufficiency conditions can also be obtained without these concavity assumptions. These are called second-order conditions for a local maximum, which require the second variation on the linearized state equation to be negative definite. For further details on second-order sufficiency conditions, the reader is referred to Maurer (1981), Malanowski (1997), and references in Hartl et al. (1995).

4.5 The Maximum Principle: Indirect Method

The main idea underlying the indirect method is that when the pure state constraint (4.7), assumed to be of order one, becomes active, we must require its first derivative $h^1(x, u, t)$ in (4.8) to be nonnegative, i.e.,

$$h^1(x, u, t) \geq 0, \text{ whenever } h(x, t) = 0. \tag{4.27}$$

While this is a mixed constraint, it is different from those treated in Chap. 3 in the sense that it is imposed only when the constraint (4.8) is tight.

Since (4.27) is a mixed constraint, it is tempting to use the maximum principle (3.12) developed in Chap. 3. This can be done provided that we can find a way to impose (4.27) only when $h(x, t) = 0$. One way to accomplish this is to append (4.27) to the Hamiltonian when forming the Lagrangian, by using a multiplier $\eta \geq 0$, i.e., append ηh^1, and require that $\eta h = 0$, which is equivalent to imposing $\eta_i h_i = 0,\ i = 1, 2, \ldots, p$. This means that when $h_i > 0$ for some i, we have $\eta_i = 0$ and it is then not a part of the Lagrangian.

Note that when we require $\eta h = 0$, we do not need to impose $\eta h^1 = 0$ as required for mixed constraints. This is because when $h_i > 0$ on an interval, then $\eta_i = 0$ and so $\eta_i h_i^1 = 0$ on that interval. On the other hand, when $h_i = 0$ on an interval, then it is because $h_i^1 = 0$, and thus, $\eta_i h_i^1 = 0$ on that interval. In either case, $\eta_i h_i^1 = 0$.

With these observations, we are ready to formulate the indirect maximum principle for the problem (4.11).

We form the Lagrangian as

$$L(x, u, \lambda, \mu, \eta, t) = H(x, u, \lambda, t) + \mu g(x, u, t) + \eta h^1(x, u, t), \tag{4.28}$$

where the Hamiltonian $H = F(x, u, t) + \lambda f(x, u, t)$ as defined in (3.8). We will now state the maximum principle which includes the discussion above and the required jump conditions.

The maximum principle states the necessary conditions for u^* (with the state trajectory x^*) to be optimal. These conditions are that there exist an adjoint function λ, which may be discontinuous at each entry or contact time, multiplier functions $\mu, \alpha, \beta, \gamma, \eta$, and a jump parameter $\zeta(\tau)$ at each τ, where λ^d is discontinuous, such that (4.29) on the following page holds.

Once again, as before, we can set $\lambda(T) = S_x(x^*(T), T)$. If $T \in [T_1, T_2]$ is a decision variable, then (4.15) with λ^d and γ^d replaced by λ and γ, respectively, must also hold.

In (4.29), we see that there are jump conditions on the adjoint variables and also the Hamiltonian in the indirect maximum principle. The remarks on the jump condition made in connection with the direct maximum principle (4.13) apply also to the jump conditions in (4.29). In (4.29), we also see a condition $\dot{\eta} \le 0$, in addition to the complimentary conditions on η. The presence of this term will become clear after we relate this multiplier to those in the direct maximum principle, which we discuss next.

In various applications that are discussed in subsequent chapters of this book, we use the indirect maximum principle. Nevertheless, it is worthwhile to provide relationships between the multipliers of the two approaches, as these will be useful when checking for the sufficiency conditions of Theorem 4.1, developed in Sect. 4.4.

$\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0$, satisfying constraints

$g(x^*, u^*, t) \geq 0,\ h(x^*, t) \geq 0$, and the terminal constraints

$a(x^*(T), T) \geq 0$ and $b(x^*(T), T) = 0$;

$\dot{\lambda} = -L_x[x^*, u^*, \lambda, \mu, \eta, t]$ with the transversality conditions

$$\lambda(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) + \gamma h_x(x^*(T), T), \text{ and}$$

$\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma \geq 0,\ \gamma h(x^*(T), T) = 0$;

the Hamiltonian maximizing condition

$H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t]$

at each $t \in [0, T]$ for all u satisfying

$g[x^*(t), u, t] \geq 0$, and

$h_i^1(x^*(t), u, t) \geq 0$ whenever $h_i(x^*(t), t) = 0, i = 1, 2, \cdots, p$; (4.29)

the jump conditions at any entry/contact time τ,

where λ is discontinuous, are

$\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau)$ and

$$H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau] - \zeta(\tau) h_t(x^*(\tau), \tau);$$

the Lagrange multipliers $\mu(t)$ are such that

$\partial L/\partial u|_{u=u^*(t)} = 0,\ dH/dt = dL/dt = \partial L/\partial t$,

and the complementary slackness conditions

$\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0$,

$\eta(t) \geq 0,\ \eta(t) h(x^*(t), t) = 0$, and

$\zeta(\tau) \geq 0,\ \zeta(\tau) h(x^*(\tau), \tau) = 0$ hold.

We now obtain the multipliers of the direct maximum principle from those in the indirect maximum principle. Since the multipliers coincide in the interior, we let $[\tau_1, \tau_2]$ denote a boundary interval and τ a contact time. It is shown in Hartl et al. (1995) that

$$\eta^d(t) = -\dot{\eta}(t),\ t \in (\tau_1, \tau_2), \tag{4.30}$$

$$\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t),\ t \in (\tau_1, \tau_2), \tag{4.31}$$

Note that $\eta^d(t) \geq 0$ in (4.13). Thus, we have $\dot{\eta} \leq 0$, which we have already included in (4.29). The jump parameter at an entry time τ_1, an exit time τ_2, or a contact time τ, respectively, satisfies

$$\zeta^d(\tau_1) = \zeta(\tau_1) - \eta(\tau_1^+),\ \zeta^d(\tau_2) = \eta(\tau_2^-),\ \zeta^d(\tau) = \zeta(\tau). \tag{4.32}$$

By comparing $\lambda^d(T^-)$ in (4.13) and $\lambda(T^-)$ in (4.29) and using (4.31), we have

$$\gamma^d = \gamma + \eta(T^-). \tag{4.33}$$

Going the other way, we have

$$\eta(t) = \int_t^{\tau_2} \eta^d(s)ds + \zeta^d(\tau_2),\ t \in (\tau_1, \tau_2),$$

$$\lambda(t) = \lambda^d(t) - \eta(t)h(x^*(t), t),\ t \in (\tau_1, \tau_2),$$

$$\zeta(\tau_1) = \zeta^d(\tau_1) + \eta(\tau_1^+),\ \zeta(\tau_2) = 0,\ \zeta(\tau) = \zeta^d(\tau),$$

$$\gamma = \gamma^d - \eta(T^-).$$

Finally, as we had mentioned earlier, the multipliers μ, α, and β are the same in both methods.

Remark 4.3 From (4.30), (4.32), and $\eta^d(t) \geq 0$ and $\zeta^d(\tau_1) \geq 0$ in (4.13), we can obtain the conditions

$$\dot{\eta}(t) \leq 0 \tag{4.34}$$

and

$$\zeta(\tau_1) \geq \eta(\tau_1^+) \text{ at each entry time } \tau_1, \tag{4.35}$$

which are useful to know about. Hartl et al. (1995) and Feichtinger and Hartl (1986) also add these conditions to the indirect maximum principle necessary conditions (4.29).

Remark 4.4 In Exercise 4.12, we discuss the indirect method for higher-order constraints. For further details, see Pontryagin et al. (1962), Bryson and Ho (1975) and Hartl et al. (1995).

Example 4.3 Consider the problem:

$$\max \left\{ J = \int_0^2 -x dt \right\}$$

subject to

$$\dot{x} = u, \; x(0) = 1, \tag{4.36}$$

$$u + 1 \geq 0, \; 1 - u \geq 0, \tag{4.37}$$

$$x \geq 0. \tag{4.38}$$

Note that this problem is the same as Example 2.3, except for the nonnegativity constraint (4.38).

Solution The Hamiltonian is

$$H = -x + \lambda u,$$

which implies the optimal control to have the form

$$u^*(x, \lambda) = \text{bang}[-1, 1; \lambda], \; \text{whenever } x > 0. \tag{4.39}$$

When $x = 0$, we impose $\dot{x} = u \geq 0$ in order to insure that (4.38) holds. Therefore, the optimal control on the state constraint boundary is

$$u^*(x, \lambda) = \text{bang}[0, 1; \lambda], \; \text{whenever } x = 0. \tag{4.40}$$

Now we form the Lagrangian

$$L = H + \mu_1(u + 1) + \mu_2(1 - u) + \eta u,$$

where μ_1, μ_2, and η satisfy the complementary slackness conditions

$$\mu_1 \geq 0, \quad \mu_1(u + 1) = 0, \tag{4.41}$$

$$\mu_2 \geq 0, \quad \mu_2(1 - u) = 0, \tag{4.42}$$

$$\eta \geq 0, \qquad \eta x = 0. \tag{4.43}$$

Furthermore, the optimal trajectory must satisfy

$$\frac{\partial L}{\partial u} = \lambda + \mu_1 - \mu_2 + \eta = 0. \tag{4.44}$$

From the Lagrangian we also get

$$\dot{\lambda} = -\frac{\partial L}{\partial x} = 1,\ \lambda(2^-) = \gamma \geq 0,\ \gamma x(2) = \ \lambda(2^-)x(2) = 0. \tag{4.45}$$

It is reasonable to guess that the optimal control u^* will be the one that keeps x^* as small as possible, subject to the state constraint (4.38). Thus,

$$u^*(t) = \begin{cases} -1, & t \in [0,1), \\ 0, & t \in [1,2]. \end{cases} \tag{4.46}$$

This gives

$$x^*(t) = \begin{cases} 1-t, & t \in [0,1), \\ 0, & t \in [1,2]. \end{cases}$$

To obtain $\lambda(t)$, let us first try $\lambda(2^-) = \gamma = 0$. Then, since $x^*(t)$ enters the boundary zero at $t = 1$, there are no jumps in the interval $(1, 2]$, and the solution for $\lambda(t)$ is

$$\lambda(t) = t - 2,\ \ t \in (1,2). \tag{4.47}$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 0$ on $(1, 2]$, we have $u^*(t) = 0$ by (4.40), as stipulated. Now let us see what must happen at $t = 1$. We know from (4.47) that $\lambda(1^+) = -1$. To obtain $\lambda(1^-)$, we see that $H(1^+) = -x^*(1^+) + \lambda(1^+)u^*(1^+) = 0$ and $H(1^-) = -x^*(1^-) + \lambda(1^-)u^*(1^-) = -\lambda(1^-)$. By equating $H(1^-)$ to $H(1^+)$ as required in (4.29), we obtain $\lambda(1^-) = 0$. Using now the jump condition on $\lambda(t)$ in (4.29), we get the value of the jump $\zeta(1) = \lambda(1^-) - \lambda(1^+) = 1 \geq 0$.

With $\lambda(1^-) = 0$, we can solve (4.45) to obtain

$$\lambda(t) = t - 1,\ \ t \in [0,1].$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 1-t > 0$ is positive on [0,1), we can use (4.39) to obtain $u^*(t) = -1$ for $0 \leq t < 1$, which is as stipulated in (4.46). In the time interval [0,1) by (4.42), $\mu_2 = 0$ since $u^* < 1$, and by (4.43), $\eta = 0$ because $x > 0$. Therefore, $\mu_1(t) = -\lambda(t) = 1 - t > 0$ for $0 \leq t < 1$, and this with $u^* = -1$ satisfies (4.41).

To complete the solution, we calculate the Lagrange multipliers in the interval [1,2]. Since $u^*(t) = 0$ on $t \in [1, 2]$, we have $\mu_1(t) = \mu_2(t) = 0$. Then, from (4.44) we obtain $\eta(t) = -\lambda(t) = 2 - t \geq 0$ which, with

$x^*(t) = 0$ satisfies (4.43). Thus, our guess $\gamma = 0$ is correct, and we do not need to examine the possibility of $\gamma > 0$. The graphs of $x^*(t)$ and $\lambda(t)$ are shown in Fig. 4.2. In Exercise 4.1, you are asked to redo Example 4.3 by guessing that $\gamma > 0$ and see that it leads to a contradiction with a condition of the maximum principle.

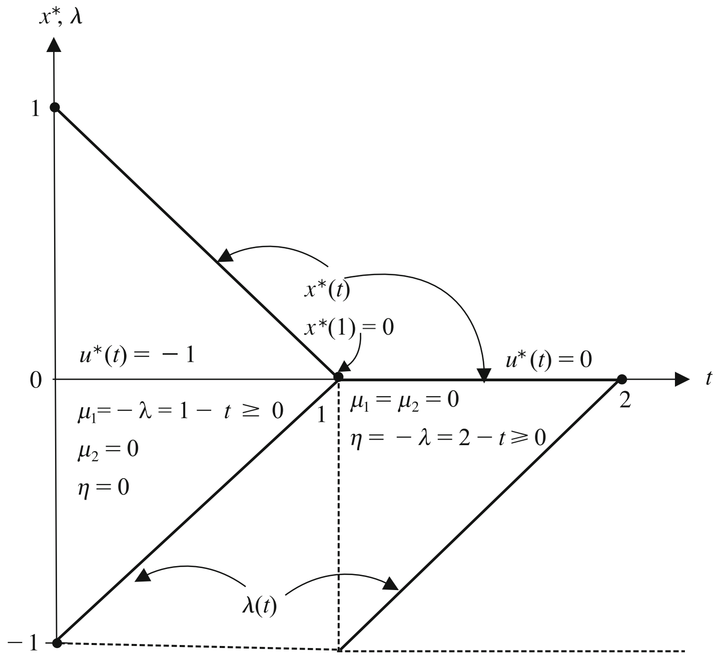

Figure 4.2: State and adjoint trajectories in Example 4.3

It should be obvious that if the terminal time were $T = 1.5$, the optimal control would be $u^*(t) = -1,\ t \in [0,1)$ and $u^*(t) = 0,\ t \in [1, 1.5]$. You are asked in Exercise 4.10 to redo the above calculations in this case and show that one now needs to have $\gamma = 1/2$. In Exercise 4.3, you are asked to solve a similar problem with $F = -u$.

Remark 4.5 Example 4.3 is a problem instance in which the state constraint is active at the terminal time. In instances where the initial state or the final state or both are on the constraint boundary, the maximum principle may *degenerate* in the sense that there is no nontrivial solution of the necessary conditions, i.e., $\lambda(t) \equiv 0,\ t \in [0, T]$, where T is the terminal time. See Arutyunov and Aseev (1997) or Ferreira and Vinter (1994) for conditions that guarantee a nontrivial solution for the multipliers.

Remark 4.6 It can easily be seen that Example 4.3 is a problem instance in which multipliers λ and μ_1 would not be unique if the jump condition on the Hamiltonian in (4.29) was not imposed. For references dealing with the issue of non-uniqueness of the multipliers and conditions under which the multipliers are unique, see Kurcyusz and Zowe (1979), Maurer (1977, 1979), Maurer and Wiegand (1992), and Shapiro (1997).

Example 4.4 The purpose here is to show that the solution obtained in Example 4.3 satisfies the sufficiency conditions of Theorem 4.1. For this we first obtain the direct adjoint variable

$$\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t) = \begin{cases} t-1, & t \in [0,1), \\ 0, & t \in [1,2). \end{cases}$$

It is easy to see that

$$H(x, u, \lambda^d(t), t) = \begin{cases} -x + (t-1)u, & t \in [0,1), \\ -x, & t \in [1,2], \end{cases}$$

is linear and hence concave in (x, u) at each $t \in [0,2]$. Functions

$$g(x, u, t) = \begin{pmatrix} u+1 \\ 1-u \end{pmatrix}$$

and

$$h(x) = x$$

are linear and hence quasiconcave in (x, u) and x, respectively. Functions $S \equiv 0$, $a \equiv 0$ and $b \equiv 0$ satisfy the conditions of Theorem 4.1 trivially. Thus, the solution obtained for Example 4.3 satisfies all conditions of Theorem 4.1, and is therefore optimal.

In Exercise 4.14, you are asked to use Theorem 4.1 to verify that the given solution there is optimal.

Example 4.5 Consider Example 4.3 with $T = 3$ and the terminal state constraint

$$x(3) = 1.$$

Solution Clearly, the optimal control u^* will be the one that keeps x as small as possible, subject to the state constraint (4.38) and the boundary condition $x(0) = x(3) = 1$. Thus,

$$u^*(t) = \begin{cases} -1, & t \in [0,1), \\ 0, & t \in [1,2], \\ 1, & t \in (2,3], \end{cases} \quad x^*(t) = \begin{cases} 1-t, & t \in [0,1), \\ 0, & t \in [1,2], \\ t-2, & t \in (2,3]. \end{cases}$$

For brevity, we will not provide the same level of detailed explanation as we did in Example 4.3. Rather, we will only compute the adjoint function and the multipliers that satisfy the optimality conditions. These are

$$\lambda(t) = \begin{cases} t-1, & t \in [0,1], \\ t-2, & t \in (1,3), \end{cases} \tag{4.48}$$

$$\mu_1(t) = \mu_2(t) = 0,\ \eta(t) = -\lambda(t),\ t \in [1,2], \tag{4.49}$$

$$\gamma = 0, \quad \beta = \lambda(2^-) = 1, \tag{4.50}$$

and the jump $\zeta(1) = 1 \geq 0$ so that

$$\lambda(1^-) = \lambda(1^+) + \zeta(1) \text{ and } H(1^-) = H(1^+). \tag{4.51}$$

Example 4.6 Introduce a discount rate $\rho > 0$ in Example 4.1 so that the objective function becomes

$$\max \left\{ J = \int_0^3 -e^{-\rho t} u dt \right\} \tag{4.52}$$

and re-solve using the indirect maximum principle (4.29).

Solution It is obvious that the optimal solution will remain the same as (4.5), shown also in Fig. 4.1.

With u^* and x^* as in (4.5), we must obtain $\lambda, \mu_1, \mu_2, \eta, \gamma$, and ζ so that the necessary optimality conditions (4.29) hold, i.e.,

$$H = -e^{-\rho t} u + \lambda u, \tag{4.53}$$

$$L = H + \mu_1 u + \mu_2 (3-u) + \eta[u + 2(t-2)], \tag{4.54}$$

$$L_u = -e^{-\rho t} + \lambda + \mu_1 - \mu_2 + \eta = 0, \tag{4.55}$$

$$\dot{\lambda} = -L_x = 0,\ \lambda(3^-) = 0, \tag{4.56}$$

$$\gamma[x^*(3) - 1 + (1-2)^2] = 0, \tag{4.57}$$

$$\mu_1 \geq 0,\ \mu_1 u = 0,\ \mu_2 \geq 0,\ \mu_2(3-u) = 0, \tag{4.58}$$

$$\eta \geq 0,\ \eta[x^*(t) - 1 + (t-2)^2] = 0, \tag{4.59}$$

and if λ is discontinuous at the entry time $\tau = 1$, then

$$\lambda(1^-) = \lambda(1^+) + \zeta(1),\ \zeta(1) \geq 0, \tag{4.60}$$

$$-e^{-\rho}u^*(1^-) + \lambda(1^-)u^*(1^-) = -e^{-\rho}u^*(1^+) + \lambda(1^+) - \zeta(1)(-2). \tag{4.61}$$

From (4.60), we obtain $\lambda(1^-) = e^{-\rho}$. This with (4.56) gives

$$\lambda(t) = \begin{cases} e^{-\rho}, & 0 \leq t < 1, \\ 0, & 1 \leq t \leq 3, \end{cases}$$

as shown in Fig. 4.3,

$$\mu_1(t) = \begin{cases} e^{-\rho t} - e^{-\rho}, & 0 \leq t < 1, \\ 0, & 1 \leq t \leq 2, \\ e^{-\rho t}, & 2 < t \leq 3, \end{cases} \quad \mu_2(t) = 0,\ 0 \leq t \leq 3,$$

and

$$\eta(t) = \begin{cases} 0, & 0 \leq t < 1, \\ e^{-\rho t}, & 1 \leq t \leq 2, \\ 0, & 2 < t \leq 3, \end{cases}$$

which, along with u^* and x^*, satisfy (4.29).

Note, furthermore, that λ is continuous at the exit time $t = 2$. At the entry time $\tau_1 = 1$, $\zeta(1) = e^{-\rho} \geq \eta(1^+) = e^{-\rho}$, so that (4.35) also holds. Finally, $\gamma = \eta(3^-) = 0$.

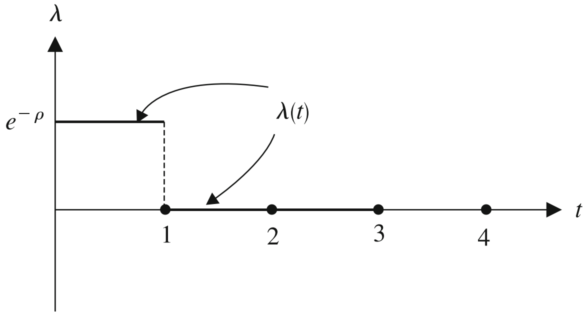

Figure 4.3: Adjoint trajectory for Example 4.4

4.6 Current-Value Maximum Principle: Indirect Method

Just as the necessary condition (3.42) represents the current-value formulation corresponding to (3.12), we can, when first-order pure state constraints are present, also state the current-value formulation of the necessary conditions (4.29). As in Sect. 3.3, with $F(x, u, t) = \phi(x, u)e^{-\rho t}$, $S(x, T) = \psi(x)e^{-\rho T}$, and $\rho > 0$, the objective function in the problem (4.11) is replaced by

$$\max \left\{ J = \int_0^T \phi(x, u)e^{-\rho t} dt + \psi[x(T)]e^{-\rho T} \right\}.$$

With the Hamiltonian H as defined in (3.35), we can write the Lagrangian as

$$L[x, u, \lambda, \mu, \eta] := H + \mu g + \eta h^1 = \phi + \lambda f + \mu g + \eta h^1.$$

We can now state the current-value form of the maximum principle, giving the necessary conditions for u^* (with the state trajectory x^*) to be optimal. These conditions are that there exist an adjoint function λ, which may be discontinuous at each entry or contact time, multiplier functions $\mu, \alpha, \beta, \gamma, \eta$, and a jump parameter $\zeta(\tau)$ at each τ where λ^d is discontinuous, such that the following (4.62) holds:

$$
\boxed{
\begin{array}{l}
\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0, \text{ satisfying constraints} \\
g(x^*, u^*, t) \geq 0,\ h(x^*(t), t) \geq 0, \text{and the terminal constraints} \\
a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0; \\
\dot{\lambda} = \rho\lambda - L_x[x^*, u^*, \lambda, \mu, \eta, t] \\
\text{with the transversality conditions} \\
\lambda(T^-) = \psi_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) \\
\qquad\qquad + \gamma h_x(x^*(T), T), \text{ and} \\
\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma \geq 0,\ \gamma h(x^*(T), T) = 0; \\
\text{the Hamiltonian maximizing condition} \\
H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \\
\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying} \\
g[x^*(t), u, t] \geq 0, \text{ and} \\
h_i^1(x^*(t), u, t) \geq 0 \text{ whenever } h_i(x^*(t), t) = 0, i = 1, 2, \cdots, p; \\
\text{the jump conditions at any entry/contact time } \tau, \\
\text{where } \lambda \text{ is discontinuous, are} \\
\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau) \text{ and} \\
H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau] \\
\qquad\qquad\qquad\qquad - \zeta(\tau) h_t(x^*(\tau), \tau); \\
\text{the Lagrange multipliers } \mu(t) \text{ are such that} \\
\partial L/\partial u|_{u=u^*(t)} = 0,\ dH/dt = dL/dt = \partial L/\partial t + \rho\lambda f, \\
\text{and the complementary slackness conditions} \\
\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0, \\
\eta(t) \geq 0,\ \eta(t) h(x^*(t), t) = 0, \text{ and} \\
\zeta(\tau) \geq 0,\ \zeta(\tau) h(x^*(\tau), \tau) = 0 \text{ hold.}
\end{array}
}
\tag{4.62}
$$

If $T \in [T_1, T_2],\ 0 \le T_1 < T_2 < \infty$, is also a decision variable, then if T^* is the optimal terminal time, then the optimal solution x^*, u^*, T^* must satisfy (4.62) with T replaced by T^* and the condition

$$H[x^*(T^*), u^*(T^{*-}), \lambda^d(T^{*-}), T^*] - \rho\psi[x^*(T^*), T^*] + \alpha a_T[x^*(T^*), T^*]$$
$$+\beta b_T[x^*(T^*), T^*] + \gamma^d h_T[x^*(T^*), T^*] \begin{cases} \le 0 & \text{if } T^* = T_1, \\ = 0 & \text{if } T^* \in (T_1, T_2), \\ \ge 0 & \text{if } T^* = T_2. \end{cases} \quad (4.63)$$

Derivation of (4.63) starting from (4.15) is similar to that of (3.44) from (3.15).

Remark 4.7 The current-value version of (4.34) in Remark 4.3 is $\dot{\eta}(t) \le \rho\eta(t)$ and (4.35).

The infinite horizon problem with pure and mixed constraints can be stated as (3.97) with an additional constraint (4.7). As in Sect. 3.6, the conditions in (4.62) except the transversality condition on the adjoint variable are still necessary for optimality. As for the sufficiency conditions, an analogue of Theorem 4.1 holds, subject to the discussion on infinite horizon transversality conditions in Sect. 3.6.

We conclude this chapter with the following cautionary remark.

Remark 4.8 While various subsets of conditions specified in the maximum principles (4.13), (4.29), or (4.62) have been proved in the literature, proofs of the entire results are still not available. For this reason, Hartl (1995) call (4.13), (4.29), or (4.62) as *informal theorems*. Seierstad and Sydsæter (1987) call them *almost necessary conditions* since, very rarely, problems arise where the optimal solution requires more complicated multipliers and adjoint variables than those specified in this chapter.

Exercises for Chapter 4

E 4.1 Rework Example 4.3 by guessing that $\gamma > 0$, and show that it leads to a contradiction with a condition of the maximum principle.

E 4.2 Rework Example 4.3 with terminal time $T = 1/2$.

E 4.3 Change the objective function of Example 4.3 as follows:

$$\max \left\{ J = \int_0^2 (-u)dt \right\}.$$

Re-solve and show that the solution is not unique.

E 4.4 Specialize the maximum principle (4.29) for the nonnegativity state constraint of the form

$$x(t) \geq 0 \text{ for all } t \text{ satisfying } 0 \leq t \leq T,$$

in place of $h(x,t) \geq 0$ in (4.7).

E 4.5 Consider the problem:

$$\max \left\{ J = \int_0^T (-x)dt \right\}$$

subject to

$$\dot{x} = -u - 1, \; x(0) = 1,$$
$$x(t) \geq 0, \; 0 \leq u(t) \leq 1.$$

Show that

(a) If $T = 1$, there is exactly one feasible and optimal solution.
(b) If $T > 1$, then there is no feasible solution.
(c) If $0 < T < 1$, then there is a unique optimal solution.
(d) If the control constraint is $0 \leq u(t) \leq K$, there is a unique optimal solution for every $K \geq 1$ and $T = 1/2$.
(e) The value of the objective in (d) increases as K increases.
(f) If the control constraint in (d) is $u(t) \geq 0$, then the optimal control is an impulse control defined by the limit of the solution in (e).

E 4.6 Impose the constraint $x \geq 0$ on Exercise 3.16(b) to obtain the problem:

$$\max \left\{ J = \int_0^4 (-x)dt \right\}$$

subject to

$$\dot{x} = u, \;\; x(0) = 1, \;\; x(4) = 1,$$

$$u + 1 \geq 0, \;\; 1 - u \geq 0,$$

$$x \geq 0.$$

Find the optimal trajectories of the control variable, the state variable, and other multipliers. Also, graph these trajectories.

E 4.7 Transform the problem (4.11) with the pure constraint of type (4.7) to a problem with the nonnegativity constraint of type (4.9).

Hint: Define $y = h(x, t)$ as an additional state variable. Recall that we have assumed (4.7) to be a first-order constraint.

E 4.8 Consider a two-reservoir system such as that shown in Fig. 4.4, where $x_i(t)$ is the volume of water in reservoir i and $u_i(t)$ is the rate of discharge from reservoir i at time t. Thus,

$$\dot{x}_1(t) = -u_1(t), \qquad x_1(0) = 4,$$

$$\dot{x}_2(t) = u_1(t) - u_2(t), \quad x_2(0) = 4.$$

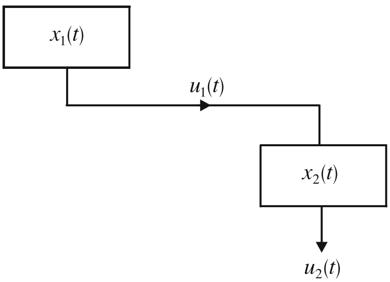

Figure 4.4: Two-reservoir system of Exercise 4.8

Solve the problem of maximizing

$$J = \int_0^{10} [(10 - t)u_1(t) + tu_2(t)]dt$$

subject to the above state equations and the constraints

$$0 \leq u_i(t) \leq 1, \;\; x_i(t) \geq 0 \text{ for all } t \in [0, 10].$$

Also compute the optimal value of the objective function.

Hint: Guess the optimal solution and verify it by using the Lagrangian form of the maximum principle.

E 4.9 *An Inventory Control Problem.* Solve

$$\max_P \int_0^T -\left(hI + \frac{P^2}{2}\right) dt$$

subject to

$$\dot{I} = P - S,\ I(0) = I_0 > \frac{S^2}{2h},$$

and the control and the pure state inequality constraints

$$P \geq 0 \text{ and } I \geq 0,$$

respectively. Assume that $S > 0$ and $h > 0$ are constants and T is sufficiently large. Note that I represents inventory, P represents production rate, and S represents demand. The constraints on P and I mean that production must be nonnegative and backlogs are not allowed, respectively.

Hint: By T being sufficiently large, we mean $T > I_0/S + S/(2h)$.

E 4.10 Redo Example 4.3 with $T = 1.5$.

E 4.11 Redo Example 4.6 using the current-value maximum principle (4.62) in Sect. 4.6.

E 4.12 For this exercise only, assume that $h(x,t) \geq 0$ in (4.7) is a second-order constraint, i.e., $r = 2$. Transform the problem to one with nonnegativity constraints. Use the result in Exercise 4.4 to derive a maximum principle for problems with second-order constraints.

Hint: As in Exercise 4.7, define $y = h$. In addition, define yet another state variable $z = \dot{y} = dh/dt$. Note further that this procedure can be generalized to handle problems with rth-order constraints for any positive integer r.

E 4.13 Re-solve Example 4.6 when $\rho < 0$.

E 4.14 Consider the following problem:

$$\min\left\{J=\int_0^5 u dt\right\}$$

subject to the state equation

$$\dot{x}=u-x,\ x(0)=1,$$

and the control and state constraints

$$0\le u\le 1,\ x(t)\ge 0.7-0.2t.$$

Use the sufficiency conditions in Theorem 4.1 to verify that the optimal control for the problem is

$$u^*(t)=\begin{cases}0, & 0\le t\le\theta,\\ 0.5-0.2t, & \theta<t\le 2.5,\\ 0, & 2.5<t\le 5,\end{cases}$$

where $\theta\approx 0.51626$. Sketch the optimal state trajectory $x^*(t)$ for the problem.

E 4.15 In Example 4.6, let $t^{\pm}(x)=2\pm\sqrt{1-x}$. Show that the value function

$$V(x,t)=\begin{cases}-\frac{2e^{-2\rho}+2(\rho\sqrt{1-x}-1)e^{-\rho(2-\sqrt{1-x})}}{\rho^2}, & \text{for } x<1, 0\le t\le t^-(x),\\ 0, & \text{for } x\ge 1 \text{ or } t^+(x)\le t\le 3.\end{cases}$$

Note that $V(x,t)$ is not defined for $x<1,\ t^-(x)<t\le 3$. Show furthermore that for the given initial condition $x(0)=0$, the marginal valuation is

$$V_x(x^*(t),t)=\lambda^d(t)=\lambda(t)+\eta(t)=\begin{cases}e^{-\rho}, & \text{for } t\in[0,1),\\ e^{-\rho t}, & \text{for } t\in[1,2],\\ 0, & \text{for } t\in(2,3].\end{cases}$$

In this case, it is interesting to note that the marginal valuation is discontinuous at the constraint exit time $t=2$.

E 4.16 Show in Example 4.3 that the value function

$$V(x,t) = \begin{cases} -x^2/2, & \text{for } x \le 2-t,\ 0 \le t \le 2, \\ -2x+2-2t+xt+t^2/2, & \text{for } x > 2-t,\ 0 \le t \le 2. \end{cases}$$

Then verify that for the given initial condition $x(0) = 1$,

$$V_x(x^*(t),t) = \lambda^d(t) = \lambda(t) + \eta(t) = \begin{cases} t-1, & \text{for } t \in [0,1), \\ 0, & \text{for } t \in [1,2]. \end{cases}$$

E 4.17 Rework Example 4.5 by using the direct maximum principle (4.13).

E 4.18 Solve the linear inventory control problem of minimizing

$$\int_0^T (cP(T) + hI(t))dt$$

subject to

$$\dot{I}(t) = P(t) - S, \quad I(0) = 1,$$
$$P(t) \ge 0 \text{ and } I(t) \ge 0, \quad t \in [0,T],$$

where $P(t)$ denotes the production rate and $I(t)$ is the inventory level at time t and where c, h and S are positive constants and the given terminal time $T > \sqrt{2S}$.

E 4.19 A machine with quality $x(t) \ge 0$ produces goods with $ax(t)$ dollars per unit time at time t. The quality deteriorates at the rate δ, but the decay can be slowed by a preventive maintenance $u(t)$ as follows:

$$\dot{x} = u - \delta x,\ x(0) = x_0 > 0.$$

Obtain the optimal maintenance rate $u(t),\ 0 \le t \le T$, so as to maximize

$$\int_0^T (ax - u)dt$$

subject to $u \in [0, \bar{u}]$ and $x \le \bar{x}$, where $\bar{u} > \delta\bar{x},\ a > \delta$, and $\bar{x} > x_0$.

Hint: Solve first the problem without the state constraint $x \le \bar{x}$. You will need to treat two cases: $\delta T \le \ln a - \ln(a-\delta)$ and $\delta T > \ln a - \ln(a-\delta)$.

E 4.20 Maximize

$$J = \int_0^3 (u - x)dt$$

subject to

$$\dot{x} = 1 - u, \ x(0) = 2,$$

$$0 \leq u \leq 3, \ x + u \leq 4, \ x \geq 0.$$

E 4.21 Maximize

$$J = \int_0^2 (1 - x)dt$$

subject to

$$\dot{x} = u, \ x(0) = 1,$$

$$-1 \leq u \leq 1, \ x \geq 0.$$

E 4.22 Maximize

$$J = \int_0^3 (4 - t)u dt$$

subject to

$$\dot{x} = u, \ x(0) = 0, \ x(3) = 3,$$

$$0 \leq u \leq 2, \ 1 + t - x \geq 0.$$

E 4.23 Maximize

$$J = -\int_0^4 e^{-t}(u - 1)^2 dt$$

subject to

$$\dot{x} = u, \quad x(0) = 0,$$

$$x \leq 2 + e^{-3}.$$

E 4.24 Solve the following problem:

$$\max\left\{J=\int_0^2 (2u-x)dt\right\}$$

$$\dot{x}=-u, \quad x(0)=e,$$

$$-3\le u\le 3, \quad x-u\ge 0, \quad x\ge t.$$

E 4.25 Solve the following problem:

$$\max\left\{J=\int_0^3 -2x_1 dt\right\}$$

$$\dot{x}_1=x_2, \quad x_1(0)=2,$$

$$\dot{x}_2=u, \quad x_2(0)=0,$$

$$x_1\ge 0.$$

E 4.26 Re-solve Example 4.6 with the control constraint (4.3) replaced by $0\le u\le 1$.

E 4.27 Solve explicitly the following problem:

$$\max\left\{J=-\int_0^2 x(t)dt\right\}$$

subject to

$$\dot{x}(t)=u(t),\ x(0)=1,$$

$$-a\ \le\ u(t)\le b,\ a>1/2,\ b>0,$$

$$x(t)\ \ge\ t-2.$$

Obtain $x^*(t),\ u^*(t)$ and all the required multipliers.

E 4.28 Minimize

$$\int_0^T \frac{1}{2}(x^2+c^2u^2)dt$$

subject to

$$\dot{x}=u,\ \ x(0)=x_0>0,\ \ x(T)=0,$$

$$h_1(x,t)=x-a_1+b_1t\ge 0,$$

$$h_2(x,t)=a_2-b_2t-x\ge 0,$$

where $a_i,b_i>0, a_2>x_0>a_1$, and $a_2/b_2>a_1/b_1$; see Fig. 4.5. The optional path must begin at x_0 on the x-axis, stay in the shaded area, and end on the t-axis.

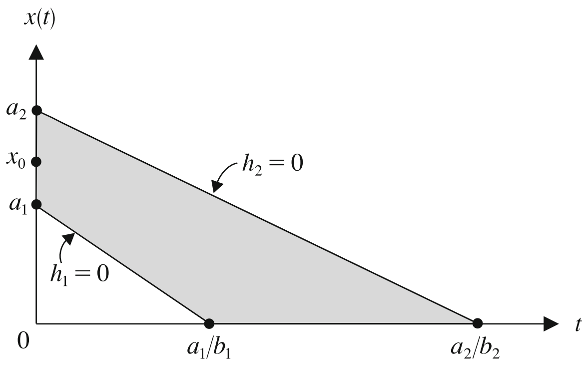

Figure 4.5: Feasible space for Exercise 4.28

(a) First, assume that the problem parameters are such that the optimal solution $x^*(t)$ satisfies $h_1(x^*(t), t) > 0$ for $t \in [0, T]$. Show that

$$x^*(t) = k_1 e^{t/c} + k_2 e^{-t/c},$$

where k_1 and k_2 are the constants to be determined. Write down the two conditions that would determine the constants. Also, illustrate graphically the optimal state trajectory.

(b) How would your solution change if the problem parameters do not satisfy the condition in (a)? Characterize and graphically illustrate the optimal state trajectory.

E 4.29 With $a > 0,\ b > 0$, and $\dot{\gamma}(t)/\gamma(t) = -\rho(t) < 0$,

$$\max_{u,T} \left\{ J = \int_0^T \frac{a}{b}(1 - e^{-bu(t)})\gamma(t)dt \right\}$$

subject to

$$\dot{x} = -u, \ \ x(0) = x_0 > 0 \text{ given},$$

and the constraint

$$x(t) \geq 0.$$

Obtain the expressions satisfied by the optimal terminal time T^*, the optimal control $u^*(t),\ 0 \leq t \leq T^*$, and the optimal state trajectory $x^*(t),\ 0 \leq t \leq T^*$. Furthermore, obtain them explicitly in the special case when $\rho(t) = \rho > 0$, a constant positive discount rate.

E 4.30 Set $\rho = 0$ in the solution of Example 4.6 and obtain $\lambda, \gamma, \eta, \zeta(1)$ for the undiscounted problem. Then use the transformation formulas (4.30)–(4.33) on these and the fact that $\zeta(2) = 0$ to obtain $\lambda^d, \gamma^d, \eta^d$, and $\zeta^d(1)$ and $\zeta^d(2)$, and show that they are the same as those obtained in Example 4.2 along with $\zeta^d(1) = 0$, which holds trivially.

E 4.31 Consider a finite-time economy in which production can be used for consumption as well as investment, but production also pollutes. The state equations for the capital stock K and stock of pollution W are

$$\dot{K} = suK,\ K(0) = K_0,$$

$$\dot{W} = uK - \delta W,\ W(0) = W_0,$$

where a fraction s of the production output uK is invested, with u denoting the capacity utilization rate. The control constraints are

$$0 \leq s \leq 1,\ 0 \leq u \leq 1,$$

and the state constraint

$$W \leq \bar{W}$$

implies that the pollution stock cannot exceed the upper bound $\bar{W}$.

The aim of the economy is to choose s and u so as to maximize the consumption utility

$$\int_0^T (1-s)uK dt.$$

Assume that $W_0 < \bar{W},\ T > 1$ and $W_0 - K_0/\delta)e^{-\delta T} + K_0/\delta < \bar{W}$, which means that even with $s(t) \equiv 0$, the pollution stock never reaches $\bar{W}$ even with $u(t) \equiv 1$.

Chapter 5

Applications to Finance

An important area of finance involves making decisions regarding investment and dividend policies over time and ways to finance them. Among the ways of financing such policies are: issuing equity, retaining earnings, borrowing money, etc. It is possible to model such situations as optimal control problems; see, for example, Davis and Elzinga (1971), Elton and Gruber (1975), and Sethi (1978b). Some of these models are simple to analyze and they yield useful insights.

In this chapter we deal with two different problems relating to a firm. The cash balance problem, in its simplest form, is a problem of controlling the level of a firm's cash balances to meet its demand for cash at minimum total cost. The problem of the optimal equity financing of a corporate firm, a central problem in finance, is that of determining the optimal dividend path along with new equity issued over time in order to maximize the value of the firm. Although we only deal with deterministic problems in this chapter, some of the more important problems in finance involve uncertainty. Thus, their optimization requires the use of stochastic optimal control theory or stochastic programming. A brief introduction to stochastic optimal control theory will be provided in Chap. 12, together with an application to a stochastic consumption-investment problem and references.

In the next section, we introduce a simple cash balance problem as a tutorial. This model is based on Sethi and Thompson (1970) and Sethi (1973d, 1978c). We will be especially interested in the financial

S. P. Sethi, *Optimal Control Theory*,
https://doi.org/10.1007/978-3-319-98237-3_5

interpretations for the various functions such as the Hamiltonian and the adjoint functions that arise in the course of the analysis.

5.1 The Simple Cash Balance Problem

Consider a firm which has a known demand for cash over time. To satisfy this cash demand, the firm must keep some cash on hand, assumed to be held in a checking account at a bank. If the firm keeps too much cash, it loses money in terms of opportunity cost, in that it can earn higher returns by buying securities such as bonds. On the other hand, if the cash balance is too small, the firm has to sell securities to meet the cash demand and thus incur a broker's commission. The problem then is to find the tradeoff between the cash and security balances.

5.1.1 The Model

To formulate the optimal control problem we introduce the following notation:

T = the time horizon,

$x(t)$ = the cash balance in dollars at time t,

$y(t)$ = the security balance in dollars at time t,

$d(t)$ = the instantaneous rate of demand for cash; $d(t)$ can be positive or negative,

$u(t)$ = the rate of sale of securities in dollars; a negative sales rate means a rate of purchase,

$r_1(t)$ = the interest rate earned on the cash balance,

$r_2(t)$ = the interest rate earned on the security balance,

α = the broker's commission in dollars per dollar's worth of securities bought or sold; $0 < \alpha < 1$.

The state equations are

$$\dot{x} = r_1 x - d + u - \alpha|u|, \quad x(0) = x_0, \tag{5.1}$$

$$\dot{y} = r_2 y - u, \quad y(0) = y_0, \tag{5.2}$$

and the control constraints are

$$-U_2 \le u(t) \le U_1, \tag{5.3}$$

where U_1 and U_2 are nonnegative constants. The objective function is:

$$\max\{J = x(T) + y(T)\} \tag{5.4}$$

subject to (5.1)–(5.3). Note that the problem is in the linear Mayer form.

5.1.2 Solution by the Maximum Principle

Introduce the adjoint variables λ_1 and λ_2 and define the Hamiltonian function

$$H = \lambda_1(r_1 x - d + u - \alpha|u|) + \lambda_2(r_2 y - u). \tag{5.5}$$

The adjoint variables satisfy the differential equations

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = -\lambda_1 r_1, \quad \lambda_1(T) = 1, \tag{5.6}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\lambda_2 r_2, \quad \lambda_2(T) = 1. \tag{5.7}$$

It is easy to solve these, respectively, as

$$\lambda_1(t) = e^{\int_t^T r_1(\tau)d\tau} \tag{5.8}$$

and

$$\lambda_2(t) = e^{\int_t^T r_2(\tau)d\tau}. \tag{5.9}$$

The interpretations of these solutions are also clear. Namely, $\lambda_1(t)$ is the future value (at time T) of one dollar held in the cash account from time t to T and, likewise, $\lambda_2(t)$ is the future value of one dollar invested in securities from time t to T. Thus, the adjoint variables have natural interpretations as the actuarial evaluations of competitive investments at each point of time.

Let us now derive the optimal policy by choosing the control variable u to maximize the Hamiltonian in (5.5). In order to deal with the absolute value function we write the control variable u as the difference of two nonnegative variables, i.e.,

$$u = u_1 - u_2, \ u_1 \geq 0, \ u_2 \geq 0. \tag{5.10}$$

Recall that this method was suggested in Remark 3.12 in Sect. 3.7. In order to make $u = u_1$ when u_1 is strictly positive, and $u = -u_2$ when u_2 is strictly positive, we also impose the quadratic constraint

$$u_1 u_2 = 0, \tag{5.11}$$

so that at most one of u_1 and u_2 can be nonzero. However, the optimal properties of the solution will automatically cause this constraint to be satisfied. The reason is that the broker's commission must be paid on every transaction, which makes it not optimal to simultaneously buy and sell securities. Given (5.10) and (5.11) we can write

$$|u| = u_1 + u_2. \tag{5.12}$$

Also, since $u \in [-U_1, U_2]$ from (5.3), we must have $u_1 \leq U_1$ and $u_2 \leq U_2$. In view of (5.10), the control constraints on the variables u_1 and u_2 are

$$0 \leq u_1 \leq U_1 \text{ and } 0 \leq u_2 \leq U_2. \tag{5.13}$$

We can now substitute (5.10) and (5.12) into the Hamiltonian (5.5) and reproduce the part that depends on control variables u_1 and u_2, and denote it by W. Thus,

$$W = u_1[(1-\alpha)\lambda_1 - \lambda_2] - u_2[(1+\alpha)\lambda_1 - \lambda_2]. \tag{5.14}$$

Maximizing the Hamiltonian (5.5) with respect to $u \in [-U_1, U_2]$ is the same as maximizing W with respect to $u_1 \in [0, U_1]$ and $u_2 \in [0, U_2]$. But W is linear in u_1 and u_2 so that the optimal strategy is bang-bang and is as follows:

$$u^* = u_1^* - u_2^*, \tag{5.15}$$

where

$$u_1^* = \text{bang}[0, U_1; (1-\alpha)\lambda_1 - \lambda_2], \tag{5.16}$$

$$u_2^* = \text{bang}[0, U_2; -(1+\alpha)\lambda_1 + \lambda_2]. \tag{5.17}$$

Since $u_1(t)$ represents the rate of sale of securities, (5.16) says that the optimal policy is: sell at the maximum allowable rate if the future value of a dollar less the broker's commission (i.e., the future value of $(1-\alpha)$ dollars) is greater than the future value of a dollar's worth of securities; and do not sell if these future values are in reverse order. In case the future value of a dollar less the commission is exactly equal to the future value of a dollar's worth of securities, then the optimal policy is undetermined. In fact, we are indifferent as to the action taken, and this is called *singular control*. Similarly, $u_2(t)$ represents the purchase of securities. Here we buy, do not buy, or are indifferent, if the future value of a dollar plus the commission is less than, greater than, or equal to the future value of a dollar's worth of securities, respectively.

Note that if

$$(1-\alpha)\lambda_1(t) \geq \lambda_2(t),$$

then

$$(1+\alpha)\lambda_1(t) > \lambda_2(t),$$

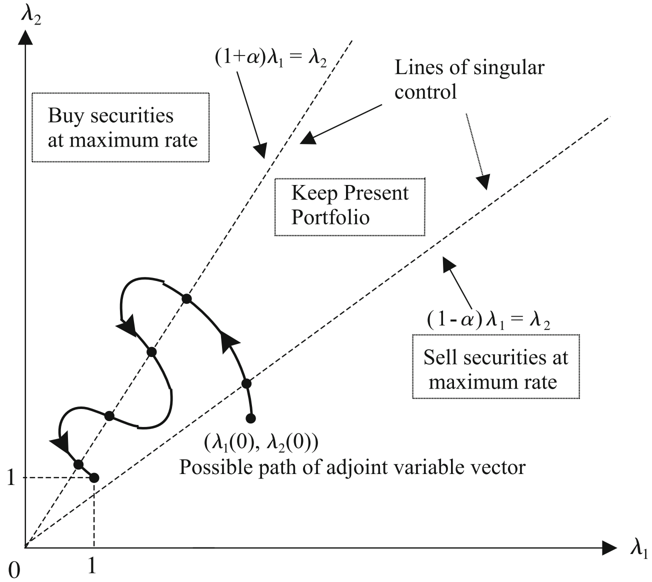

Figure 5.1: Optimal policy shown in (λ_1, λ_2) space

so that if $u_1(t) > 0$, then $u_2(t) = 0$. Similarly, if

$$(1 + \alpha)\lambda_1(t) \leq \lambda_2(t),$$

then

$$(1 - \alpha)\lambda_1(t) < \lambda_2(t),$$

so that if $u_2(t) > 0$, then $u_1(t) = 0$. Hence, with the optimal policy, the relation (5.11) is always satisfied.

Figure 5.1 illustrates the optimal policy at time t. The first quadrant is divided into three areas which represent different actions (including no action) to be taken. The dotted lines represent the singular control manifolds. A possible path of the vector $(\lambda_1(t), \lambda_2(t))$ of the adjoint variables is shown in Fig. 5.1 also. Note that on this path, there is one period of selling, two periods of buying, and three periods of inactivity. Note also that the final point on the path is $(1, 1)$, since the terminal values $\lambda_1(T) = \lambda_2(T) = 1$, and therefore, the last interval is always characterized by inactivity.

Another way to represent the optimal path is in the $(t, \lambda_2/\lambda_1)$ space. The path of $(\lambda_1(t), \lambda_2(t))$ shown in Fig. 5.1 corresponds to the path of $\lambda_2(t)/\lambda_1(t)$ over time shown in Fig. 5.2.

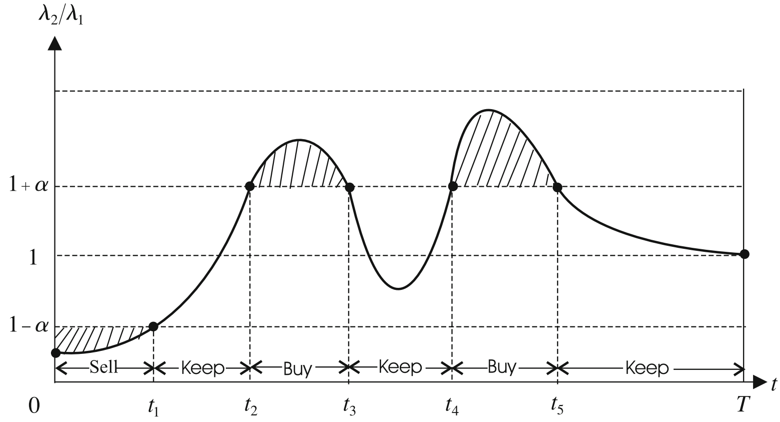

Figure 5.2: Optimal policy shown in $(t, \lambda_2/\lambda_1)$ space

Perhaps a more realistic version of the cash balance problem is to disallow overdraft on the bank account. This means imposing the pure state constraint $x(t) \geq 0$. In addition, if short selling of securities is not permitted, then we must also have $y(t) \geq 0$. These extensions give rise to pure state constraints treated in Chap. 4. In Exercise 5.2 you are asked to formulate such an extension and write the indirect maximum principle (4.29) for it. Exercises 5.3 and 5.4 present instances where it is easy to guess the optimal solutions. In Exercise 5.5, you are asked to show if the guessed solution in Exercise 5.4 satisfies the maximum principle (4.29). It is in Chap. 6 that we discuss in detail an application of the indirect maximum principle (4.29) for solving a problem called the wheat trading model.

5.2 Optimal Financing Model

In the present section, we discuss a model of a corporate firm which must finance its investments by an optimal combination of retained earnings and external equity. The model to be discussed is due to Krouse and Lee (1973), with corrections and extensions due to Sethi (1978b). The problem of the optimal financing of the firm can be formulated as an optimal control problem. The formulations, such as those of Davis (1970), Krouse (1972), and Krouse and Lee (1973), permit the firm to finance its investments by retained earnings, debt, and/or external equity in various proportions which may vary over time. Note that earnings not retained are paid out as dividends to the firm's stockholders.

For reasons of simplicity and ease of its solution, the model analyzed here does not permit debt as a source of financing, but does permit retained earnings and external equity to be used in any proportions.

5.2.1 The Model

In order to formulate the model, we use the following notation:

$y(t)$	=	the value of the firm's assets or invested capital at time t,
$x(t)$	=	the current earnings rate in dollars per unit time at time t,
$u(t)$	=	the external or new equity financing expressed as a multiple of current earnings; $u \geq 0$,
$v(t)$	=	the fraction of current earnings retained, i.e., $1 - v(t)$ represents the rate of dividend payout; $0 \leq v(t) \leq 1$,
$1 - c$	=	the proportional floatation (i.e., transaction) cost for external equity; c a constant, $0 \leq c < 1$,
ρ	=	the continuous discount rate (assumed constant); known commonly as the stockholder's required rate of return, or the cost of capital,
r	=	the actual rate of return (assumed constant) on the firm's invested capital; $r > \rho$,
g	=	the upper bound on the growth rate of the firm's assets,
T	=	the planning horizon; $T < \infty$ ($T = \infty$ in Sect. 5.2.4) .

Given these definitions, the current earnings rate is $x = ry$. The rate of change in the current earnings rate is given by

$$\dot{x} = r\dot{y} = r(cu + v)x, \; x(0) = x_0. \tag{5.18}$$

Furthermore, the upper bound on the rate of growth of the assets implies the following constraint on the control variables:

$$\dot{y}/y = (cu + v)x/(x/r) = r(cu + v) \leq g. \tag{5.19}$$

Finally, the objective of the firm is to maximize its value, which is taken to be the present value of the future dividend stream accruing to the shares outstanding at time zero. To derive this expression, note that

$$\int_0^T (1 - v)xe^{-\rho t} dt$$

represents the present value of total dividends issued by the firm. A portion of these dividends go to the new equity, which under the assumption of an efficient market will get a rate of return exactly equal to the discount rate ρ. This should therefore be equal to the present value

$$\int_0^T uxe^{-\rho t}dt$$

of the external equity raised over time.

Thus, the net present value of the total future dividends that accrue to the initial shares is the difference of the previous two expressions, i.e.,

$$J = \int_0^T e^{-\rho t}(1 - v - u)xdt; \tag{5.20}$$

see Miller and Modigliani (1961) and Sethi (1996) for further discussion. Note that in the case of a finite horizon, a more realistic objective function would include a salvage value or bequest term $S[x(T)]$. This is not very difficult to incorporate. See Exercise 5.12 where the bequest function is linear. We will also solve the infinite horizon problem (i.e., $T = \infty$) after we have solved the finite horizon problem.

Remark 5.1 An intuitive interpretation of (5.20) is that the value J of the firm is the present value of the cash flows (dividends) going out from the firm to the society less the present value of the cash flows (new equity) coming from the society into the firm.

The optimal control problem is to choose u and v over time so as to maximize J in (5.20) subject to (5.18), the constraints (5.19), $u \geq 0$, and $0 \leq v \leq 1$. For convenience, we restate this problem as

$$\begin{cases} \max_{u,v} \left\{ J = \int_0^T e^{-\rho t}(1 - v - u)xdt \right\} \\ \text{subject to} \\ \dot{x} = r(cu + v)x, \quad x(0) = x_0, \\ \text{and the control constraints} \\ cu + v \leq g/r, \quad u \geq 0, \quad 0 \leq v \leq 1. \end{cases} \tag{5.21}$$

5.2.2 Application of the Maximum Principle

This is a bilinear problem with two control variables which is a special case of Row (f) in Table 3.3. The current-value Hamiltonian is

$$\begin{aligned} H &= (1-v-u)x + \lambda r(cu+v)x \\ &= [(cr\lambda - 1)u + (r\lambda - 1)v + 1]x, \end{aligned} \tag{5.22}$$

where the current-value adjoint variable λ satisfies

$$\dot{\lambda} = \rho\lambda - (1-v-u) - \lambda r(cu+v) \tag{5.23}$$

with the transversality condition

$$\lambda(T) = 0. \tag{5.24}$$

The first term in the Hamiltonian in (5.22) is the dividend payout rate to stockholders of record at time t. According to Sect. 2.2.1, λ is the marginal value (in time t dollars) of a unit change in the earnings rate at time t. Thus, $\lambda r(cu+v)x$ is the value at time t of the incremental earnings rate due to the investment of retained earnings vx and the net amount of external financing cux. This explains why we should maximize H with respect to u and v at each t. To interpret (5.23) as in Sect. 2.2.4, consider an earnings rate of one dollar at time t. It is worth λ, on which the stockholders expect a return of $\rho\lambda dt$ at time dt. In equilibrium this must be equal to the "capital gain" $d\lambda$, plus the immediate dividend $(1-v)dt$ less udt, the "claims" of the new stockholders, plus the value $\lambda r(cu+v)dt$ of the incremental earnings rate $r(cu+v)dt$ at time $t+dt$.

To specify the form of optimal policy, we rewrite the Hamiltonian as

$$H = [W_1 u + W_2 v + 1]x, \tag{5.25}$$

where

$$W_1 = cr\lambda - 1, \tag{5.26}$$

$$W_2 = r\lambda - 1. \tag{5.27}$$

Note first that the state variable x factors out so that the optimal controls are independent of the state variable. Second, since the Hamiltonian is linear in the two control variables, the optimal policy is a combination of generalized bang-bang and singular controls. Of course, the characterization of these optimal controls in terms of the adjoint variable λ will require solving a parametric linear programming problem at each

Table 5.1: Characterization of optimal controls with $c < 1$

	Conditions on W_1, W_2	Case A: $g \le r$ Subcases	Case B: $g > r$ Subcases	Optimal controls	Characterization
(1)	$W_2 < 0$	A1	B1	$u^* = 0,\ v^* = 0$	Generalized bang-bang
(2)	$W_2 = 0$	A2	B2	$u^* = 0,$ $0 \le v^* \le \min[1, g/r]$	Singular
(3)	$W_2 > 0$	A3	–	$u^* = 0,\ v^* = g/r$	Generalized bang-bang
(4)	$W_1 < 0,\ W_2 > 0$	–	B3	$u^* = 0, v^* = 1$	Generalized bang-bang
(5)	$W_1 = 0$	–	B4	$0 \le u^* \le (g-r)/rc,$ $v^* = 1$	Singular
(6)	$W_1 > 0$	–	B5	$u^* = (g-r)/rc, v^* = 1$	Generalized bang-bang

instant of time t. The Hamiltonian maximization problem can be stated as follows:

$$\begin{cases} \max\limits_{u,v} \{W_1 u + W_2 v\} \\ \text{subject to} \\ u \ge 0, \quad 0 \le v \le 1, \quad cu + v \le g/r. \end{cases} \tag{5.28}$$

Obviously, the constraint $v \le 1$ becomes redundant if $g/r < 1$. Therefore, we have two cases:

Case A: $g \le r$ and **Case B:** $g > r$,

under each of which, we can solve the linear programming problem (5.28) graphically in a closed form. This is done in Figs. 5.3 and 5.4.

There are seven subcases shown in Fig. 5.3 and nine subcases on Fig. 5.4, but some of these subcases cannot occur. To see this, we note from our assumption $c < 1$ that

$$W_1 = cr\lambda - 1 < cr\lambda - c = cW_2,$$

which also gives $W_2 > 0$ if $W_1 = 0$. Thus, subcases A4–A7 and B6–B9 are ruled out. The remaining Subcases A1–A3 and B1–B5 are shown adjacent to the darkened lines in Figs. 5.3 and 5.4, respectively. In addition to $W_1 < cW_2$ and $W_1 = 0$ implying $W_2 > 0$, we see that $W_2 \leq 0$ implies $W_1 < 0$. In view of these, we can simply characterize Subcases A1 and B1 by $W_2 < 0$, A2 and B2 by $W_2 = 0$, A3 by $W_2 > 0$, B4 by $W_1 = 0$, and B5 by $W_1 > 0$, and use these simpler characterizations in our subsequent discussion. Keep in mind that Subcase B3 remains characterized as $W_1 < 0, W_2 > 0$.

In Table 5.1, we list the feasible cases, shown along the darkened lines in Figs. 5.3 and 5.4 and provide the form of the optimal control in each of these cases. The catalog of possible optimal control regimes shown in Table 5.1 gives the potential time-paths for the firm. What must be done to obtain the optimal path (given an initial condition) is to synthesize these subcases into an optimal sequence. This is carried out in the following section.

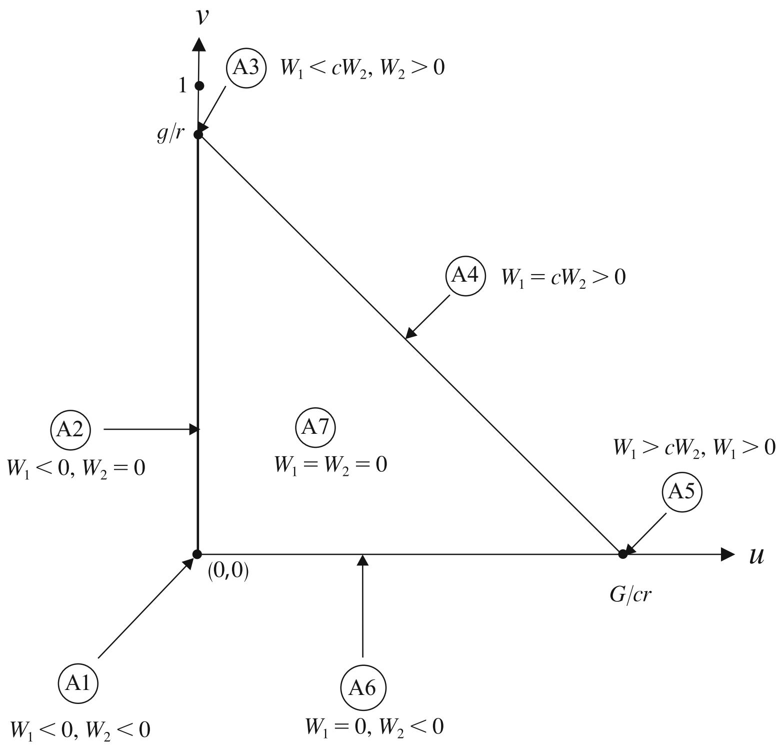

Figure 5.3: Case A: $g \leq r$

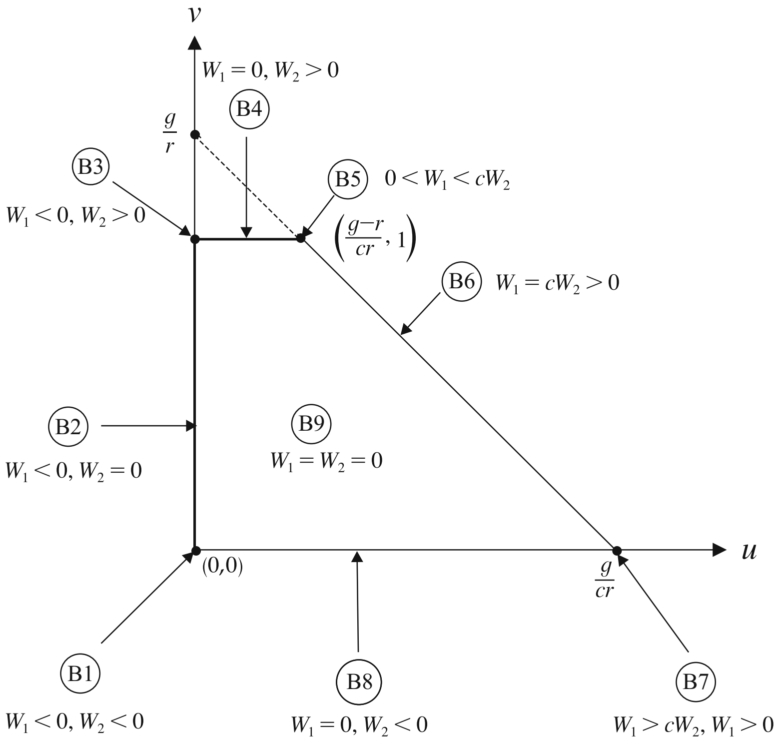

Figure 5.4: Case B: $g > r$

5.2.3 Synthesis of Optimal Control Paths

To obtain an optimal path, we must synthesize an optimal sequence of subcases. The usual procedure employed is that of the reverse-time construction, first developed by Isaacs (1965). Reverse time can only be defined for finite horizon problems. However, the infinite horizon solution can usually be inferred from the finite horizon solution if sufficient care is exercised. This will be done in Sect. 5.2.4.

Our analysis of the finite horizon problem (5.21) proceeds with the assumption that the terminal time T is assumed to be sufficiently large. We will make this assumption precise during our analysis. Moreover, we will discuss the solution when T is not sufficiently large in Remarks 5.2 and 5.4.

Define the reverse-time variable τ as

$$\tau = T - t,$$

so that

$$\overset{\circ}{y} = \frac{dy}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau} = -\dot{y}.$$

As a consequence, $\overset{\circ}{y} = -\dot{y}$, and the reverse-time versions of the state and adjoint equations (5.18) and (5.23), respectively, can be obtained by simply replacing $\dot{y}$ by $\overset{\circ}{y}$ and changing the signs of the right-hand sides. The transversality condition on the adjoint variable

$$\lambda(t = T) = \lambda(\tau = 0) = 0 \tag{5.29}$$

becomes the initial condition in the reverse-time sense. Furthermore, let us parameterize the terminal state by assuming that

$$x(t = T) = x(\tau = 0) = \alpha_A, \tag{5.30}$$

where α_A is a parameter to be determined.

From now on in this section, *everything is expressed in the reverse-time sense unless otherwise specified*. Using the definitions of $\overset{\circ}{x}$ and $\overset{\circ}{\lambda}$ and the conditions (5.30) and (5.29), we can write reverse-time versions of (5.18) and (5.23) as follows:

$$\overset{\circ}{x} = -r(cu + v)x, \; x(0) = \alpha_A, \tag{5.31}$$

$$\overset{\circ}{\lambda} = (1 - u - v) - \lambda\{\rho - r(cu + v)\}, \;\; \lambda(0) = 0. \tag{5.32}$$

This is the starting point for our switching point synthesis. First, we consider Case A.

Case A: $g \leq r$.

Note that the constraint $v \leq 1$ is superfluous in this case and the only feasible subcases are A1, A2, and A3. Since $\lambda(0) = 0$, we have $W_1(0) = W_2(0) = -1$ and, therefore, Subcase A1.

Subcase A1: $W_2 = r\lambda - 1 < 0$.

From Row (1) of Table 5.1, we have $u^* = v^* = 0$, which gives the state equation (5.31) and the adjoint equation (5.32) as

$$\overset{\circ}{x} = 0 \text{ and } \overset{\circ}{\lambda} = 1 - \rho\lambda. \tag{5.33}$$

With the initial conditions given in (5.29), the solutions for x and λ are

$$x(\tau) = \alpha_A \text{ and } \lambda(\tau) = (1/\rho)[1 - e^{-\rho\tau}]. \tag{5.34}$$

It is easy to see that because of the assumption $0 \le c < 1$, it follows that if $W_2 = r\lambda - 1 < 0$, then $W_1 = cr\lambda - 1 < 0$. Therefore, to remain in this subcase as τ increases, $W_2(\tau)$ must remain negative for some time as τ increases. From (5.34), however, $\lambda(\tau)$ is increasing asymptotically toward the value $1/\rho$ and therefore, $W_2(\tau)$ is increasing asymptotically toward the value $r/\rho - 1$. Since, we have assumed $r > \rho$, there exists a τ_1 such that $W_2(\tau_1) = (1 - e^{-\rho\tau_1})r/\rho - 1 = 0$. It is easy to compute

$$\tau_1 = (1/\rho)\ln[r/(r - \rho)]. \tag{5.35}$$

From this expression, it is clear that the firm leaves Subcase A1 provided $\tau_1 < T$. Moreover, this observation also makes precise the notion of a sufficiently large T in Case A by having $T > \tau_1$.

Remark 5.2 When T is not sufficiently large, i.e., when $T \le \tau_1$ in Case A, the firm stays in Subcase A1. The optimal solution in this case is $u^* = 0$ and $v^* = 0$, i.e., a policy of no investment.

Remark 5.3 Note that if we had assumed $r < \rho$, the firm would never have exited from Subcase A1 regardless of the value of T. Obviously, there is no use investing if the rate of return is less than the discount rate.

At reverse time τ_1, we have $W_2 = 0$ and $W_1 < 0$ and the firm, therefore, is in Subcase A2. Also, $\lambda(\tau_1) = 1/r$ since $W_2(\tau_1) = 0$.

Subcase A2: $W_2 = r\lambda - 1 = 0$.

In this subcase, the optimal controls

$$u^* = 0,\ 0 \le v^* \le g/r \tag{5.36}$$

from Row (3) of Table 5.1 are singular with respect to v. This case is termed singular because the Hamiltonian maximizing condition does not yield a unique value for the control v. In such cases, the optimal controls are obtained by conditions required to sustain $W_2 = 0$ for a finite time interval. This means we must have $\overset{\circ}{W} = 0$, which in turn implies $\overset{\circ}{\lambda} = 0$. To compute $\overset{\circ}{\lambda}$, we substitute (5.36) into (5.32) and obtain

$$\overset{\circ}{\lambda} = (1 - v^*) - \lambda[\rho - rv^*]. \tag{5.37}$$

Substituting $\lambda = 1/r$, its value at τ_1, in (5.37) and equating the right-hand side to zero we obtain

$$r = \rho \tag{5.38}$$

as a necessary condition required to maintain singularity over a finite time interval following τ_1. Condition (5.38) is fortuitous and will not generally hold. In fact we have assumed $r > \rho$. Thus, the firm will not stay in Subcase A2 for a nonzero time interval. Furthermore, since $r > \rho$, we have $\overset{\circ}{\lambda}(\tau_1) = (1 - \rho/r) > 0$. Therefore, W_2 is increasing from zero and becomes positive after τ_1. Thus, at τ_1^+ the firm switches to Subcase A3.

Subcase A3: $W_2 = r\lambda - 1 > 0$.

The optimal controls in this subcase from Row (2) of Table 5.1 are

$$u^* = 0, \;\; v^* = g/r. \tag{5.39}$$

The state and the adjoint equations are

$$\overset{\circ}{x} = -gx, \; x(\tau_1) = \alpha_A, \tag{5.40}$$

$$\overset{\circ}{\lambda} = (1 - g/r) - \lambda(\rho - g), \; \lambda(\tau_1) = 1/r, \tag{5.41}$$

with values at $\tau = \tau_1$ deduced from (5.34) and (5.35).

Since $\overset{\circ}{\lambda}(\tau_1) > 0$, λ is increasing at τ_1 from its value of $1/r$. A further examination of the behavior of $\lambda(\tau)$ as τ increases will be carried out under two different possible conditions: (i) $\rho > g$ and (ii) $\rho \leq g$.

(i) $\rho > g$: Under this condition, as λ increases, $\overset{\circ}{\lambda}$ decreases and becomes zero at a value obtained by equating the right-hand side of (5.41) to zero, i.e., at

$$\bar{\lambda} = \frac{1 - g/r}{\rho - g}. \tag{5.42}$$

This value $\bar{\lambda}$ is, therefore, an asymptote to the solution of (5.41) starting at $\lambda(\tau_1) = 1/r$. Since $r > \rho > g$ in this case,

$$\overline{W_2} = r\bar{\lambda} - 1 = \frac{r(1 - g/r)}{\rho - g} - 1 = \frac{r - \rho}{\rho - g} > 0, \tag{5.43}$$

which implies that the firm continues to stay in Subcase A3.

(ii) $\rho \leq g$: Under this condition, as $\lambda(\tau)$ increases, $\overset{\circ}{\lambda}(\tau)$ increases. So $W_2(\tau) = r\lambda(\tau) - 1$ continues to be greater than zero and the firm continues to remain in Subcase A3.

Remark 5.4 With $\rho \leq g$, note that $\lambda(\tau)$ increases to infinity as τ increases to infinity. This has important implications later when we deal with the solution of the infinite horizon problem.

Since the optimal decisions for $\tau \geq \tau_1$ have been found to be independent of α_A for T sufficiently large, we can sketch the solution for Case A in Fig. 5.5 starting with x_0. This also gives the value of

$$\alpha_A = x_0 e^{g(T-\tau_1)} = x_0 e^{gT}[1-\rho/r]^{g/\rho},$$

as shown in Fig. 5.5.

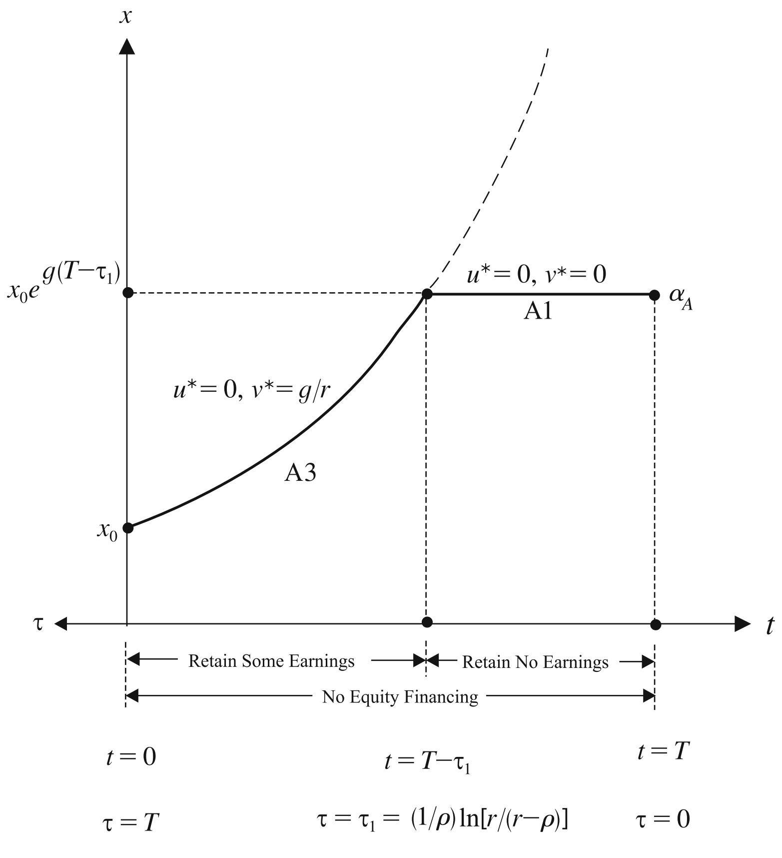

Figure 5.5: Optimal path for case A: $g \leq r$

Mathematically, we can now express the optimal controls and the optimal state, now in forward time, as

$$u^*(t) = 0,\ v^*(t) = g/r,\ x^*(t) = x_0 e^{gt},\ t \in [0, T-\tau_1], \tag{5.44}$$

$$u^*(t) = 0,\ v^*(t) = 0,\ x^*(t) = x_0 e^{g(T-\tau_1)},\ t \in (T-\tau_1, T], \tag{5.45}$$

As for $\lambda(t)$, from (5.34) we have

$$\lambda(t) = \frac{1}{\rho}[1 - e^{-\rho(T-t)}],\ t \in (T-\tau_1, T]. \tag{5.46}$$

For $t \in [0, T-\tau_1]$, we have from (5.41),

$$\dot{\lambda}(t) = \lambda(\rho - g) - (1 - g/r),\ \lambda(T-\tau_1) = 1/r. \tag{5.47}$$

Following Sect. A.1, we can solve this equation as

$$\lambda(t) = \frac{1}{r}e^{-(\rho-g)(T-\tau_1-t)} + \frac{1-g/r}{\rho-g}[1 - e^{-(\rho-g)(T-\tau_1)}],\ t \in [0, T-\tau_1]. \tag{5.48}$$

In this solution for Case A, there is only one switching point provided that T is sufficiently large (i.e., $T > \tau_1$ in this case). The switching time $t = T - \tau_1$ has an interesting economic interpretation. Namely, it requires at least τ_1 units of time to retain a dollar of earnings to be worthwhile for investment. That means, it pays to invest as much of the earnings as feasible before $T - \tau_1$, and it does not pay to invest any earnings after $T - \tau_1$. Thus, $T - \tau_1$ is the point of indifference between retaining earnings or paying dividends out of earnings. To see this directly, let us suppose the firm retains one dollar of earnings at $T - \tau_1$. Since this is the last time that any of the earnings invested will be worthwhile, it is obvious (because all earnings are paid out) that the dollar just invested at $T - \tau_1$ yields dividends at the rate r from $T - \tau_1$ to T. The value of this dividend stream in terms of $(T - \tau_1)$-dollars is

$$\int_0^{\tau_1} re^{-\rho s} ds = \frac{r}{\rho}[1 - e^{-\rho\tau_1}], \tag{5.49}$$

which must be equated to one dollar to find the indifference point. Equating (5.49) to 1 yields precisely the value of τ_1 given in (5.35).

With this interpretation of τ_1, we conclude that enough earnings must be retained so as to make the firm grow exponentially at the maximum rate of g until $t = T - \tau_1$. After this time, all of the earnings are paid out and the firm stops growing. Since $g \le r$ (assumed for Case A), the growth in the first part of the solution can be financed entirely from retained earnings. Thus, there is no need to resort to more expensive external equity financing. The latter will not be true, however, in Case B when $g > r$, which we now discuss.

Case B: $g > r$.

Since $g/r > 1$, the constraint $v \leq 1$ in Case B is relevant. The feasible subcases are B1, B2, B3, B4, and B5 shown adjacent to the darkened lines in Fig. 5.4. As in Case A, it is obvious that the firm starts (in the reverse-time sense) in Subcase B1. Recall that T is assumed to be sufficiently large here as well. This statement in Case B will be made precise in the course of our analysis. Furthermore, the solution when T is not sufficiently large in Case B will be discussed in Remark 5.4.

Subcase B1: $W_2 = r\lambda - 1 < 0$.

The analysis of this subcase is the same as Subcase A1. As in that subcase, the firm switches out at time $\tau = \tau_1$ to Subcase B2.

Subcase B2: $W_2 = r\lambda - 1 = 0$.

In this subcase, the optimal controls

$$u^* = 0, \;\; 0 \leq v^* \leq 1 \tag{5.50}$$

from Row (3) of Table 5.1 are singular with respect to v. As before in Subcase A2, the singular case cannot be sustained for a finite time because of our assumption $r > \rho$. As in Subcase A2, W_2 is increasing at τ_1 from zero and becomes positive after τ_1. Thus, at τ_1^+, the firm finds itself in Subcase B3.

Subcase B3: $W_1 = cr\lambda - 1 < 0, W_2 = r\lambda - 1 > 0$.

The optimal controls in this subcase are

$$u^* = 0, \;\; v^* = 1, \tag{5.51}$$

as shown in Row (5) of Table 5.1. The state and the adjoint equations are

$$\overset{\circ}{x} = -rx, \;\; x(\tau_1) = \alpha_B \tag{5.52}$$

with α_B, a parameter to be determined, and

$$\overset{\circ}{\lambda} = \lambda(r - \rho), \;\; \lambda(\tau_1) = 1/r. \tag{5.53}$$

Obviously, earnings are growing exponentially at rate r and $\lambda(\tau)$ is increasing at rate $(r - \rho)$ as τ increases from τ_1. Since $\lambda(\tau_1) = 1/r$, we have

$$\lambda(\tau) = (1/r)e^{(r-\rho)(\tau-\tau_1)} \text{ for } \tau \geq \tau_1. \tag{5.54}$$

As λ increases, W_1 increases and becomes zero at a time τ_2 defined by

$$W_1(\tau_2) = cr\lambda(\tau_2) - 1 = ce^{(r-\rho)(\tau-\tau_1)} - 1 = 0, \tag{5.55}$$

which, in turn, gives

$$\tau_2 = \tau_1 + \frac{1}{r-\rho} \ln\left(\frac{1}{c}\right). \tag{5.56}$$

At τ_2^+, the firm switches to Subcase B4.

Before proceeding to Subcase B4, let us observe that in Case B, we can now define T to be sufficiently large when $T > \tau_2$. See Remark 5.4 when $T \leq \tau_2$.

Subcase B4: $W_1 = cr\lambda - 1 = 0$.

In Subcase B4, the optimal controls are

$$0 \leq u^* \leq (g-r)/rc, \quad v^* = 1. \tag{5.57}$$

From Row (6) in Table 5.1, these controls are singular with respect to u. To maintain this singular control over a finite time period, we must keep $W_1 = 0$ in the interval. This means we must have $\overset{\circ}{W}_1(\tau_2) = 0$, which, in turn, implies $\overset{\circ}{\lambda}(\tau_2) = 0$. To compute $\overset{\circ}{\lambda}$, we substitute (5.57) into (5.32) and obtain

$$\overset{\circ}{\lambda} = -u^* - \lambda\{\rho - r(cu^* + 1)\}. \tag{5.58}$$

At $\tau_2, W_1(\tau_2) = 0$ gives $\lambda(\tau_2) = 1/rc$. With this in (5.58), its right-hand side equals zero only when $r = \rho$. But we have assumed $r > \rho$ throughout Sect. 5.2, and therefore a singular path cannot be sustained for $\tau_2 > 0$, and the firm will not stay in Subcase B4 for a finite amount of time. Furthermore, from (5.58), we have

$$\overset{\circ}{\lambda}(\tau_2) = \frac{r-\rho}{rc} > 0, \tag{5.59}$$

which implies that λ is increasing and therefore, W_1 is increasing. Thus at τ_2^+, the firm switches to Subcase B5.

Subcase B5: $W_1 = cr\lambda - 1 > 0$.

The optimal controls in this subcase from Row (4) of Table 5.1 are

$$u^* = \frac{g-r}{rc}, \ \ v^* = 1. \tag{5.60}$$

Then from (5.31) and (5.32), the reverse-time state and the adjoint equations are

$$\overset{\circ}{x} = -gx, \tag{5.61}$$

$$\overset{\circ}{\lambda} = -(\frac{g-r}{rc}) + \lambda(g-\rho). \tag{5.62}$$

Since $\overset{\circ}{\lambda}(\tau_2) > 0$ from (5.59), $\lambda(\tau)$ is increasing at τ_2 from its value $\lambda(\tau_2) = 1/rc > 0$. Furthermore, we have $g > r$ in Case B, which together with $r > \rho$, assumed throughout Sect. 5.2, makes $g > \rho$. This implies that the second term on the right-hand side of (5.62) is increasing. Moreover, the second term dominates the first term for $\tau > \tau_2$, since $\lambda(\tau_2) = 1/(rc) > 0$, and $r > \rho$ and $g > r$ imply $g - \rho > g - r > 0$. Thus, $\overset{\circ}{\lambda}(\tau) > 0$ for $\tau > \tau_2$, and $\lambda(\tau)$ increases with τ. Therefore, the firm continues to stay in Subcase B5.

Remark 5.5 Note that $\lambda(\tau)$ in Case B increases without bound as τ becomes large. This will have important implications when dealing with the infinite horizon problem in Sect. 5.2.4.

As in Case A, we can obtain this optimal solution explicitly in forward time, and we ask you to do this in Exercise 5.9. We now can sketch the complete solution for Case B in Fig. 5.6. In this solution, there are two switching points instead of just one as in Case A. The reason for two switching points becomes quite clear when we interpret the significance of τ_1 and τ_2. It is obvious that τ_1 has the same meaning as before. Namely, if τ_1 is the remaining time to the horizon, the firm is indifferent between retaining a dollar of earnings or paying it out as dividends. Intuitively, it seems that since external equity is more expensive than retained earnings as a source of financing, investment financed by external equity requires more time to be worthwhile. That is,

$$\tau_2 - \tau_1 = \frac{1}{r-\rho} \ln\left(\frac{1}{c}\right) > 0 \tag{5.63}$$

as obtained in (5.56), should be the time required to compensate for the floatation cost of external equity. Let us see why.

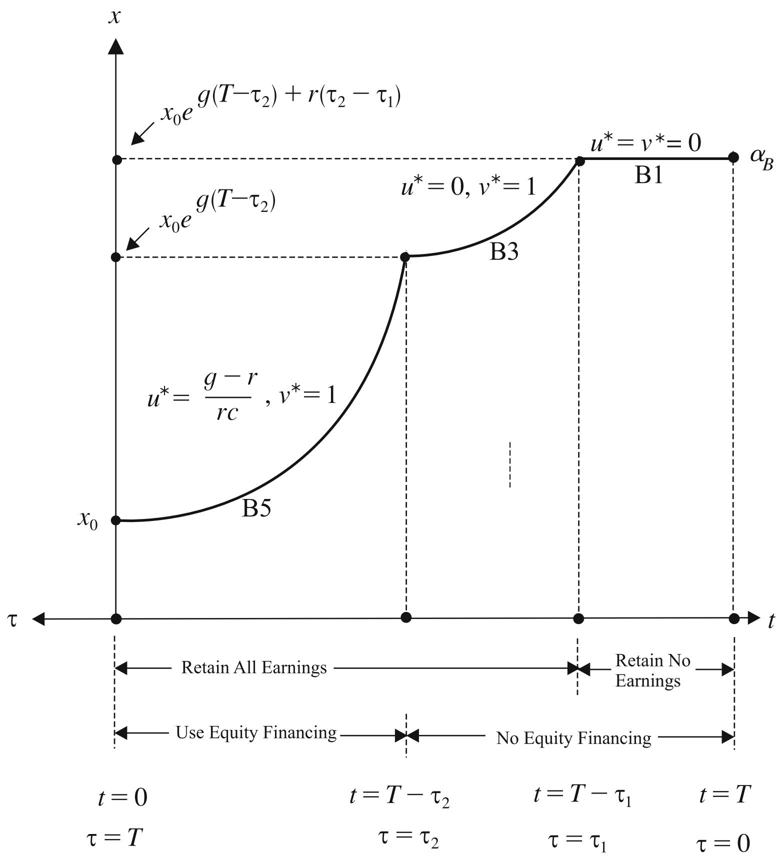

Figure 5.6: Optimal path for case B: $g > r$

When the firm issues a dollar's worth of stock at time $t = T - \tau_2$, it incurs a future dividend obligation in the amount of one $(T - \tau_2)$-dollar, even though the capital acquired is only c dollars because of the floatation cost $(1 - c)$. Since we are attempting to find the breakeven time for external equity, it is obvious that retaining all of the earnings for investment is still profitable. Thus, there is no dividend from $(T - \tau_2)$ to $(T - \tau_1)$, and the firm grows at the rate r. Therefore, this investment of c dollars at time $(T - \tau_2)$ grows into $ce^{r(\tau_2 - \tau_1)}$ dollars at time $(T - \tau_1)$. From the point of view of a buyer of the stock at time $(T - \tau_2)$, since no dividend is paid until time $(T - \tau_1)$ and since the stockholder's required rate of return is ρ, the firm's future dividend obligation at time $(T - \tau_1)$ is $e^{\rho(\tau_2 - \tau_1)}$ in terms of $(T - \tau_1)$-dollars. But then we must have

$$e^{\rho(\tau_2 - \tau_1)} = ce^{r(\tau_2 - \tau_1)}, \tag{5.64}$$

which can be rewritten precisely as (5.63). Moreover, the firm is marginally indifferent between investing any costless retained earnings at time $(T-\tau_1)$ or paying it all out as dividends. This also means that the firm will be indifferent between having the new available capital of $ce^{r(\tau_2-\tau_1)}$ dollars at time $(T-\tau_1)$ as a result of issuing a dollar's worth of stock at time $(T-\tau_2)$, or not having it. Thus, we can conclude that the firm is indifferent between issuing a dollar's worth of stock at time $(T-\tau_2)$ or not issuing it. This means that before time $(T-\tau_2)$, it pays to issue stocks at as large a rate as feasible, and after time $(T-\tau_2)$, it does not pay to issue any external equity at all.

We have now provided an intuitive justification of (5.63) and concluded that all earnings must be retained from time $(T-\tau_2)$ to $(T-\tau_1)$. Because $r>\rho$, it follows that the excess return on the proceeds c from the new stock issue is $ce^{r(\tau_2-\tau_1)}-ce^{\rho(\tau_2-\tau_1)}$ at time $(T-\tau_1)$. When discounted this amount back to time $(T-\tau_2)$, we can use (5.63) or (5.64) to see that

$$\left[ce^{r(\tau_2-\tau_1)}-ce^{\rho(\tau_2-\tau_1)}\right]e^{-\rho(\tau_2-\tau_1)}=ce^{\ln(1/c)}-c=1-c.$$

Thus, the excess return from time $(T-\tau_2)$ to $(T-\tau_1)$ recovers precisely the floatation cost.

Remark 5.6 When T is not sufficiently large, i.e., when $T<\tau_2$ in Case B, the optimal solution is the same as in Remark 5.1 when $T\le\tau_1$. If $\tau_1<T\le\tau_2$, then the optimal solution is $u^*=0$ and $v^*=1$ until $t=T-\tau_1$. For $t>T-\tau_1$, the optimal solution is $u^*=0$ and $v^*=0$.

Having completely solved the finite horizon case, we now turn to the infinite horizon case.

5.2.4 Solution for the Infinite Horizon Problem

As indicated in Sect. 3.6 for the infinite horizon case, the transversality condition must be changed to

$$\lim_{t\to\infty}e^{-\rho t}\lambda(t)=0. \tag{5.65}$$

Furthermore, this condition may no longer be a necessary condition; see Sect. 3.6. It is a sufficient condition for optimality however, in conjunction with the other sufficiency conditions stated in Theorem 2.1.

As demonstrated in Example 3.7, a common method of solving an infinite horizon problem is to take the limit as $T \to \infty$ of the finite horizon solution and then prove that the limiting solution obtained solves the infinite horizon problem. The proof is important because the limit of the solution may or may not solve the infinite horizon problem. The proof is usually based on the sufficiency conditions of Theorem 2.1, modified slightly as indicated above for the infinite horizon case.

We now analyze the infinite horizon case following the above procedure. We start with Case A.

Case A: $g \le r$.

Let us first consider the case $\rho > g$ and examine the solution in forward time obtained in (5.44)–(5.48) as T goes to infinity. Clearly (5.45) and (5.46) disappear, and (5.44) and (5.48) can be written as

$$u^*(t) = 0,\ v^*(t) = g/r,\ x^*(t) = x_0 e^{gt}, t \ge 0, \tag{5.66}$$

$$\lambda(t) = \frac{1 - g/r}{\rho - g} = \bar{\lambda},\ t \ge 0. \tag{5.67}$$

Clearly $\lambda(t)$ satisfies (5.65). Furthermore,

$$W_2(t) = r\bar{\lambda} - 1 = \frac{r - \rho}{\rho - g} > 0,\ t \ge 0,$$

which implies that the firm is in Subcase A3 for $t \ge 0$. The maximum principle holds, and (5.66) and (5.67) represent an optimal solution for the infinite horizon problem. Note that the assumption $\rho > g$ together with our overall assumption that $\rho < r$ gives $g < r$ so that $1 - v^* > 0$, which means a constant fraction of earnings is being paid as dividends.

Note that the value of the adjoint variable $\bar{\lambda}$ in this case is a constant and its form is reminiscent of Gordon's classic formula; see Gordon (1962). In the control theory framework, the value of $\bar{\lambda}$ represents the marginal worth per additional unit of earnings. Obviously, a unit increase in earnings will mean an increase of $1 - v^*$ or $1 - g/r$ units in dividends. This, of course, should be capitalized at a rate equal to the discount rate less the growth rate (i.e., $\rho - g$), which is precisely Gordon's formula.

For $\rho \le g$, it is clear from (5.48) that $\lambda(t)$ does not satisfy (5.65). A moment's reflection shows that for $\rho \le g$, the objective function can be made infinite. For example, any control policy with earnings growing at rate q, $\rho \leq q \leq g$, coupled with a partial dividend payout, i.e., a *constant* v such that $0 < v < 1$, gives an infinite value for the objective function. That is, with $u^* = 0, v^* = q/r < 1$, we have

$$J = \int_0^\infty e^{-\rho t}(1 - u^* - v^*)x^* dt = \int_0^\infty e^{-\rho t}(1 - q/r)x_0 e^{qt} = \infty.$$

Since there are many policies which give an infinite value to the objective function, the choice among them may be decided on subjective grounds. We will briefly discuss only the constant (over time) optimal policies. If $g < r$, then the rate of growth q may be chosen in the closed interval $[\rho, g]$; if $g = r$, then q may be chosen in the half-open interval $[\rho, r)$. In either case, the choice of a low rate of growth (i.e., a high proportional dividend payout) would mean a higher dividend rate (in dollars per unit time) early in time, but a lower dividend rate later in time because of the slower growth rate. Similarly the choice of high growth rate means the opposite in terms of dividend payments in dollars per unit time.

To conclude, we note that for $\rho \leq g$ in Case A, the limiting solution of the finite case is an optimal solution for the infinite horizon problem in the sense that the objective function becomes infinite. However, this will not be the situation in Case B; see also Remark 5.7.

Case B: $g > r$.

The limit of the finite horizon optimal solution is to grow at the maximum allowable growth rate with

$$u = \frac{g - r}{rc} \text{ and } v = 1$$

all the way. Since τ_1 disappears in the limit, the stockholders will never collect dividends. The firm has become an infinite *sink* for investment. In fact, the limiting solution is a *pessimal* solution because the value of the objective function associated with it is zero. From the point of view of optimal control theory, this can be explained as before in Case A when $\rho \leq g$. In Case B, we have $g > r$ so that (since $r > \rho$ throughout the chapter) we have $\rho < g$. For this, as noted in Remark 5.5, $\lambda(\tau)$ increases without bound as τ increases and, therefore, (5.64) does not have a solution.

As in Case A with $\rho < g$, any control policy with earnings growing at rate $q \in [\rho, g]$ coupled with a constant v, $0 < v < 1$, has an infinite value for the objective function.

In summary, we note that the only nondegenerate case in the infinite horizon problem is when $\rho > g$. In this case, which occurs only in Case A, the policy of maximum allowable growth is optimal. On the other hand, when $\rho \le g$, whether in Case A or B, the infinite horizon problem has nonunique policies with infinite values for the objective function.

Before solving a numerical example, we will make an interesting remark concerning Case B.

Remark 5.7 Let (u_T^*, v_T^*) denote the optimal control for the finite horizon problem in Case B. Let (u_∞^*, v_∞^*) denote any optimal control for the infinite horizon problem in Case B. We already know that $J(u_\infty^*, v_\infty^*) = \infty$. Define an infinite horizon control (u_∞, v_∞) by extending (u_T^*, v_T^*) as follows:

$$(u_\infty, v_\infty) = \lim_{T\to\infty} (u_T^*, v_T^*).$$

We now note that for our model in Case B, we have

$$\lim_{T\to\infty} J(u_T^*, v_T^*) = \infty \text{ and } J(\lim_{T\to\infty} (u_T^*, v_T^*)) = J(u_\infty, v_\infty) = 0. \quad (5.68)$$

Obviously (u_∞, v_∞) is *not* an optimal control for the infinite horizon problem. Since the two terms in (5.68) are not equal, we can say in technical terms that $J(u, v)$, regarded as a mapping, is not a *closed* mapping. However, if we introduce a salvage value $Bx(T),\ B > 0$, for the finite horizon problem, then the new objective function,

$$J_B(u,v) = \begin{cases} \int_0^T e^{-\rho t}(1-u-v)xdt + Bx(T)e^{-\rho T}, & \textit{if } T < \infty, \\ \int_0^\infty e^{-\rho t}(1-u-v)xdt + \lim_{T\to\infty}\{Bx(T)e^{-\rho T}\}, & \textit{if } T = \infty, \end{cases}$$

is a closed mapping in the sense that

$$\lim_{T\to\infty} J_B(u_T^*, v_T^*) = \infty \text{ and } J_B(\lim_{T\to\infty} (u_T^*, v_T^*)) = J_B(u_\infty, v_\infty) = \infty$$

for the modified model.

Example 5.1 We will now assign numbers to the various parameters in the optimal financing problem in order to compute the optimal solution. Let

$$x_0 = 1000/\text{month},\ T = 60 \text{ months},$$

$$r = 0.15,\ \rho = 0.10,\ g = 0.05,\ c = 0.98.$$

Solution Since $g \leq r$, the problem belongs to Case A. We compute

$$\tau_1 = \frac{1}{\rho}\ln[r/(r-\rho)] = 10\ln 3 \approx 11 \text{ months.}$$

The optimal controls for the problem are

$$u^* = 0, \quad v^* = g/r = 1/3, \quad t \in [0, 49),$$
$$u^* = 0, \quad v^* = 0, \quad t \in [49, 60],$$

and the optimal state trajectory is

$$x(t) = \begin{cases} 1000e^{0.05t}, & t \in [0, 49), \\ 1000e^{2.45}, & t \in [49, 60]. \end{cases}$$

The value of the objective function is

$$\begin{aligned} J^* &= \int_0^{49} e^{-0.1t}(1 - 1/3)(1000)e^{0.05t}dt + \int_{49}^{60} 1000e^{2.45} \cdot e^{-0.1t}dt \\ &= 12,578.75. \end{aligned}$$

Note that the infinite horizon problem is well defined in this case, since $g < \rho$ and $g < r$. The optimal controls are

$$u^* = 0, v^* = g/r = 1/3,$$

and

$$J = \int_0^{\infty} e^{-0.1t}(2/3)(1000)e^{0.05t}dt = 2000/0.15 = 13,333\frac{1}{3}.$$

In Exercise 5.14, you are asked to extend the optimal financing model to allow for debt financing. Exercise 5.15 requires you to reformulate the optimal financing model (5.21) with decisions expressed in dollars per unit of time rather than in terms relative to x. Exercise 5.16 extends the model to allow the rate of return on the assets to decrease as the assets grow.

Exercises for Chapter 5

E 5.1 Find the optimal policies for the simple cash balance model (Sects. 5.1.1 and 5.1.2) with $x_0 = 2$, $y_0 = 2$, $U_1 = U_2 = 5$, $T = 1$, $\alpha = 0.01$, and the following specifications for the interest rates:

(a) $r_1(t) = 1/2,\ r_2(t) = 1/3$.
(b) $r_1(t) = t/2,\ r_2(t) = 1/3$.
(c) Sketch the optimal policy in (b) in the $(t,\ \lambda_2/\lambda_1)$ space, like in Fig. 5.2.

E 5.2 Formulate the extension of the model in Sect. 5.1.1 when overdraft and short selling are disallowed in the following two cases: (a) $\alpha = 0$ and (b) $\alpha > 0$. State the maximum principle (4.29) as it applies to these cases.

Hint: Adjoin the control constraints to the Hamiltonian in forming the Lagrangian. For (b), write $u = u_1 - u_2$ as in (5.10).

E 5.3 It is possible to guess the optimal solution for Exercise 5.2 when $\alpha = 0,\ T = 10,\ x_0 = 0,\ y_0 = 3$,

$$r_1(t) = \begin{cases} 0 & \text{for } 0 \le t < 5, \\ 0.3 & \text{for } 5 \le t \le 10, \end{cases}$$

$$r_2(t) = 0.1 \quad \text{for } 0 \le t \le 10,$$

and $U_1 = U_2 = \infty$ (allowing for impulse controls). Show that the optimum policy remains the same for each $\alpha \in [0, 1 - 1/e]$.

Hint: Use an elementary compound interest argument.

E 5.4 Do the following for Exercise 5.3 with $U_1 = U_2 = 1$, so that the control constraints are $-1 \le u \le 1$.

(a) Give reasons why the solution shown in Fig. 5.7 is optimal.
(b) Compute $f(t^*)$ in terms of t^*.
(c) Compute J in terms of t^*.
(d) Find t^* that maximizes J by setting $dJ/dt^* = 0$.

Hint: Because this is a long and tedious calculus problem, you may wish to use *Mathematica* or MAPLE to solve this problem.

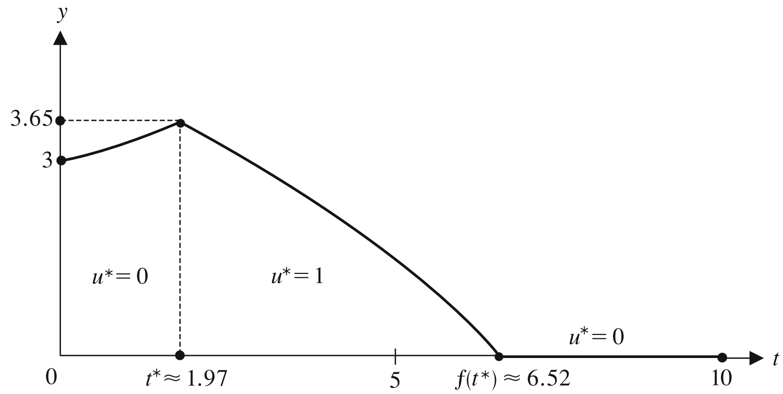

Figure 5.7: Solution for Exercise 5.4

E 5.5 For the solution found in Exercise 5.4, show by using the maximum principle (4.29) that the adjoint trajectories are:

$$\lambda_1(t) = \begin{cases} \lambda_1(0) = e^{1.5}, & 0 \le t \le 5, \\ \lambda_1(5)e^{-0.3(t-5)} = e^{3-0.3t}, & 5 \le t \le 10, \end{cases}$$

and

$$\lambda_2(t) = \begin{cases} \lambda_2(0)e^{-0.1t^*} = e^{1.5+0.1(t^*-t)}, & 0 \le t \le f(t^*) \approx 6.52, \\ \frac{2}{3} + \frac{1}{3}e^{3-0.3t}, & f(t^*) < t \le 10, \end{cases}$$

where $t^* \approx 1.97$. Sketches of these functions are shown in Fig. 5.8.

E 5.6 Argue that as the lower and upper bounds on u go to $-\infty$ and $+\infty$ in Exercise 5.4, respectively, t^* goes to 0 and $f(t^*)$ goes to 5. Show that this solution is consistent with the guess in Exercise 5.3. Finally, find the corresponding impulse solution and show that it satisfies the maximum principle as applied in Exercise 5.2.

E 5.7 Discuss the optimal equity financing model of Sect. 5.2.1 when $c = 1$. Show that only one control variable is needed. Then solve the problem.

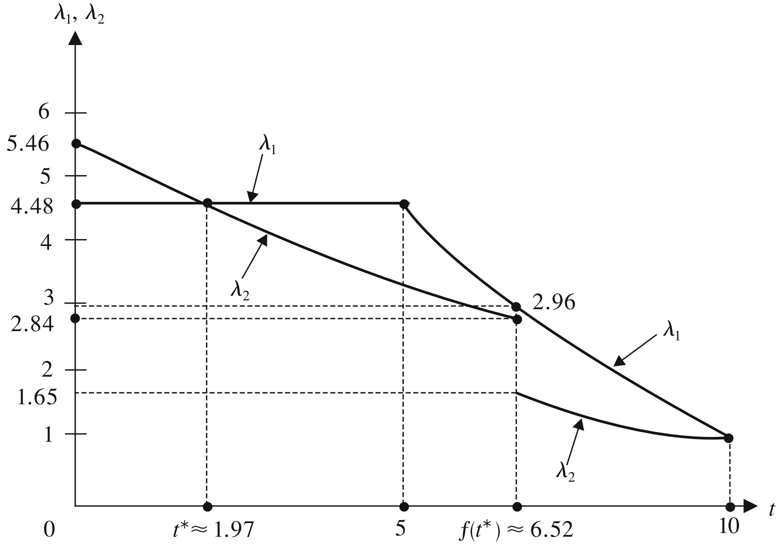

Figure 5.8: Adjoint trajectories for Exercise 5.5

E 5.8 What happens in the optimal equity financing model when $r < \rho$? Guess the optimal solution (without actually solving it).

E 5.9 In Sect. 5.2.3, we obtained the optimal solution in Case B. Express the corresponding control, state, and adjoint trajectories in forward time.

E 5.10 Let $g = 0.12$ in Example 5.1. Re-solve the finite horizon problem with this new value of g. Also, for the infinite horizon problem, state a policy which yields an infinite value for the objective function.

E 5.11 Reformulate and solve the simple cash balance problem of Sects. 5.1.1 and 5.1.2, if the earnings on bonds are paid in cash.

E 5.12 Add a salvage value function

$$e^{-\rho T} Bx(T),$$

where $B \geq 0$, to the objective function in the problem (5.21) and analyze the modified problem due to Sethi (1978b). Show how the solution changes as B varies from 0 to $1/rc$.

E 5.13 Suppose we extend the model of Exercise 5.12 to include debt. For this let z denote the total debt at time t and $w \geq 0$ denote the amount of debt issued expressed as a proportion of current earnings. Then the state equation for z is

$$\dot{z} = wx,\ y(0) = y_0.$$

How would you modify the objective function, the state equation for x, and the growth constraint (5.19)? Assume i to be the constant interest rate on debt, and $i < r$.

E 5.14 Remove the assumption of an arbitrary upper bound g on the growth rate in the financing model of Sect. 5.2.1 by introducing a convex cost associated with the growth rate. With r re-interpreted now as the gross rate of return, obtain the net increase in rate of earnings by the rate of increase in gross earnings less the cost associated with the growth rate. Also assume $c = 1$ as in Exercise 5.7. Formulate the resulting model and apply the maximum principle to find the form of the optimal policy. You may assume the cost function to be quadratic in the growth rate to get an explicit form for the solution.

E 5.15 Reformulate the optimal financing model (5.21) with $y(t)$ as the state variable, $U(t)$ as the new equity financing rate in dollars per unit of time, and $V(t)$ as the retained earnings in dollars per unit of time.

Hint: This formulation has mixed constraints requiring the Lagrangian formulation of the maximum principle (3.42) introduced in Chap. 3. Note further that it can be converted into the form (5.21) by setting $U = ux$, $V = vx$, and $x = ry$.

E 5.16 In Exercise 5.15, we assume a constant rate of return r on the assets so that the total earnings rate at time t is $ry(t)$ dollars per unit of time. Extend this formulation to allow for a decreasing marginal rate of return as the assets grow. More specifically, replace ry by an increasing, strictly concave function $R(y) > 0$ with $R'(0) = r$ and $R'(\bar{y}) = \rho$ for some $\bar{y} > y_0 > 0$. Obtain the optimal solution in the case when $r > g > \rho$, $0 < c < 1$, T sufficiently large, and $y_0 < y_1 < \bar{y}$, where y_1 is defined by the relation $R(y_1)/y_1 = g$. See Perrakis (1976).

E 5.17 Find the form of the optimal policy for the following model due to Davis and Elzinga (1971):

$$\max_{u,v} \left\{ J = \int_0^T e^{-\rho t}(1 - v)Erdt + P(T)e^{-\rho T} \right\}$$

subject to

$$\dot{P} = k[rE(1-v) - \rho P], \qquad P(0) = P_0,$$

$$\dot{E} = rE[v + u(c - E/P)], \quad E(0) = E_0,$$

and the control constraints

$$u \geq 0,\ v \geq 0,\ cu + v \leq g/r.$$

Here P denotes the price of a stock, E denotes equity per stock and $k > 0$ is a constant. Also, assume $r > \rho > g$ and $1/c < r/\rho < 1/c + (ck + 1)g/(\rho ck)$. This example requires the use of the generalized Legendre-Clebsch condition (D.69) in Appendix D.8.

Chapter 6

Applications to Production and Inventory

Applications of optimization methods to production and inventory problems date back at least to the classical EOQ (Economic Order Quantity) model or the lot size formula of Harris (1913). The EOQ is essentially a static model in the sense that the demand is constant and only a stationary solution is sought. A dynamic version of the lot size model was analyzed by Wagner and Whitin (1958). The solution methodology used there was dynamic programming.

An important dynamic production planning model was developed by Holt et al. (1960). In their model, referred to as the HMMS model, they considered both production costs and inventory holding costs over time. They used calculus of variations techniques to solve the continuous-time version of their model. In Sect. 6.1, a model of Thompson and Sethi (1980), similar to the HMMS model, is formulated and completely solved using optimal control theory. The turnpike solution is also obtained when the horizon is infinite.

In Sect. 6.2, we introduce the wheat trading model of Ijiri and Thompson (1970), in which a wheat speculator must buy and sell wheat in an optimal way in order to take advantage of changes in the price of wheat over time. In Sects. 6.2.1–6.2.3, we solve the model when the short-selling of wheat is allowed. In Sect. 6.2.4, we follow Norström (1978) to solve a simple example that disallows short-selling.

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In Sect. 6.3, we introduce a warehousing constraint, i.e., an upper bound on the amount of wheat that can be stored, in the wheat trading model. In addition to being realistic, the introduction of the warehousing constraint helps us to illustrate the concepts of decision and forecast horizons by means of examples. This section is expository in nature, but theoretical developments of these ideas are available in the literature.

6.1 Production-Inventory Systems

Many manufacturing enterprises use a production-inventory system to manage fluctuations in consumer demand for their products. Such a system consists of a manufacturing plant and a finished goods warehouse to store products which are manufactured but not immediately sold. Once a product is made and put into inventory, it incurs inventory holding costs of two kinds: (1) costs of physically storing the product, insuring it, etc.; and (2) opportunity cost of having the firm's money invested or tied up in the unsold inventory. The advantages of having products in inventory are: first, that they are immediately available to meet demand; second, that excess production during low demand periods can be stored in the warehouse so it will be available for sale during high demand periods. This usually permits the use of a smaller manufacturing plant than would otherwise be necessary, and also reduces the difficulties of managing the system.

The optimization problem is to balance the benefits of production smoothing versus the costs of holding inventory. Works that apply control theory to production and inventory problems have been reviewed in Sethi (1978a, 1984).

6.1.1 The Production-Inventory Model

We consider a factory producing a single homogeneous good and having a finished goods warehouse. To state the model we define the following quantities:

$I(t)$ = the inventory level at time t (state variable),

$P(t)$ = the production rate at time t (control variable),

$S(t)$ = the exogenously given sales rate at time t; assumed to be bounded and differentiable for $t \geq 0$,

T = the length of the planning period,

$\hat{I}$ = the inventory goal level,

I_0 = the initial inventory level,

$\hat{P}$ = the production goal level,

h = the inventory holding cost coefficient; $h > 0$,

c = the production cost coefficient; $c \geq 0$,

ρ = the constant nonnegative discount rate; $\rho \geq 0$.

The interpretation of the inventory goal level $\hat{I}$ is that it is a *safety stock* that the company wants to keep on hand. For example, $\hat{I}$ could be 2 months of average sales or $\hat{I}$ could be 100 units of the finished goods. Similarly, the production goal level $\hat{P}$ can be interpreted as the most efficient level at which it is desired to run the factory.

With this notation, the state equation is given by the stock-flow differential equation

$$\dot{I}(t) = P(t) - S(t),\ I(0) = I_0, \tag{6.1}$$

which says that the inventory at time t is increased by the production rate and decreased by the sales rate. The objective function of the model is:

$$\min \left\{ J = \int_0^T e^{-\rho t}[\frac{h}{2}(I - \hat{I})^2 + \frac{c}{2}(P - \hat{P})^2]dt \right\}. \tag{6.2}$$

The interpretation of the objective function is that we want to keep the inventory as close as possible to its goal level $\hat{I}$, and also to keep the production rate P as close as possible to its goal level $\hat{P}$. The quadratic terms $(h/2)(I - \hat{I})^2$ and $(c/2)(P - \hat{P})^2$ impose "penalties" for having either I or P not being close to its corresponding goal level.

Next we apply the maximum principle to solve the optimal control problem specified by (6.1) and (6.2). A stochastic extension of this problem will be carried out in Sect. 12.2.

6.1.2 Solution by the Maximum Principle

We now associate an adjoint function λ with Eq. (6.1) and can write the current-value Hamiltonian function as

$$H = \lambda(P - S) - \frac{h}{2}(I - \hat{I})^2 - \frac{c}{2}(P - \hat{P})^2. \tag{6.3}$$

In (6.3), we have used the negative of the (undiscounted) integrand in (6.2), since the minimization of J in (6.2) is equivalent to the maximization of $-J$.

To apply the Pontryagin maximum principle, we differentiate (6.3) and set the resulting expression equal to 0, which gives

$$\frac{\partial H}{\partial P} = \lambda - c(P - \hat{P}) = 0. \tag{6.4}$$

From this we obtain the optimal production rate

$$P^*(t) = \hat{P} + \lambda(t)/c. \tag{6.5}$$

We should mention that in writing (6.5), we are allowing negative production (or disposal). Of course, the situation of a disposal will not arise if we assume a sufficiently large $\hat{P}$ and a sufficiently small I_0.

Remark 6.1 If P is constrained to be nonnegative, then the form of the optimal control will be

$$P^*(t) = \max\{\hat{P} + \lambda(t)/c, 0\}. \tag{6.6}$$

This case will be treated in Sect. 6.1.6.

By substituting (6.5) into (6.1), we obtain

$$\dot{I} = \hat{P} + \lambda/c - S,\ I(0) = I_0. \tag{6.7}$$

The equation for the adjoint variable is easily found to be

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial I} = \rho\lambda + h(I - \hat{I}),\ \lambda(T) = 0. \tag{6.8}$$

We see that (6.7) has the initial boundary specified and (6.8) has the terminal boundary specified, so together these give a two-point boundary value problem. We will employ a method to solve these two equations simultaneously, which works only in some special cases including the present case. The method is the well-known trick used to solve simultaneous differential equations by differentiation and substitution until one of the variables is eliminated. Specifically, we differentiate (6.7) with respect to t, which creates an equation with $\dot{\lambda}$ in it. We then use (6.8) to eliminate $\dot{\lambda}$ and (6.7) to eliminate λ from the resulting equation as follows:

$$\begin{aligned}\ddot{I} = \dot{\lambda}/c - \dot{S} &= \rho(\lambda/c) + (h/c)(I - \hat{I}) - \dot{S} \\ &= \rho(\dot{I} - \hat{P} + S) + (h/c)(I - \hat{I}) - \dot{S}.\end{aligned}$$

We rewrite this as

$$\ddot{I} - \rho\dot{I} - \alpha^2 I = -\alpha^2\hat{I} - \dot{S} - \rho(\hat{P} - S), \tag{6.9}$$

where the constant α is given by

$$\alpha = \sqrt{h/c}. \tag{6.10}$$

We can now solve (6.9) by using the standard method described in Appendix A. The auxiliary equation for (6.9) is

$$m^2 - \rho m - \alpha^2 = 0,$$

which has the two real roots

$$m_1 = (\rho - \sqrt{\rho^2 + 4\alpha^2})/2, \;\; m_2 = (\rho + \sqrt{\rho^2 + 4\alpha^2})/2; \tag{6.11}$$

note that $m_1 < 0$ and $m_2 > 0$. We can therefore write the general solution to (6.9) as

$$I(t) = a_1 e^{m_1 t} + a_2 e^{m_2 t} + Q(t), \; I(0) = I_0, \tag{6.12}$$

where $Q(t)$ is a particular integral of (6.9).

We will say that $Q(t)$ is a special particular integral of (6.9) if it has no additive terms involving $e^{m_1 t}$ and $e^{m_2 t}$. From now on we will always assume that $Q(t)$ is a special particular integral.

Although (6.12) has two arbitrary constants a_1 and a_2, it has only one boundary condition. To get the other boundary condition we differentiate (6.12), substitute the result into (6.7), and solve for λ. We obtain

$$\lambda(t) = c(m_1 a_1 e^{m_1 t} + m_2 a_2 e^{m_2 t} + \dot{Q} + S - \hat{P}), \; \lambda(T) = 0. \tag{6.13}$$

Note that we have imposed the boundary condition on λ so that we can determine the constants a_1 and a_2.

For ease of expressing a_1 and a_2, let us define two constants

$$b_1 = I_0 - Q(0), \tag{6.14}$$
$$b_2 = \hat{P} - \dot{Q}(T) - S(T). \tag{6.15}$$

We now impose the boundary conditions in (6.12) and (6.13) and solve for a_1 and a_2 as follows:

$$a_1 = \frac{b_2 e^{m_1 T} - m_2 b_1 e^{(m_1+m_2)T}}{m_1 e^{2m_1 T} - m_2 e^{(m_1+m_2)T}}, \tag{6.16}$$

$$a_2 = \frac{b_1 m_1 e^{2m_1 T} - b_2 e^{m_1 T}}{m_1 e^{2m_1 T} - m_2 e^{(m_1+m_2)T}}. \tag{6.17}$$

If we recall that m_1 is negative and m_2 is positive, then when T is sufficiently large so that e^{m_1T} and e^{2m_1T} are negligible, we can write

$$a_1 \approx b_1, \tag{6.18}$$

$$a_2 \approx \frac{b_2}{m_2}e^{-m_2T}. \tag{6.19}$$

Note that for a large T, e^{-m_2T} is close to zero and, therefore, a_2 is close to zero. However, the reason for retaining the exponential term in (6.19) is that a_2 is multiplied by e^{m_2t} in (6.13), which, while small when t is small, becomes large and important when t is close to T.

With these values of a_1 and a_2 and with (6.5), (6.12), and (6.13), we now write the expressions for I^*, P^*, and λ. We will break each expression into three parts: the first part labeled *Starting Correction* is important only when t is small; the second part labeled *Turnpike Expression* is significant for all values of t; and the third part labeled *Ending Correction* is important only when t is close to T.

Starting Correction	Turnpike Expression	Ending Correction	
$I^* = (b_1e^{m_1t})+$	$(Q)+$	$\left(\frac{b_2}{m_2}e^{m_2(t-T)}\right)$	(6.20)
$P^* = (m_1b_1e^{m_1t})+$	$(\dot{Q}+S)+$	$\left(b_2e^{m_2(t-T)}\right)$	(6.21)
$\lambda = c(m_1b_1e^{m_1t})+$	$c\left(\dot{Q}+S-\hat{P}\right)+$	$c\left(b_2e^{m_2(t-T)}\right)$	(6.22)

Note that if $b_1 = 0$, which by (6.14) means $I_0 = Q(0)$, then there is no starting correction. In other words, $I_0 = Q(0)$ is a starting inventory that causes the solution to be on the turnpike initially. In the same way, if $b_2 = 0$, then the ending correction vanishes in each of these formulas, and the solution stays on the turnpike until the end.

Expressions (6.20) and (6.21) represent approximate closed-form solutions for the optimal inventory and production functions I^* and P^* as long as S is such that the special particular integral Q can be found explicitly. For such examples of S; see Sect. 6.1.4.

6.1.3 The Infinite Horizon Solution

It is important to show that this solution also makes sense when $T \to \infty$. In this case it is usual to assume that the discount rate $\rho > 0$ and the sales rate S does not grow too fast so that the objective function (6.2) remains finite. One can then show that the limit of the finite horizon solution as $T \to \infty$ also solves the infinite horizon problem. Note that as $T \to \infty$, the ending correction terms in (6.20)–(6.22) disappear because $e^{-m_2 T}$ goes to 0. We now have

$$\lambda(t) = c[m_1 b_1 e^{m_1 t} + \dot{Q} + S - \hat{P}]. \tag{6.23}$$

Since we would like

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t) = 0, \tag{6.24}$$

we would require that $S + \dot{Q}$ grows slower asymptotically than the discount rate ρ. One can easily verify that this condition holds for the demand terms discussed in Sect. 6.1.4 that follows. Moreover, the condition is easy to check for any given specific demand $S(t)$ for which the particular integral $Q(t)$ is known.

By the sufficiency of the maximum principle conditions (Sect. 2.4), it can be verified that the limiting solution

$$I^*(t) = b_1 e^{m_1 t} + Q,\ \ P^*(t) = m_1 b_1 e^{m_1 t} + \dot{Q} + S \tag{6.25}$$

is optimal. If $I(0) = Q(0)$, the solution is always on the turnpike. Note that the triple $\{\bar{I}, \bar{P}, \bar{\lambda}\} = \{Q, \dot{Q} + S, c(\dot{Q} + S - \hat{P})\}$ represents a nonstationary turnpike. If $I(0) \neq Q(0)$, then $b_1 \neq 0$ and the expressions (6.25) imply that the paths of inventory and production only approach the turnpike but never attain it.

6.1.4 Special Cases of Time Varying Demands

In this section, we provide some important cases of time varying demands including seasonal demands. These involve polynomial or sinusoidal demand functions. We then solve some numerical examples of the model described in Sect. 6.1.1 for $\rho = 0$ and $T < \infty$.

For the first example, we assume that $S(t)$ is a polynomial of degree $2p$ or $2p-1$ so that $S^{(2p+1)} = 0$, where $S^{(k)}$ denotes the kth time derivative of S with respect to t. In other words,

$$S(t) = C_0 t^{2p} + C_1 t^{2p-1} + \ldots + C_{2p}, \tag{6.26}$$

where at least one of C_0 and C_1 is not zero. Then, from Zwillinger (2003), a particular integral of (6.9) is

$$Q(t) = \hat{I} + \frac{1}{\alpha^2}S^{(1)} + \frac{1}{\alpha^4}S^{(3)} + \cdots + \frac{1}{\alpha^{2p}}S^{(2p-1)}. \tag{6.27}$$

In Exercise 6.2 the reader is asked to verify this by direct substitution.

For the second example, we assume that $S(t)$ is a sinusoidal demand function of form

$$S(t) = A\sin(\pi Bt + C) + D, \tag{6.28}$$

where A, B, C, and D are constants. In Exercise 6.3 you are asked to verify that a particular integral of (6.9) for S in (6.28) is

$$Q(t) = \hat{I} + \frac{\pi AB}{\alpha^2 + \pi^2 B^2}\cos(\pi Bt + C). \tag{6.29}$$

It is well known in the theory of differential equations that demands that are sums of functions of the form (6.26) and/or (6.28) give rise to solutions that are sums of functions of form (6.27) and/or (6.29).

Example 6.1 Assume $\hat{P} = 30,\ \hat{I} = 15,\ T = 8,\ \rho = 0,$ and $h = c = 1$ so that $\alpha = 1,\ m_1 = -1,$ and $m_2 = 1$. Assume

$$S(t) = t(t-4)(t-8) + 30 = t^3 - 12t^2 + 32t + 30.$$

Solution It is then easy to show from (6.27) that

$$Q(t) = 3t^2 - 24t + 53 \text{ and } \dot{Q}(t) = 6t - 24.$$

Also from (6.14), (6.15), and (6.16), we have $a_1 \approx b_1 = I_0 - 53$ and $b_2 = -24$. Then, from (6.20) and (6.21),

$$I^*(t) = (I_0 - 53)e^{-t} + Q(t) - 24e^{t-8},$$

$$P^*(t) = -(I_0 - 53)e^{-t} + \dot{Q}(t) + S(t) - 24e^{t-8}.$$

In Fig. 6.1 the graphs of sales, production, and inventory are drawn with $I_0 = 10$ (a small starting inventory), which makes $b_1 = -43$. In Fig. 6.2 the same graphs are drawn with $I_0 = 50$ (a large starting inventory), which makes $b_1 = -3$. In Fig. 6.3 the same graphs are drawn with $I_0 = 30$, which makes $b_1 = -23$. Note that initially during the time from 0 to 4, the three cases are quite different, but during the time from 4 to 8, they are nearly identical. The ending inventory ends up being 29 in all three cases.

Example 6.2 Assume that

$$S(t) = A + Bt + \sum_{k=1}^{K} C_k \sin(\pi D_k t + E_k), \tag{6.30}$$

where the constants A, B, C_k, D_k, and E_k are estimated from future demand data by means of one of the standard forecasting techniques such as those in Brown (1959, 1963).

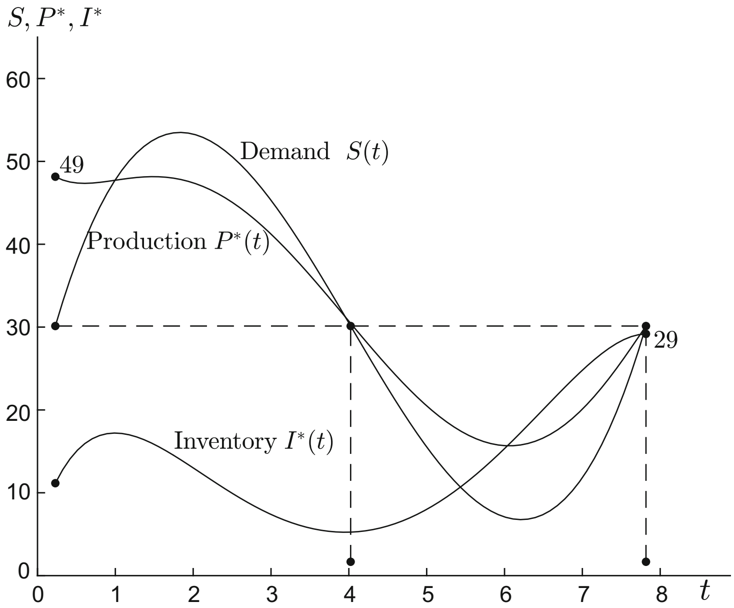

Figure 6.1: Solution of Example 6.1 with $I_0 = 10$

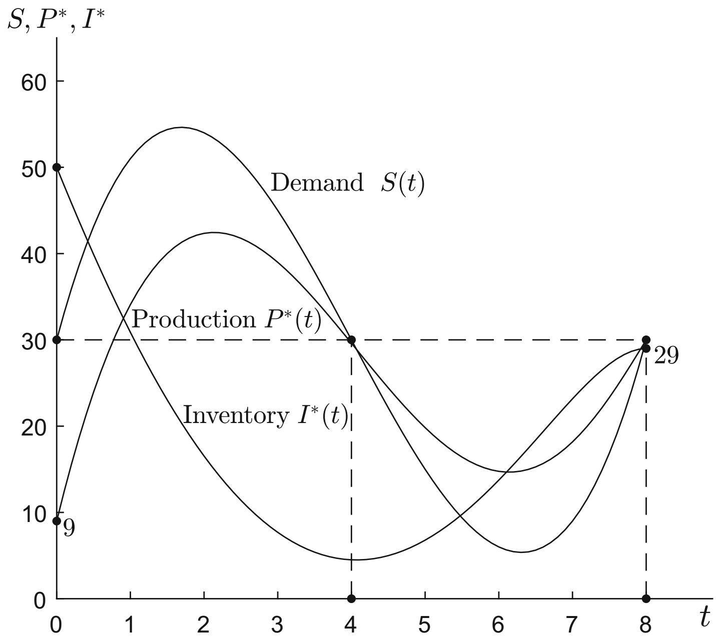

Figure 6.2: Solution of Example 6.1 with $I_0 = 50$

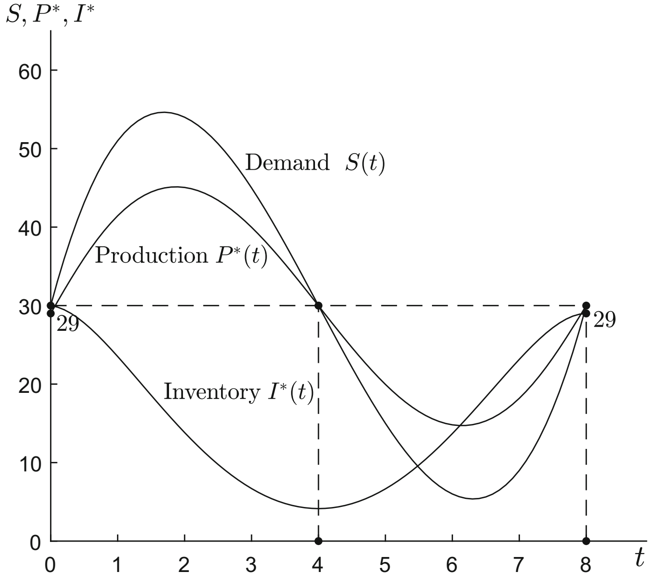

Figure 6.3: Solution of Example 6.1 with $I_0 = 30$

Solution By using formulas (6.27) and (6.29), we obtain the particular integral

$$Q(t) = \hat{I} + \frac{1}{\alpha^2}B + \sum_{k=1}^{K} \frac{\pi C_k D_k}{\alpha^2 + (\pi D_k)^2} \cos(\pi D_k t + E_k). \tag{6.31}$$

6.1.5 Optimality of a Linear Decision Rule

In Sect. 6.1.2, our emphasis was to explore the turnpike nature of the solution of the inventory model of Sect. 6.1.1. For this purpose, we made some asymptotic approximations when solving the state and adjoint differential equations under the assumption that the horizon is long. Here our focus is to solve the undiscounted version (i.e., $\rho = 0$) of the model exactly to find its optimal feedback solution, and show that it is a linear decision rule as reported in the classical work of Holt et al. (1960).

Since the two-point boundary value problem given by (6.7) and (6.8) is a linear system of differential equations, it is known via its fundamental solution matrix that λ can be expressed in terms of I in a linear way as follows:

$$\lambda(t) = \psi(t) - s(t)I(t), \tag{6.32}$$

where $\psi(t)$ and $s(t)$ are continuously differentiable in t. Differentiating (6.32) with respect to t and substituting for $\dot{I}$ and $\dot{\lambda}$ from (6.7) and (6.8) with $\rho = 0$, respectively, we obtain

$$I(h - s^2/c + \dot{s}) + (\hat{P} + \psi/c - S)s - h\hat{I} - \dot{\psi} = 0.$$

Since the above relation must hold for any value of the initial inventory I_0, we must have

$$\dot{s} = s^2/c - h \text{ and } \dot{\psi} = (\hat{P} + \psi/c - S)s - h\hat{I}. \tag{6.33}$$

Also from $\lambda(T) = 0$ in (6.8) and (6.32), we have $0 = \psi(T) - s(T)I(T)$, a relation that must hold regardless of the value of $I(T)$. Thus, we can conclude that

$$s(T) = 0 \text{ and } \psi(T) = 0. \tag{6.34}$$

Clearly, the solution of the differential equation given by (6.33) and (6.34) will give us the optimal control (6.5) in terms of $S(t)$ and $\psi(t)$. In particular, the differential equation

$$\dot{s} = s^2/c - h, \quad s(T) = 0 \tag{6.35}$$

is known as the Riccati equation, whose solution is given by

$$s(t) = \sqrt{hc}\tanh\left(\sqrt{\frac{h}{c}}(T - t)\right). \tag{6.36}$$

Using (6.32) and (6.36) in (6.5), the optimal production rate $P^*(t)$ is

$$P^*(t) = \hat{P} - \sqrt{\frac{h}{c}}\tanh\left(\sqrt{\frac{h}{c}}(T - t)\right)I^*(t) + \frac{\psi(t)}{c}. \tag{6.37}$$

This says that the optimal production rate equals the production goal level $\hat{P}$ plus two adjustment terms. The first term implies ceteris paribus that the higher the current inventory level, the lower the production rate is. Furthermore, this dependence is linear with the linear effect decreasing as t increases, reaching zero at $t = T$. The second term depends on all the model parameters including the demand rate from time t to T.

Because of the linear dependence of the optimal production rate on the inventory level in (6.37), this rule is known as a linear decision rule as reported by Holt et al. (1960). More generally, this rule can be extended to linear quadratic problems as listed in Table 3.3(c). In Appendix D.4, we derive this rule for the problems given in Table 3.3(c), but without the forcing function d. Furthermore, the rule can be extended to a class of stochastic linear-quadratic problems that include the stochastic production planning problem treated in Sect. 12.2.

6.1.6 Analysis with a Nonnegative Production Constraint

Thus far in this chapter, we have ignored the production constraint $P \geq 0$ and used (6.5) and (6.37) as the optimal decision rules. Here we will solve the production-inventory problem subject to $P \geq 0$, and use (6.6) as the optimal production rule. For simplicity of analysis and exposition, we will assume also that S is a positive constant, $T = \infty$, and $\rho > 0$. These specifications make $\dot{S} = 0$, making the right hand side $-\alpha\hat{I} - \rho(\hat{P} - S)$ a constant, $a_1 = b_1$ in (6.16), and $a_2 = 0$ in (6.17).

In view of its constant right-hand side, we can use Row (3) of Table A.2 to obtain its particular integral as

$$Q = \frac{\rho}{\alpha^2}(\hat{P} - S) + \hat{I}, \tag{6.38}$$

which is a constant and thus $\dot{Q} = 0$. From (6.14) and (6.15), we now have

$$b_1 = I_0 - Q = I_0 - \hat{I} - (\rho/\alpha^2)(\hat{P} - S) \text{ and } b_2 = \hat{P} - S.$$

The turnpike is defined by the triple $\{\bar{I}, \bar{P}, \bar{\lambda}\} = \{(\rho/\alpha^2)(\hat{P} - S) + \hat{I}, S, c(S - \hat{P})\}$ formed from the turnpike expressions in (6.20), (6.21), and (6.22), respectively. Note that we could have obtained the turnpike levels directly by applying the conditions (3.108), which in this case are

$$\dot{\bar{I}} = 0, \ \dot{\bar{\lambda}} = 0, \text{ and } \bar{P} = \hat{P} + \bar{\lambda}/c = S. \tag{6.39}$$

If $I_0 = Q$, then the optimal solution stays on the turnpike. If $I_0 \neq Q$, we must obtain the transient solution. It should be clear that the control in (6.25) may become negative, especially when the initial inventory is high. Let us complete the solution of the problem by considering three cases: $I_0 \leq Q$, $Q < I_0 \leq Q - S/m_1$, and $I_0 > Q - S/m_1$.

If $I_0 \leq Q$, then the control in (6.25) with $b_1 = I_0 - Q_0$ is clearly positive. Thus, the optimal production rate is given by

$$P^*(t) = m_1 b_1 e^{m_1 t} + S = m_1(I_0 - Q)e^{m_1 t} + S \geq 0. \tag{6.40}$$

Moreover, from the state in (6.25), we can obtain the corresponding $I^*(t)$ as

$$I^*(t) = (I_0 - Q)e^{m_1 t} + Q. \tag{6.41}$$

It is easy to see that $I^*(t)$ increases monotonically to Q as $t \to \infty$, as shown in Fig. 6.4.

If $Q < I_0 \leq Q - S/m_1$, we can easily see from (6.40) that $P^*(0) \geq 0$. Furthermore, $\dot{P}^*(t) \geq 0$, and therefore the optimal production rate is once again given by (6.40). We also have $I^*(t)$ as in (6.41) and conclude that $I^*(t) \to Q$ monotonically as $t \to \infty$, as shown in Fig. 6.4.

Finally, if $I_0 > Q - S/m_1$, (6.40) would have a negative value for the initial production which is infeasible. By (6.6), $P^*(0) = 0$. We can now depict this situation in Fig. 6.4. The time $\hat{t}$ shown in the figure is the time at which $P^*(\hat{t}) = \hat{P} + \lambda(\hat{t})/c = 0$. We already know from (6.40) that in the case when $I_0 = Q - S/m_1$, $P^*(0) = 0$. This suggests that

$$I^*(\hat{t}) = Q - \frac{S}{m_1}. \tag{6.42}$$

For $t \leq \hat{t}$, we have $P^*(t) = 0$ so that $\dot{I}^* = -S$, which gives

$$I^*(t) = I_0 - St, \; t \leq \hat{t}. \tag{6.43}$$

As for the adjoint equation (6.7), we now need the boundary condition at $\hat{t}$. For this, we can use (6.4) to obtain $\lambda(\hat{t}) = -c\hat{P}$. Thus, the adjoint equation in the interval $[0, \hat{t}\,]$ is

$$\dot{\lambda} = \rho\lambda + h(I - \hat{I}), \; \lambda(\hat{t}) = -c\hat{P}. \tag{6.44}$$

We can substitute $I_0 - St$ for I in Eq. (6.44) and solve for λ. Note that we can easily obtain $\hat{t}$ as

$$I_0 - S\hat{t} = Q - \frac{S}{m_1} \Rightarrow \hat{t} = \frac{I_0 - Q}{S} + \frac{1}{m_1}. \tag{6.45}$$

We can now specify the complete solution in the case when $I_0 > Q - S/m_1$. With $\hat{t}$ specified in (6.45), the solution is as follows.

For $0 \le t \le \hat{t} : P^*(t) = 0, I^*(t) = I_0 - St$, and $\lambda(t)$ is the solution of

$$\dot{\lambda} = \rho\lambda + h(I_0 - St - \hat{I}), \lambda(\hat{t}) = -c\hat{P}.$$

For $t > \hat{t}$: we replace I_0 by $Q - S/m_1$ and t by $t - \hat{t}$ on the right hand side of (6.40) to obtain $P^*(t) = -Se^{m_1(t-\hat{t})}$. The same replacements in (6.41) gives us the corresponding $I^*(t) = -(S/m_1)e^{m_1 t}$. Finally, $\lambda(t)$ can be obtained by solving

$$\dot{\lambda} = \rho\lambda - h\left(\frac{S}{m_1}e^{m_1 t} + \hat{I}\right), \lambda(\hat{t}) = -c\hat{P}.$$

We have thus solved the problem in every case of the initial condition I_0. These solutions are sketched in Fig. 6.4 for $\hat{I} = 8, \hat{P} = 5, S = 6, h = 1, c = 4$, and $\rho = 0.1$, for three different values of I_0, namely, 25, 15, and 1. In Exercise 6.7, you are asked to solve the problem for these values and obtain Fig. 6.4.

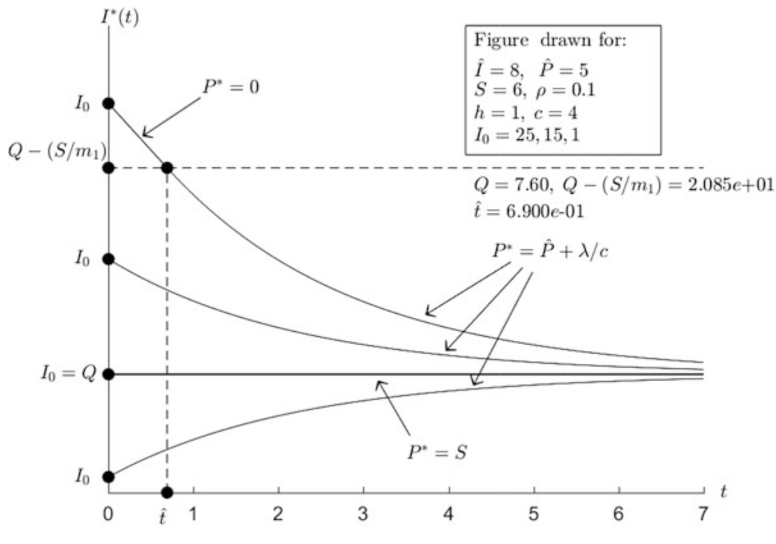

Figure 6.4: Optimal production rate and inventory level with different initial inventories

6.2 The Wheat Trading Model

Consider a firm that buys and sells wheat. The firm's only assets are cash and wheat, and the price of wheat over time is known with certainty. The objective of this firm is to buy and sell wheat in order to maximize the total value of its assets at the horizon time T. The problem here is similar to the simple cash balance model of Sect. 5.1 except that there are nonlinear holding costs associated with storing wheat. An extension of this model to one having two control variables appears in Ijiri and Thompson (1972).

6.2.1 The Model

We introduce the following notation:

$$
\begin{aligned}
T &= \text{the horizon time,} \\
x(t) &= \text{the cash balance in dollars at time } t, \\
y(t) &= \text{the wheat balance in bushels at time } t, \\
v(t) &= \text{the rate of purchase of wheat in bushels per unit time;} \\
&\quad\ \text{a negative purchase means a sale,} \\
p(t) &= \text{the price of wheat in dollars per bushel at time } t, \\
r &= \text{the constant positive interest rate earned on the cash} \\
&\quad\ \text{balance,} \\
h(y) &= \text{the cost of holding } y \text{ bushels per unit time.}
\end{aligned}
$$

In this section we permit x and y to go negative, meaning that borrowing money and short-selling wheat are both allowed. In the next section we disallow the short-selling of wheat.

The state equations are:

$$\dot{x} = rx - h(y) - pv, \quad x(0) = x_0, \tag{6.46}$$

$$\dot{y} = v, \quad y(0) = y_0, \tag{6.47}$$

and the control constraints are

$$-V_2 \leq v(t) \leq V_1, \tag{6.48}$$

where V_1 and V_2 are nonnegative constants. The objective function is:

$$\max\{J = x(T) + p(T)y(T)\} \tag{6.49}$$

subject to (6.46)–(6.48). Note that the problem is in the linear Mayer form.

6.2.2 Solution by the Maximum Principle

Introduce the adjoint variables λ_1 and λ_2 and define the Hamiltonian function

$$H = \lambda_1[rx - h(y) - pv] + \lambda_2 v. \tag{6.50}$$

The adjoint equations are:

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_1 r, \;\; \lambda_1(T) = 1, & (6.51) \\ \dot{\lambda}_2 &= h'(y)\lambda_1, \;\; \lambda_2(T) = p(T). & (6.52) \end{aligned}$$

It is easy to solve (6.51) as

$$\lambda_1(t) = e^{r(T-t)} \tag{6.53}$$

and (6.52) as

$$\lambda_2(t) = p(T) - \int_t^T h'(y(\tau))e^{r(T-\tau)}d\tau. \tag{6.54}$$

The interpretation of $\lambda_1(t)$ is that it is the future value (at time T) of one dollar held as cash from t to T. The interpretation of $\lambda_2(t)$ is the price at time T of a bushel of wheat less the total future value (at time T) of the stream of storage costs incurred to store that bushel of wheat from t to T.

From (6.50) the optimal control is

$$v^*(t) = \text{bang}[-V_2, V_1; \lambda_2(t) - \lambda_1(t)p(t)]. \tag{6.55}$$

In Exercise 6.8 you are asked to provide the interpretation of this optimal policy.

Equations (6.46), (6.47), (6.54), and (6.55) determine the two-point boundary value problem which usually requires a numerical solution procedure. In the next section we assume a special form for the storage function $h(y)$ to be able to obtain a closed-form solution.

6.2.3 Solution of a Special Case

For this special case we assume $h(y) = \frac{1}{2}|y|,\ r = 0,\ x(0) = 10,\ y(0) = 0,$ $V_1 = V_2 = 1,\ T = 6$, and

$$p(t) = \begin{cases} 3 & \text{for } 0 \le t \le 3, \\ 4 & \text{for } 3 < t \le 6. \end{cases} \tag{6.56}$$

We will apply the maximum principle (2.31) developed in Chap. 2 to this problem even though $h(y)$ is not differentiable at $y = 0$. The answer can be obtained rigorously by using the maximum principle for models involving nondifferentiable functions discussed, e.g., in Clarke (1989, Chapter 4) and Feichtinger and Hartl (1985b).

For this case with $r = 0$, we have $\lambda_1(t) = 1$ for all t from (6.53) so that the TPBVP is

$$\dot{x} = -\frac{1}{2}|y| - pv, \; x(0) = 10, \tag{6.57}$$

$$\dot{y} = v, \; y(0) = 0, \tag{6.58}$$

$$\dot{\lambda}_2(t) = \frac{1}{2}\,\text{sgn}(y), \; \lambda_2(6) = 4. \tag{6.59}$$

For this simple problem it is easy to guess a solution. From the fact that $\lambda_1 = 1$, the optimal policy (6.55) reduces to

$$v^*(t) = \text{bang}[-1, 1; \lambda_2(t) - p(t)]. \tag{6.60}$$

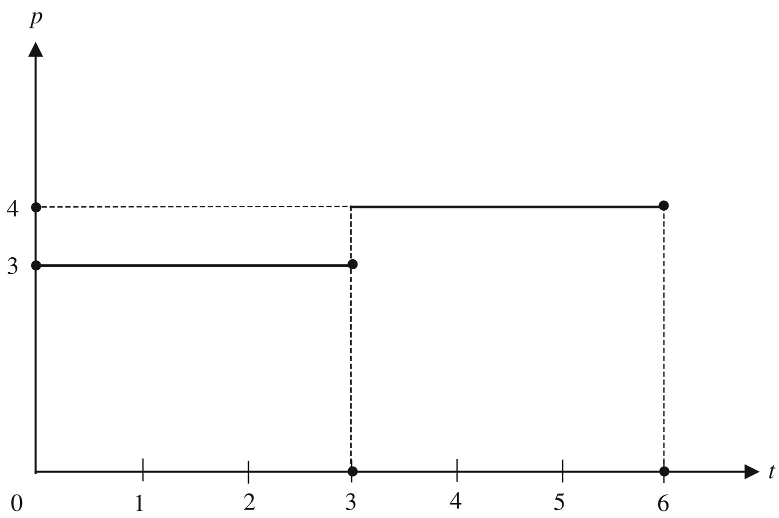

Figure 6.5: The price trajectory (6.56)

The graph of the price function is shown in Fig. 6.5. Since $p(t)$ is increasing, short-selling is never optimal. Since the storage cost is 1/2 per unit per unit time and the wheat price jumps by 1 unit at $t = 3$, it never pays to store wheat for more than 2 time units. Because $y(0) = 0$, we have $v^*(t) = 0$ for $0 \leq t \leq 1$. This obviously must be a singular control. Suppose we start buying wheat at $t^* > 1$. From (6.60) the rate of buying is 1; clearly buying will continue at this rate until $t = 3$, and not longer. In order to not lose money on the storage of wheat, it must be sold within 2 time units of its purchase. Clearly we should start selling at $t = 3^+$ at the maximum rate of 1, and continue until a last sale time t^{**}. In order to sell exactly all of the wheat purchased, we must have

$$3 - t^* = t^{**} - 3. \tag{6.61}$$

Thus, $v^*(t) = 0$ in the interval $[t^{**}, 6]$, which is also a singular control. With this policy, $y(t) > 0$ for all $t \in (t^*, t^{**})$. From (6.59), $\dot{\lambda}_2 = 1/2$ in the interval (t^*, t^{**}). In order to have a singular control in the interval $[t^{**}, 6]$, we must have $\lambda_2(t) = 4$ in that interval. Also, in order to have a singular control in $[0, t^*]$, we must have $\lambda_2(t) = 3$ in that interval. Thus, $\lambda_2(t^{**}) - \lambda_2(t^*) = 1$, which with $\dot{\lambda}_2 = 1/2$ allows us to conclude that

$$t^{**} - t^* = 2, \tag{6.62}$$

and therefore $t^* = 2$ and $t^{**} = 4$. Thus from (6.59) and (6.60),

$$\lambda_2(t) = \begin{cases} 3, & 0 \le t \le 2, \\ 2 + t/2, & 2 \le t \le 4, \\ 4, & 4 \le t \le 6. \end{cases} \tag{6.63}$$

We can now sketch graphs for $\lambda_2(t)$, $v^*(t)$, and $y^*(t)$ as shown in Fig. 6.6. In Exercise 6.13 you are asked to show that these trajectories are optimal by verifying that the maximum principle necessary conditions hold and that they are also sufficient.

6.2.4 The Wheat Trading Model with No Short-Selling

We next consider the wheat trading problem in the case when short-selling is not permitted, i.e., we impose the state constraint $y \ge 0$. Moreover, for simplicity in exposition we consider the following special case of Norström (1978). Specifically, we assume $h(y) = y/2$, $r = 0$, $x(0) = 10$, $y(0) = 1$, $V_1 = V_2 = 1$, $T = 3$, and

$$p(t) = \begin{cases} -2t + 7 & \text{for } 0 \le t < 2, \\ t + 1 & \text{for } 2 \le t \le 3. \end{cases} \tag{6.64}$$

The statement of the problem is:

$$\begin{cases} \max\{J = x(3) + p(3)y(3) = x(3) + 4y(3)\} \\ \text{subject to} \\ \dot{x} = -\frac{1}{2}y - pv, \ x(0) = 10, \\ \dot{y} = v, \ y(0) = 1, \\ v + 1 \geq 0, \ 1 - v \geq 0, \ y \geq 0. \end{cases} \tag{6.65}$$

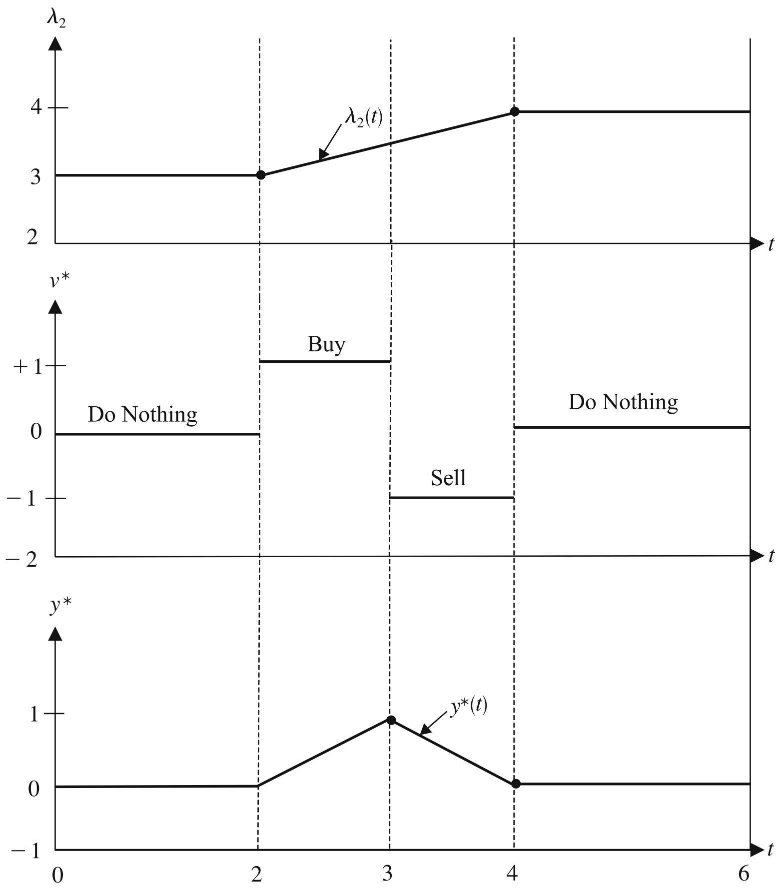

Figure 6.6: Adjoint variable, optimal policy and inventory in the wheat trading model

To solve this problem, we use the Lagrangian form of the indirect maximum principle given in (4.29). The Hamiltonian is

$$H = \lambda_1(-y/2 - pv) + \lambda_2 v. \tag{6.66}$$

The optimal control is

$$v^*(t) = \text{bang}[-1, 1; \lambda_2(t) - \lambda_1(t)p(t)] \text{ when } y > 0. \tag{6.67}$$

Whenever $y = 0$ we must impose $\dot{y} = v \geq 0$ in order to insure that no short-selling occurs. Therefore,

$$v^*(t) = \text{bang}[0, 1; \lambda_2(t) - \lambda_1(t)p(t)] \text{ when } y = 0. \tag{6.68}$$

Next we form the Lagrangian

$$L = H + \mu_1(v + 1) + \mu_2(1 - v) + \eta v, \tag{6.69}$$

where μ_1, μ_2, and η satisfy the complementary slackness conditions:

$$\mu_1 \geq 0, \quad \mu_1(v + 1) = 0, \tag{6.70}$$
$$\mu_2 \geq 0, \quad \mu_2(1 - v) = 0, \tag{6.71}$$
$$\eta \geq 0, \quad \eta y = 0. \tag{6.72}$$

Furthermore, the optimal trajectory must satisfy

$$\frac{\partial L}{\partial v} = \lambda_2 - p\lambda_1 + \mu_1 - \mu_2 + \eta = 0. \tag{6.73}$$

With $r = 0$, we get $\lambda_1 = 1$ as before, and

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial y} = 1/2, \ \lambda_2(3^-) = 4 + \gamma, \tag{6.74}$$

with

$$\gamma \geq 0, \gamma y(3) = 0. \tag{6.75}$$

Let us first try $\gamma = 0$. Then $\lambda_2(3^-) = 4$, and if we let $\hat{t}$ denote the time of the last jump before the terminal time, then there is no jump in the interval $(\hat{t}, 3)$. Then, from (6.74) we have

$$\lambda_2(t) = t/2 + 5/2 \text{ for } \hat{t} \leq t < 3, \tag{6.76}$$

and the optimal control from (6.67) or (6.68) is $v^* = 1$, i.e., buy wheat at the maximum rate of 1, so long as $\lambda_2(t) > p(t)$. Also, this will give

$y(3) > 0$, so that (6.75) holds. Let us next find the time $\hat{t}$ of the last jump before the terminal time. Clearly, this value will not be larger than the time at which $\lambda_2(t) = p(t)$. Thus,

$$\hat{t} \leq \{t|t/2 + 5/2 = -2t + 7\} = 1.8. \tag{6.77}$$

Since $p(t)$ is decreasing at the start of the problem, it appears that selling at the maximum rate of 1, i.e., $v^* = -1$, should be optimal at the start. Since the beginning inventory is $y(0) = 1$, selling at the rate of 1 can continue only until $t = 1$, at which time the inventory $y(1)$ becomes 0. Suppose that we do nothing, i.e., $v^*(t) = 0$ in the interval $(1, 1.8]$. Then, $t = 1$ is an entry time (see Sect. 4.2) and $t = 1.8$ is not an entry time, and $\hat{t} = 1$. Hence, according to the maximum principle (4.29), $\lambda_2(t)$ is continuous at $t = 1.8$, and therefore $\lambda_2(t)$ is given by (6.76) in the interval $[1, 3)$, i.e.,

$$\lambda_2(t) = t/2 + 5/2 \text{ for } 1 \leq t < 3. \tag{6.78}$$

Using (6.73) with $\lambda_1 = 1$ in the interval $(1, 1.8]$ and $v^* = 0$ so that $\mu_1 = \mu_2 = 0$, we have

$$\lambda_2 - p + \mu_1 - \mu_2 + \eta = \lambda_2 - p + \eta = 0,$$

and consequently

$$\eta(t) = p(t) - \lambda_2(t) = -5t/2 + 9/2, \ t \in (1, 1.8]. \tag{6.79}$$

Since $h_t = 0$, the jump condition in (4.29) for the Hamiltonian at $\tau = 1$ reduces to

$$H[x^*(1),\, u^*(1^-),\, \lambda(1^-),\, 1] = H[x^*(1),\, u^*(1^+),\, \lambda(1^+),\, 1].$$

From the definition of the Hamiltonian H in (6.66), we can rewrite the condition as

$$\lambda_1(1^-)[-y(1)/2 - p(1^-)v^*(1^-)] + \lambda_2(1^-)v^*(1^-) =$$

$$\lambda_1(1^+)[-y(1)/2 - p(1^+)v^*(1^+)] + \lambda_2(1^+)v^*(1^+).$$

Since $\lambda_1(t) = 1$ for all t, the above condition reduces to

$$-p(1^-)v^*(1^-) + \lambda_2(1^-)v^*(1^-) = -p(1^+)v^*(1^+) + \lambda_2(1^+)v^*(1^+).$$

Substituting the values of $p(1^-) = p(1^+) = 5$ from (6.64), $\lambda_2(1^+) = 3$ from (6.78), and $v^*(1^+) = 0$ and $v^*(1^-) = -1$ from the above discussion, we obtain

$$-5(-1) + \lambda_2(1^-)(-1) = -5(0) + 3(0) = 0 \Rightarrow \lambda_2(1^-) = 5. \tag{6.80}$$

We can now use the jump condition in (4.29) on the adjoint variables to obtain

$$\lambda_2(1^-) = \lambda_2(1^+) + \zeta(1) \Rightarrow \zeta(1) = \lambda_2(1^-) - \lambda_2(1^+) = 5 - 3 = 2 \geq 0.$$

It is important to note that in the interval $[1, 1.8]$, the optimal control condition (6.68) holds, justifying our supposition that $v^* = 0$ in this interval. Furthermore, using (6.80) and (6.74),

$$\lambda_2(t) = t/2 + 9/2 \text{ for } t \in [0, 1), \tag{6.81}$$

and the optimal control condition (6.67) holds, justifying our supposition that $v^* = -1$ in this interval. Also, we can conclude that our guess $\gamma = 0$

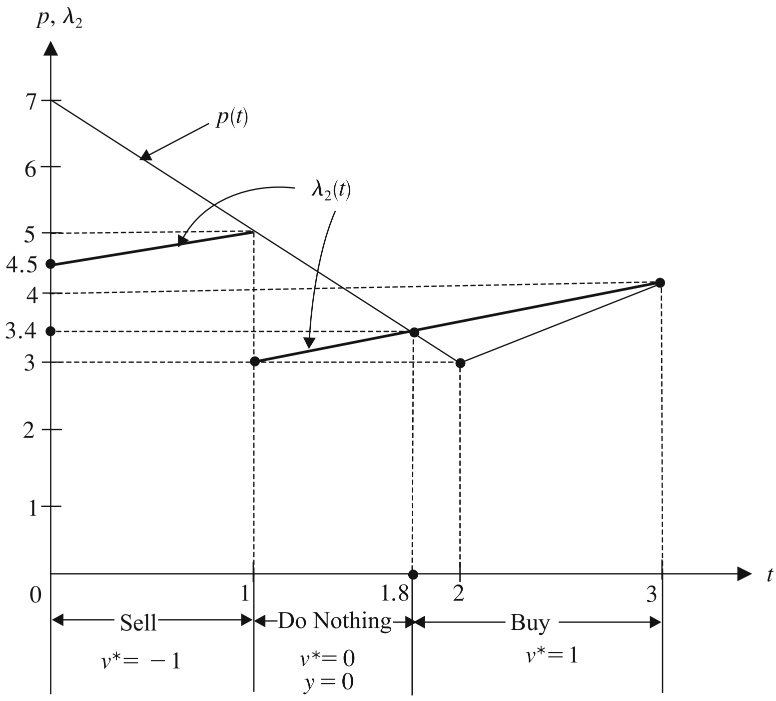

Figure 6.7: Adjoint trajectory and optimal policy for the wheat trading model

is correct. The graphs of $\lambda_2(t)$, $p(t)$, and $v^*(t)$ are displayed in Fig. 6.7. To complete the solution of the problem, you are asked to determine the values of μ_1, μ_2, and η in these various intervals.

6.3 Decision Horizons and Forecast Horizons

In some dynamic problems it is possible to show that the optimal decisions during an initial positive time interval are either partially or wholly independent of the data from some future time onwards. In such cases, a forecast of the future data needs to be made only as far as that time to make optimal decisions in the initial time interval. The initial time interval is called the *decision horizon* and the time up to which data is required to make the optimal decisions during the decision horizon is called the *forecast horizon*; see Bes and Sethi (1988), Bensoussan et al. (1983), and Haurie and Sethi (1984) for details on these concepts. Whenever they exist, these horizons naturally decompose the problem into a series of smaller problems.

If the optimal decisions during the decision horizon are completely independent of the data beyond the forecast horizon, then the latter is called a *strong forecast horizon*. If, on the other hand, some mild restrictions on the data after the forecast horizon are required in order to keep the optimal decisions during the decision horizon unaffected, then it is called a *weak forecast horizon*.

In this section we demonstrate these concepts in the context of the wheat trading model of the previous section. In Sect. 6.3.1 we obtain a decision horizon for the model of Sect. 6.2.4 which is also a weak forecast horizon. In Sect. 6.3.2 we modify the wheat trading model by adding a warehousing constraint. For the new problem we obtain a decision horizon and a strong forecast horizon. See also Sethi and Thompson (1982), Rempala (1986) and Hartl (1986a, 1988a) for further research in the context of the wheat trading model.

In what follows we obtain these horizons and verify them for some examples with different forecast data. For more details and proofs in other situations including more general ones, see Modigliani and Hohn (1955), Lieber (1973), Pekelman (1974, 1975, 1979), Kleindorfer and Lieber (1979), Vanthienen (1975), Morton (1978), Lundin and Morton (1975), Rempala and Sethi (1988, 1992), Hartl (1989a), and Sethi (1990).

6.3.1 Horizons for the Wheat Trading Model with No Short-Selling

For the model of Sect. 6.2.4, we will demonstrate that $t = 1$ is a decision horizon as well as a weak forecast horizon. In Fig. 6.8 we have redrawn Fig. 6.7 with a new price trajectory in the time interval $[1, 3]$. Also in the figure, we have extended the initial λ_2 trajectory and labeled it the *price shield*. Its significance is that, as long as the new price trajectory in the interval $[1, 3]$ stays below the price shield, the optimal solution in the interval $[0, 1]$, which is the decision horizon, remains unchanged. That is, it is optimal to sell throughout the interval. The restriction that $p(t)$ must stay below the price shield in $[1, 3]$ is the reason that $t = 1$ is a *weak* forecast horizon. The optimality of the control shown in Fig. 6.8 can be concluded by obtaining the adjoint trajectory in the interval $[1, 3]$ as a straight line with slope 1/2 and the terminal value $\lambda_2(3^-) = p(3)$. This way of drawing the adjoint trajectory is correct as long as the corresponding policy does not violate the inventory constraint $y(t) \geq 0$ in the interval $[1, 3]$. For example, this will be the case if the buy interval in Fig. 6.8 is shorter than the sell interval at the end. On the other hand, if the inventory constraint is violated, then the $\lambda_2(t)$ trajectory may jump in the interval $[1, 3)$, and it will be more complicated to obtain it. Nevertheless, the decision horizon and weak forecast horizon still occur at $t = 1$. Moreover, if we let $T > 1$ be any finite horizon and assume that $p(t)$ in the interval $[1, T]$ is always below the price shield line of Fig. 6.8 extended to T, then the policy of selling at the maximum rate in the interval $[0, 1]$ remains optimal.

6.3.2 Horizons for the Wheat Trading Model with No Short-Selling and a Warehousing Constraint

In order to give an example in which a strong forecast horizon occurs, we modify the example of Sect. 6.2.4 by adding the warehousing constraint $y \leq 1$ or

$$1 - y \geq 0, \tag{6.82}$$

changing the terminal time to $T = 4$, and defining the price trajectory to be

$$p(t) = \begin{cases} -2t + 7 & \text{for } t \in [0, 2), \\ t + 1 & \text{for } t \in [2, 4]. \end{cases} \tag{6.83}$$

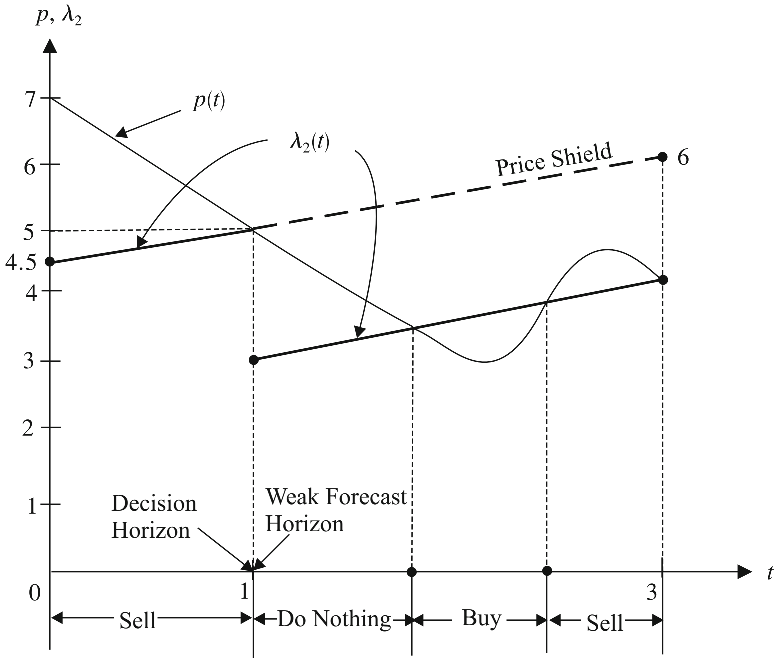

Figure 6.8: Decision horizon and optimal policy for the wheat trading model

The Hamiltonian of the new problem is unchanged and is given in (6.66). Furthermore, $\lambda_1 = 1$. The optimal control is defined in three parts as:

$$v^*(t) \;=\; \text{bang}[-1, 1; \lambda_2(t) - p(t)] \text{ when } 0 < y < 1, \tag{6.84}$$

$$v^*(t) \;=\; \text{bang}\,[0, 1; \lambda_2(t) - p(t)] \text{ when } y = 0, \tag{6.85}$$

$$v^*(t) \;=\; \text{bang}[-1, 0; \lambda_2(t) - p(t)] \text{ when } y = 1. \tag{6.86}$$

Defining a Lagrange multiplier η_1 for the derivative of (6.82), i.e., for $-\dot{y} = -v \geq 0$, we form the Lagrangian

$$L = H + \mu_1(v + 1) + \mu_2(1 - v) + \eta v + \eta_1(-v), \tag{6.87}$$

where μ_1, μ_2, and η satisfy (6.70)–(6.72) and η_1 satisfies

$$\eta_1 \geq 0,\ \eta_1(1 - y) = 0,\ \dot{\eta}_1 \leq 0. \tag{6.88}$$

Furthermore, the optimal trajectory must satisfy

$$\frac{\partial L}{\partial v} = \lambda_2 - p + \mu_1 - \mu_2 + \eta - \eta_1 = 0. \tag{6.89}$$

As before, $\lambda_1 = 1$ and λ_2 satisfies

$$\dot{\lambda}_2 = 1/2,\ \lambda_2(4^-) = p(4) + \gamma_1 - \gamma_2 = 5 + \gamma_1 - \gamma_2, \tag{6.90}$$

where

$$\gamma_1 \geq 0,\ \gamma_1 y(4) = 0,\ \gamma_2 \geq 0,\ \gamma_2(1 - y(4)) = 0. \tag{6.91}$$

Let us first try $\gamma_1 = \gamma_2 = 0$. Let $\hat{t}$ be the time of the last jump of the adjoint function $\lambda_2(t)$ before the terminal time $T = 4$. Then,

$$\lambda_2(t) = t/2 + 3 \text{ for } \hat{t} \leq t < 4. \tag{6.92}$$

The graph of (6.92) intersects the price trajectory at $t = 8/5$ as shown in Fig. 6.9. It also stays above the price trajectory in the interval $[8/5, 4]$ so that, if there were no warehousing constraint (6.82), the optimal decision in this interval would be to buy at the maximum rate. However, with the constraint (6.82), this is not possible. Thus $\hat{t} > 8/5$, since λ_2 will have a jump in the interval $[8/5, 4]$.

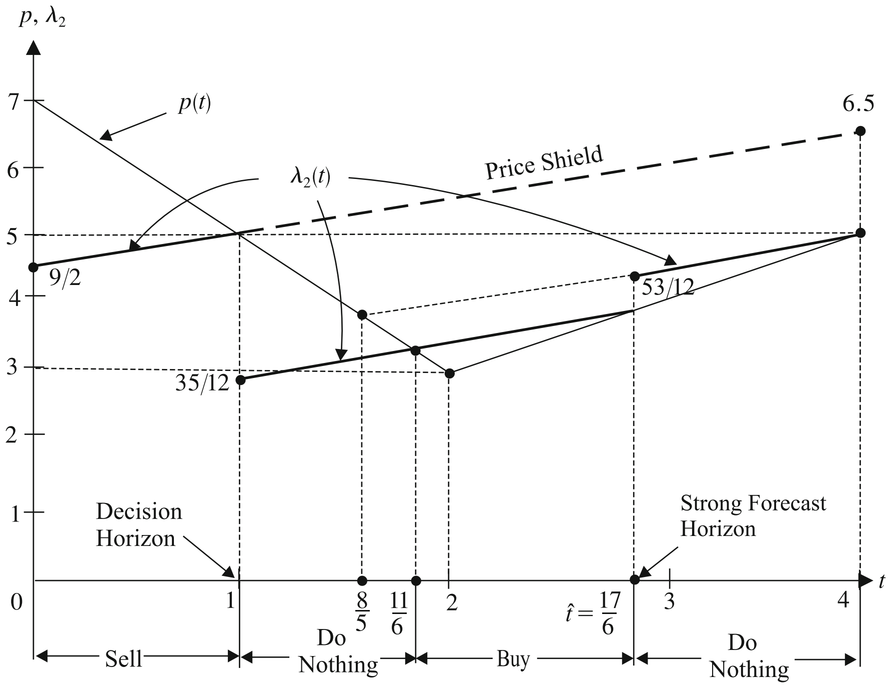

Figure 6.9: Optimal policy and horizons for the wheat trading model with no short-selling and a warehouse constraint

To find the actual value of $\hat{t}$ we must insert a line of slope 1/2 above the minimum price at $t = 2$ in such a way that its two intersection points with the price trajectory are exactly one time unit (the time required to fill up the warehouse) apart. Thus using (6.83), $\hat{t}$ must satisfy

$$-2(\hat{t} - 1) + 7 + (1/2)(1) = \hat{t} + 1,$$

which yields $\hat{t} = 17/6$.

The rest of the analysis for determining λ_2 including the jump conditions is similar to that given in Sect. 6.2.4. Thus,

$$\lambda_2(t) = \begin{cases} t/2 + 9/2 & \text{for } t \in [0, 1), \\ t/2 + 29/12 & \text{for } t \in [1, 17/6), \\ t/2 + 3 & \text{for } t \in [17/6, 4]. \end{cases} \tag{6.93}$$

This makes $\gamma_1 = \gamma_2 = 0$ the correct guess.

Given (6.93), the optimal policy is given by (6.84)–(6.86) and is shown in Fig. 6.9. To complete the maximum principle we must derive expressions for the Lagrange multipliers in the four intervals shown in Fig. 6.9.

Interval $[0, 1)$: $\mu_2 = \eta = \eta_1 = 0,\ \mu_1 = p - \lambda_2 > 0;$

$$v^* = -1,\ 0 < y^* < 1.$$

Interval $[1, 11/6)$: $\mu_1 = \mu_2 = \eta_1 = 0,\ \eta = p - \lambda_2 > 0,\ \dot{\eta} \leq 0;$

$$v^* = 0,\ y^* = 0.$$

Interval $[11/16, 17/6)$: $\mu_1 = \eta = \eta_1 = 0,\ \mu_2 = \lambda_2 - p > 0;$

$$v^* = 1,\ 0 < y^* < 1.$$

Interval $[17/6, 4]$: $\mu_1 = \mu_2 = \eta = 0,\ \eta_1 = \lambda_2 - p > 0,\ \dot{\eta}_1 \leq 0,$
$\gamma_1 = \gamma_2 = 0;$

$$v^* = 0,\ y^* = 1.$$

In Exercise 6.17 you are asked to solve another variant of this problem.

For the example in Fig. 6.9 we have labeled $t = 1$ as a decision horizon and $\hat{t} = 17/6$ as a strong forecast horizon. By this we mean that the optimal decision in $[0,1]$ continues to be to sell at the maximum rate regardless of the price trajectory $p(t)$ for $t > 17/6$. Because $\hat{t} = 17/6$ is a strong forecast horizon, we can terminate the price shield at that time as shown in the figure.

In order to illustrate the statements in the previous paragraph, we consider two examples of price changes after $\hat{t} = 17/6$.

Example 6.3 Assume the price trajectory to be

$$p(t) = \begin{cases} -2t + 7 & \text{for } t \in [0, 2), \\ t + 1 & \text{for } t \in [2, 17/6), \\ 25t/7 - 44/7 & \text{for } t \in [17/6, 4], \end{cases}$$

which is sketched in Fig. 6.10. Note that the price trajectory up to time 17/6 is the same as before, and the price after time 17/6 goes above the extension of the price shield in Fig. 6.9.

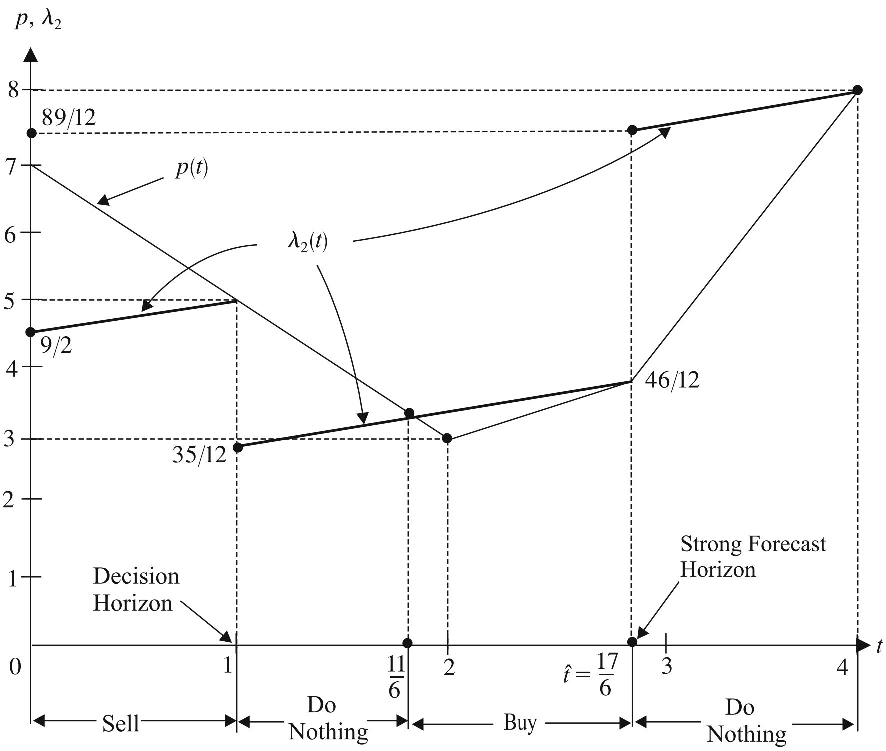

Figure 6.10: Optimal policy and horizons for Example 6.3

Solution The new λ_2 trajectory is shown in Fig. 6.10, which is the same as before for $t < 17/6$, and after that it is $\lambda_2(t) = t/2+6$ for $t \in [17/6,\ 4]$. The optimal policy is as shown in Fig. 6.10, and as previously asserted, the optimal policy in [0,1) remains unchanged. In Exercise 6.17 you are asked to verify the maximum principle for the solution of Fig. 6.10.

Example 6.4 Assume the price trajectory to be

$$p(t) = \begin{cases} -2t+7 & \text{for } t \in [0,2), \\ t+1 & \text{for } t \in [2,17/6), \\ -t/2+21/4 & \text{for } t \in [17/6,4], \end{cases}$$

which is sketched in Fig. 6.11.

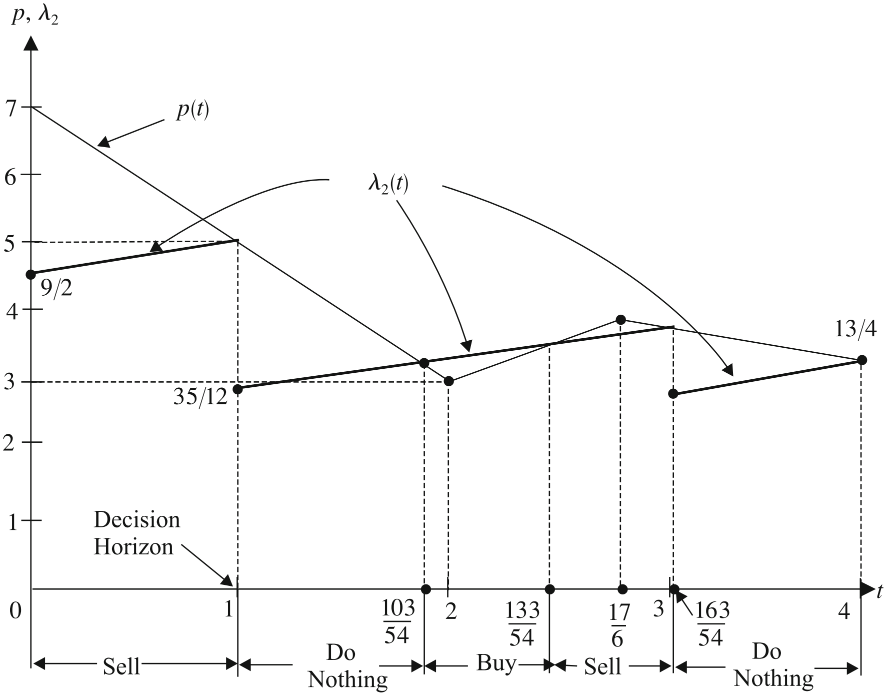

Figure 6.11: Optimal policy and horizons for Example 6.4

Solution Again the price trajectory is the same up to time 17/6, but the price after time 17/6 is declining. This changes the optimal policy in the time interval $[1, 17/6)$, but the optimal policy will still be to sell in $[0, 1)$.

As in the beginning of the section, we solve (6.90) to obtain $\lambda_2(t) = t/2 + 5/4$ for $\hat{t}_1 \le t \le 4$, where $\hat{t}_1 \ge 1$ is the time of the last jump which is to be determined. It is intuitively clear that some profit can be made by buying and selling to take advantage of the price rise between $t = 2$ and $t = 17/6$. For this, the $\lambda_2(t)$ trajectory must cross the price trajectory between times 2 and 17/6 as shown in Fig. 6.11, and the inventory y must go to 0 between times 17/6 and 4 so that λ_2 can jump downward to satisfy the ending condition $\lambda_2(4^-) = p(4) = 13/4$. Since we must buy and sell equal amounts, the point of intersection of the λ_2 trajectory with the rising price segment, i.e., $\hat{t}_1 - \alpha$, must be exactly in the middle of the two other intersection points, $\hat{t}_1$ and $\hat{t}_1 - 2\alpha$, of λ_2 with the two declining price trajectories. Thus, $\hat{t}_1$ and α must satisfy:

$$\begin{aligned} -2(\hat{t}_1 - 2\alpha) + 7 + \alpha/2 &= (\hat{t}_1 - \alpha) + 1, \\ (\hat{t}_1 - \alpha) + 1 + \alpha/2 &= -\hat{t}_1/2 + 21/4. \end{aligned}$$

These can be solved to yield $\hat{t}_1 = 163/54$ and $\alpha = 5/9$. The times $\hat{t}_1, \hat{t}_1 - \alpha$, and $\hat{t}_1 - 2\alpha$ are shown in Fig. 6.11. The λ_2 trajectory is given by

$$\lambda_2(t) = \begin{cases} t/2 + 9/2 & \text{for } t \in [0, 1), \\ t/2 + 241/108 & \text{for } t \in [1, 163/54), \\ t/2 + 5/4 & \text{for } t \in [163/54, 4]. \end{cases}$$

Evaluation of the Lagrange multipliers and verification of the maximum principle is similar to that for the case in Fig. 6.9.

In Sect. 6.3 we have given several examples of decision horizons and weak and strong forecast horizons. In Sect. 6.3.1 we found a decision horizon which was also a weak forecast horizon, and it occurred exactly when $y(t) = 0$. We also introduced the idea of a price shield in that section. In Sect. 6.3.2 we imposed a warehousing constraint and obtained the same decision horizon and a strong forecast horizon, which occurred when $y(t) = 1$.

Note that if we had solved the problem with $T = 1$, then $y^*(1) = 0$; and if we had solved the problem with $T = 17/6$, then $y^*(1) = 0$ and $y^*(17/6) = 1$. The latter problem has the smallest T such that both $y^* = 0$ and $y^* = 1$ occur for $t > 0$, given the price trajectory. This is one of the ways that time $t = 17/6$ can be found to be a forecast horizon along with the decision horizon at time $t = 1$. There are other ways to find strong forecast horizons. For a survey of the literature, see Chand et al. (2002).

Exercises for Chapter 6

E 6.1 Verify the expressions for a_1 and a_2 given in (6.16) and (6.17).

E 6.2 Verify (6.27). Note that $\rho = 0$ is assumed in Sect. 6.1.4.

E 6.3 Verify (6.29). Again assume $\rho = 0$.

E 6.4 Given the demand function

$$S = t(t-4)(t-8)(t-12)(t-16) + 30,$$

$\rho = 0, \hat{I} = 15, T = 16,$ and $\alpha = 1$, obtain $Q(t)$ from (6.27).

E 6.5 Complete the solution of Example 6.2 in Sect. 6.1.4.

E 6.6 For the model of Sect. 6.1.6, derive the turnpike triple by using the conditions in (6.39).

E 6.7 Solve the production-inventory model of Sect. 6.1.6 for the parameter values listed on Fig. 6.4, and draw the figure using MATLAB or another suitable software.

E 6.8 Give an intuitive interpretation of (6.55).

E 6.9 Assume that there is a transaction cost cv^2 when v units of wheat are bought or sold in the model of Sect. 6.2.1. Derive the form of the optimal policy.

E 6.10 In Exercise 6.9, assume $T = 10$, $x(0) = 10$, $y(0) = 0$, $c = 1/18$, $h(y) = (1/2)y^2$, $V_1 = V_2 = \infty$, $r = 0$, and $p(t) = 10 + t$. Solve the resulting TPBVP to obtain the optimal control in closed form.

E 6.11 Set up the two-point boundary value problem for Exercise 6.9 with $c = 0.05$, $h(y) = (1/2)y^2$, and the remaining values of parameters as in the model of Sect. 6.2.3.

E 6.12 Use Excel, as illustrated in Sect. 2.5, to solve the TPBVP of Exercise 6.11.

E 6.13 Show that the solution obtained for the problem in Sect. 6.2.3 satisfies the necessary conditions of the maximum principle. Conclude the optimality of the solution by showing that the maximum principle conditions are also sufficient.

E 6.14 Re-solve the problem of Sect. 6.2.3 with $V_1 = 2$ and $V_2 = 1$.

E 6.15 Compute the optimal trajectories for $\mu_1,\ \mu_2$, and η for the model in Sect. 6.2.4.

E 6.16 Solve the model in Sect. 6.2.4 with each of the following conditions:

$$(a) \quad y(0) = 2.$$

$$(b) \quad T = 10 \text{ and } p(t) = 2t - 2 \text{ for } 3 \le t \le 10.$$

E 6.17 Verify that the solutions shown in Figs. 6.10 and 6.11 satisfy the maximum principle.

E 6.18 Re-solve the model of Sect. 6.3.2 with $y(0) = 1/2$ and with the warehousing constraint $y \le 1/2$ in place of (6.82).

E 6.19 Solve and interpret the following production planning problem with linear inventory holding costs:

$$\begin{cases} \max \left\{ J = \displaystyle\int_0^T -[hI + \frac{c}{2}P^2]dt \right\} \\ \text{subject to} \\ \dot{I} = P,\ I(0) = 0,\ I(T) = B;\ 0 < B < hT^2/2c, \\ P \ge 0 \text{ and } I \ge 0. \end{cases} \tag{6.94}$$

E 6.20 Re-solve Exercise 6.19 with the state equation $\dot{I}(t) = P(t) - S(t)$, where $I(0) = I_0 \ge 0$ and $I(T)$ is not fixed. Assume the demand $S(t)$ to be continuous in t and non-negative. Keep the state constraint $I \ge 0$, but drop the production constraint $P \ge 0$ for simplicity. For specificity, you may assume $S = -\sin \pi t + C$ with the constant $C \ge 1$ and $T = 4$. (Note that negative production can and will occur when initial inventory I_0 is too large. Specifically, how large is too large depends on the parameters of the problem.)

E 6.21 Re-solve Exercise 6.19 with the state equation $\dot{I}(t) = P(t) - S$, where $S > 0$ and $h > 0$ are constants, $I(0) = I_0 > cS^2/2h$, and $I(T)$ is not fixed. Assume that T is sufficiently large. Also, graph the optimal $P^*(t)$ and $I^*(t),\ t \in [0, T]$.

Chapter 7

Applications to Marketing

Over the years, a number of applications of optimal control theory have been made to the field of marketing. Many of these applications deal with the problem of finding or characterizing the optimal advertising rate over time. Others deal with the problem of determining the optimal price and quality over time, in addition to or without advertising. The reader is referred to Sethi (1977a) and Feichtinger et al. (1994a) for comprehensive reviews on dynamic optimal control problems in advertising and related problems. In this chapter we discuss optimal advertising policies for two of the well-known models called the Nerlove-Arrow model and the Vidale-Wolfe model.

To describe the specific problems under consideration, let us assume that a firm has some way of knowing or estimating the dynamics of sales and advertising. Such knowledge is expressed in terms of a differential equation with either goodwill or the rate of sales as the state variable and the rate of advertising expenditures as the control variable. We assume that the firm wishes to maximize an objective function (the criterion function) which reflects its profit motives expressed in terms of sales and advertising rates. The optimal control problem is to find an advertising policy which maximizes the firm's objective function.

The plan of this chapter is as follows. Section 7.1 will cover the Nerlove-Arrow model as well as a nonlinear extension of it. Section 7.2 deals with the Vidale-Wolfe advertising model and its detailed analysis using Green's theorem in conjunction with the maximum principle. The switching-point analysis for this problem is a good example of the reverse-time construction technique used earlier in Chaps. 4 and 5. Ex-

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tensions of these models to multi-state problems are treated in Turner and Neuman (1976) and Srinivasan (1976).

7.1 The Nerlove-Arrow Advertising Model

The belief that advertising expenditures by a firm affect its present and future sales, and hence its present and future net revenues, has led a number of economists including Nerlove and Arrow (1962) to treat advertising as an investment in building up some sort of advertising capital, usually called *goodwill*. Furthermore, the stock of goodwill depreciates over time. Vidale and Wolfe (1957), Palda (1964), and others present empirical evidence that the effects of advertising linger but diminish over time.

Goodwill may be created by adding new customers or by altering the tastes and preferences of consumers and thus changing the demand function for the firm's product. Goodwill depreciates over time because consumers "drift" to other brands as a result of advertising by competing firms and the introduction of new products and/or new brands, etc.

7.1.1 The Model

Let $G(t) \geq 0$ denote the stock of goodwill at time t. The price of (or cost of producing) one unit of goodwill is one dollar so that a dollar spent on current advertising increases goodwill by one unit. It is assumed that the stock of goodwill depreciates over time at a constant proportional rate δ, so that

$$\dot{G} = u - \delta G, \quad G(0) = G_0, \tag{7.1}$$

where $u = u(t) \geq 0$ is the advertising effort at time t measured in dollars per unit time. In economic terms, Eq. (7.1) states that the net investment in goodwill is the difference between gross investment $u(t)$ and depreciation $\delta G(t)$.

To formulate the optimal control problem for a monopolistic firm, assume that the rate of sales $S(t)$ depends on the stock of goodwill $G(t)$, the price $p(t)$, and other exogenous factors $Z(t)$, such as consumer income, population size, etc. Thus,

$$S = S(p, G; Z). \tag{7.2}$$

Assuming the rate of total production cost is $c(S)$, we can write the total revenue net of production cost as

$$R(p, G; Z) = pS(p, G; Z) - c(S(p, G; Z)). \tag{7.3}$$

The revenue net of advertising expenditure is therefore $R(p, G; Z) - u$. We assume that the firm wants to maximize the present value of net revenue streams discounted at a fixed rate ρ, i.e.,

$$\max_{u \geq 0, p \geq 0} \left\{ J = \int_0^\infty e^{-\rho t}[R(p, G; Z) - u]\, dt \right\} \tag{7.4}$$

subject to (7.1).

Since the only place that p occurs is in the integrand, we can maximize J by first maximizing R with respect to price p while holding G fixed, and then maximize the result with respect to u. Thus,

$$\frac{\partial R}{\partial p} = S + p\frac{\partial S}{\partial p} - c'(S)\frac{\partial S}{\partial p} = 0, \tag{7.5}$$

which implicitly gives the optimal price $p^*(t) = p(G(t); Z(t))$. Defining $\eta = -(p/S)(\partial S/\partial p)$ as the elasticity of demand with respect to price, we can rewrite condition (7.5) as

$$p^* = \frac{\eta c'(S)}{\eta - 1}, \tag{7.6}$$

which is the usual price formula for a monopolist, known sometimes as the Amoroso-Robinson relation. You are asked to derive this relation in Exercise 7.2. In words, the relation means that the marginal revenue $(\eta - 1)p/\eta$ must equal the marginal cost $c'(S)$. See, e.g., Cohen and Cyert (1965, p. 189).

Defining $\Pi(G; Z) = R(p^*, G; Z)$, the objective function in (7.4) can be rewritten as

$$\max_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t}[\Pi(G; Z) - u]\, dt \right\}.$$

For convenience, we assume Z to be a given constant. Thus, we can define $\pi(G) = \Pi(G; Z)$ and restate the optimal control problem which we have just formulated:

$$\begin{cases} \max\limits_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t}[\pi(G) - u]\, dt \right\} \\ \text{subject to} \\ \dot{G} = u - \delta G, \;\; G(0) = G_0. \end{cases} \tag{7.7}$$

Furthermore, it is reasonable to assume the functions introduced in (7.2) and (7.3) to satisfy conditions so that $\pi(G)$ is increasing and concave in goodwill G. More specifically, we assume that $\pi'(G) \geq 0$ and $\pi''(G) < 0$.

7.1.2 Solution by the Maximum Principle

While Nerlove and Arrow (1962) used calculus of variations, we use Pontryagin's maximum principle to derive their results. We form the current-value Hamiltonian

$$H = \pi(G) - u + \lambda[u - \delta G] \tag{7.8}$$

with the current-value adjoint variable λ satisfying the differential equation

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{d\pi}{dG} \tag{7.9}$$

and the condition that

$$\lim_{t \to +\infty} e^{-\rho t}\lambda(t) = 0. \tag{7.10}$$

Recall from Sect. 3.6 that this limit condition is only a sufficient condition.

The adjoint variable $\lambda(t)$ is the shadow price associated with the goodwill at time t. Thus, the Hamiltonian in (7.8) can be interpreted as the dynamic profit rate which consists of two terms: (1) the current net profit rate $(\pi(G) - u)$ and (2) the value $\lambda\dot{G} = \lambda[u - \delta G]$ of the goodwill rate $\dot{G}$ created by advertising at rate u. Also, Eq. (7.9) corresponds to the usual equilibrium relation for investment in capital goods; see Arrow and Kurz (1970) and Jacquemin (1973). It states that the marginal opportunity cost $\lambda(\rho + \delta)dt$ of investment in goodwill, by spending on advertising, should equal the sum of the marginal profit $\pi'(G)dt$ from the increased goodwill due to that investment and the capital gain $d\lambda := \dot{\lambda}dt$ on the unit price of goodwill.

We use (3.108) to obtain the optimal long-run stationary equilibrium or turnpike $\{\bar{G}, \bar{u}, \bar{\lambda}\}$. That is, we obtain $\lambda = \bar{\lambda} = 1$ from (7.8) by using $\partial H/\partial u = 0$. We then set $\lambda = \bar{\lambda} = 1$ and $\dot{\lambda} = 0$ in (7.9) to obtain

$$\pi'(\bar{G}) = \rho + \delta. \tag{7.11}$$

In order to obtain a strictly positive equilibrium goodwill level $\bar{G}$, we may assume $\pi'(0) > \rho + \delta$ and $\pi'(\infty) < \rho + \delta$.

Before proceeding further to obtain the optimal advertising policy, let us relate (7.11) to the equilibrium condition for $\bar{G}$ obtained by Jacquemin (1973). For this we define $\beta = (G/S)(\partial S/\partial G)$ as the elasticity of demand with respect to goodwill. We can now use (7.3), (7.5), (7.6), and (7.9) with $\dot{\lambda} = 0$ and $\bar{\lambda} = 1$ to derive, as you will in Exercise 7.3,

$$\frac{\bar{G}}{pS} = \frac{\beta}{\eta(\rho + \delta)}. \tag{7.12}$$

The interpretation of (7.12) is that in the equilibrium, the ratio of goodwill to sales revenue pS is directly proportional to the goodwill elasticity, inversely proportional to the price elasticity, and inversely proportional to the cost of maintaining goodwill given by the marginal opportunity cost $\lambda(\rho + \delta)$ of investment in goodwill.

The property of $\bar{G}$ is that the optimal policy is to go to $\bar{G}$ as fast as possible. If $G_0 < \bar{G}$, it is optimal to jump instantaneously to $\bar{G}$ by applying an appropriate impulse at $t = 0$ and then set $u^*(t) = \bar{u} = \delta\bar{G}$ for $t > 0$. If $G_0 > \bar{G}$, the optimal control $u^*(t) = 0$ until the stock of goodwill depreciates to the level $\bar{G}$, at which time the control switches to $u^*(t) = \delta\bar{G}$ and stays at this level to maintain the level $\bar{G}$ of goodwill. This optimal policy is graphed in Fig. 7.1 for these two different initial conditions.

Of course, if we had imposed an upperbound $M > 0$ on the control so that $0 \le u \le M$, then for $G_0 < \bar{G}$, we would use $u^*(t) = M$ until the goodwill stock reaches $\bar{G}$ and switch to $u^*(t) = \bar{u}$ thereafter. This is shown as the dotted curve in Fig. 7.1.

Problem (7.7) is formulated with the assumption that a dollar spent on current advertising increases goodwill by one unit. Suppose, instead, that we need to spend m dollars on current advertising to increase goodwill by one unit. We can then define u as advertising effort costing the firm mu dollars, and reformulate problem (7.7) by replacing $[\pi(G) - u]$

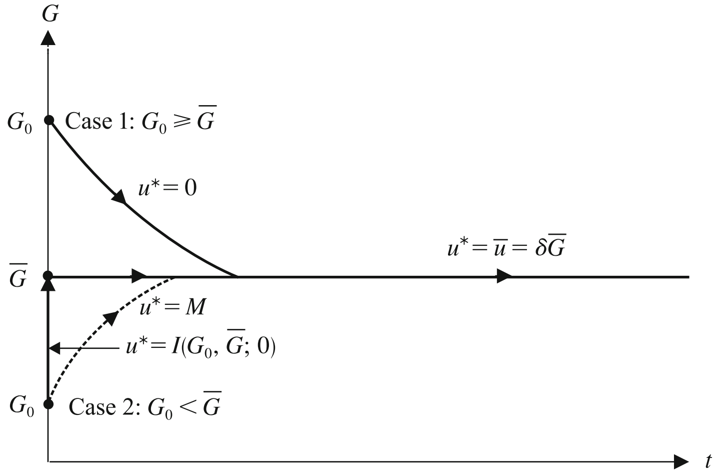

Figure 7.1: Optimal policies in the Nerlove-Arrow model

in its integrand by $[\pi(G) - mu]$. In Exercise 7.4, you are asked to solve problem (7.7) with its objective function and the control constraint replaced by

$$\max_{0 \le u \le M} \left\{ J = \int_0^\infty e^{-\rho t}[\pi(G) - mu]dt \right\}, \tag{7.13}$$

and show that the equilibrium goodwill level formula (7.11) changes to

$$\pi'(\bar{G}) = (\rho + \delta)m. \tag{7.14}$$

With $\bar{G}$ thus defined, the optimal solution is as shown in Fig. 7.1 with the dotted curve representing the solution in Case 2: $G_0 < \bar{G}$.

For a time-dependent Z, however, $\bar{G}(t) = G(Z(t))$ will be a function of time. To maintain this level of $\bar{G}(t)$, the required control is $\bar{u}(t) = \delta\bar{G}(t) + \dot{\bar{G}}(t)$. If $\bar{G}(t)$ is decreasing sufficiently fast, then $\bar{u}(t)$ may become negative and thus infeasible. If $\bar{u}(t) \ge 0$ for all t, then the optimal policy is as before. However, suppose $\bar{u}(t)$ is infeasible in the interval $[t_1, t_2]$ shown in Fig. 7.2. In such a case, it is feasible to set $u(t) = \bar{u}(t)$ for $t \le t_1$; at $t = t_1$ (which is point A in Fig. 7.2) we can no longer stay on the turnpike and must set $u(t) = 0$ until we hit the turnpike again (at point B in Fig. 7.2). However, such a policy is not necessarily optimal. For instance, suppose we leave the turnpike at point C anticipating the infeasibility at point A. The new path CDEB may be better than the old path CAB. Roughly the reason this may happen is that path CDEB is "nearer" to the turnpike than CAB. The picture in Fig. 7.2 illustrates such a case. The optimal policy is the one that is "nearest" to the turnpike. This discussion will become clearer in Sect. 7.2.2, when a similar situation arises in connection with the Vidale-Wolfe model. For further details; see Sethi (1977b) and Breakwell (1968).

The Nerlove-Arrow model is an example involving bang-bang and impulse controls followed by a singular control, which arises in a class of optimal control problems of Model Type (b) in Table 3.3 that are linear in control.

Nonlinear extensions of the Nerlove-Arrow model have been offered in the literature. These amount to making the objective function nonlinear in advertising. Gould (1970) extended the model by assuming a

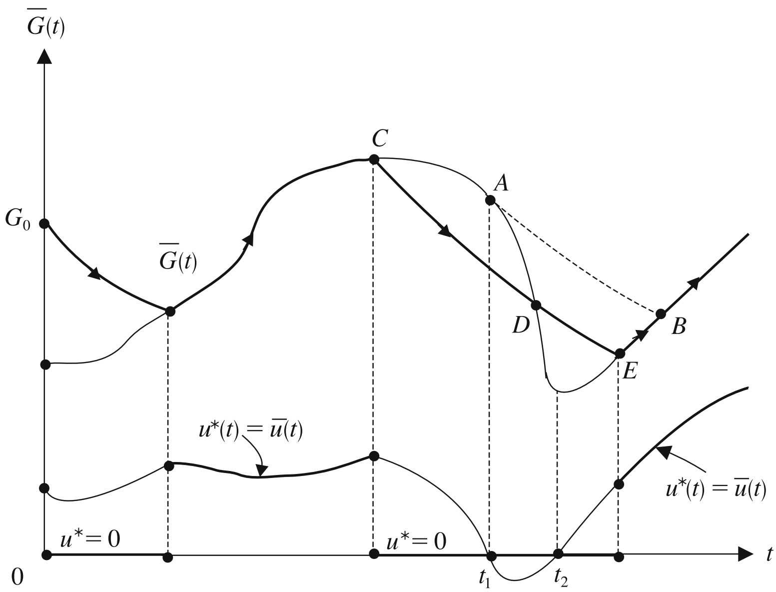

Figure 7.2: A case of a time-dependent turnpike and the nature of optimal control

convex cost of advertising effort, which implies a marginally diminishing effect of advertising expenditures. Jacquemin (1973) assumed that the current demand function S in (7.2) also depends explicitly on the current advertising effort u. In Exercise 11.6, you are asked to analyze Gould's extension via the phase diagram analysis introduced in Chap. 11. The analysis of Jacquemin's extension is similar.

7.1.3 Convex Advertising Cost and Relaxed Controls

Another nonlinear extension of the Nerlove-Arrow model would involve a concave advertising cost resulting from quantity discounts that may be available in the purchase of advertising. Such an extension results in an optimal control problem with a profit rate that is convex in advertising, and this has a possibility of rendering the problem without an optimal solution within the class of admissible controls discussed thus far. What is then required is an enlargement of the class to include what are known as *relaxed controls*. To introduce such controls, we formulate and solve a convex optimal control problem involving the Nerlove-Arrow model.

Let $c(u)$ be a strictly concave advertising cost function with $c(0) = 0$, $c'(u) > 0$ and $c''(u) < 0$ for $0 \leq u \leq M$, where $M > 0$ denotes an upperbound on the advertising rate. Let us also consider $T > 0$ to be the fixed terminal time. Then, our problem is the following modification of problem (7.7):

$$\begin{cases} \max\limits_{0\leq u\leq M} \left\{ J_1 = \displaystyle\int_0^T e^{-\rho t}[\pi(G) - c(u)]dt \right\} \\ \text{subject to} \\ \dot{G} = u - \delta G, \quad G(0) = G_0. \end{cases} \tag{7.15}$$

Note that with concave $c(u)$, the profit rate $\pi(G) - c(u)$ is convex in u. Thus, its maximum over u would occur at the boundary 0 or M of the set $[0, M]$. It should be clear that if we replace $c(u)$ by the linear function mu with $m = c(M)/M$, then

$$\pi(G) - c(u) < \pi(G) - mu, \; u \in (0, M). \tag{7.16}$$

This means that if problem (7.15) with mu in place of $c(u)$, i.e., the problem

$$\begin{cases} \max\limits_{0\leq u\leq M} \left\{ J_2 = \displaystyle\int_0^T e^{-\rho t}\left[\pi(G) - mu\right]dt \right\} \\ \text{subject to} \\ \dot{G} = u - \delta G, \quad G(0) = G_0 \end{cases} \tag{7.17}$$

has only the bang-bang solution, then the solution of problem (7.17) would also be the solution of the convex problem (7.15). Given the similarity of problem (7.17) to problem (7.7), we can see that for a sufficiently small value of T, the solution of (7.17) will be bang-bang only, and therefore, it will also solve (7.15). However, if T is large or infinity, then the solution of (7.17) will have a singular portion, and it will not solve (7.17).

In particular, let us consider problems (7.15) and (7.17) when $T = \infty$ and $G_0 < \bar{G}$. Note that problem (7.17) is the same as the problem in Exercise 7.4, and its optimal solution is as shown in Fig. 7.1 with $\bar{G}$ given by (7.14) and the optimal trajectory given by the dotted line followed by the solid horizontal line representing the singular part of the solution.

Let u_2^* denote the optimal control of problem (7.17). Since the singular control is in the open interval $(0, M)$, then in view of (7.16),

$$J_1(u_2^*) < J_2(u_2^*). \tag{7.18}$$

Thus, for sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we can "chatter" between $G_1 = (\bar{G} + \varepsilon_1)$ and $G_2 = (\bar{G} - \varepsilon_2)$ by using controls M and 0 alternately, as shown in Fig. 7.3, to obtain a near-optimal control of problem (7.15). Clearly, in the limit as ε_1 and ε_2 go to 0, the objective function of problem (7.15) will converge to $J_2(u_2^*)$.

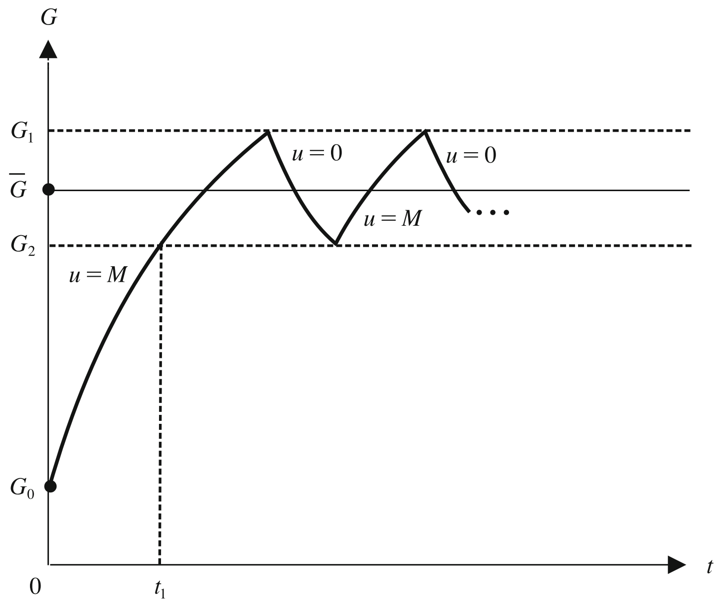

Figure 7.3: A near-optimal control of problem (7.15)

This is an intuitive explanation that there does not exist an optimal control of problem (7.15) in the class of controls discussed thus far. However, when the class of controls is enlarged to include relaxed or generalized controls, which are the limits of the approximating controls like the ones constructed above, we can recover the existence of an optimal solution; see Gamkrelidze (1978) and Lou (2007) for details.

The manner in which the theory of relaxed controls manifests itself for our problem is to provide a probability measure on the boundary values $\{0, M\}$. Thus, let v be the probability that control M is used, so that the probability of using control 0 is $(1-v)$. With this, we transform problem (7.15) with $T = \infty$ as follows:

$$\begin{cases} \max\limits_{v \in [0,1]} \left\{ J_3 = \int_0^\infty e^{-\rho t} \left[\pi(G) - vc(M)\right] dt \right\} \\ \text{subject to} \\ \dot{G} = vM - \delta G, \quad G(0) = G_0. \end{cases} \tag{7.19}$$

We can now use the maximum principle to solve problem (7.19). Thus, the Hamiltonian

$$H = \pi G - vc(M) + \lambda(vM - \delta G)$$

with the adjoint equation as defined by (7.9) and (7.10). The optimal control is given by

$$v^* = \text{bang}[0, 1; \lambda M - c(M)]. \tag{7.20}$$

The singular control is given by

$$\bar{\lambda} = m, \ \pi'(\bar{G}) = (\rho + \delta)m, \ \bar{v} = \delta \bar{G}/M. \tag{7.21}$$

The way we interpret this control is by use of a biased coin with the probability of heads being $\bar{v}$. We flip this coin infinitely fast, and use the maximum control M when heads comes up and the minimum control 0 when tails comes up. Because the control will chatter infinitely fast according to the outcome of the coin tosses, such a control is also referred to as a *chattering control.*

While such a chattering control cannot be implemented, it can be arbitrarily approximated by using alternately $u^* = M$ for $\tau\bar{v}$ periods and $u^* = 0$ for $\tau(1 - \bar{v})$ periods for a small $\tau > 0$. With reference to Fig. 7.3 and with G_1 and G_2 to be determined for the given τ, this approximate policy of rapidly switching the control between M and 0 can be said to begin at time t_1, when the goodwill reaches G_2. After that goodwill goes up to G_1 and then back down to G_2, and so on. The values of G_1 and G_2, corresponding to the given τ, are specified in Exercise 7.8, and you are asked to derive them.

In marketing parlance, advertising rates that alternate between maximum and zero are known as a *pulsing policy*. While there are other reasons for pulsing that are known in the advertising literature, the convex cost of advertising is one of them; see Feinberg (1992, 2001) for details.

Another example of relaxed control appears in Haruvy et al. (2003) in connection with open-source software development. This is given as Exercise 7.14.

7.2 The Vidale-Wolfe Advertising Model

We now present the analysis of the Vidale-Wolfe advertising model which, in contrast to the Nerlove-Arrow model, does not make use of the idea of advertising goodwill; see Vidale and Wolfe (1957) and Sethi (1973a, 1974b). Instead the model exploits the closely related notion that the effect of advertising tends to persist, but diminishes over subsequent time periods. This carryover effect is modeled explicitly by means of a differential equation that gives the relationship between sales and advertising.

Vidale and Wolfe argued that changes in the rate of sales of a product depend on two effects: the action of advertising (via the response constant a) on the unsold portion of the market and the loss of sales (via the decay constant b) from the sold portion of the market. Let $M(t)$, known as the saturation level or market potential, denote the maximum potential rate of sales at time t. Let $S(t)$ be the actual rate of sales at time t. Then, the Vidale-Wolfe model for a monopolistic firm can be stated as

$$\dot{S} = au(1 - \frac{S}{M}) - bS. \tag{7.22}$$

The important feature of this equation, which distinguishes it from the Nerlove-Arrow equation (7.1), is the idea of the finite saturation level M. The Vidale-Wolfe model exhibits diminishing returns to the level of advertising as a direct consequence of this saturation phenomenon. Note that when M is infinitely large, the saturation phenomenon disappears, reducing (7.22) to the equation (with constant returns to advertising) similar to the Nerlove-Arrow equation (7.1). Nerlove and Arrow, on the other hand, include the idea of diminishing returns to advertising in their model by making the sales S in (7.2) a concave function of goodwill.

Vidale and Wolfe based their model on the results of several experimental studies of advertising effectiveness, which are described in Vidale and Wolfe (1957).

7.2.1 Optimal Control Formulation for the Vidale-Wolfe Model

Whereas Vidale and Wolfe offered their model primarily as a description of actual market phenomena represented by cases which they had observed, we obtain the optimal advertising expenditures for the model in order to maximize a certain objective function over the horizon T, while also attaining a terminal sales target; see Sethi (1973a). For this, it is convenient to transform (7.22) by making the change of variable

$$x = \frac{S}{M}. \tag{7.23}$$

Thus, x represents the market share (or more precisely, the rate of sales expressed as a fraction of the saturation level M). Furthermore, we define

$$r = \frac{a}{M},\ \delta = b + \frac{\dot{M}}{M}. \tag{7.24}$$

Now we can rewrite (7.22) as

$$\dot{x} = ru(1 - x) - \delta x,\ x(0) = x. \tag{7.25}$$

From now on we assume M, and hence δ and r, to be positive constants. It would not be difficult to extend the analysis when M depends on t, but we do not carry it out here. In Exercise 7.35 you are asked to partially analyze the time-dependent case.

To define the optimal control problem arising from the Vidale-Wolfe model, we let π denote the maximum sales revenue corresponding to $x = 1$, with πx denoting the revenue function for $x \in [0, 1]$. Also let Q be the maximum allowable rate of advertising expenditure and let ρ denote the continuous discount rate. With these definitions the optimal control problem can be stated as follows:

$$\begin{cases} \max\left\{ J = \int_0^T e^{-\rho t}(\pi x - u)dt \right\} \\ \text{subject to} \\ \dot{x} = ru(1-x) - \delta x,\ x(0) = x_0, \\ \text{the terminal state constraint} \\ x(T) = x_T, \\ \text{and the control constraint} \\ 0 \le u \le Q. \end{cases} \tag{7.26}$$

Here Q can be finite or infinite and the target market share x_T is in $[0, 1]$. Note that the problem is a fixed-end-point problem. It is obvious that the requirement $0 \le x \le 1$ holds without being imposed, where $x_0 \in [0, 1]$ is the initial market share.

It is possible to solve this problem by an application of the maximum principle; see Exercise 7.18. However, we will use instead a method based on Green's theorem which does not make use of the maximum principle. This method provides a convenient procedure for solving fixed-end-point problems having one state variable and one control variable, and where the control variable appears linearly in both the state equation and the objective function; see Miele (1962) and Sethi (1977b). Problem (7.26) has these properties, and therefore it is also a good example with which to illustrate the method. For the application of Green's theorem we require that Q be large. In particular we can let $Q = \infty$.

7.2.2 Solution Using Green's Theorem When Q Is Large

In this section we will solve the fixed-end-point problem starting with x_0 and ending with x_T, under the assumption that Q is either unbounded or very large. The places where these assumptions are needed will be indicated.

To make use of Green's theorem, it is convenient to consider times τ and θ, where $0 \le \tau < \theta \le T$, and the problem:

$$\max\left\{ J(\tau, \theta) = \int_\tau^\theta e^{-\rho t}(\pi x - u)dt \right\} \tag{7.27}$$

subject to

$$\dot{x} = ru(1-x) - \delta x, \ \ x(\tau) = A, \ \ x(\theta) = B, \tag{7.28}$$

$$0 \le u \le Q. \tag{7.29}$$

To change the objective function in (7.27) into a line integral along any feasible arc Γ_1 from (τ, A) to (θ, B) in (t, x)-space as shown in Fig. 7.4, we multiply (7.28) by dt and obtain the formal relation

$$udt = \frac{dx + \delta x dt}{r(1-x)},$$

which we substitute into the objective function (7.27). Thus,

$$J_{\Gamma_1} = \int_{\Gamma_1} \left\{ \left[\pi x - \frac{\delta x}{r(1-x)} \right] e^{-\rho t} dt - \frac{1}{r(1-x)} e^{-\rho t} dx \right\}.$$

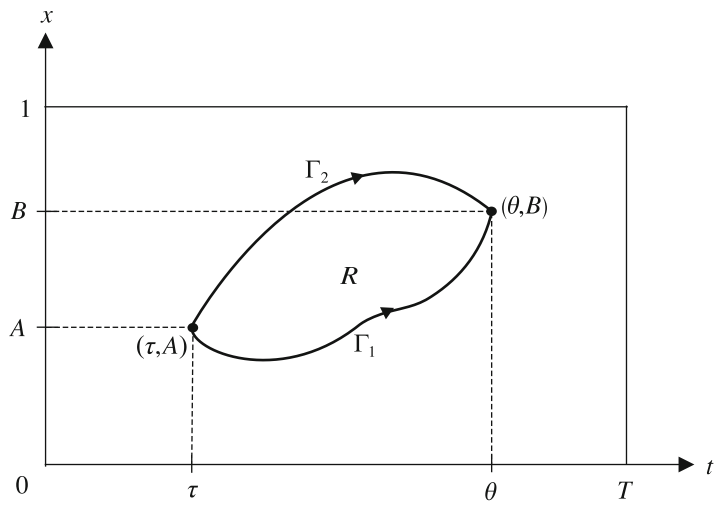

Figure 7.4: Feasible arcs in (t, x)-space

Consider another feasible arc Γ_2 from (τ, A) to (θ, B) lying above Γ_1 as shown in Fig. 7.4. Let $\Gamma = \Gamma_1 - \Gamma_2$, where Γ is a simple closed curve traversed in the counter-clockwise direction. That is, Γ goes along Γ_1 in the direction of its arrow and along Γ_2 in the direction opposite to its arrow. We now have

$$J_{\Gamma} = J_{\Gamma_1 - \Gamma_2} = J_{\Gamma_1} - J_{\Gamma_2}. \tag{7.30}$$

Since Γ is a simple closed curve, we can use Green's theorem to express J_Γ as an area integral over the region R enclosed by Γ. Thus, treating x and t as independent variables, we can write

$$\begin{aligned} J_\Gamma &= \oint_\Gamma \left\{ \left[\pi x - \frac{\delta x}{r(1-x)} \right] e^{-\rho t} dt - \frac{1}{r(1-x)} e^{-\rho t} dx \right\} \\ &= \int\int_R \left\{ \frac{\partial}{\partial t} \left[\frac{-e^{-\rho t}}{r(1-x)} \right] - \frac{\partial}{\partial x} \left[(\pi x - \frac{\delta x}{r(1-x)}) e^{-\rho t} \right] \right\} dt dx \\ &= \int\int_R \left[\frac{\delta}{(1-x)^2} + \frac{\rho}{(1-x)} - \pi r \right] \frac{e^{-\rho t}}{r} dt dx. \end{aligned} \tag{7.31}$$

Denote the term in brackets of the integrand of (7.31) by

$$I(x) = \frac{\delta}{(1-x)^2} + \frac{\rho}{(1-x)} - \pi r. \tag{7.32}$$

Note that the sign of the integrand is the same as the sign of $I(x)$.

Lemma 7.1 (Comparison Lemma) *Let Γ_1 and Γ_2 be the lower and upper feasible arcs as shown in Fig. 7.4. If $I(x) \geq 0$ for all $(x,t) \in R$, then the lower arc Γ_1 is at least as profitable as the upper arc Γ_2. Analogously, if $I(x) \leq 0$ for all $(x,t) \in R$, then Γ_2 is at least as profitable as Γ_1.*

Proof If $I(x) \geq 0$ for all $(x,t) \in R$, then $J_\Gamma \geq 0$ from (7.31) and (7.32). Hence from (7.30), $J_{\Gamma_1} \geq J_{\Gamma_2}$. The proof of the other statement is similar. □

To make use of this lemma to find the optimal control for the problem stated in (7.26), we need to find regions where $I(x)$ is positive and where it is negative. For this, note first that $I(x)$ is an increasing function of x in $[0,1]$. Solving $I(x) = 0$ will give that value of x, above which $I(x)$ is positive and below which $I(x)$ is negative. Since $I(x)$ is quadratic in $1/(1-x)$, we can use the quadratic formula (see Exercise 7.16) to get

$$x = 1 - \frac{2\delta}{-\rho \pm \sqrt{\rho^2 + 4\pi r \delta}}. \tag{7.33}$$

To keep x in the interval $[0,1]$, we must choose the positive sign before the radical. The optimal x must be nonnegative so we have

$$x^s = \max\left\{ 1 - \frac{2\delta}{-\rho + \sqrt{\rho^2 + 4\pi r \delta}}, 0 \right\}, \tag{7.34}$$

where the superscript s is used because this will turn out to be a singular trajectory. Since x^s is nonnegative, the control

$$u^s = \frac{\delta x^s}{r(1 - x^s)} \tag{7.35}$$

corresponding to (7.34) will always be nonnegative. Also since Q is assumed to be large, u^s will always be feasible. Moreover, in Exercise 7.17, you will be asked to show that $x^s = 0$ and $u^s = 0$ if, and only if, $\pi r \leq \delta + \rho$.

We now have enough machinery to obtain the optimal solution for (7.26) when Q is assumed to be sufficiently large, i.e., $Q \geq u^s$, where u^s is given in (7.35). We state these in the form of two theorems: Theorem 7.1 refers to the case in which T is *large*; Theorem 7.2 refers to the case in which T is *small*. To define these concepts, let t_1 be the shortest time to go from x_0 to x^s and similarly let t_2 be the shortest time to go from

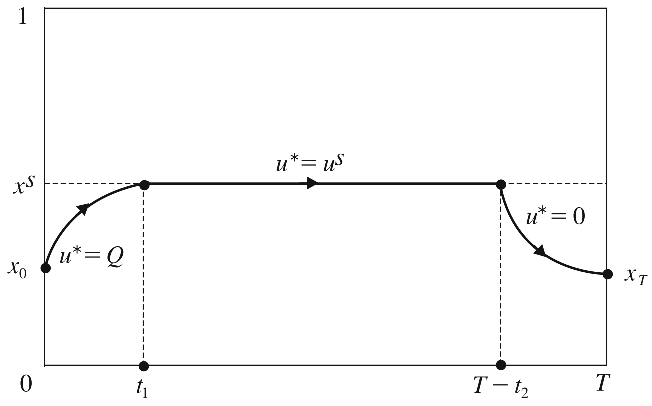

Figure 7.5: Optimal trajectory for Case 1: $x_0 \leq x^s$ and $x_T \leq x^s$

x^s to x_T. Then, we say T is *large* if $T \geq t_1 + t_2$; otherwise T is *small*. Figures 7.5, 7.6, 7.7, and 7.8 show cases for which T is large, while Figs. 7.10 and 7.11 show cases for which T is small. In Exercise 7.21 you are asked to determine whether T is large or small in specific cases. In the statements of the theorems we will assume that x_0 and x_T are such that x_T is reachable from x_0. In Exercise 7.15 you are asked to find the reachable set for any given initial condition x_0.

In Figures 7.5, 7.6, 7.7, and 7.8, the quantities t_1 and t_2 are case dependent and not necessarily the same; see Exercise 7.20.

Theorem 7.1 *Let T be large and let x_T be reachable from x_0. For the Cases* 1–4 *of inequalities relating x_0 and x_T to x^s, the optimal trajectories are given in Figures 7.5, 7.6, 7.7, and 7.8, respectively.*

Proof We give details for Case 1 only. The proofs for the other cases are similar. Figure 7.9 shows the optimal trajectory for Fig. 7.5 together with an arbitrarily chosen feasible trajectory, shown dotted. It should be clear that the dotted trajectory cannot cross the arc x_0 to C, since $u = Q$ on that arc. Similarly the dotted trajectory cannot cross the arc G to x_T, because $u = 0$ on that arc.

We subdivide the interval $[0, T]$ into subintervals over which the dotted arc is either above, below, or identical to the solid arc. In Fig. 7.9

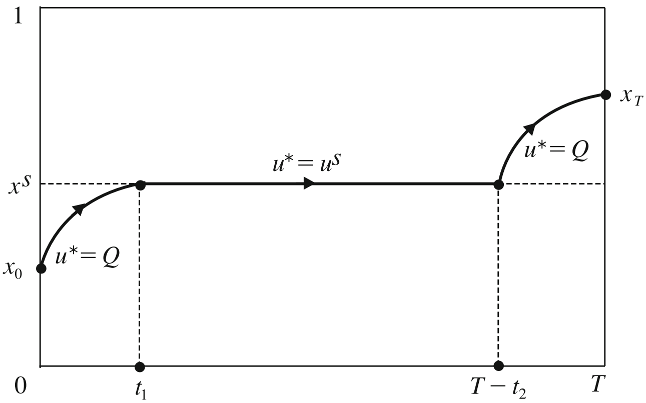

Figure 7.6: Optimal trajectory for Case 2: $x_0 < x^s$ and $x_T > x^s$

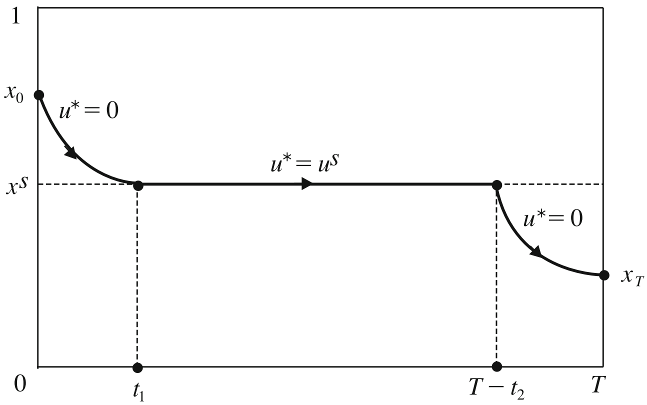

Figure 7.7: Optimal trajectory for Case 3: $x_0 > x^s$ and $x_T < x^s$

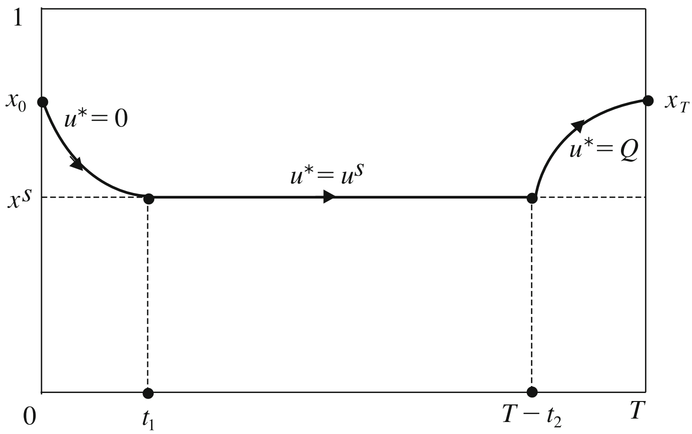

Figure 7.8: Optimal trajectory for Case 4: $x_0 > x^s$ and $x_T > x^s$

these subintervals are $[0, d]$, $[d, e]$, $[e, f]$, and $[f, T]$. Because $I(x)$ is positive for $x > x^s$ and $I(x)$ is negative for $x < x^s$, the regions enclosed by the two trajectories have been marked with a + or − sign depending on whether $I(x)$ is positive or negative on the regions, respectively. By Lemma 7.1, the solid arc is better than the dotted arc in the subintervals $[0, d]$, $[d, e]$, and $[f, T]$; in interval $[e, f]$, they have identical values. Hence the dotted trajectory is inferior to the solid trajectory. This proof can be extended to any (countable) number of crossings of the trajectories; see Sethi (1977b). □

Figures 7.5, 7.6, 7.7, and 7.8 are drawn for the situation when $T > t_1 + t_2$. In Exercise 7.25, you are asked to consider the case when $T = t_1 + t_2$. The following theorem deals with the case when $T < t_1 + t_2$.

Theorem 7.2 *Let T be small, i.e., $T < t_1 + t_2$, and let x_T be reach- able from x_0. For the two possible Cases* 1 *and* 2 *of inequalities relating x_0 and x_T to x^s, the optimal trajectories are given in Figs.* 7.10 *and* 7.11*, respectively.*

Proof The requirement of feasibility when T is small rules out cases where x_0 and x_T are on opposite sides of or equal to x^s. The proofs of optimality of the trajectories shown in Figs. 7.10 and 7.11 are similar to the proofs of the parts of Theorem 7.1, and are left as Exercise 7.25. In Figs. 7.10 and 7.11, it is possible to have either $t_1 \geq T$ or $t_2 \geq T$. Try sketching some of these special cases. □

All of the previous discussion has assumed that Q was finite and sufficiently large, but we can easily extend this to the case when $Q = \infty$.

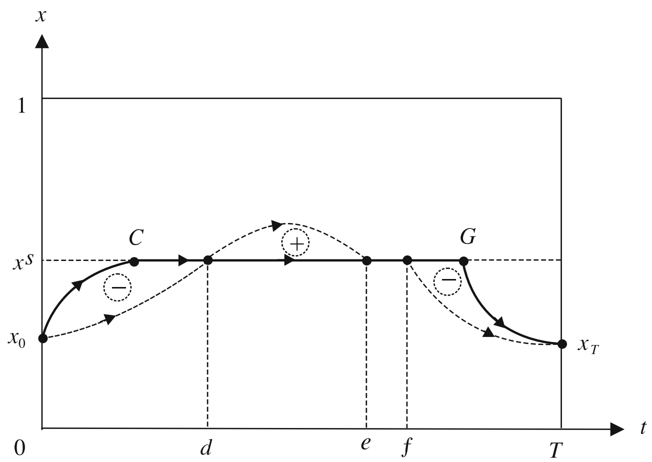

Figure 7.9: Optimal trajectory (solid lines)

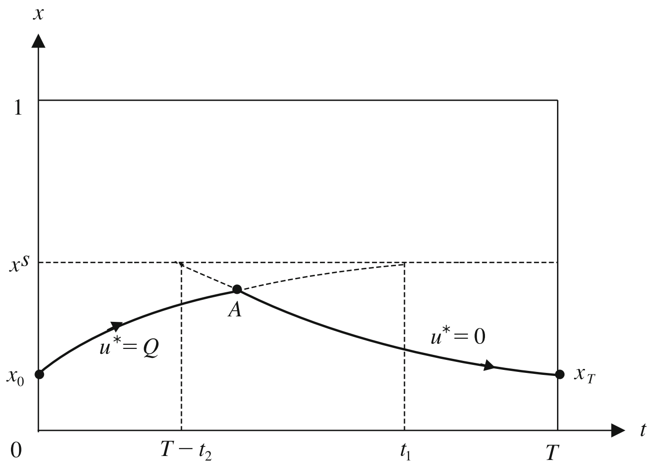

Figure 7.10: Optimal trajectory when T is small in Case 1: $x_0 < x^s$ and $x_T > x^s$

This possibility makes the arcs in Figs. 7.5, 7.6, 7.7, 7.8, 7.9, and 7.10, corresponding to $u^* = Q$, become vertical line segments corresponding to impulse controls. For example, Fig. 7.6 becomes Fig. 7.12 when $Q = \infty$ and we apply the *impulse control* $\text{imp}(x_0, x^s; 0)$ when $x_0 < x^s$.

Next we compute the cost of $\text{imp}(x_0, x^s; 0)$ by assessing its effect on the objective function of (7.26). For this, we integrate the state equation in (7.26) from 0 to ε with the initial condition x_0 and u treated

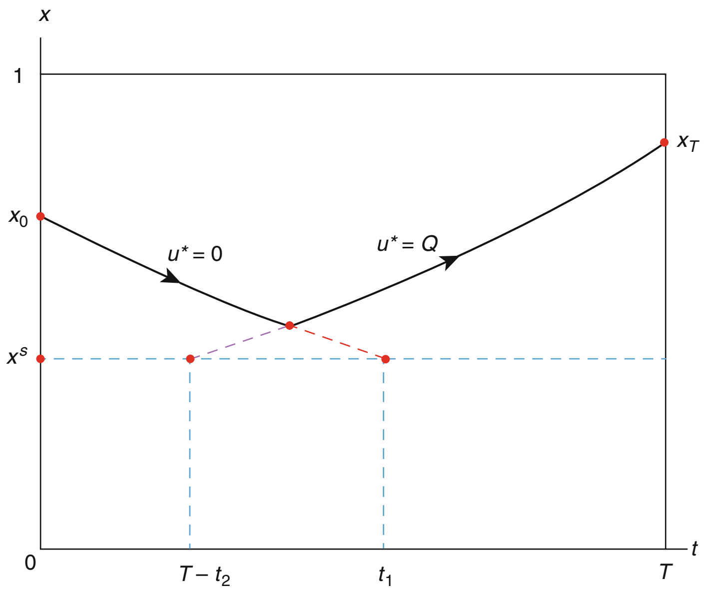

Figure 7.11: Optimal trajectory when T is small in Case 2: $x_0 > x^s$ and $x_T > x^s$

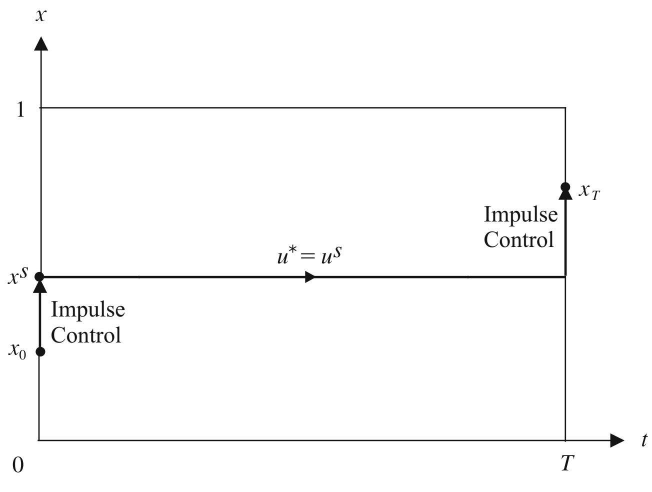

Figure 7.12: Optimal trajectory for Case 2 of Theorem 7.1 for $Q = \infty$

as constant. By using (A.7), we can write the solution as

$$\begin{aligned} x(\varepsilon) &= x_0 e^{-(\delta+ru)\varepsilon} + \int_0^{\varepsilon} e^{(\delta+ru)(\tau-\varepsilon)} ru d\tau \\ &= \left(x_0 - \frac{ru}{\delta+ru}\right) e^{-(\delta+ru)\varepsilon} + \frac{ru}{\delta+ru}. \end{aligned}$$

According to the procedure given in Sect. 1.4, we must, for u, choose $u(\varepsilon)$ so that $x(\varepsilon)$ is x^s. It should be clear that $u(\varepsilon) \to \infty$ as $\varepsilon \to 0$. With $F(x,u,\tau) = \pi x(\tau) - u(\tau)$ and $t = 0$ in (1.23), we have the impulse

$$I = \text{imp}(x_0, x^s; 0) = \lim_{\varepsilon \to 0}[-u(\varepsilon)\varepsilon].$$

It is possible to solve for I by letting $\varepsilon \to 0, -u(\varepsilon)\varepsilon \to I, u(\varepsilon) \to \infty$, and $x(\varepsilon) = x^s$ in the expression for $x(\varepsilon)$ obtained above. This gives

$$x(0+) = e^{rI}(x_0 - 1) + 1.$$

Therefore,

$$\text{imp}(x_0, x^s; 0) = -\frac{1}{r} \ln \left[\frac{1 - x_0}{1 - x^s}\right]. \tag{7.36}$$

We remark that this formula holds for any time t, as well as $t = 0$. Hence it can also be used at $t = T$ to compute the impulse at the end of the period; see Fig. 7.12 and Exercise 7.28.

7.2.3 Solution When Q Is Small

When Q is small, it is not possible to go along the turnpike x^s, so the arguments based on Green's theorem become difficult to apply. We therefore return to the maximum principle approach to analyze the problem. By "Q is small" we mean $Q < u^s$, where u^s is defined in (7.35). Another characterization of the phrase "Q is small" in terms of the problem parameters is given in Exercise 7.30.

We now apply the current-value maximum principle (3.42) to the fixed-end-point problem given in (7.26). We form the current-value Hamiltonian as

$$\begin{aligned} H &= \pi x - u + \lambda[ru(1-x) - \delta x] \\ &= \pi x - \delta\lambda x + u[-1 + r\lambda(1-x)], \end{aligned} \tag{7.37}$$

and the Lagrangian function as

$$L = H + \mu(Q - u). \tag{7.38}$$

The adjoint variable λ satisfies

$$\dot{\lambda} = \rho\lambda - \frac{\partial L}{\partial x} = \rho\lambda + \lambda(ru + \delta) - \pi, \tag{7.39}$$

where $\lambda(T)$ is a constant, as in Row 2 of Table 3.1, that must be determined. Furthermore, the Lagrange multiplier μ in (7.38) must satisfy

$$\mu \geq 0, \quad \mu(Q - u) = 0. \tag{7.40}$$

From (7.37) we notice that the Hamiltonian is linear in the control. So the optimal control is

$$u^*(t) = \text{bang}[0, Q; W(t)], \tag{7.41}$$

where

$$W(t) = W(x(t), \lambda(t)) = r\lambda(t)(1 - x(t)) - 1. \tag{7.42}$$

We remark that the sufficiency conditions of Sect. 2.4, which require concavity of the derived Hamiltonian H^0, do not apply here; see Exercise 7.33. However, the sufficiency of the maximum principle for this kind of problem has been established in the literature; see, for example, Lansdowne (1970).

When $W = r\lambda(1 - x) - 1 = 0$, we have the possibility of a singular control, provided we can maintain this equality over a finite time interval. For the case when Q is large, we showed in the previous section that the optimal trajectory contains a segment on which $x = x^s$ and $u^* = u^s$, where $0 \leq u^s \leq Q$. (See Exercise 7.30 for the condition that Q is small.) This can obviously be a singular control. Further discussion of singular control is given in Sect. D.6.

A complete solution of problem (7.26) when Q is small requires a lengthy switching point analysis. The details are too voluminous to give here, but an interested reader can find the details in Sethi (1973a).

7.2.4 Solution When T Is Infinite

In Sects. 7.2.1 and 7.2.2, we assumed that T was finite. We now formulate the infinite horizon version of (7.26):

$$\begin{cases} \max\left\{J = \int_0^\infty e^{-\rho t}(\pi x - u)dt\right\} \\ \text{subject to} \\ \dot{x} = ru(1-x) - \delta x,\ x(0) = x_0, \\ 0 \le u \le Q. \end{cases} \tag{7.43}$$

We divide the analysis of this problem into the same two cases defined as before, namely, "Q is large" and "Q is small".

When Q is large, the results of Theorem 7.1 suggest the solution when T is infinite. Because of the discount factor, the ending parts of the solutions shown in Figs. 7.5, 7.6, 7.7, and 7.8 can be shown to be irrelevant (i.e., the discounted profit accumulated during the interval $(T - t_2, T)$ goes to 0 as T goes to ∞). Therefore, we only have two cases: (a) $x_0 \le x^s$, and (b) $x_0 \ge x^s$. The optimal control in Case (a) is to use $u^* = Q$ in the interval $[0, t_1)$ and $u^* = u^s$ for $t \ge t_1$. Similarly, the optimal control in Case (b) is to use $u^* = 0$ in the interval $[0, t_1)$ and $u^* = u^s$ for $t \ge t_1$.

An alternate way to see that the above solutions give $u^* = u^s$ for $t \ge t_1$ is to check that they satisfy the turnpike conditions (3.107). To do this we need to find the values of the state, control, and adjoint variables and the Lagrange multiplier along the turnpike. It can be easily shown that $x = x^s$, $u = u^s$, $\lambda^s = \pi/(\rho + \delta + ru^s)$, and $\mu^s = 0$ satisfy the turnpike conditions (3.107).

When Q is small, i.e., $Q < u^s$, it is not possible to follow the turnpike $x = x^s$, because that would require $u = u^s$, which is not a feasible control. Intuitively, it seems clear that the "nearest" stationary path to x^s that we can follow is the path obtained by setting $\dot{x} = 0$ and $u = Q$, the largest possible control, in the state equation of (7.43). This gives

$$\bar{x} = \frac{rQ}{rQ + \delta}, \tag{7.44}$$

and correspondingly we obtain

$$\bar{\lambda} = \frac{\pi}{\rho + \delta + rQ} \tag{7.45}$$

by setting $u = Q$ and $\dot{\lambda} = 0$ in (7.39) and solving for λ.

To find an optimal solution from any given initial x_0, the approach we take is to find a feasible path that is "nearest" to x^s; See Sethi (1977b) for further discussion. As we shall see, for $x_0 < x^s$, such a path is obtained by using the maximum possible control Q all the way. For $x_0 > x^s$, the situation is more difficult. Nevertheless, the following two theorems give the turnpike as well as the optimal path starting from any given initial x_0. Let us define $\hat{x}$ and $\bar{\mu}$ such that $W(\hat{x}, \bar{\lambda}) = r\bar{\lambda}(1 - \hat{x}) - 1 = 0$ and $L_u(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}) = W(\bar{x}, \bar{\lambda}) - \bar{\mu} = 0$. Thus,

$$\hat{x} = 1 - 1/r\bar{\lambda}, \tag{7.46}$$

$$\bar{\mu} = r\bar{\lambda}(1 - \bar{x}) - 1. \tag{7.47}$$

Theorem 7.3 *When Q is small, the quadruple $\{\bar{x}, Q, \bar{\lambda}, \bar{\mu}\}$ forms a turnpike.*

Proof We show that the turnpike conditions (3.107) hold for the quadruple. The first two conditions of (3.107) are (7.44) and (7.45). By Exercise 7.31 we know $\bar{x} \leq \hat{x}$, which, from definitions (7.46) and (7.47), implies $\bar{\mu} \geq 0$. Furthermore $\bar{u} = Q$, so (7.40) holds and the third condition of (3.107) also holds. Finally because $W = \bar{\mu}$ from (7.42) and (7.47), it follows that $W \geq 0$, so the Hamiltonian maximizing condition of (3.107) holds with $\bar{u} = Q$. □

Theorem 7.4 *When Q is small, the optimal control at any time $\tau \geq 0$ is given by:*

(a) *If $x(\tau) \leq \hat{x}$, then $u^*(\tau) = Q$.*

(b) *If $x(\tau) > \hat{x}$, then $u^*(\tau) = 0$.*

Proof (a) We set $\lambda(t) = \bar{\lambda}$ for all $t \geq \tau$ and note that λ satisfies the adjoint equation (7.39) and the transversality condition (3.99).

By Exercise 7.31 and the assumption that $x(\tau) \leq \hat{x}$, we know that $x(t) \leq \hat{x}$ for all t. The proof that (7.40) and (7.41) hold for all $t \geq \tau$ relies on the fact that $x(t) \leq \hat{x}$ and on an argument similar to the proof of the previous theorem.

Figure 7.13 shows the optimal trajectories when $x_0 < \hat{x}$ for two different starting values of x_0, one above and the other below $\bar{x}$. Note that in this figure we are always in Case (a) since $x(\tau) \leq \hat{x}$ for all $\tau \geq 0$.

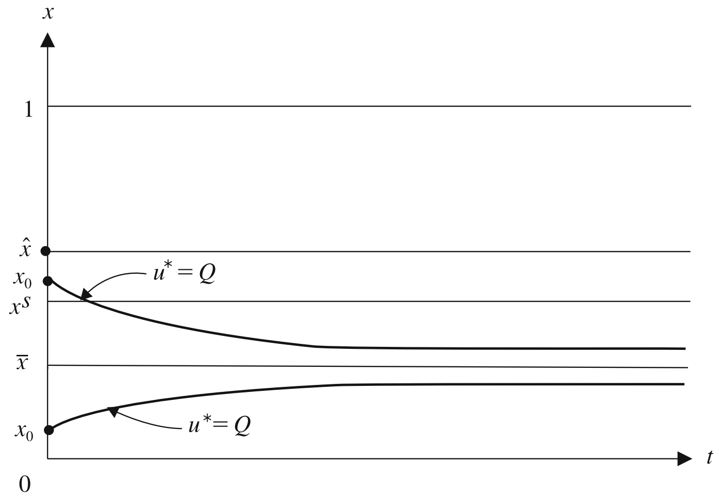

Figure 7.13: Optimal trajectories for $x(0) < \hat{x}$

(b) Assume $x_0 > \hat{x}$. In this case we will show that the optimal trajectory is as shown in Fig. 7.14, which is obtained by applying $u = 0$ until $x = \hat{x}$ and $u = Q$ thereafter. Using this policy we can find the time t_1 at which $x(t_1) = \hat{x}$, by solving the state equation in (7.43) with $u = 0$. This gives

$$t_1 = \frac{1}{\delta} \ln \frac{x_0}{\hat{x}}. \tag{7.48}$$

Clearly for $t \geq t_1$, the policy $u = Q$ is optimal because Case (a) applies. We now consider the interval $[0, t_1]$, where we set $u = 0$. Let τ be any time in this interval as shown in Fig. 7.14, and let $x(\tau)$ be the corresponding value of the state variable. Then $x(\tau) = x_0 e^{-\delta\tau}$. With $u = 0$ in (7.39), the adjoint equation on $[0, t_1]$ becomes

$$\dot{\lambda} = (\rho + \delta)\lambda - \pi.$$

We also know that $x(t_1) = \hat{x}$. Thus, Case (a) applies at time t_1, and we would like to have $\lambda(t_1) = \bar{\lambda}$. So, we solve the adjoint equation with $\lambda(t_1) = \bar{\lambda}$ and obtain

$$\lambda(\tau) = \frac{\pi}{\rho + \delta} + \left(\bar{\lambda} - \frac{\pi}{\rho + \delta} \right) e^{(\rho+\delta)(\tau - t_1)}, \ \tau \in [0, t_1]. \tag{7.49}$$

Now, with the values of $x(\tau)$ and $\lambda(\tau)$ in hand, we can use (7.42) to obtain the switching function value $W(\tau)$. In Exercise 7.34, you are

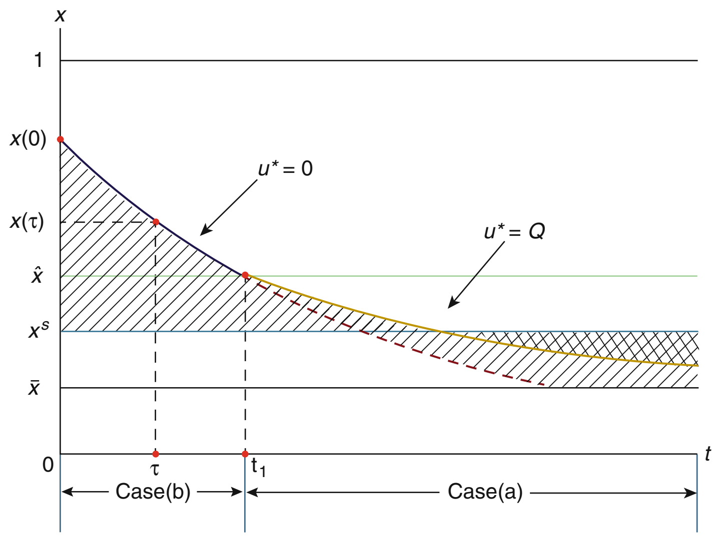

Figure 7.14: Optimal trajectory for $x(0) > \hat{x}$

asked to show that the switching function $W(\tau)$ is negative for each τ in the interval $[0, t_1)$ and $W(t_1) = 0$. Therefore by (7.41), the policy $u = 0$ used in deriving (7.48) and (7.49) satisfies the maximum principle. This policy "joins" the optimal policy after t_1 because $\lambda(t_1) = \bar{\lambda}$.

In this book the sufficiency of the transversality condition (3.99) is stated under the hypothesis that the derived Hamiltonian is concave; see Theorem 2.1. In the present example, this hypothesis does not hold. However, as mentioned in Sect. 7.2.3, for this simple bilinear problem it can be shown that (3.99) is sufficient for optimality. Because of the technical nature of this issue we omit the details. □

Exercises for Chapter 7

E 7.1 In Eqs. (7.2) and (7.3), assume $S(p, G) = 1000 - 5p + 2G$ and $c(S) = 5S$. Substitute into (7.5) and solve for the optimal price p^* in terms of G.

E 7.2 Derive the optimal monopoly price formula in (7.6) from (7.5).

E 7.3 Derive the equilibrium goodwill level formula (7.12).

E 7.4 Re-solve problem (7.7) with its objective function and the control constraint replaced by (7.13), and show that the only possible singular level of goodwill (which can be maintained over a finite time interval) is the value $\bar{G}$ obtained in (7.14).

E 7.5 Show that the total cost of advertising required to go from $G_0 < \bar{G}$ to $\bar{G}$ instantaneously (by an impulse) is $\bar{G} - G_0$.

Hint: Integrate $\dot{G} = u - \delta G$, $G(0) = G_0$, from 0 to ε and equate $\bar{G} = \lim G(\varepsilon)$, where the limit is taken as $\varepsilon \to 0$ and $u \to \infty$ in such a way that $u\varepsilon \to \text{cost} = -\text{imp}(G_0, \bar{G}; 0)$. See also the derivation of (7.36).

E 7.6 Assume the effect of the exogenous variable $Z(t)$ is seasonal so that the goodwill $\bar{G}(t) = 2 + \sin t$. Assume $\delta = 0.1$. Sketch the graph of $\bar{u}(t) = \delta\bar{G} + \dot{\bar{G}}$, similar to Fig. 7.2, and identify intervals in which maintaining the singular level $\bar{G}(t)$ is infeasible.

E 7.7 In the Nerlove-Arrow Model of Sects. 7.1.1 and 7.1.2, assume $S(p, A, Z) = \alpha p^{-\eta} G^{\beta} Z^{\gamma}$ and $c(S) = cS$. Show that the optimal stationary policy gives $\bar{u}/pS$ = constant, i.e., that the optimal advertising level is a constant fraction of sales regardless of the value of Z. (Such policies are followed by many industries.)

E 7.8 Verify that G_1 and G_2, which are shown in Fig. 7.3 for the pulsing policy derived from solving problem (7.19) as a near-optimal solution of problem (7.17) with $T = \infty$, are given by

$$G_1 = \frac{M}{\delta}\left[\frac{1 - e^{-\delta\tau\bar{v}}}{1 - e^{-\delta\tau}}\right], \quad G_2 = \frac{M}{\delta}\left[\frac{e^{-\delta\tau(1-\bar{v})} - e^{-\delta\tau}}{1 - e^{-\delta\tau}}\right].$$

E 7.9 Extend the Nerlove-Arrow Model and its results by introducing the additional capital stock variable

$$\dot{K} = v - \gamma K, \quad K(0) = K_0,$$

where v is the research expenditure. Assume the new cost function to be $C(S, K)$. Note that this model allows the firm to manipulate its cost function. See Dhrymes (1962).

E 7.10 Analyze an extension of a finite horizon Nerlove-Arrow Model subject to a budget constraint. That is, introduce the following isoperimetric constraint:

$$\int_0^T u e^{-\rho t} dt = B.$$

Also assume $\pi(G) = \alpha\sqrt{G}$ where $\alpha > 0$ is a constant. See Sethi (1977c).

E 7.11 Introduce a budget constraint in a different way into the Nerlove-Arrow model as follows. Let $B(t)$ be the budget at time t, and let $\gamma > 0$ be a constant. Assume B satisfies

$$\dot{B} = e^{-\rho t}(-u + \gamma G), \quad B(0) = B_0$$

and $B(t) \geq 0$ for all t. Solve only the infinite horizon model. See Sethi and Lee (1981).

E 7.12 Maximize the present value of total sales in the Nerlove-Arrow model, i.e.,

$$\max_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t} pS(p, G) dt \right\}$$

subject to (7.1) and the isoperimetric profit constraint

$$\int_0^\infty e^{-\rho t}[pS(p, G) - C(S) - u]dt = \hat{\pi}.$$

See Tsurumi and Tsurumi (1971).

E 7.13 *A Logarithmic Advertising Model* (Sethi 1975).

(a) With $\pi r > \rho + \delta$, solve

$$\begin{cases} \max \left\{ J = \int_0^T e^{-\rho t}(\pi x - u) dt \right\} \\ \text{subject to} \\ \dot{x} = r \log u - \delta x, \quad x(0) = x_0, \\ \text{and the control constraint} \\ u \geq 1. \end{cases}$$

(b) Find the value of T for which the minimum advertising is optimal throughout, i.e., $u^*(t) = 1, \ 0 \leq t \leq T$.

(c) Let $T = \infty$. Obtain the long-run stationary equilibrium $(\bar{x}, \bar{u}, \bar{\lambda})$.

E 7.14 Let

$$Q(t) \quad = \quad \text{the quality of the software at time } t; \ Q(0) \geq 0,$$

$$\begin{aligned}
P(t) &= \text{the price of the software at time } t;\ P(t) \geq 0, \\
D(P,Q) &= \text{the demand; } D(P,Q) \geq 0,\ D_Q \geq 0,\ D_P \leq 0, \\
g(x) &= \text{a decreasing function; } g(x) \geq 0,\ g'(x) \leq 0,\ g''(x) \geq 0, \\
&\quad\ \text{and } g(x) \to 0 \text{ as } x \to \infty, \\
\rho &= \text{the discount rate; } \rho > 0, \\
\delta &= \text{the obsolescence rate for software quality; } \delta > 0.
\end{aligned}$$

Assume that

$$\lim_{P \to 0} PD(P,Q) = 0, \text{ for each } Q.$$

Furthermore, we assume that there is a price that maximizes the revenue (in the case when there is more than one global maximum, we will choose the largest of these) and denote it as $P^m(Q)$.

We assume that $0 < P^m(Q) < \infty$ and define

$$R(Q) = P^m(Q)D(P^m(Q), Q).$$

By the envelope theorem (see Derzko et al. 1984), we have

$$R_Q(Q) = P^m(Q)D_Q(P^m(Q), Q) \geq 0.$$

In an open-source approach to software development, the improvement in software quality is proportional to the number of volunteer programmers participating at any point in time. The volunteer programmers' willingness to contribute to software quality is driven by fairness considerations.

To capture the loss of motivation that results from the profit making of the firm, we formulate the motivations of the programmers based on the current or projected future profit of the firm. Then, let $g(PD)$ be the quality improvement affected by the volunteer programmers. The optimal dynamic price and quality paths can be obtained by solving the following problem due to Haruvy et al. (2003):

$$\begin{aligned}
&\max_{P(t) \geq 0} \left\{ J = \int_0^\infty e^{-rt} PD dt \right\}, \\
&s.t. \qquad dQ/dt = g(PD) - \delta Q,\ \ Q(0) = Q_0.
\end{aligned}$$

Because of the convexity of function g in this case, argue that the problem would require the inclusion of chattering controls. Then reformulate the problem as

$$\max_{0 \leq v \leq 1} \left\{ J = \int_0^\infty e^{-rt} vR(Q) dt \right\},$$

$$s.t. \qquad dQ/dt = (1-v)g(0) + vg(R(Q)) - \delta Q, \;\; Q(0) = Q_0.$$

Apply the Green's theorem approach to solve this problem.

E 7.15 For problem (7.26), find the reachable set for a given initial x_0 and horizon time T.

E 7.16 Solve the quadratic equation $I(x) = 0$, where $I(x)$ is defined in (7.32), to obtain its solution as shown in (7.33).

E 7.17 Show that both x^s in (7.34) and u^s in (7.35) are 0 if, and only if, $\pi r \le \delta + \rho$.

E 7.18 For problem (7.26) with $\pi r > \delta + \rho$ and Q sufficiently large, derive the turnpike $\{\bar{x}, \bar{u}, \bar{\lambda}\}$ by using the maximum principle. Check to see that $\bar{x}$ and $\bar{u}$ correspond, respectively, to x^s and u^s derived by Green's theorem. Show that when $\rho = 0$, $\bar{x}$ reduces to the golden path rule.

E 7.19 Let x^s denote the solution of $I(x) = 0$ and let $A < x^s < B$ in Fig. 7.4. Assume that $I(x) > 0$ for $x > x^s$ and $I(x) < 0$ for $x < x^s$. Construct a path Γ_3 such that $J_{\Gamma_3} \ge J_{\Gamma_1}$ and $J_{\Gamma_3} \ge J_{\Gamma_2}$.

Hint: Use Lemma 7.1.

E 7.20 For the problem in (7.26), suppose x_0 and x_T are given and define x^s as in (7.34). Let t_1 be the shortest time to go from x_0 to x^s, and t_2 be the shortest time to go from x^s to x_T.

(a) If $x_0 < x^s$ and $x^s > x_T$, show that

$$t_1 = \frac{1}{rQ+\delta} \ln\left[\frac{\bar{x}-x_0}{\bar{x}-x^s}\right], t_2 = \frac{1}{\delta}\ln\left[\frac{x^s}{x_T}\right],$$

where $\bar{x} = rQ/(rQ+\delta)$; assume $\bar{x} > x^s$.

(b) Using the form of the answers in (a), find t_1 and t_2 when $x_0 > x^s$ and $x^s < x_T < \bar{x}$.

E 7.21 For Exercise 7.20(a), write the condition that T is large, i.e., $T \ge t_1 + t_2$, in terms of all the parameters of problem (7.26).

E 7.22 Perform the following:

(a) For problem (7.26), assume $r = 0.2$, $\delta = 0.05$, $\rho = 0.1$, $Q = 5$, $\pi = 2$, $x_0 = 0.2$ and $x_T = 0.3$. Use Exercises 7.20(a) and 7.21 to show that $T = 13$ is large and $T = 8$ is small. Sketch the optimal trajectories for $T = 13$ and $T = 8$.

(b) Redo (a) when $x_T = 0.7$. Show that both $T = 13$ and $T = 8$ are large.

E 7.23 Prove Theorem 7.1 for Case 3.

E 7.24 Draw four figures for the case $T = t_1 + t_2$ corresponding to Figs. 7.5, 7.6, 7.7, and 7.8.

E 7.25 Prove Theorem 7.2.

E 7.26 Sketch one or two other possible curves for the case when T is small.

E 7.27 An intermediate step in the derivation of (7.36) is to establish that

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon} e^{-\rho t}[\pi x(t) - u(t)]dt = \lim_{\varepsilon \to 0}[-u(\varepsilon)\varepsilon].$$

Show how to accomplish this by using the Mean Value Theorem.

E 7.28 Obtain the impulse function, $\text{imp}(x^s, x_T; T)$, required to take the state from x^s up to x_T instantaneously at time T as shown in Fig. 7.12 for the Vidale-Wolfe model in Sect. 7.2.2.

E 7.29 Perform the following:

(a) Re-solve Exercise 7.22(a) with $Q = \infty$. Show $T = 10.5$ is no longer small.

(b) Show that $T > 0$ is large for Exercise 7.22(b) when $Q = \infty$. Find the optimal value of the objective function when $T = 8$.

E 7.30 Show that Q is small if, and only if,

$$\frac{\pi r \delta}{(\delta + \rho + rQ)(\delta + rQ)} > 1.$$

E 7.31 Perform the following:

(a) Show that $\bar{x} < x^s < \hat{x}$ when Q is small, where $\hat{x}$ is defined in (7.46).

(b) Show that $\bar{x} > x^s$ when Q is large.

E 7.32 Derive (7.48).

E 7.33 Show the derived Hamiltonian H^0 corresponding to (7.37) and (7.41) is not concave in x for any given $\lambda > 0$.

E 7.34 Show that the switching function defined in (7.42) is concave in t, and then verify that the policy in Fig. 7.14 satisfies (7.41).

E 7.35 In (7.25), assume r and δ are positive, differentiable functions of time. Derive expressions similar to (7.31)–(7.35) in order to get the new turnpike values.

E 7.36 Write the equation satisfied by the turnpike level $\bar{x}$ for the model

$$\begin{cases} \max\limits_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t}(\pi x - u^2)dt \right\} \\ \text{subject to} \\ \dot{x} = ru(1-x) - \delta x, \quad x(0) = x_0. \end{cases}$$

Show that the turnpike reduces to the golden path when $\rho = 0$.

E 7.37 Obtain the optimal long-run stationary equilibrium for the following modification of the model (7.26), due to Sethi (1983b):

$$\begin{cases} \max \int_0^\infty e^{-\rho t}(\pi x - u^2)dt \\ \text{subject to} \\ \dot{x} = ru\sqrt{(1-x)} - \delta x, \ x_0 \in [0,1], \\ u \geq 0. \end{cases} \tag{7.50}$$

In particular, show that the turnpike triple $(\bar{x}, \bar{\lambda}, \bar{u})$ is given by

$$\bar{x} = \frac{r^2\bar{\lambda}/2}{r^2\bar{\lambda}/2 + \delta}, \quad \bar{u} = \frac{r\bar{\lambda}\sqrt{1-\bar{x}}}{2}, \tag{7.51}$$

and

$$\bar{\lambda} = \frac{\sqrt{[(\rho+\delta)^2 + r^2\pi]} - (\rho+\delta)}{r^2/2}. \quad (7.52)$$

Show that the optimal value of the objective function is

$$J^*(x_0) = \bar{\lambda}x_0 + \frac{r^2\bar{\lambda}^2}{4\rho}. \quad (7.53)$$

E 7.38 Consider (7.43) with the state equation replaced by

$$\dot{x} = ru(1-x) + \mu x(1-x) - \delta x, \;\; x(0) = x_0,$$

where the constant $\mu > 0$ reflects *word-of-mouth* communication between buyers (represented by x) and non-buyers (represented by $(1-x)$) of the product. Assume Q is infinite for convenience. Obtain the turnpike for this problem. See Sethi (1974b).

E 7.39 *The Ozga Model* (Ozga 1960; Gould 1970). Suppose the information spreads by word of mouth rather than by an impersonal advertising medium, i.e., individuals who are already aware of the product inform individuals who are not, at a certain rate, influenced by advertising expenditure. What we have now is the Ozga model

$$\dot{x} = ux(1-x) - \delta x, \;\; x(0) = x_0.$$

The optimal control problem is to maximize

$$J = \int_0^\infty e^{-\rho t}[\pi(x) - w(u)]dt$$

subject to the Ozga model. Assume that $\pi(x)$ is concave and $w(u)$ is convex. See Sethi (1979c) for a Green's theorem application to this problem.

Chapter 8

The Maximum Principle: Discrete Time

For many purposes it is convenient to assume that time is represented by a discrete variable, $k = 0, 1, 2, \ldots, T$, rather than by a continuous variable $t \in [0, T]$. This is particularly true when we wish to solve a large control theory problem by means of a computer. It is also desirable, even when solving small problems which have state or adjoint differential equations whose solutions cannot be expressed in closed form, to formulate them as discrete problems and let the computer solve them in a stepwise manner.

We will see that the maximum principle, which is to be derived in this chapter, is not valid for the discrete-time problem in as wide a sense as for the continuous-time problem. In fact, we will reduce it to a nonlinear programming problem and state necessary conditions for its solution by using the well-known Kuhn-Tucker theorem. In order to follow this procedure, we have to make some simplifying assumptions and hence will obtain only a restricted form of the discrete maximum principle. In Sect. 8.3, we state without proof a more general form of the discrete maximum principle.

8.1 Nonlinear Programming Problems

We begin by stating a general form of a nonlinear programming problem. Let x be an n-component column vector, a an r-component column vector, and b an s-component column vector. Let the functions

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$h : E^n \to E^1$, $f : E^n \to E^r$, and $g : E^n \to E^s$ be continuously differentiable. We assume functions f and g to be column vectors with r and s components, respectively. We consider the nonlinear programming problem:

$$\max h(x) \tag{8.1}$$

subject to r equality constraints and s inequality constraints given, respectively, by

$$f(x) = a, \tag{8.2}$$
$$g(x) \geq b. \tag{8.3}$$

Next we develop necessary conditions, called the Kuhn-Tucker conditions, which a solution x^* to this problem must satisfy. We start with simpler problems and work up to the statement of these conditions for the general problem in a heuristic fashion. References are given for rigorous developments of these results.

In this chapter, whenever we take derivatives of functions, we assume that those derivatives exist and are continuous. It would be also helpful to recall the notation developed in Sect. 1.4.

8.1.1 Lagrange Multipliers

Suppose we want to solve (8.1) without imposing constraints (8.2) or (8.3). The problem is now the classical unconstrained maximization problem of calculus, and the first-order necessary conditions for its solution are

$$h_x = 0. \tag{8.4}$$

The points satisfying (8.4) are called *critical points*. Critical points which are maxima, minima, or saddle points are of interest in this book. Additional higher-order conditions required to determine whether a critical point is a maximum or a minimum are stated in Exercise 8.2. In an important case when the function h is concave, condition (8.4) is also sufficient for a global maximum of h.

Suppose we want to solve (8.1) while imposing just the equality constraints (8.2). The method of Lagrange multipliers permits us to obtain the necessary conditions that a solution to the constrained maximization problem (8.1) and (8.2) must satisfy. We define the Lagrangian function

$$L(x, \lambda) = h(x) + \lambda[f(x) - a], \tag{8.5}$$

where λ is an r-component row vector. The necessary condition for x^* to be a (maximum) solution to (8.1) and (8.2) is that there exists an r-component row vector λ such that (x^*, λ) satisfy the equations

$$L_x = h_x + \lambda f_x = 0, \tag{8.6}$$

$$L_\lambda = f(x) - a = 0. \tag{8.7}$$

Note that (8.7) states simply that x^* is feasible according to (8.2).

The system of $n + r$ Eqs. (8.6) and (8.7) has $n + r$ unknowns. Since some or all of the equations are nonlinear, the solution method will, in general, involve nonlinear programming techniques, and may be difficult. In other cases, e.g., when h is linear and f is quadratic, it may only involve the solution of linear equations. Once a solution (x^*, λ) is found satisfying the necessary conditions (8.6) and (8.7), the solution must still be checked to see whether it satisfies sufficient conditions for a global maximum. Such sufficient conditions will be stated in Sect. 8.1.4.

Suppose (x^*, λ) is in fact a solution to equations (8.6) and (8.7). Note that x^* depends on a and we can show this dependence by writing $x^* = x^*(a)$. Now $h^* = h^*(a) = h(x^*(a))$ is the optimum value of the objective function. By differentiating $h^*(a)$ with respect to a and using (8.6), we obtain

$$h_a^* = h_x \frac{dx^*}{da} = -\lambda f_x \frac{dx^*}{da}.$$

But by differentiating (8.7) with respect to a at $x = x^*(a)$, we get

$$f_x \frac{dx^*}{da} = 1,$$

and therefore we have

$$h_a^* = -\lambda. \tag{8.8}$$

We can see that the Lagrange multipliers have an important managerial interpretation, namely, λ_i is the negative of the imputed value or *shadow price* of having one unit more of the resource a_i. In Exercise 8.4 you are asked to provide a proof of (8.8).

Example 8.1 Consider the two-dimensional problem:

$$\begin{cases} \max\{h(x,y) = -x^2 - y^2\} \\ \text{subject to} \\ 2x + y = 10. \end{cases}$$

Solution We form the Lagrangian

$$L(x, y, \lambda) = (-x^2 - y^2) + \lambda(2x + y - 10).$$

The necessary conditions for an optimal solution (x^*, y^*) are that (x^*, y^*, λ) satisfy the equations

$$\begin{aligned} L_x &= -2x + 2\lambda = 0, \\ L_y &= -2y + \lambda = 0, \\ L_\lambda &= 2x + y - 10 = 0. \end{aligned}$$

From the first two equations we get $\lambda = x = 2y$. Solving this with the last equation yields the quantities

$$x^* = 4,\ y^* = 2,\ \lambda = 4,\ h^* = -20,$$

which can be seen to give a maximum value to h, since h is concave and the constraint set is convex. The interpretation of the Lagrange multiplier $\lambda = 4$ can be obtained to verify (8.8) by replacing the constant 10 by $10 + \epsilon$ and expanding the objective function in a Taylor series; see Exercise 8.5.

8.1.2 Equality and Inequality Constraints

Now suppose we want to solve the problem defined by (8.1)–(8.3). As before, we define the Lagrangian

$$L(x, \lambda, \mu) = h(x) + \lambda[f(x) - a] + \mu[g(x) - b]. \tag{8.9}$$

The Kuhn-Tucker necessary conditions for this problem cannot be as easily derived as for the equality-constrained problem in the preceding section. We will write them first, and then give interpretations to make them plausible. The necessary conditions for x^* to be a solution of (8.1)–(8.3) are that there exist an r-dimensional vector λ and an s-dimensional row vector μ such that

$$L_x = h_x + \lambda f_x + \mu g_x = 0, \tag{8.10}$$

$$f = a, \tag{8.11}$$

$$g \geq b, \tag{8.12}$$

$$\mu \geq 0, \quad \mu(g - b) = 0. \tag{8.13}$$

Note that g is appended in (8.10) in the same way f is appended in (8.6). Also (8.12) repeats the inequality constraint (8.3) in the same way that (8.11) repeats the equality constraint (8.2). However, the conditions in (8.13) are new and particular to the inequality-constrained problem. We will see that they include some of the boundary points of the feasible set of points as well as unconstrained maximum solution points, as candidates for the solution to the maximum problem. This is best brought out by examples.

Example 8.2 Solve the problem:

$$\begin{cases} \max\{h(x) = 8x - x^2\} \\ \text{subject to} \\ x \geq 2. \end{cases}$$

Solution We form the Lagrangian

$$L(x, \mu) = 8x - x^2 + \mu(x - 2).$$

The necessary conditions (8.10)–(8.13) become

$$L_x = 8 - 2x + \mu = 0, \tag{8.14}$$

$$x - 2 \geq 0, \tag{8.15}$$

$$\mu \geq 0, \ \mu(x - 2) = 0. \tag{8.16}$$

Observe that the constraint $\mu(x-2) = 0$ in (8.16) can be phrased as: *either* $\mu = 0$ or $x = 2$. We treat these two cases separately.

Case 1: $\mu = 0$. From (8.14) we get $x = 4$, which also satisfies (8.15). Hence, this solution, which makes $h(4) = 16$, is a possible candidate for the maximum solution.

Case 2: $x = 2$. Here from (8.14) we get $\mu = -4$, which does not satisfy the inequality $\mu \geq 0$ in (8.16).

From these two cases we conclude that the optimum solution is $x^* = 4$ and $h^* = h(x^*) = 16$.

Example 8.3 Solve the problem:

$$\begin{cases} \max\{h(x) = 8x - x^2\} \\ \text{subject to} \\ x \geq 6. \end{cases}$$

Solution The Lagrangian is

$$L(x, \mu) = 8x - x^2 + \mu(x - 6).$$

The necessary conditions are

$$L_x = 8 - 2x + \mu = 0, \tag{8.17}$$

$$x - 6 \geq 0, \tag{8.18}$$

$$\mu \geq 0, \ \mu(x - 6) = 0. \tag{8.19}$$

Again, the condition $\mu(x - 6) = 0$ is an either-or relation which gives two cases.

Case 1: $\mu = 0$. From (8.17) we obtain $x = 4$, which does not satisfy (8.18), so this case is infeasible.

Case 2: $x = 6$. Obviously (8.18) holds. From (8.17) we get $\mu = 4$, so (8.19) holds as well. The optimal solution is then

$$x^* = 6, \ h^* = h(x^*) = 12,$$

since it is the only solution satisfying the necessary conditions.

The examples above involve only one variable, and are relatively obvious. The next example, which is two-dimensional, will reveal more of the power and the difficulties of applying the Kuhn-Tucker conditions.

Example 8.4 Find the shortest distance between the point (2,2) and the upper half of the semicircle of radius one with its center at the origin, shown as the curve in Fig. 8.1. In order to simplify the calculation, we minimize h, the square of the distance. Hence, the problem can be stated as the following nonlinear programming problem:

$$\begin{cases} \max\left\{-h(x,y) = -(x-2)^2 - (y-2)^2\right\} \\ \text{subject to} \\ x^2 + y^2 = 1, \\ y \geq 0. \end{cases}$$

The Lagrangian function for this problem is

$$L = -(x-2)^2 - (y-2)^2 + \lambda(x^2 + y^2 - 1) + \mu y. \tag{8.20}$$

The necessary conditions are

$$-2(x-2) + 2\lambda x = 0, \tag{8.21}$$
$$-2(y-2) + 2\lambda y + \mu = 0, \tag{8.22}$$
$$x^2 + y^2 - 1 = 0, \tag{8.23}$$
$$y \geq 0, \tag{8.24}$$
$$\mu \geq 0, \quad \mu y = 0. \tag{8.25}$$

First, we conclude that $\lambda \neq 0$, since otherwise $\lambda = 0$ would imply $x = 2$ from (8.21), which would contradict (8.23). Next, from (8.25) we conclude that either $\mu = 0$ or $y = 0$. If $\mu = 0$, then from (8.21) and (8.22), we get $x = y$. Solving the equation $x = y$ together with $x^2+y^2 = 1$ gives:

(a) $(1/\sqrt{2}, 1/\sqrt{2})$ and $h = -(9 - 4\sqrt{2})$.

If $y = 0$, then solving with $x^2 + y^2 = 1$ gives:

(b) $(1, 0)$ and $h = -5$,

(c) $(-1, 0)$ and $h = -13$.

These three points are shown in Fig. 8.1. Of the three points found that satisfy the necessary conditions, clearly the point $(1/\sqrt{2}, 1/\sqrt{2})$ found in (a) is the nearest point and solves the closest-point problem. The point $(-1, 0)$ in (c) is in fact the farthest point; and the point $(1, 0)$ in (b) is neither the closest nor the farthest point. The associated multiplier values can be easily computed, and these are: (a) $\lambda = 1 - 2\sqrt{2},\ \mu = 0$; (b) $\lambda = -1,\ \mu = 4$; and (c) $\lambda = 3,\ \mu = 4$.

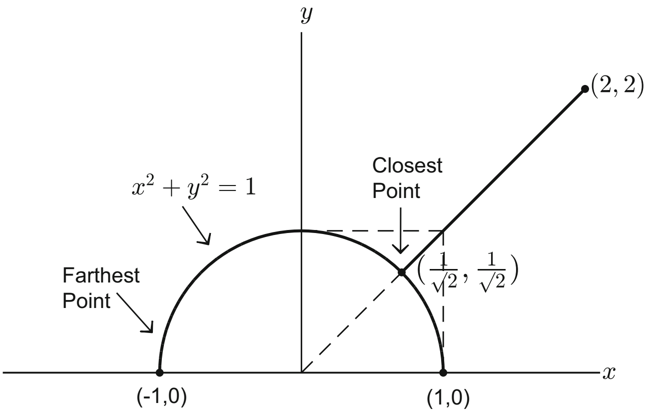

Figure 8.1: Shortest distance from point (2,2) to the semicircle

The fact that there are three points satisfying the necessary conditions, and only one of them actually solves the problem at hand, emphasizes that the conditions are only necessary and not sufficient. In every case it is important to check the solutions to the necessary conditions to see which of the solutions provides the optimum.

Next we work two examples that show some technical difficulties that can arise in the application of the Kuhn-Tucker conditions.

Example 8.5 Consider the problem:

$$\max\{h(x,y) = y\} \tag{8.26}$$

subject to

$$(1-y)^3 - x^2 \geq 0, \tag{8.27}$$

$$x \geq 0 \tag{8.28}$$

$$y \geq 0. \tag{8.29}$$

The set of points satisfying the constraints is shown shaded in Fig. 8.2. From the figure it is obvious that the solution point (0,1) maximizes the value of y.

Hence, the optimum solution is $(x^*, y^*) = (0, 1)$ and $h^* = 1$. Let us see if we can find it using the above procedure. The Lagrangian is

$$L = y + \lambda[(1-y)^3 - x^2] + \mu x + \nu y, \tag{8.30}$$

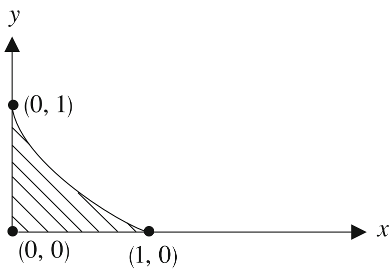

Figure 8.2: Graph of Example 8.5

so that the necessary conditions are

$$L_x = -2x\lambda + \mu = 0, \tag{8.31}$$

$$L_y = 1 - 3\lambda(1-y)^2 + \nu = 0, \tag{8.32}$$

$$\lambda \geq 0,\ \lambda[(1-y)^3 - x^2] = 0, \tag{8.33}$$

$$\mu \geq 0,\ \mu x = 0, \tag{8.34}$$

$$\nu \geq 0,\ \nu y = 0, \tag{8.35}$$

together with (8.27)–(8.29). Let us check if these conditions hold at the point (0,1). At $y = 1$, the constraint $y \geq 0$ is not active, and we have $\nu = 0$. With $\nu = 0$ and $y = 1$, (8.32) cannot be satisfied.

The reason for failure of the method in Example 8.5 is that the constraints do not satisfy what is called the *constraint qualification*. A complete study of the topic is beyond the scope of this book, but we state in the next section a constraint qualification sufficient for our purposes. For further information, see Mangasarian (1969).

8.1.3 Constraint Qualification

Example 8.5 shows the need for imposing some kind of condition to rule out features such as the *cusp* at $(0,1)$ in Fig. 8.2 on the boundary of the constraint set. One way to accomplish this is to assume that the gradients of the equality constraints and of the active inequality constraints at the candidate point under consideration are linearly independent. Equivalently, we say that the constraints (8.2) and (8.3) satisfy the *constraint qualification* at x if the following full-rank condition holds at x, that is,

$$\text{rank}\begin{bmatrix} \partial g/\partial x & \text{diag}(g) \\ \partial f/\partial x & 0 \end{bmatrix} = \min(s+r, s+n), \tag{8.36}$$

where $\partial g/\partial x$ and $\partial f/\partial x$ are $s \times n$ and $r \times n$ gradient matrices, respectfully, as defined in Sect. 1.4.3, the notation diag(g) refers to the diagonal $s \times s$ matrix

$$\begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & g_s \end{bmatrix},$$

and therefore the matrix in (8.36) is an $(s+r) \times (s+n)$ matrix.

Let us now return to Example 8.5 and examine whether the constraints (8.27)–(8.29) satisfy the constraint qualification at point (0,1). In this example, $s = 3$, $r = 0$ and $n = 2$, and the matrix in (8.36) is

$$\begin{bmatrix} -2x & -3(1-y)^2 & (1-y)^3 - x^2 & 0 & 0 \\ 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

at point $(x, y) = (0, 1)$. It has a null vector in the first row, and therefore its rows are not linearly independent; see Sect. 1.4.10. Thus, it does not have a full rank of three, and the condition (8.36) does not hold. Alternatively, note that the inequality constraints (8.27) and (8.28) are active at point $(x, y) = (0, 1)$, and their respective gradients $(-2x, -3(1-y)^2) = (0, 0)$ and $(1, 0)$ at that point are clearly not linearly independent.

8.1.4 Theorems from Nonlinear Programming

In order to derive our version of the discrete maximum principle, we use two well-known results from nonlinear programming. These provide sufficient and necessary conditions for the problem given by (8.1)–(8.3). The Lagrangian function for this problem is

$$L(x, \lambda, \mu) = h + \lambda(f(x) - a) + \mu(g(x) - b), \tag{8.37}$$

where λ and μ are row vectors of multipliers associated with the constraints (8.2) and (8.3), respectively. We now state two theorems whose proofs can be found in Mangasarian (1969).

Theorem 8.1 (Necessary Conditions) *If $h, f,$ and g are differentiable, x^* solves (8.1)–(8.3), and the constraint qualification (8.36) holds at x^*, then there exist multipliers λ and μ such that (x^*, λ, μ) satisfy the Kuhn-Tucker conditions*

$$L_x(x^*, \lambda, \mu) = h_x(x^*) + \lambda f_x(x^*) + \mu g_x(x^*) = 0, \tag{8.38}$$

$$f(x^*) = a, \tag{8.39}$$

$$g(x^*) \geq b, \tag{8.40}$$

$$\mu \geq 0, \ \mu(g(x^*) - b) = 0, \tag{8.41}$$

Theorem 8.2 (Sufficient Conditions) *If $h, f,$ and g are differentiable, f is affine, g is concave, and (x^*, λ, μ) satisfy the conditions* (8.38)–(8.41)*, then x^* is a solution to the maximization problem* (8.1)–(8.3)*.*

8.2 A Discrete Maximum Principle

We will now use the nonlinear programming results of the previous section to derive a special form of the discrete maximum principle. Some references in this connection are Luenberger (1972), Mangasarian and Fromovitz (1967), and Ravn (1999). A more general discrete maximum principle will be stated in Sect. 8.3.

8.2.1 A Discrete-Time Optimal Control Problem

In order to state a discrete-time optimal control problem over the periods $0, 1, 2, \ldots, T$, we define the following:

$$\begin{aligned}
\Theta &= \text{the set } \{0, 1, 2, \ldots, T-1\}, \\
x^k &= \text{an } n\text{-component column state vector; } k = 0, 1, \ldots, T, \\
u^k &= \text{an } m\text{-component column control vector; } k = 0, 1, 2, \ldots, T-1, \\
b^k &= \text{an } s\text{-component column vector of constants; } k=0, 1, \ldots, T-1.
\end{aligned}$$

Here, the state x^k is assumed to be measured at the beginning of period k and control u^k is implemented during period k. This convention is depicted in Fig. 8.3.

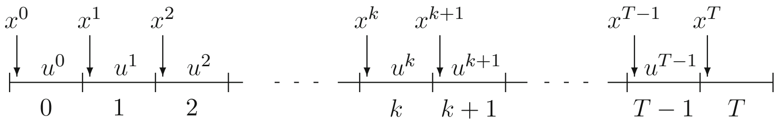

Figure 8.3: Discrete-time conventions

We also define continuously differentiable functions $f : E^n \times E^m \times \Theta \to E^n$, $F : E^n \times E^m \times \Theta \to E^1$, $g : E^m \times \Theta \to E^s$, and $S : E^m \times \Theta \cup \{T\} \to E^1$.

Then, a discrete-time optimal control problem in the Bolza form (see Sect. 2.1.4) is:

$$\max \left\{ J = \sum_{k=0}^{T-1} F(x^k, u^k, k) + S(x^T, T) \right\} \tag{8.42}$$

subject to the difference equation

$$\triangle x^k = x^{k+1} - x^k = f(x^k, u^k, k),\ k = 0, \ldots, T-1,\ x^0 \text{ given}, \tag{8.43}$$

and the constraints

$$g(u^k, k) \geq b^k,\ k = 0, \ldots, T-1. \tag{8.44}$$

In (8.43) the term $\triangle x^k = x^{k+1} - x^k$ is known as the *difference operator*.

This problem is clearly a special case of the nonlinear programming problem (8.1)–(8.3) with $x = (x^1, x^2, \ldots, x^T, u^0, u^1, \ldots, u^{T-1})$ as the $(n+m)T$ vector of variables, nT equality constraints (8.43), and sT inequality constraints (8.44).

8.2.2 A Discrete Maximum Principle

We now apply the nonlinear programming theory of Sect. 8.1 to find necessary conditions for the solution to the Mayer form of the control problem of Sect. 8.2.1.

We let λ^{k+1} be an n-component row vector of Lagrange multipliers, which we rename *adjoint variables* and associate with Eq. (8.43). Similarly, we let μ^k be an s-component row vector of Lagrange multipliers associated with constraint (8.44). These multipliers are defined for each time $k = 0, \ldots, T-1$.

The Lagrangian function of the problem is

$$\begin{aligned} L = \sum_{k=0}^{T-1} F(x^k, u^k, k) + S(x^T, T) \quad &+ \quad \sum_{k=0}^{T-1} \lambda^{k+1}[f(x^k, u^k, k) - x^{k+1} + x^k] \\ &+ \quad \sum_{k=0}^{T-1} \mu^k [g(u^k, k) - b^k]. \end{aligned} \tag{8.45}$$

We now define the Hamiltonian function H^k to be

$$H^k = H(x^k, u^k, k) = F(x^k, u^k, k) + \lambda^{k+1} f(x^k, u^k, k). \tag{8.46}$$

Using (8.46) we can rewrite (8.45) as

$$\begin{aligned} L = S(x^T, T) \quad &+ \quad \sum_{k=0}^{T-1} [H^k - \lambda^{k+1}(x^{k+1} - x^k)] \\ &+ \quad \sum_{k=0}^{T-1} \mu^k [g(u^k, k) - b^k]. \end{aligned} \tag{8.47}$$

We can now apply the Kuhn-Tucker conditions (8.38)–(8.41). Conditions (8.39) and (8.40) in this case give (8.43) and (8.44), respectively. Application of (8.38) results in (8.48)–(8.50) below and application of (8.41) gives the complimentary slackness conditions (8.51) below.

By differentiating (8.47) with respect to x^k for $k = 1, 2, \ldots, T-1$, we obtain

$$\frac{\partial L}{\partial x^k} = \frac{\partial H^k}{\partial x^k} - \lambda^k + \lambda^{k+1} = 0,$$

which upon rearranging terms becomes

$$\triangle \lambda^k \quad = \quad \lambda^{k+1} - \lambda^k = -\frac{\partial H^k}{\partial x^k}, \; k = 0, 1, \ldots, T-1. \tag{8.48}$$

By differentiating (8.47) with respect to x^T, we get

$$\frac{\partial L}{\partial x^T} = \frac{\partial S}{\partial x^T} - \lambda^T = 0, \text{ or } \lambda^T = \frac{\partial S}{\partial x^T}. \tag{8.49}$$

The difference equations (8.48) with terminal boundary conditions (8.49) are called the *adjoint equations*.

By differentiating L with respect to u^k and stating the corresponding Kuhn-Tucker conditions for the multiplier μ^k and constraint (8.44), we have

$$\frac{\partial L}{\partial u^k} = \frac{\partial H^k}{\partial u^k} + \mu^k \frac{\partial g}{\partial u^k} = 0$$

or

$$\frac{\partial H^k}{\partial u^k} = -\mu^k \frac{\partial g}{\partial u^k}, \tag{8.50}$$

and

$$\mu^k \geq 0,\ \mu^k[g(u^k,k) - b^k] = 0. \tag{8.51}$$

We note that, provided H^k is concave in u^k, $g(u^k,k)$ is concave in u^k, and the constraint qualification holds, then conditions (8.50) and (8.51) are precisely the necessary and sufficient conditions for solving the following Hamiltonian maximization problem:

$$\begin{cases} \max\limits_{u^k} H^k \\ \text{subject to} \\ g(u^k,k) \geq b^k. \end{cases} \tag{8.52}$$

We have thus derived the following restricted form of the discrete maximum principle.

Theorem 8.3 *If for every k, H^k in (8.46) and $g(u^k,k)$ are concave in u^k, and the constraint qualification holds, then the necessary conditions for $u^{k*}, k = 0,1,\ldots,T-1$, to be an optimal control for the problem (8.42)–(8.44), with the corresponding state x^{k*}, $k = 0,1,\ldots,T$, are*

$$\begin{cases} \triangle x^{k*} = f(x^{k*},u^{k*},k),\ x^0 \text{ given}, \\ \triangle \lambda^k = -\frac{\partial H^k}{\partial x^k}[x^{k*},u^{k*},\lambda^{k+1},k],\ \lambda^T = \frac{\partial S(x^{T*},T)}{\partial x^T}, \\ H^k(x^{k*},u^{k*},\lambda^{k+1},k) \geq H^k(x^{k*},u^k,\lambda^{(k+1)},k), \\ \text{ for all } \ u^k \ \text{ such that } \ g(u^k,k) \geq b^k, \quad k = 0,1,\ldots,T-1. \end{cases} \tag{8.53}$$

Section 8.2.3 gives examples of the application of this maximum principle (8.53). In Sect. 8.3 we state a more general discrete maximum principle.

8.2.3 Examples

Our first example will be similar to Example 2.4 and it will be solved completely. The reader will note that the solutions of the continuous and discrete problems are very similar. The second example is a discrete version of the production-inventory problem of Sect. 6.1.

Example 8.6 Consider the discrete-time optimal control problem:

$$\max \left\{ J = \sum_{k=1}^{T-1} -\frac{1}{2}(x^k)^2 \right\} \tag{8.54}$$

subject to

$$\triangle x^k = u^k, x^0 = 5, \tag{8.55}$$

$$u^k \in \Omega = [-1, 1]. \tag{8.56}$$

We will solve this problem for $T = 6$ and $T \geq 7$.

Solution The Hamiltonian is

$$H^k = -\frac{1}{2}(x^k)^2 + \lambda^{k+1} u^k, \tag{8.57}$$

from which it is obvious that the optimal policy is bang-bang. Its form is

$$u^{k*} = \text{bang}[-1, 1; \lambda^{k+1}] = \begin{cases} 1 & \text{if } \lambda^{k+1} > 0, \\ \text{singular} & \text{if } \lambda^{k+1} = 0, \\ -1 & \text{if } \lambda^{k+1} < 0. \end{cases} \tag{8.58}$$

Let us assume, as we did in Example 2.4, that $\lambda^k < 0$ as long as x^k is positive so that $u^k = -1$. Given this assumption, (8.55) becomes $\triangle x^k = -1$, whose solution is

$$x^{k*} = -k + 5 \text{ for } k = 1, 2, \ldots, T - 1. \tag{8.59}$$

By differentiating (8.57), we obtain the adjoint equation

$$\triangle \lambda^k = -\frac{\partial H^k}{\partial x^k}\Big|_{x^{k*}} = x^{k*},\ \lambda^T = 0. \tag{8.60}$$

Let us assume $T = 6$. Substitute (8.59) into (8.60) to obtain

$$\triangle \lambda^k = -k + 5,\ \lambda^6 = 0.$$

From Sect. A.5, we find the solution to be

$$\lambda^k = -\frac{1}{2}k^2 + \frac{11}{2}k + c,$$

where c is a constant. Since $\lambda^6 = 0$, we can obtain the value of c by setting $k = 6$ in the above equation. Thus,

$$\lambda^6 = -\frac{1}{2}(36) + \frac{11}{2}(6) + c = 0 \Rightarrow c = -15,$$

so that

$$\lambda^k = -\frac{1}{2}k^2 + \frac{11}{2}k - 15. \tag{8.61}$$

A sketch of the values for λ^k and x^k appears in Fig. 8.4. Note that $\lambda^5 = 0$, so that the control u^4 is singular. However, since $x^4 = 1$ we choose $u^4 = -1$ in order to bring x^5 down to 0.

The solution of the problem for $T \geq 7$ is carried out in the same way that we solved Example 2.4. Namely, observe that $x^{5*} = 0$ and $\lambda^5 = \lambda^6 = 0$, so that the control is singular. We simply make $\lambda^k = 0$ for $k \geq 7$ so that $u^{k*} = 0$ for all $k \geq 7$. It is clear without a formal proof that this maximizes (8.54).

Example 8.7 Let us consider a discrete version of the production-inventory example of Sect. 6.1; see Kleindorfer et al. (1975). Let I^k, P^k, and S^k be the inventory, production, and demand at time k, respectively. Let I^0 be the initial inventory, let $\hat{I}$ and $\hat{P}$ be the goal levels of inventory and production, and let h and c be inventory and production cost coefficients. The problem is:

$$\max_{P^k \geq 0} \left\{ J = \sum_{k=0}^{T-1} -\frac{1}{2}[h(I^k - \hat{I})^2 + c(P^k - \hat{P})^2] \right\} \tag{8.62}$$

subject to

$$\triangle I^k = P^k - S^k,\ k = 0, 1, \ldots, T-1,\ I^0 \text{ given.} \tag{8.63}$$

Form the Hamiltonian

$$H^k = -\frac{1}{2}[h(I^k - \hat{I})^2 + c(P^k - \hat{P})^2] + \lambda^{k+1}(P^k - S^k), \tag{8.64}$$

where the adjoint variable satisfies

$$\triangle \lambda^k = -\frac{\partial H^k}{\partial I^k} = h(I^k - \hat{I}),\ \lambda^T = 0. \tag{8.65}$$

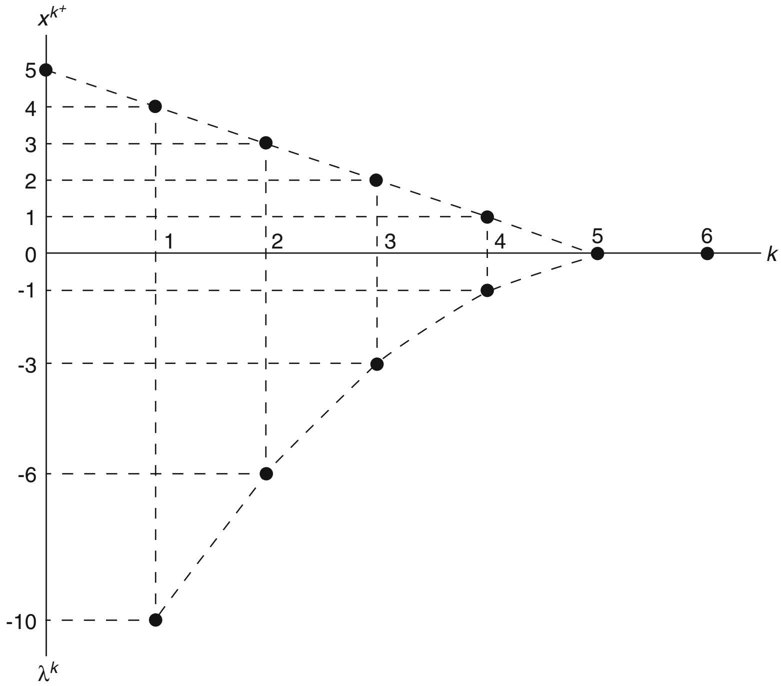

Figure 8.4: Optimal state x^{k^*} and adjoint λ^k

To maximize the Hamiltonian, let us differentiate (8.64) to obtain

$$\frac{\partial H^k}{\partial P^k} = -c(P^k - \hat{P}) + \lambda^{k+1} = 0.$$

Since production must be nonnegative, we obtain the optimal production as

$$P^{k*} = \max[0, \hat{P} + \lambda^{k+1}/c]. \tag{8.66}$$

Expressions (8.63), (8.65), and (8.66) determine a two-point boundary value problem. For a given set of data, it can be solved numerically by using spreadsheet software like Excel; see Sect. 2.5 and Exercise 8.21. If the constraint $P^k \geq 0$ is dropped it can be solved analytically by the method of Sect. 6.1, with difference equations replacing the differential equations used there.

8.3 A General Discrete Maximum Principle

For the maximum principle (8.53) we assumed that H^k and g were concave in u_k so that the set of admissible controls was convex. These are fairly strong assumptions which will now be relaxed and a general maximum principle stated. The proof can be found in Canon et al. (1970). Other references on discrete maximum principles are Halkin (1966) and Holtzman (1966). The problem to be solved is:

$$\max \left\{ J = \sum_{k=0}^{T-1} F(x^k, u^k, k) \right\} \tag{8.67}$$

subject to

$$\begin{aligned} &\triangle x^k = f(x^k, u^k, k), \ \ x^0 \text{ given} \\ &u^k \in \Omega_k, \ k = 0, 1, \ldots, (T-1). \end{aligned} \tag{8.68}$$

Assumptions required are:

(i) $F(x^k, u^k, k)$ and $f(x^k, u^k, k)$ are continuously differentiable in x^k for every u^k and k.

(ii) The sets $\{-F(x, \Omega^k, k), f(x, \Omega^k, k)\}$ are *b-directionally convex* for every x and k, where $b = (-1, 0, \ldots, 0)$. That is, given v and w in Ω^k and $0 \leq \lambda \leq 1$, there exists $u(\lambda) \in \Omega^k$ such that

$$F(x, u(\lambda), k) \geq \lambda F(x, v, k) + (1 - \lambda) F(x, w, k)$$

and

$$f(x, u(\lambda), k) = \lambda f(x, v, k) + (1 - \lambda) f(x, w, k)$$

for every x and k. It should be noted that convexity implies b-directional convexity, but not the converse.

(iii) Ω^k satisfies the Kuhn-Tucker constraint qualification.

With these assumptions replacing the assumptions of Theorem 8.3, and since there is no salvage value term in (8.67) meaning that $S(x^T, T) \equiv 0$, the maximum principle (8.53) with $\lambda^T = 0$ holds with control constraint set $g(u^k, k) \geq b^k$ replaced by $u^k \in \Omega$. When the salvage function $S(x^T, T)$ is not identically zero, the objective function in

(8.67) is replaced by the Bolza form objective function (8.42). In Exercise 8.20, you are asked to convert the problem defined by (8.42) and (8.68) to its Lagrange form, and then obtain the corresponding assumptions on the salvage value function $S(x^T, T)$ for the results of this section to apply. For a fixed-end-point problem, i.e., when x^T is also given in (8.68), the more general maximum principle holds with λ^T a constant to be determined. Exercise 8.17 is an example of a fixed-end-point problem. Finally, when there are lags in the system dynamics, i.e., when the state of the system in a period depends not only on the state and the control in the previous period, but also on the values of these variables in prior periods, it is easy to adapt the discrete maximum principle to deal with such systems; see Burdet and Sethi (1976). Exercise 8.22 presents an advertising model containing lags in its sales-advertising dynamics.

Some concluding remarks on the applications of discrete-time optimal control problems are appropriate. Real-life examples that can be modeled as such problems include the following: payments of principal and interest on loans; harvesting of crops; production planning for monthly demands; etc. Such problems would require efficient computational procedures for their solution. Some references dealing with computational methods for discrete optimal control problems are Murray and Yakowitz (1984), Dunn and Bertsekas (1989), Pantoja and Mayne (1991), Wright (1993), and Dohrmann and Robinett (1999). Another reason that makes the discrete optimal control theory important arises from the fact that computers are being used increasingly in the control of dynamic systems.

Finally, Pepyne and Cassandras (1999) have explored an optimal control approach to treat discrete event dynamic systems (DEDS). They also apply the approach to a transportation problem, modeled as a polling system.

Exercises for Chapter 8

E 8.1 Determine the critical points of the following functions:

(a) $h(y, z) = -5y^2 - z^2 + 10y + 6z + 27$,

(b) $h(y, z) = 5y^2 - yz + z^2 - 10y - 18z + 17$.

E 8.2 Let h be twice differentiable with its Hessian matrix defined to be $H = h_{xx}$. Let $\bar{x}$ be a critical point, i.e., a solution of $h_x = 0$. Let H_j be the jth principal minor, i.e., the $j \times j$ submatrix found in the first j

rows and the first j columns of H. Let $|H_j|$ be the determinant of H_j. Then, y^0 is a local maximum of h if

$$H_1 < 0,\ |H_2| > 0,\ |H_3| < 0, \ldots, (-1)^n |H_n| = (-1)^n |H| > 0$$

evaluated at $\bar{x}$, and $\bar{x}$ is a local minimum of h if

$$H_1 > 0,\ |H_2| > 0,\ |H_3| > 0, \ldots, |H_n| = |H| > 0$$

evaluated at $\bar{x}$. Apply these conditions to Exercise 8.1 to identify local minima and maxima of the functions in (a) and (b).

E 8.3 Find the optimal speed in cases (a) and (b) below:

(a) During times of an energy crisis, it is important to economize on fuel consumption. Assume that when traveling x mile/hour in high gear, a truck burns fuel at the rate of

$$\frac{1}{500}\left[\frac{2500}{x} + x\right] \text{ gallons/mile.}$$

If fuel costs 50 cents per gallon, find the speed that will minimize the cost of fuel for a 1000 mile trip. Check the second-order condition.

(b) When the government imposed this optimal speed in 1974, truck drivers became so angry that they staged blockades on several freeways around the country. To explain the reason for these blockades, we found that a crucial figure was the hourly wage of the truckers, estimated at \$3.90 per hour at that time. Recompute a speed that will minimize the total cost of fuel and the driver's wages for the same trip. You do not need to check for the second-order condition.

E 8.4 Use (8.5)–(8.7) to derive Eq. (8.8).

E 8.5 Verify Eq. (8.8) in Example 8.1 by determining $h^*(a)$ and expanding the function $h^*(10 + \epsilon)$ in a Taylor series around the value 10.

E 8.6 Maximize $h(x) = (1/3)x^3 - 6x^2 + 32x + 5$ subject to each of the following constraints:

(a) $x \leq 6$

(b) $x \leq 20$.

E 8.7 Rework Example 8.4 by replacing $(2, 2)$ with each of the following points:

(a) $(0, -1)$

(b) $(1/2, 1/2)$.

E 8.8 Add the equality constraint $2x = y$ to the problem in Example 8.4 and solve it.

E 8.9 Solve the problem:

$$\begin{cases} \max h(x, y) \\ \text{subject to} \\ x^2 \leq (2 - y)^3, \ y \geq 0, \end{cases}$$

for (a) $h(x, y) = x + y$, (b) $h(x, y) = x + 2y$, and (c) $h(x, y) = x + 3y$. Comment on the solution in each of the cases (a), (b), and (c).

E 8.10 *Constraint Qualification.* Show that the feasible region in two dimensions, determined by the constraints $(1 - x)^3 - y \geq 0, \ x \geq 0$, and $y \geq 0$, does not satisfy the constraint qualification (8.36) at the boundary point (1,0). Also sketch the feasible region to see the presence of a cusp at point $(1, 0)$.

E 8.11 *Constraint Qualification.* Show that the feasible region in two dimensions, determined by the constraints $x^2 + y^2 \leq 1, \ x \geq 0$, and $y \geq 0$, satisfies the constraint qualification (8.36) at the boundary point (1,0). Also sketch the feasible region to contrast it with that in Exercise 8.10.

E 8.12 Solve graphically the problem of minimizing x subject to the constraints

$$1 - x \geq 0, \quad y \geq 0, \quad x^3 - y \geq 0.$$

Show that the constraints do not satisfy the constraint qualification (8.36) at the optimal point.

E 8.13 Rewrite the maximum principle (8.53) for the special case of the linear Mayer form problem obtained when $F \equiv 0$ and $S(x^T, T) = cx^T$, where c is an n-component row vector of constants.

E 8.14 Show that the necessary conditions for u^k to be an optimal solution for (8.52) are given by (8.50) and (8.51).

E 8.15 Prove Theorem 8.3.

E 8.16 Formulate and solve a discrete-time version of the cash balance model of Sect. 5.1.1.

E 8.17 *Minimum Fuel Problem.* Consider the problem:

$$\begin{cases} \min\left\{J = \sum_{k=0}^{T-1} |u^k|\right\} \\ \text{subject to} \\ \triangle x^k = Ax^k + bu^k, \quad x^0 \text{ and } x^T \text{ given} \\ u^k \in [-1, 1], \quad k = 0, 1, \ldots, T-1, \end{cases}$$

where A is a given matrix. Obtain the expression for the adjoint variable and the form of the optimal control.

E 8.18 *Current-Value Formulation.* Obtain the current-value formulation of the discrete maximum principle. Assume that r is the discount rate, i.e., $1/(1+r)$ is the discount factor.

E 8.19 Convert the Bolza form problem (8.42)–(8.44) to the equivalent linear Mayer form; see Sect. 2.1.4 for a similar conversion in the continuous-time case.

E 8.20 Convert the problem defined by (8.42) and (8.68) to its Lagrange form. Then, obtain the assumptions on the salvage value function $S(x^T, T)$ so that the results of Sect. 8.3 apply. Under these assumptions, state the maximum principle for the Bolza form problem defined by (8.42) and (8.68).

E 8.21 Use Excel to solve the production planning problem given by (8.62) and (8.63) with $I^0 = 1,\ \hat{P} = 30,\ \hat{I} = 15,\ h = c = 1,\ T = 8$, and $S^k = k^3 - 12k^2 + 32k + 30,\ k = 0, 1, 2, \ldots, (T-1)$. This is a discrete time version of Example 6.1 so that you can compare your solution with Fig. 6.1.

E 8.22 *An Advertising Model* (Burdet and Sethi 1976). Let x^k denote the sale and $u^k, k = 1, 2, \ldots, T-1$, denote the amount of advertising in period k. Formulate the sales-advertising dynamics as

$$\triangle x^k = -\delta x^k + r \sum_{l=0}^{k} f_k^l(x^l, u^l), x^0 \text{ given},$$

where δ and r are decay and response constants, respectively, and $f_k^l(x^l, u^l)$ is a nonnegative function that decreases with x^l and increases with u^l. In the special case when

$$f_k^l(x^l, u^l) = \gamma_k^l u^l, \gamma_k^l > 0,$$

obtain optimal advertising amounts to maximize the total discounted profit given by

$$\sum_{k=1}^{T-1} (\pi x^k - u^k)(1+\rho)^{-k},$$

where, as in Sect. 7.2.1, π denotes per unit sales revenue, ρ denotes the discount rate, and the inequalities $0 \le u^k \le Q^k$ represent the restrictions on the advertising amount u^k. For the continuous-time version of problems with lags, see Hartl and Sethi (1984b).

Chapter 9

Maintenance and Replacement

The problem of simultaneously determining the lifetime of an asset or an activity along with its management during that lifetime is an important problem in practice. The most typical example is the problem of optimal maintenance and replacement of a machine; see Rapp (1974) and Pierskalla and Voelker (1976). Other examples occur in forest management, such as in Näslund (1969), Clark (1976), and Heaps (1984), and in advertising copy management, such as in Pekelman and Sethi (1978).

The first major work dealing with machine replacement problems appeared in 1949 as a MAPI (Machinery and Applied Products Institute) study by Terborgh (1949). For the most part, this study was confined to those problems where the optimization was carried out only with respect to the replacement lives of the machines under consideration. Boiteux (1955) and Massé (1962) extended the single machine replacement problem to include the optimal timing of a partial replacement of the machine before its actual retirement. Näslund (1966) was the first to solve a generalized version of the Boiteux problem by using the maximum principle. He considered optimal preventive maintenance applied continuously over the entire period instead of a single optimal partial replacement before the machine is retired. Thompson (1968) presented a modification of Näslund's model which is described in the following section.

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9.1 A Simple Maintenance and Replacement Model

Consider a single machine whose resale value gradually declines over time. Its output is assumed to be proportional to its resale value. By applying preventive maintenance, it is possible to slow down the rate of decline of the resale value. The control problem consists of simultaneously determining the optimal rate of preventive maintenance and the sale date of the machine. Clearly this is an optimal control problem with unspecified terminal time; see Sect. 3.1 and Example 3.6.

9.1.1 The Model

In order to define Thompson's model, we use the following notation:

T = the sale date of the machine to be determined,

ρ = the constant discount rate,

$x(t)$ = the resale value of the machine in dollars at time t; let $x(0) = x_0$,

$u(t)$ = the preventive maintenance rate at time t (maintenance here means money spent over and above the minimum required for necessary repairs),

$g(t)$ = the maintenance effectiveness function at time t (measured in dollars added to the resale value per dollar spent on preventive maintenance),

$d(t)$ = the obsolescence function at time t (measured in terms of dollars subtracted from x at time t),

π = the constant production rate in dollars per unit time per unit resale value; assume $\pi > \rho$ or else it does not pay to produce.

It is assumed that $g(t)$ is a nonincreasing function of time and $d(t)$ is a nondecreasing function of time, and that for all t

$$u(t) \in \Omega = [0, U], \tag{9.1}$$

where U is a positive constant.

The present value of the machine is the sum of two terms, the discounted income (production minus maintenance) stream during its life plus the discounted resale value at T:

$$J = \int_0^T [\pi x(t) - u(t)]e^{-\rho t}dt + x(T)e^{-\rho T}. \tag{9.2}$$

The state variable x is affected by the obsolescence factor, the amount of preventive maintenance, and the maintenance effectiveness function. Thus,

$$\dot{x}(t) = -d(t) + g(t)u(t), \quad x(0) = x_0. \tag{9.3}$$

In the interests of realism we assume that

$$-d(t) + g(t)U \leq 0, \; t \geq 0. \tag{9.4}$$

The assumption implies that preventive maintenance is not so effective as to enhance the resale value of the machine over its previous values; rather, it can at most slow down the decline of the resale value, even when preventive maintenance is performed at the maximum rate U. A modification of (9.3) is given in Arora and Lele (1970). See also Hartl (1983b).

The optimal control problem is to maximize (9.2) subject to (9.1) and (9.3).

9.1.2 Solution by the Maximum Principle

This problem is similar to Model Type (a) of Table 3.3 with the free-end-point condition as in Row 1 of Table 3.1. Therefore, we follow the steps for solution by the maximum principle stated in Chap. 3.

The standard Hamiltonian as formulated in Sect. 2.2 is

$$H = (\pi x - u)e^{-\rho t} + \lambda(-d + gu), \tag{9.5}$$

where the adjoint variable λ satisfies

$$\dot{\lambda} = -\pi e^{-\rho t}, \;\; \lambda(T) = e^{-\rho T}. \tag{9.6}$$

Since T is unspecified, the required additional terminal condition (3.15) for this problem is

$$-\rho e^{-\rho T} x(T) = -H, \tag{9.7}$$

which must hold on the optimal path at time T.

The adjoint variable λ can be easily obtained by integrating (9.6), i.e.,

$$\lambda(t) = e^{-\rho T} + \int_t^T \pi e^{-\rho\tau} d\tau = e^{-\rho T} + \frac{\pi}{\rho}[e^{-\rho t} - e^{-\rho T}]. \tag{9.8}$$

The interpretation of $\lambda(t)$ is as follows. It gives, in present value terms, the marginal profit per dollar of gain in resale value at time t.

The first term represents the present value of one dollar of additional salvage value at T brought about by one dollar of additional resale value at the current time t. The second term represents the present value of incremental production from t to T brought about by the extra productivity of the machine due to the additional one dollar of resale value at time t.

Since the Hamiltonian is linear in the control variable u, the optimal control for a problem with any fixed T is bang-bang as in Model Type (a) in Table 3.3. Thus,

$$u^*(t) = \text{bang}\left[0, U; \{e^{-\rho T} + \frac{\pi}{\rho}(e^{-\rho t} - e^{-\rho T})\}g(t) - e^{-\rho t}\right]. \tag{9.9}$$

To interpret this optimal policy, we see that the term

$$\{e^{-\rho T} + \frac{\pi}{\rho}(e^{-\rho t} - e^{-\rho T})\}g(t)$$

is the present value of the marginal return from increasing the preventive maintenance by one dollar at time t. The last term $e^{-\rho t}$ in the argument of the bang function is the present value of that one dollar spent for preventive maintenance at time t. Thus, in words, the optimal policy means the following: if the marginal return of one dollar of additional preventive maintenance is more than one dollar, then perform the maximum possible preventive maintenance, otherwise do not perform any at all.

To find how the optimal control switches, we need to examine the switching function in (9.9). Rewriting it as

$$e^{-\rho t}\left[\frac{\pi g(t)}{\rho} - (\frac{\pi}{\rho} - 1)e^{\rho(t-T)}g(t) - 1\right] \tag{9.10}$$

and taking the derivative of the bracketed terms with respect to t, we can conclude that the expression inside the square brackets in (9.10) is monotonically decreasing with time t on account of the assumptions that $\pi/\rho > 1$ and that $g(t)$ is nonincreasing with t (see Exercise 9.1). It follows that there will not be a singular control for any finite interval of time. Furthermore, since $e^{-\rho t} > 0$ for all t, we can conclude that the switching function can only go from positive to negative and not vice versa. Thus, the optimal control will be either U, or zero, or U followed by zero. The switching time t^s is obtained as follows: equate (9.10) to zero and solve for t. If the solution is negative, let $t^s = 0$, and if the solution is greater than T, let $t^s = T$, otherwise set t^s equal to the solution. It is clear that the optimal control in (9.9) can now be rewritten as

$$u^*(t) = \begin{cases} U & t \le t^s, \\ 0 & t > t^s. \end{cases} \tag{9.11}$$

Note that all of the above calculations were made on the assumption that T was fixed, i.e., without imposing condition (9.7). On an optimal path, this condition, which uses (9.5), (9.7), and (9.8), can be restated as

$$\begin{aligned} -\rho e^{-\rho T^*} x^*(T^*) &= -\{\pi x^*(T^*) - u^*(T^*)\}e^{-\rho T^*} \\ &\quad -e^{-\rho T^*}\{-d(T^*) + g(T^*)u(T^*)\}. \end{aligned} \tag{9.12}$$

This means that when $u^*(T^*) = 0$ (i.e., $t^s < T^*$), we have

$$x^*(T^*) = \frac{d(T^*)}{\pi - \rho}, \tag{9.13}$$

and when $u^*(T^*) = U$ (i.e., $t^s = T^*$), we have

$$x^*(T^*) = \frac{d(T^*) - [g(T^*) - 1]U}{\pi - \rho}. \tag{9.14}$$

Since $d(t)$ is nondecreasing, $g(t)$ is nonincreasing, and $x(t)$ is nonincreasing, Eq. (9.13) or Eq. (9.14), whichever the case may be, has a solution for T^*.

9.1.3 A Numerical Example

It is instructive to work an example of this model in which specific values are assumed for the various functions. Examples that illustrate other kinds of qualitatively different behavior are left as Exercises 9.3–9.5.

Suppose $U = 1$, $x(0) = 100$, $d(t) = 2$, $\pi = 0.1$, $\rho = 0.05$, and $g(t) = 2/(1+t)^{1/2}$. Then (9.3) specializes to

$$\dot{x}(t) = -2 + \frac{2u(t)}{\sqrt{1+t}}, \; x(0) = 100. \tag{9.15}$$

First, we write the condition on t^s by equating (9.10) to 0, which gives

$$\pi - (\pi - \rho)e^{-\rho(T - t^s)} = \frac{\rho}{g(t^s)}. \tag{9.16}$$

In doing so, we have assumed that the solution of (9.16) lies in the open interval $(0, T)$. As we will indicate later, special care needs to be exercised if this is not the case.

Substituting the data in (9.16) we have

$$0.1 - 0.05e^{-0.05(T-t^s)} = 0.025(1+t^s)^{1/2},$$

which simplifies to

$$(1+t^s)^{1/2} = 4 - 2e^{-0.05(T-t^s)}. \tag{9.17}$$

Then, integrating (9.15), we find

$$x(t) = -2t + 4(1+t)^{1/2} + 96, \text{ if } t \leq t^s,$$

and hence

$$\begin{aligned} x(t) &= -2t^s + 4(1+t^s)^{1/2} + 96 - 2(t - t^s) \\ &= 4(1+t^s)^{1/2} + 96 - 2t, \text{ if } t > t^s. \end{aligned}$$

Since we have assumed $0 < t^s < T$, we substitute $x(T)$ into (9.13), and obtain

$$4(1+t^s)^{1/2} + 96 - 2T = 2/0.05 = 40,$$

which simplifies to

$$T = 2(1+t^s)^{1/2} + 28. \tag{9.18}$$

We must solve (9.17) and (9.18) simultaneously. Substituting (9.18) into (9.17), we find that t^s must be a zero of the function

$$h(t^s) = (1+t^s)^{1/2} - 4 + 2e^{-[2(1+t^s)^{1/2} - t^s + 28]/20}. \tag{9.19}$$

A simple binary search program was written to solve this equation, which obtained the value $t^s = 10.6$. Substitution of this into (9.18) yields $T = 34.8$. Since this satisfies our supposition that $0 < t^s < T$, we can conclude our computations. Thus, if we let the unit of time be 1 month, then the optimal solution is to perform preventive maintenance at the maximum rate during the first 10.6 months, and thereafter not at all. The sale date is at 34.8 months after purchase. Figure 9.1 gives the functions $x(t)$ and $u(t)$ for this optimal maintenance and sale date policy.

If, on the other hand, the solution of (9.17) and (9.18) did not satisfy our supposition, we would need to follow the procedure outlined earlier in the section. This would result in $t^s = 0$ or $t^s = T$. If $t^s = 0$, we would obtain T from (9.18), and conclude $u^*(t) = 0,\ 0 \leq t \leq T$. Alternatively, if $t^s = T$, we would need to substitute $x(T)$ into (9.14) to obtain T. In this case the optimal control would be $u^*(t) = U,\ 0 \leq t \leq T$.

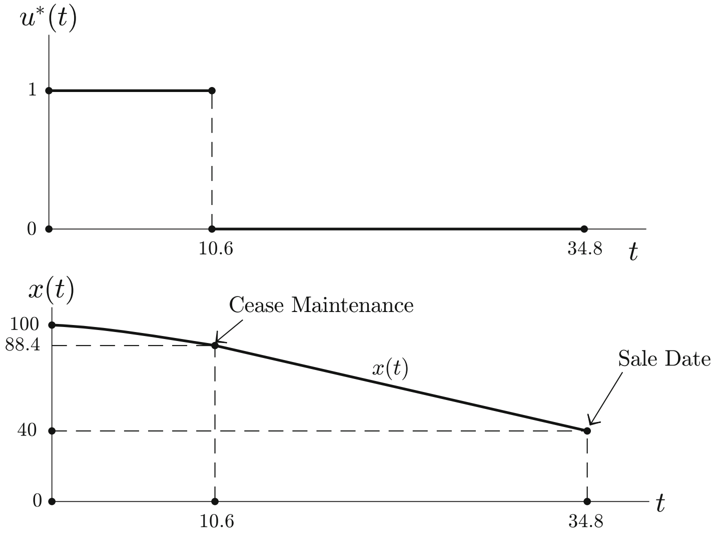

Figure 9.1: Optimal maintenance and machine resale value

9.1.4 An Extension

The pure bang-bang result in the model developed above is a result of the linearity in the problem. The result can be enriched as in Sethi (1973b) by generalizing the resale value equation (9.3) as follows:

$$\dot{x}(t) = -d(t) + g(u(t), t), \tag{9.20}$$

where g is nondecreasing and concave in u. For this section, we will assume the sale date T to be fixed for simplicity and g to be strictly concave in u, i.e., $g_u \geq 0$ and $g_{uu} < 0$ for all t. Also, $g_t \leq 0,\ g_{ut} \leq 0$, and $g(0, t) = 0$; see Exercise 9.7 for an example of the function $g(u, t)$.

The standard Hamiltonian is

$$H = (\pi x - u)e^{-\rho t} + \lambda[-d + g(u, t)], \tag{9.21}$$

where λ is given in (9.8). To maximize the Hamiltonian, we differentiate it with respect to u and equate the result to zero. Thus,

$$H_u = -e^{-\rho t} + \lambda g_u = 0. \tag{9.22}$$

If we let $u^0(t)$ denote the solution of (9.22), then $u^0(t)$ maximizes the Hamiltonian (9.21) because of the concavity of g in u. Thus, for a fixed T, the optimal control is

$$u^*(t) = \text{sat}[0, U; u^0(t)]. \tag{9.23}$$

To determine the direction of change in $u^*(t)$, we obtain $\dot{u}^0(t)$. For this, we use (9.22) and the value $\lambda(t)$ from (9.8) to obtain

$$g_u = \frac{e^{-\rho t}}{\lambda(t)} = \frac{1}{\frac{\pi}{\rho} - (\frac{\pi}{\rho} - 1)e^{\rho(t-T)}}. \tag{9.24}$$

Since $\pi > \rho$, the denominator on the right-hand side of (9.24) is monotonically decreasing with time. Therefore, the right-hand side of (9.24) is increasing with time. Taking the time derivative of (9.24), we have

$$g_{ut} + g_{uu}\dot{u}^0 = \frac{\rho^2(\pi - \rho)e^{\rho(t-T)}}{[\pi - (\pi - \rho)e^{\rho(t-T)}]^2} > 0.$$

But $g_{ut} \leq 0$ and $g_{uu} < 0$, it is therefore obvious that $\dot{u}^0(t) < 0$. In order now to sketch the optimal control $u^*(t)$ specified in (9.23), let us define $0 \leq t_1 \leq t_2 \leq T$ such that $u^0(t) \geq U$ for $t \leq t_1$ and $u^0(t) \leq 0$ for $t \geq t_2$. Then, we can rewrite the sat function in (9.23) as

$$u^*(t) = \begin{cases} U & \text{for } t \in [0, t_1], \\ u^0(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in [t_2, T]. \end{cases} \tag{9.25}$$

In (9.25), it is possible to have $t_1 = 0$ and/or $t_2 = T$. In Fig. 9.2 we have sketched a case when $t_1 > 0$ and $t_2 < T$.

Note that while $u^0(t)$ in Fig. 9.2 is decreasing over time, the way it will decrease will depend on the nature of the function g. Indeed, the shape of $u^0(t)$, while always decreasing, can be quite general. In particular, you will see in Exercise 9.7 that the shape of $u^0(t)$ is concave and, furthermore, $u^0(t) > 0,\ t \geq 0$, so $t_2 = T$ in that case.

9.2 Maintenance and Replacement for a Machine Subject to Failure

In Kamien and Schwartz (1971a), a related model is developed which has somewhat different assumptions. They assume that the production rate of the machine is independent of its age, while its probability of failure increases with its age. Consistent with this assumption, the purpose of preventive maintenance in the Kamien-Schwartz model is to influence

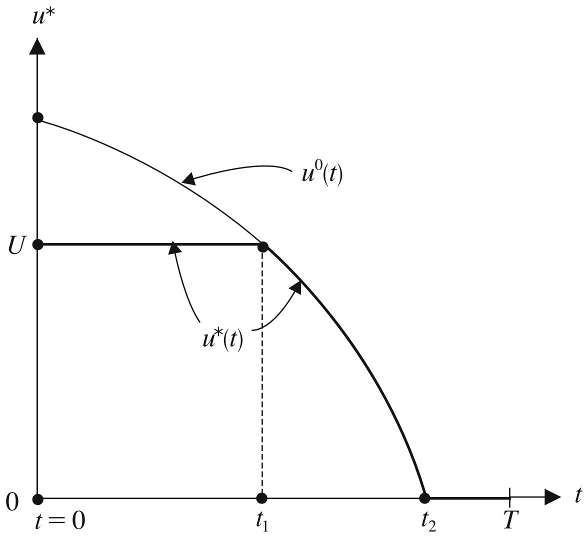

Figure 9.2: Sat function optimal control

the failure rate of the machine rather than arrest the deterioration in the resale value as before. Furthermore, their model also allows for sale of the machine at any time, provided it is still in running condition, and for its disposal as junk if it breaks down for good. The optimal control problem is therefore to find an optimal maintenance policy for the period of ownership and an optimal sale date at which the machine should be sold, provided that it has not yet failed. Other references to related models are Alam et al. (1976), Alam and Sarma (1974, 1977), Sarma and Alam (1975), Gaimon and Thompson (1984a, 1989), Dogramaci and Fraiman (2004), Dogramaci (2005), Bensoussan and Sethi (2007), and Bensoussan et al. (2015a).

9.2.1 The Model

In order to define the Kamien-Schwartz model, we use the following notation:

T = the sale date of a machine to be determined,

$u(t)$ = the preventive maintenance rate at time t; $0 \leq u(t) \leq 1$,

R = the constant positive rate of revenue produced by a functioning machine independent of its age at any time, net of all costs except preventive maintenance,

ρ = the constant discount rate,

L = the constant positive junk value of the failed machine independent of its age at failure,

$B(t)$ = the (exogenously specified) resale value of the machine at time t, if it is still functioning; $\dot{B}(t) \leq 0$,

$h(t)$ = the natural failure rate (also termed the natural hazard rate in the reliability theory); $h(t) \geq 0,\ \dot{h}(t) \geq 0$,

$F(t)$ = the cumulative probability that the machine has failed by time t,

$C(u;h)$ = the cost function depending on the preventive maintenance u when the natural failure rate is h.

To make economic sense, an operable machine must be worth at least as much as an inoperable machine and its resale value should not exceed the present value of the potential revenue generated by the machine if it were to function forever. Thus,

$$0 \leq L \leq B(t) \leq R/\rho, \quad t \geq 0. \tag{9.26}$$

Also for all $t > 0$,

$$u(t) \in \Omega = [0,1]. \tag{9.27}$$

Finally, when the natural failure rate is h and a controlled failure rate of $h(1-u)$ is sought, the action of achieving this reduction will cost $C(u;h)$ dollars. For simplicity, we assume that $C(u;h) = C(u)h$ with

$$C(0) = 0,\ C_u > 0,\ C_{uu} > 0,\ \text{for } u \in [0,1]. \tag{9.28}$$

Thus, the cost of reducing the failure rate increases more than proportionately as the fractional reduction increases. But the cost of a given fractional reduction increases linearly with the natural failure rate. Hence, these conditions imply that a given absolute reduction becomes increasingly more costly as the machine gets older.

To derive the state equation for $F(t)$, we note that $\dot{F}/(1-F)$ denotes the conditional probability density for the failure of the machine at time t, given that it has survived to time t. This is assumed to depend on two things, namely (i) the natural failure rate that governs the machine in the absence of preventive maintenance, and (ii) the current rate of preventive maintenance.

Thus,

$$\frac{\dot{F}(t)}{1-F(t)} = h(t)[1-u(t)], \tag{9.29}$$

which gives the state equation

$$\dot{F} = h(1-u)(1-F),\ F(0) = 0. \tag{9.30}$$

Thus, the controlled failure rate at time t is $h(t)(1-u(t))$. If $u = 0$, the failure rate assumes its natural value h. As u increases, the failure rate decreases and drops to zero when $u = 1$.

The expected present value of the machine is the sum of the expected present values of (i) the total revenue it produces less the total cost of maintenance, (ii) its junk value if it should fail before it is sold, and (iii) the salvage value if it does not fail and is sold. That is,

$$J = \int_0^T e^{-\rho t}\left\{[R - C(u)h](1-F) + L\dot{F}\right\} dt + e^{-\rho T}B(T)[1-F(T)].$$

Using (9.30), we can rewrite J as follows:

$$J = \int_0^T e^{-\rho t}[R - C(u)h + L(1-u)h]\,(1-F)dt + e^{-\rho T}B(T)\,[1-F(T)]\,. \tag{9.31}$$

The optimal control problem is to maximize J in (9.31) subject to (9.30) and (9.27).

Remark 9.1 In the absence of discounting, the expected junk value term $\int_0^T L\dot{F}(t)dt$ reduces to $LF(T)$, i.e., the junk value times the probability that the machine fails by time T.

Remark 9.2 While the maintenance and replacement problem of Kamien and Schwartz is stochastic, they formulate and solve it as a deterministic optimal control problem. Bensoussan and Sethi (2007) formulate the underlying stochastic problem as a stochastic optimal control problem, and show how their solution relates to that of the Kamien-Schwartz model. They also provide a sufficient condition for an optimal maintenance and replacement policy.

9.2.2 Optimal Policy

The problem is similar to Model Type (f) in Table 3.3 subject to the free-end-point condition as in Row 1 of Table 3.1. Therefore, we follow the steps for solution by the maximum principle stated in Chap. 3. The standard Hamiltonian is

$$H = e^{-\rho t}[R - C(u)h + L(1-u)h](1-F) + \lambda(1-u)h(1-F), \tag{9.32}$$

and the adjoint variable satisfies

$$\begin{cases} \dot{\lambda} & = e^{-\rho t}[R - C(u)h + L(1-u)h] + \lambda h(1-u), \\ \lambda(T) & = -e^{-\rho T} B(T). \end{cases} \tag{9.33}$$

Since $T \geq 0$ is also to be decided, we require the additional transversality condition (3.77) for an optimal T^* to satisfy.

$$\begin{aligned} & R - C[u^*(T^*)]h(T^*) + L[1-u^*(T^*)]h(T^*) \\ & \qquad -[1-u^*(T^*)]h(T^*)B(T^*) - \rho B(T^*) + B_T(T^*) = 0. \end{aligned} \tag{9.34}$$

In Exercise 9.8, you are asked to derive this condition by using (9.31)–(9.33) in (3.77).

While we know from (3.79) that (9.34) has a standard economic interpretation of having zero marginal profit of changing T^*, it is still illuminating to flesh out a more detailed interpretation of each term in what looks like a fairly complex expression. A good way to accomplish that is totally what we get if we decide to sell the machine at time $T^* + \delta$ in comparison to selling it at T^*. We will do this only for a small $\delta > 0$, and leave it as Exercise 9.9 for a small $\delta < 0$.

First we note that in solving Exercise 9.8 to obtain (9.34) from (3.77), a simplification involved canceling the common factor $e^{-\rho T^*}(1 - F(T^*)) > 0$. Removing $e^{-\rho T^*}$ brings the revenue and cost terms from present-value dollars to dollars at time T^*. The presence of the probability term $1 - F(T^*)$ means that the machine will be replaced at T^* if it has not failed by time T^* with that probability. Its removal means that (9.34) can be interpreted as if we are at T^* and we find the machine to be working, which is tantamount to interpreting (9.34) with $F(T^*) = 0$.

Now consider keeping the machine to $T^* + \delta$. Clearly we lose its selling price $B(T^*)$ in doing so. But then we gain the following amounts discounted to time T^*:

$$\{R - C(u^*(T^*))h(T^*)\}\delta e^{-\rho\delta} = \{R - C(u^*(T^*))h(T^*)\}\delta + o(\delta), \tag{9.35}$$

$$L(1-u^*(T^*))h(T^*)\delta e^{-\rho\delta} = L(1-u^*(T^*))h(T^*)\delta + o(\delta), \tag{9.36}$$

$$\begin{aligned} B(T^*+\delta)(1-F(T^*+\delta))e^{-\rho\delta} &= B(T^*) - B(T^*)(1-u^*(T^*))h(T^*)\delta \\ & \quad -\rho B(T^*)\delta + B_T(T^*)\delta + o(\delta). \end{aligned} \tag{9.37}$$

The RHS of these equations can be obtained by noting that $e^{-\rho\delta} = 1 - \rho\delta + o(\delta)$, $B(T^* + \delta) = B(T^*) + B_T(T^*)\delta + o(\delta)$ and $F(T^* + \delta) = F(T^*) + \dot{F}(T^*)\delta = F(T^*) + (1 - u^*(T^*))h(T^*)(1 - F(T^*))\delta = (1 - u^*(T^*))h(T^*)\delta + o(\delta)$, since we had set $F(T^*) = 0$ for interpreting (9.34) upon arrival at T^* and finding the machine to be working. The net gain is the sum of (9.35), (9.36) and (9.37) less $B(T^*)$, where (9.35) gives the net cash flow (revenue—cost of preventative maintenance from T^* to $T^* + \delta$), (9.36) represents the junk value L multiplied by the probability $[1 - u(T^*)]h(T^*)\delta$ that the machine fails during the short time δ when the machine is found to be working at T^*, and (9.37) less $B(T^*)$ has three terms $-\rho B(T^*)\delta + B_T(T^*)\delta - B(T^*)(1 - u^*(T^*))h(T^*)\delta$: the first of which is the loss of interest $\rho B(T^*)\delta$ on the resale value $B(T^*)$ not obtained when deciding to keep the machine to $T^* + \delta$, the second term $B_T(T^*) < 0$ is the decrease in the resale value from T^* to $T^* + \delta$, and the third term represents the loss of the entire resale value if the machine fails with the probability $(1 - u^*(T^*))h(T^*)\delta$ given that the machine was found to be working at time T^*. Moreover, if we divide the net gain by δ and then let $\delta \to 0$, we obtain the marginal profit of keeping the machine from time T^* to $T^* + \delta$, and setting it equal to zero gives precisely the transversality condition (9.34). If we separate the revenue and cost terms in the resulting expression of the marginal profit, then (9.34) determining the optimal sale date T^* is the usual economic condition equating marginal revenue to marginal cost.

Next, we analyze the problem to obtain the optimal maintenance policy for a fixed T. If the optimal solution is in the interior, i.e., $u^* \in (0, 1)$, then the Hamiltonian maximizing condition gives

$$H_u = -e^{-\rho t}h(1 - F)[C_u + L + e^{\rho t}\lambda] = 0. \tag{9.38}$$

In the trivial cases in which the natural failure rate $h(t)$ is zero or when the machine fails with certainty by time t (i.e., $F(t) = 1$), then $u^*(t) = 0$. Assume therefore $h > 0$ and $F < 1$. Under these conditions, we can infer from (9.28) and (9.38) that

$$\left.\begin{array}{ll} \text{(i)} & C_u(0) + L + \lambda e^{\rho t} > 0 \Rightarrow u^*(t) = 0, \\ \text{(ii)} & C_u(1) + L + \lambda e^{\rho t} < 0 \Rightarrow u^*(t) = 1. \\ \text{(iii)} & \text{Otherwise, } C_u + L + \lambda e^{\rho t} = 0 \text{determines } u^*(t). \end{array}\right\} \tag{9.39}$$

Using the terminal condition $\lambda(T) = -e^{-\rho T}B(T)$ from (9.33), we can derive $u^*(T)$ satisfying (9.39):

$$\left.\begin{array}{ll} \text{(i)} & C_u(0) > B(T) - L \text{ and } u^*(T) = 0, \\ \text{(ii)} & C_u(1) < B(T) - L \text{ and } u^*(T) = 1. \\ \text{(iii)} & \text{Otherwise, } C_u = B(T) - L \Rightarrow u^*(T). \end{array}\right\} \quad (9.40)$$

Next we determine how $u^*(t)$ changes over time. Kamien and Schwartz (1971a, 1992) have shown that $u^*(t)$ is nonincreasing; see Exercise 9.10. Thus, there exists $T \geq t_2 \geq t_1 \geq 0$ such that

$$u^*(t) = \begin{cases} 1 & \text{for } t \in [0, t_1], \\ u^0(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in (t_2, T], \end{cases} \quad (9.41)$$

where $u^0(t)$ is the solution of (9.39)(iii). Clearly, it must also be shown that $\dot{u}^0(t) \leq 0$ as part of Exercise 9.10. Of course, $u^*(T)$ is immediately known from (9.40). If $u^*(T) \in (0, 1)$, it implies $t_2 = T$; and if $u^*(T) = 1$, it implies $t_1 = t_2 = T$.

For this model, the sufficiency of the maximum principle follows from Theorem 2.1; see Exercise 9.11.

9.2.3 Determination of the Sale Date

For a fixed T, we know that the terminal optimal control $u^*(T)$ is determined by (9.40). If this $u^*(T)$ also satisfies (9.34), we have determined an optimal trajectory as well as the optimal life of the machine. This, of course, is subject to the second-order condition since (9.34) is only a necessary condition for an optimal T^* to satisfy. It is clear that the determination of T^*, in most cases, will require numerical computations. The algorithm needs only to be a simple search method because it requires consideration of the single variable T.

Before we go to the next section, we remark that a business is usually a continuing entity and does not end at the sale date of one machine. Normally, an existing machine will be replaced by another, which in turn will be replaced by another, and so on. The technology of the newer machines will generally be different from that of the existing machine. In what follows, we address these issues. We will choose the discrete-time setting and illustrate the use of the discrete-time maximum principle developed in Chap. 8.

9.3 Chain of Machines

We now extend the problem of maintenance and replacement to a chain of machines. By this we mean that given the time periods $0, 1, 2, \ldots, T-1$, we begin with a machine purchase at the beginning of period zero. Then, we find an optimal number of machines, say ℓ, and optimal times $0 < t_1 < t_2, \ldots, t_{\ell-1} < t_\ell < T$ of their replacements such that the existing machine will be replaced by a new machine at time $t_j,\ j = 1, 2, \ldots, \ell$. At the end of the horizon defined by the beginning of period T, the last machine purchased will be salvaged. Moreover, the optimal maintenance policy for each of the machines in the chain must be found.

Two approaches to this problem have been developed in the literature. The first attempts to solve for an infinite horizon ($T = \infty$) with a simplifying assumption of identical machine lives, i.e.,

$$t_j - t_{j-1} = t_{j+1} - t_j \tag{9.42}$$

for all $j \geq 1$; see Sethi (1973b) as well as Exercise 9.16. In this case $\ell = \infty$ as well. The second relaxes the assumption (9.42) of identical machine lives, but then, it can only solve a finite horizon problem involving a finite chain of machines, i.e., ℓ is finite; see Sethi and Morton (1972) and Tapiero (1973). For a decision horizon formulation of this problem, see Sethi and Chand (1979), Chand and Sethi (1982), and Bylka et al. (1992).

In this section, we will deal with the latter problem as analyzed by Sethi and Morton (1972). The problem is solved by a mixed optimization technique. The subproblems dealing with the maintenance policy are solved by appealing to the discrete maximum principle. These subproblem solutions are then incorporated into a Wagner and Whitin (1958) model formulation for solution of the full problem. The procedure is illustrated by a numerical example.

9.3.1 The Model

Consider buying a machine at the beginning of period s and salvaging it at the beginning of period $t > s$. Let J_{st} denote the present value of all net earnings associated with the machine. To calculate J_{st} we need the following notation:

x_s^k = the resale value of the machine at the beginning of period $k,\ k = s, s+1, \ldots, t,$

P_s^k = the production quantity (in dollar value) during period $k,\ k = s, s+1, \ldots, t-1,$

E_s^k = the necessary expense of the ordinary maintenance (in dollars) during period k,

R_s^k = $P_s^k - E_s^k,\ k = s, s+1, \ldots, t-1,$

u^k = the rate of preventive maintenance (in dollars) during period $k,\ k = s, s+1, \ldots, t-1,$

C_s = the cost of purchasing the machine at the beginning of period s,

ρ = the periodic discount rate.

It is required that

$$0 \le u^k \le U^{sk},\ k \in [s, t-1]. \tag{9.43}$$

We can calculate J_{st} in terms of the variables and functions defined above:

$$J_{st} = \sum_{k=s}^{t-1} R_s^k (1+\rho)^{-k} - \sum_{k=s}^{t-1} u^k (1+\rho)^{-k} - C_s (1+\rho)^{-s} + x_s^t (1+\rho)^{-t}. \tag{9.44}$$

We must also have functions that will provide us with the ways in which states change due to the age of the machine and the amount of preventive maintenance. Also, assuming that at time s, the only machines available are those that are up-to-date with respect to the technology prevailing at s, we can subscript these functions by s to reflect the effect of the machine's technology on its state at a later time k. Let $\Psi_s(u^k, k)$ and $\Phi_s(u^k, k)$ be such concave functions so that we can write the following state equations:

$$\Delta R_s^k = R_s^{k+1} - R_s^k = \Psi_s(u^k, k),\ \ R_s^s \tag{9.45}$$

given,

$$\Delta x_s^k = \Phi_s(u^k, k),\ x_s^s = (1-\delta) C_s, \tag{9.46}$$

where δ is the fractional depreciation immediately after the purchase of the machine at time s.

To convert the problem into the Mayer form, define

$$A_s^k = \sum_{i=s}^{k-1} R_s^i (1+\rho)^{-i}, \tag{9.47}$$

$$B_s^k = \sum_{i=s}^{k-1} u^i (1+\rho)^{-i}. \tag{9.48}$$

Using Eqs. (9.47) and (9.48), we can write the optimal control problem as follows:

$$\max_{\{u_k\}} [J_{st} = A_s^t - B_s^t - C_s(1+\rho)^{-s} + x_s^t(1+\rho)^{-t}] \tag{9.49}$$

subject to

$$\Delta A_s^k = R_s^k (1+\rho)^{-k}, \;\; A_s^s = 0, \tag{9.50}$$

$$\Delta B_s^k = u^k (1+\rho)^{-k}, \;\; B_s^s = 0, \tag{9.51}$$

and the constraints (9.45), (9.46), and (9.43).

9.3.2 Solution by the Discrete Maximum Principle

We associate the adjoint variables $\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1}$, and λ_4^{k+1}, respectively with the state equations (9.50), (9.51), (9.45), and (9.46). Therefore, the Hamiltonian becomes

$$H = \lambda_1^{k+1} R_s^k (1+\rho)^{-k} + \lambda_2^{k+1} u^k (1+\rho)^{-k} + \lambda_3^{k+1} \Psi_s + \lambda_4^{k+1} \Phi_s, \tag{9.52}$$

where the adjoint variables $\lambda_1, \lambda_2, \lambda_3$, and λ_4 satisfy the following difference equations and terminal boundary conditions:

$$\Delta \lambda_1^k = -\frac{\partial H}{\partial A_s^k} = 0, \;\; \lambda_1^t = 1, \tag{9.53}$$

$$\Delta \lambda_2^k = -\frac{\partial H}{\partial B_s^k} = 0, \;\; \lambda_2^t = -1, \tag{9.54}$$

$$\Delta \lambda_3^k = -\frac{\partial H}{\partial R_s^k} = -\lambda_1^{k+1}(1+\rho)^{-k}, \;\; \lambda_3^t = 0, \tag{9.55}$$

$$\Delta \lambda_4^k = -\frac{\partial H}{\partial x^k} = 0, \;\; \lambda_4^t = (1+\rho)^{-t}. \tag{9.56}$$

The solutions of these equations are

$$\lambda_1^k = 1, \tag{9.57}$$

$$\lambda_2^k = -1, \tag{9.58}$$

$$\lambda_3^k = \sum_{i=k}^{t-1}(1+\rho)^{-i}, \tag{9.59}$$

$$\lambda_4^k = (1+\rho)^{-t}. \tag{9.60}$$

Note that λ_1^k, λ_2^k, and λ_4^k are constants for a fixed machine salvage time t. To apply the maximum principle, we substitute (9.57)–(9.60) into the Hamiltonian (9.52), collect terms containing the control variable u^k, and rearrange and decompose H as

$$H = H_1 + H_2(u^k), \tag{9.61}$$

where H_1 is that part of H which is independent of u^k and

$$H_2(u^k) = -u^k(1+\rho)^{-k} + \sum_{i=k+1}^{t-1}(1+\rho)^{-i}\Psi_s + (1+\rho)^{-t}\Phi_s. \tag{9.62}$$

Next we apply the maximum principle to obtain the necessary condition for the optimal schedule of preventive maintenance expenditures in dollars. The condition of optimality is that H should be a maximum along the optimal path. If u^k were unconstrained, this condition, given the concavity of Ψ_s and Φ_s, would be equivalent to setting the partial derivative of H with respect to u equal to zero, i.e.,

$$H_{u^k} = [H_2]_{u^k} = -(1+\rho)^{-k} + (\Psi_s)_{u^k}\sum_{i=k+1}^{t-1}(1+\rho)^{-i} + (\Phi_s)_{u^k}(1+\rho)^{-t} = 0. \tag{9.63}$$

Equation (9.63) is an equation in u^k with the exception of the particular case when Ψ_s and Φ_s are linear in u^k (which will be treated later in this section). In general, (9.63) may or may not have a unique solution. For our case we will assume Ψ_s and Φ_s to be of the form such that they give a unique solution for u^k. One such case occurs when Ψ_s and Φ_s are quadratic in u^k. In this case, (9.63) is linear in u^k and can be solved explicitly for a unique solution for u^k. Whenever a unique solution does exist, let this be

$$u^k = U_{st}^k. \tag{9.64}$$

The optimal control u^{k*} is given as

$$u^{k*} = \begin{cases} 0 & \text{if } U_{st}^k \leq 0, \\ U_{st}^k & \text{if } 0 \leq U_{st}^k \leq U^{sk}, \\ U^{sk} & \text{if } U_{st}^k \geq U^{sk}. \end{cases} \tag{9.65}$$

9.3.3 Special Case of Bang-Bang Control

We now treat the special case in which the problem, and therefore H, is linear in the control variable u^k. In this case, H can be maximized simply by having the control at its maximum when the coefficient of u^k in H is positive, and minimum when it is negative, i.e., the optimal control is of bang-bang type.

In our problem, we obtain the special case if Ψ_s and Φ_s assume the form

$$\Psi_s(u^k, k) = u^k \psi_s^k \tag{9.66}$$

and

$$\Phi_s(u^k, k) = u^k \phi_s^k, \tag{9.67}$$

respectively, where ψ_s^k and ϕ_s^k are given constants. Then, the coefficient of u^k in H, denoted by $W_s(k,t)$, is

$$W_s(k,t) = -(1+\rho)^{-k} + \psi_s^k \sum_{i=k+1}^{t-1} (1+\rho)^{-i} + \phi_s^t (1+\rho)^{-t}, \tag{9.68}$$

and the optimal control u^{k*} is given by

$$u^{k*} = \text{bang}[0, U^{sk}; W_s(k,t)],\ k = s, s+1, \ldots, t-1. \tag{9.69}$$

9.3.4 Incorporation into the Wagner-Whitin Framework for a Complete Solution

Once u^{k*} has been obtained as in (9.65) or (9.69), we can substitute it into (9.45) and (9.46) to obtain R_s^{k*} and x_s^{k*}, which in turn can be used in (9.44) to obtain the optimal value of the objective function denoted by J_{st}^*. This can be done for each pair of machine purchase time s and sale time $t > s$.

Let g_s denote the present value of the profit (discounted to period 0) of an optimal replacement and preventive maintenance policy for periods $s, s+1, \ldots, T-1$. Then,

$$g_s = \max_{t=s+1,\ldots,T} [J_{st}^* + g_t], \ 0 \leq s \leq T-1 \tag{9.70}$$

with the boundary condition

$$g_T = 0. \tag{9.71}$$

The value of g_0 will give the required maximum.

The mixed optimization technique presented here avoids many of the shortcomings of either pure dynamic programming or pure control theory formulations. Since the solution technique used to optimize a given machine represents a submodule of the overall method, the pure dynamic programming approach may be recognized as a special case. It should be advantageous, however, to be able to use a methodology for the submodule that is most efficient for a given particular problem. Previous control theory formulations do not seem to be easily adaptable to the situation of an existing initial machine; see Sethi and Morton (1972) for other similar asymmetries.

The mixed technique can also be adapted to the case of probabilistic technological breakthroughs (Exercise 9.17). Here the path of technological growth is assumed to be a tree with probabilities associated with its branches. The subproblems can be solved by using the maximum principle for stochastic networks given in Sethi and Thompson (1977). However, the number of subproblems that must be solved increases rapidly with the number of branches, thus putting computational limitations on the general usefulness of this extension.

Another application of the mixed technique has been used by Pekelman and Sethi (1978) to obtain the optimal durations of advertising copies, and the optimal level of advertising expenditures for each copy.

9.3.5 A Numerical Example

To illustrate the procedure, a simple three-period example will be presented and solved for the case where there is no existing machine at time zero.

Machines may be bought at times 0, 1, and 2. The cost of a machine bought at time s is assumed to be

$$C_s = 1,000 + 500s^2.$$

The discount rate, the fractional instantaneous depreciation at purchase, and the maximum preventive maintenance per period are assumed to be

$$\rho = 0.06,\ \delta = 0.25,\ \text{and } U = \$100,$$

respectively.

Let R_s^s be the net return (net of necessary maintenance) of a machine purchased at the beginning of period s and operated during period s. We assume

$$R_0^0 = \$600,\ R_1^1 = \$1,000,\ \text{and } R_2^2 = \$1,100.$$

In a period k subsequent to the period s of machine purchase, the returns R_s^k, $k > s$, depend on the preventive maintenance performed on the machine in the periods prior to period k. The incremental return function is given by $\Psi_s(u,k)$, which we assume to be linear. Specifically,

$$\Delta R_s^k = \Psi_s(u^k, k) = -d_s + a_s u^k,$$

where

$$d_0 = 200,\ d_1 = 50,\ d_2 = 100,\ \text{and } a_s = 0.5 + 0.1s^3.$$

This means that, in the absence of any preventative maintenance, the return in period k on a machine purchased in period s goes down by an amount d_s every period from s to k, including s, in which there is no preventive maintenance. This decrease can be offset by an amount proportional to the amount of preventive maintenance.

Note that the function Ψ_s is assumed to be stationary over time in order to simplify the example.

Let x_s^k be the salvage value at time k of a machine purchased at s. We assume

$$x_s^s = (1-\delta)C_s = 0.75[1,000 + 500s^2].$$

The incremental salvage value function is given by

$$\Delta x_s^k = -\ell_s C_s + b_s u^k,$$

where

$$\ell_s = \begin{cases} 0.1 & \text{when } s = 0, 1, \\ 0.2 & \text{when } s = 2, \end{cases}$$

and

$$b_s = (0.5 - 0.05s).$$

That is, the decrease in salvage value is a constant percentage of the purchase price if there is no preventive maintenance. With preventive maintenance, the salvage value can be enhanced by a proportional amount.

Let J^*_{st} be the optimal value of the objective function associated with a machine purchased at s and sold at $t \geq s+1$. We will now solve for $J^*_{st},\ s = 0, 1, 2$, and $s < t \leq 3$, where t is an integer.

Before we proceed, we will as in (9.68) denote by $W_s(k,t)$, the coefficient of u^k in the Hamiltonian H, i.e.,

$$W_s(k,t) = -(1+\rho)^{-k} + a_s \sum_{i=k+1}^{t-1} (1+\rho)^{-i} + b_s(1+\rho)^{-t}. \tag{9.72}$$

The optimal control is given by (9.69).

It is noted in passing that

$$W_s(k+1,t) - W_s(k,t) = (1+\rho)^{-(k+1)}(\rho - a_s),$$

so that

$$\text{sgn}[W_s(k+1,t) - W_s(k,t)] = \text{sgn}[\rho - a_s]. \tag{9.73}$$

This implies that

$$u^{(k+1)*} - u^{k*} \begin{cases} \geq 0 & \text{if } (\rho - a_s) > 0, \\ = 0 & \text{if } (\rho - a_s) = 0, \\ \leq 0 & \text{if } (\rho - a_s) < 0. \end{cases} \tag{9.74}$$

In this example $\rho - a_s < 0$, which means that if there is a switching in the preventive maintenance trajectory of a machine, the switch must be from \$100 to \$0.

Solution of Subproblems We now solve the subproblems for various values of s and $t (s < t)$ by using the discrete maximum principle.

Subproblem: $s = 0,\ t = 1$.

$$W_0(0,1) = -1 + 0.5(1.06)^{-1} < 0.$$

From (9.69) we have

$$u^{0*} = 0.$$

Now,

$$\begin{aligned} R_0^0 &= 600, \\ R_0^1 &= 600 - 200 = 400, \\ x_0^0 &= 0.75 \times 1,000 = 750, \\ x_0^1 &= 750 - 0.1 \times 1,000 = 650, \\ J_{01}^* &= 600 - 1,000 + 650 \times (1.06)^{-1} = \$213.2. \end{aligned}$$

Similar calculations can be carried out for other subproblems. We will list these results.

Subproblem: $s = 0,\ t = 2.$

$$W_0(0,2) < 0, \quad W_0(1,2) < 0,$$

$$u^{0*} = 0, \qquad u^{1*} = 0,$$

$$J_{02}^* = 466.9.$$

Subproblem: $s = 0,\ t = 3.$

$$W_0(0,3) > 0, \quad W_0(1,3) < 0, \quad W_0(2,3) < 0,$$

$$u^{0*} = 100, \qquad u^{1*} = 100, \qquad u^{2*} = 0,$$

$$J_{03}^* = 639.$$

Subproblem: $s = 1,\ t = 2.$

$$\begin{aligned} W_1(1,2) &< 0, \\ u^{1*} &= 0, \\ J_{12}^* &= 559.9. \end{aligned}$$

Subproblem: $s = 1,\ t = 3.$

$$W_1(1,3) > 0, \quad W_1(2,3) < 0,$$

$$u^{1*} = 100, \qquad u^{2*} = 0,$$

$$J_{13}^* = 1024.2.$$

Subproblem: $s = 2,\ t = 3.$

$$\begin{aligned} W_2(2,3) &< 0, \\ u^{2*} &= 0, \\ J^*_{23} &= 80. \end{aligned}$$

Wagner-Whitin Solution of the Entire Problem With reference to the dynamic programming equation in (9.70) and (9.71), we have

$$\begin{aligned} g_3 &= 0, \\ g_2 &= J^*_{23} = \$80, \\ g_1 &= \max\,[J^*_{13}, J^*_{12} + g_2] \\ &= \max\,[1024.2, 559.9 + 80] \\ &= \$1024.2, \\ g_0 &= \max\,[J^*_{03}, J^*_{01} + g_1, J^*_{02} + g_2] \\ &= \max\,[639.0, 213.2 + 1024.2, 466.9 + 80] \\ &= \$1237.4. \end{aligned}$$

Now we can summarize the optimal solution. The optimal number of machines is 2, and their optimal purchase times, maintenance rates, and sell times are as follows:

First Machine Optimal Policy: Purchase at $s = 0$ and sell at $t = 1$. The optimal preventive maintenance policy is $u^{0*} = 0$.

Second Machine Optimal Policy: Purchase at $s = 1$ and sell at $t = 3$. The optimal preventive maintenance policy is $u^{1*} = 100, u^{2*} = 0$.

The associated value of the objective function is J^*= \$1237.4.

Exercises for Chapter 9

E 9.1 Show that the bracketed expression in (9.10) is monotonically decreasing in t.

E 9.2 Change the values of U and $d(t)$ in Sect. 9.1.3 to the new values $U = 1/2$ and $d(t) = 3$ and re-solve the problem.

E 9.3 Show for the model in Sect. 9.1.1 that if it is optimal to have the maximum maintenance throughout the life of the machine, then its optimal life T must satisfy $g(T) - 1 \geq 0$. In particular, for the example in Sect. 9.1.3, show $T \leq 3$.

E 9.4 Re-solve the example in Sect. 9.1.3 with $x(0) = 40$.

E 9.5 Replace the maintenance effectiveness function in Sect. 9.1.3 by

$$g(t) = 2/(16 + t)^{1/2}$$

and solve the resulting problem.

E 9.6 Re-solve Exercise 2.20 when T is unspecified and it denotes the sale date of the machine to be determined.

E 9.7 Let the maintenance effectiveness function in the model of Sect. 9.1.4 be

$$g(u,t) = \frac{2u^{1/2}}{(1+t)^{1/2}}.$$

Derive the formula for $u^0(t)$ for this case. Furthermore, solve the problem with $T = 34.8$, $U = 1$, $x(0) = 100$, $d(t) = 2$, $\pi = 0.1$ and $\rho = 0.05$, and compare its solution to that of the numerical example in Sect. 9.1.3. Note that the sale date T is assumed to be fixed in Sect. 9.1.4 for simplicity in exposition.

E 9.8 Derive the formula in (9.34) by using (3.77).

E 9.9 Redo the analysis providing the detailed economic interpretation of (9.34) when selling the machine at time $T^* + \delta$, which is earlier than time T^* when the small $\delta < 0$.

Hint: The salvage value function required in (3.77) for the problem here is $S(F(T), T) = e^{-\rho T} B(T)(1 - F(T))$ as given in (9.31). Its partial derivative with respect to T is $[-\rho e^{-\rho T} B(T) + e^{-\rho T} B_T(T)(1 - F(T))$.

E 9.10 To show that the singular control in the third alternative in (9.39) can be sustained, we set $dH_u/dt = 0$ for all t for which a singular control obtains. That is, $u^0(t)$ satisfies

$$C_{uu}\dot{u}^0 = C_u[\rho + (1 - u^0)h] + \rho L - R + C(u^0)h. \tag{9.75}$$

Show that $\dot{u}^0(t) \leq 0$. Furthermore, show that $u^*(t)$ is nonincreasing over time.

E 9.11 For the model of Sect. 9.2, prove that the derived Hamiltonian H is concave in F for each given λ and t, so that the Sufficiency Theorem 2.1 holds.

E 9.12 A firm wants to price its product to maximize the stream of discounted profits. If it maximizes current profits, the high price and profits may attract the entry of rivals, which in turn will reduce future profit possibilities. Let the current profit rate $R_1(p)$ be a strictly concave function of price p with $R_1''(p) < 0$. The profit rate that the firm believes will be available to it after rival entry is $R_2 < \max_p R_1(p)$ (independent of current price and lower than current monopoly profits). Whether, or when, a rival will enter is not known, but let $F(t)$ denote the probability that entry will occur by time t, with $F(0) = 0$. The conditional probability density of entry at time t, given its nonoccurrence prior to t, is $\dot{F}(t)/[1 - F(t)]$. We assume that this conditional entry probability density is a strictly increasing, convex function $h(p)$ of product price p. This specification reflects the supposition that as price rises, the profitability of potential entrants of a given size increases and so does their likelihood of entry. Thus, we assume

$$\dot{F}(t)/[1 - F(t)] = h(p(t))$$

where

$$h(0) = 0, \quad h'(p) > 0, \quad h''(p) \geq 0.$$

Discounting future profits at rate ρ, the firm seeks a price policy $p(t)$ to

$$\max \int_0^\infty e^{-\rho t}\{R_1(p(t))[1 - F(t)] + R_2 F(t)\}dt$$

subject to

$$\dot{F}(t) = h(p(t))[1 - F(t)], \;\; F(0) = 0.$$

The integrand represents the expected profits at t, composed of R_1 if no rival has entered by t, and otherwise R_2.

(a) Show that the maximum principle necessary conditions are satisfied by $p(t) = p^*$, where p^* is a constant. Obtain the equation satisfied by p^* and show that it has a unique solution.

(b) Let p^m denote the monopoly price (in the absence of any rival), i.e., $R_1(p^m) = \max_p R_1(p)$. Show that $p^* < p^m$ and $R_1(p^m) > R_1(p^*) > R_2$. Provide an intuitive explanation of the result.

(c) Verify the sufficiency condition for optimality by showing that the *maximized* Hamiltonian is concave.

E 9.13 Let us define the state of a machine to be '0' if it is working and '1' if it is being repaired. Let λ be the breakdown rate and μ be the service rate as in waiting-line theory, so that we have

$$\dot{P}_0 = -\lambda P_0 + \mu(1 - P_0), \quad P_0(0) = 1,$$

where $P_0(t)$ is the probability that the machine is in the state 0 at time t. Let $P_1(t) = 1 - P_0(t)$, which is the probability that the machine is in state 1 at time t. This equation along with (9.3) gives us two state equations. In view of the equation for $\dot{P}_0$, we modify the objective function (9.2) to

$$J = \int_0^T [\pi x(t)P_0(t) - u(t) - kP_1(t)]e^{-\rho t}dt + x(T)e^{-\rho T},$$

where k characterizes the additional expenditure rate while the machine is being repaired. Solve this model to obtain the optimal control. See Alam and Sarma (1974).

E 9.14 Starting from $W_s(k, t)$ in (9.72), derive the result in (9.74).

E 9.15 Extend the Thompson model in Sect. 9.1 to allow for process discontinuities. An example of this type of machine is an airplane assigned to passenger transportation which may, after some deterioration or obsolescence, be assigned to freight transportation before its eventual retirement. Formulate and analyze the problem. See Tapiero (1971).

E 9.16 Extend the Thompson model in Sect. 9.1 to allow for a chain of machines with identical lines. See Sethi (1973b) for an analysis of a similar model.

E 9.17 Extend the formulation of the Sethi-Morton model in Sect. 9.3 to allow for probabilistic technological breakthroughs. See Sethi and Morton (1972) and Sethi and Thompson (1977).

Chapter 10

Applications to Natural Resources

The increase in world population is causing a corresponding increase in the demand for consumption of natural resources. As a consequence the optimal management and utilization of natural resources is becoming increasingly important. There are two main kinds of natural resource models: those involving renewable resources such as fish, food, timber, etc., and those involving nonrenewable or exhaustible resources such as petroleum, minerals, etc.

In Sect. 10.1 we deal with a fishery resource model, the sole owner of which is considered to be a regulatory agency. The management problem of the agency is to control the rate of fishing over time so that an appropriate objective function is maximized over an infinite horizon. A differential game extension known as the common property fishery resource model is discussed in Sect. 13.2.3. For other applications of optimal control theory to renewable resource models including those involving predator-prey relationships, see Clark (1976), Goh et al. (1974), Jørgensen and Kort (1997), and Munro and Scott (1985).

Section 10.2 deals with an optimal forest thinning model, where thinning is the process of removing some trees from a forest to improve its growth rate and quality. An extension to a chain of forests model is presented in Sect. 10.2.3.

The final model presented in Sect. 10.3 deals with an exhaustible resource such as petroleum, which must be utilized optimally over a

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given horizon under the assumption that when its price reaches a given high threshold, a substitute will be used instead. Therefore, the analysis of this section can also be viewed as a problem of optimally phasing in an expensive substitute.

10.1 The Sole-Owner Fishery Resource Model

With the establishment of 200-mile territorial zones in the ocean for most countries having coastlines, the control of fishing in these zones has become highly regulated by these countries. In this sense, fishing in territorial waters can be considered as a sole owner fishery problem. On the other hand, if the citizens and commercial fishermen of a given country are permitted to fish freely in their territorial waters, the problem becomes that of an open access fishery. The solutions of these two extreme problems are quite different, as will be shown in this section.

10.1.1 The Dynamics of Fishery Models

We introduce the following notation and terminology which is due to Clark (1976):

$$\begin{aligned}
\rho &= \text{the discount rate,}\\
x(t) &= \text{the biomass of fish population at time } t,\\
g(x) &= \text{the natural growth function,}\\
u(t) &= \text{the rate of fishing effort at time } t;\ 0 \le u \le U,\\
q &= \text{the catchability coefficient,}\\
p &= \text{the unit price of landed fish,}\\
c &= \text{the unit cost of effort.}
\end{aligned}$$

Assume that the growth function g is differentiable and concave, and it satisfies

$$g(0) = 0,\ \ g(X) = 0,\ \ g(x) > 0 \text{ for } 0 < x < X, \tag{10.1}$$

where X denotes the *carrying capacity*, i.e., the maximum sustainable fish biomass.

The state equation due to Gordon (1954) and Schaefer (1957) is

$$\dot{x} = g(x) - qux,\ \ x(0) = x_0, \tag{10.2}$$

where qux is the catch rate assumed to be proportional to the biomass as well as the rate of fishing effort. The instantaneous profit rate is

$$\pi(x, u) = pqux - cu = (pqx - c)u. \tag{10.3}$$

From (10.1) and (10.2), it follows that x will stay in the closed interval $0 \leq x \leq X$ provided x_0 is in the same interval.

An *open access fishery* is one in which exploitation is completely uncontrolled. Gordon (1954) analyzed this model, also known as the Gordon-Schaefer model, and showed that the fishing effort tends to reach an equilibrium, called a *bionomic equilibrium*, at the level where total revenue equals total cost. In other words, the so-called economic rent is completely dissipated. From (10.3) and (10.2), this level is simply

$$x_b = \frac{c}{pq} \text{ and } u_b = \frac{g(x_b)p}{c}. \tag{10.4}$$

Let $U > g(c/pq)p/c$ so that u_b is in the interior of $[0, U]$. The economic basis for (10.4) is as follows: If the fishing effort $u > u_b$ is made, then total costs exceed total revenues so that at least some fishermen will lose money, and eventually some will drop out, thus reducing the level of the fishing effort. On the other hand, if the fishing effort $u < u_b$ is made, then total revenues exceed total costs, thereby attracting additional fishermen, and increasing the fishing effort.

The Gordon-Schaefer model does not maximize the present value of the total profits that can be obtained from the fish resources. This is done next.

10.1.2 The Sole Owner Model

The bionomic equilibrium solution obtained from the open access fishery model usually implies severe biological overfishing. Suppose a fishing regulatory agency is established to improve the operation of the fishing industry. In determining the objective of the agency, it is convenient to think of it as a sole owner who has complete rights to exploit the fishing resource. It is reasonable to assume that the agency attempts to maximize

$$J = \int_0^\infty e^{-\rho t}(pqx - c)u dt \tag{10.5}$$

subject to (10.2). This is the optimal control problem to be solved.

10.1.3 Solution by Green's Theorem

The solution method presented in this section generalizes the one based on Green's theorem used in Sect. 7.2.2. Solving (10.2) for u we obtain

$$u = \frac{g(x) - \dot{x}}{qx}, \tag{10.6}$$

which we substitute into (10.3), giving

$$J = \int_0^\infty e^{-\rho t}(pqx - c)\frac{g(x) - \dot{x}}{qx} dt. \tag{10.7}$$

Rewriting, we have

$$J = \int_0^\infty e^{-\rho t}[M(x) + N(x)\dot{x}]dt, \tag{10.8}$$

where

$$N(x) = -p + \frac{c}{qx} \text{ and } M(x) = (p - \frac{c}{qx})g(x). \tag{10.9}$$

We note that we can write $\dot{x}dt = dx$ so that (10.8) becomes the following line integral

$$J_B = \int_B [e^{-\rho t}M(x)dt + e^{-\rho t}N(x)dx], \tag{10.10}$$

where B is a state trajectory in (x, t) space, $t \in [0, \infty)$.

In this section we are only interested in the infinite horizon solution. The Green's theorem method achieves such a solution by first solving a finite horizon problem as in Sect. 7.2.2, and then determining the infinite horizon solution for which you are asked to verify that the maximum principle holds in Exercise 10.1. See also Sethi (1977b).

In order to apply Green's Theorem to (10.10), let Γ denote a simple closed curve in the (x, t) space surrounding a region R in the space. Then,

$$\begin{aligned} J_\Gamma &= \oint_\Gamma [e^{-\rho t}M(x)dt + e^{-\rho t}N(x)dx] \\ &= \iint_R \left\{ \frac{\partial}{\partial t}[e^{-\rho t}N(x)] - \frac{\partial}{\partial x}[e^{-\rho t}M(x)] \right\} dt dx \\ &= \iint_R -e^{-\rho t}[\rho N(x) + M'(x)]dt dx. \end{aligned} \tag{10.11}$$

If we let

$$\begin{aligned} I(x) &= -[\rho N(x) + M'(x)] \\ &= (\rho - g'(x))(p - \frac{c}{qx}) - \frac{cg(x)}{qx^2}, \end{aligned}$$

we can rewrite (10.11) as

$$J_\Gamma = \int\int_R e^{-\rho t} I(x) dt dx.$$

We can now conclude, as we did in Sects. 7.2.2 and 7.2.4, that the turnpike level $\bar{x}$ is given by setting the integrand of (10.11) to zero. That is,

$$-I(x) = [g'(x) - \rho](p - \frac{c}{qx}) + \frac{cg(x)}{qx^2} = 0. \tag{10.12}$$

In addition, a second-order condition must be satisfied for the solution $\bar{x}$ of (10.12) to be a turnpike solution; see Lemma 7.1 and the subsequent discussion there. The required second-order condition can be stated as

$$I(x) < 0 \text{ for } x < \bar{x} \text{ and } I(x) > 0 \text{ for } x > \bar{x}.$$

Let $\bar{x}$ be the unique solution to (10.12) satisfying the second-order condition. The procedure can be extended to the case of nonunique solutions as in Sethi (1977b); see Appendix D.8 on the Sethi-Skiba points.

The corresponding value $\bar{u}$ of the control which would maintain the fish stock level at $\bar{x}$ is $g(\bar{x})/q\bar{x}$. In Exercise 10.2 you are asked to show that $\bar{x} \in (x_b, X)$ and also that $\bar{u} < U$. In Fig. 10.1 optimal trajectories are shown for two different initial values: $x_0 < \bar{x}$ and $x_0 > \bar{x}$.

Let

$$\pi(x) = \frac{g(x)(pqx - c)}{qx}. \tag{10.13}$$

With $\pi'(x)$ obtained from (10.13), condition (10.12) can be rewritten as

$$\frac{d\pi(x)}{dx} = \rho\left(\frac{pqx - c}{qx}\right), \tag{10.14}$$

which facilitates the following economic interpretations.

The interpretation of $\pi(x)$ is that it is the *sustainable economic rent* at fish stock level x. This can be seen by substituting $u = g(x)/qx$ into (10.3), where $u = g(x)/qx$, obtained using (10.2), is the fishing effort required to maintain the fish stock at level x. Suppose we have attained

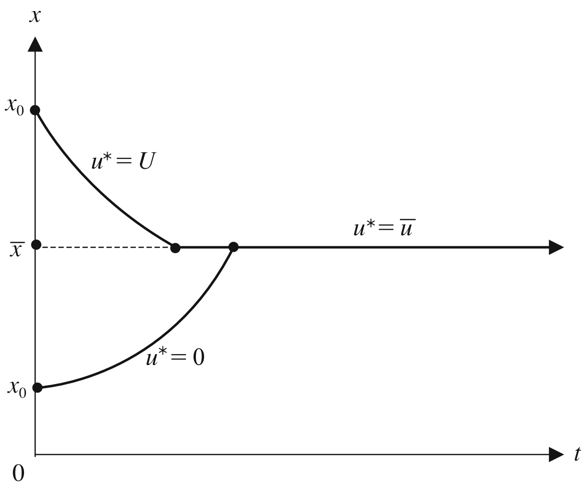

Figure 10.1: Optimal policy for the sole owner fishery model

the equilibrium level $\bar{x}$ given by (10.12), and suppose we reduce this level to $\bar{x} - \varepsilon$ by removing ε amount of fish instantaneously from the fishery, which can be accomplished by an impulse fishing effort of $\varepsilon / q\bar{x}$. The immediate marginal revenue MR from this action is

$$MR = (pq\bar{x} - c)\frac{\varepsilon}{q\bar{x}}.$$

However, this causes a decrease in the sustainable economic rent which equals

$$\pi'(\bar{x})\varepsilon.$$

Over the infinite future, the present value of this stream is

$$\int_0^\infty e^{-\rho t}\pi'(\bar{x})\varepsilon dt = \frac{\pi'(\bar{x})\varepsilon}{\rho}.$$

Adding to this the cost $c\varepsilon / q\bar{x}$ of the additional fishing effort $\varepsilon / q\bar{x}$, we get the marginal cost

$$MC = \frac{\pi'(\bar{x})\varepsilon}{\rho} + \frac{c\varepsilon}{q\bar{x}}.$$

Equating MR and MC, we obtain (10.14), which is also (10.12).

When the discount rate $\rho = 0$, Eq. (10.14) reduces to

$$\pi'(x) = 0,$$

so that it gives the equilibrium fish stock level $\bar{x}\,|_{\rho=0}$. On account of this level satisfying the above first-order condition, one can show that it maximizes the instantaneous profit rate $\pi(x)$. In economics, such a level is called the *golden rule level.* On the other hand, when $\rho = \infty$, we can conclude from (10.12) that $pqx - c = 0$. This gives

$$\bar{x}\,|_{\rho=\infty} = x_b = c/pq.$$

The latter is the bionomic equilibrium attained in the open access fishery solution; see (10.4). Finally, by denoting $\bar{x}$ obtained from (10.12) for any given $\rho > 0$ as $\bar{x}\,|_{\rho}$, you are asked in Exercise 10.3 to show that

$$\bar{x}\,|_{\rho=0} > \bar{x}\,|_{\rho>0} > \bar{x}\,|_{\rho=\infty} = x_b. \tag{10.15}$$

The sole owner solution $\bar{x}$ satisfies $\bar{x} > x_b = c/pq$. If we regard a government regulatory agency as the sole owner responsible for operating the fishery at level $\bar{x}$, then it can impose restrictions, such as gear regulations, catch limitations, etc. that will increase the fishing cost c. If c is increased to the level $pq\bar{x}$, then the fishery can be turned into an open access fishery subject to those regulations, and it will attain the bionomic equilibrium at level $\bar{x}$.

10.2 An Optimal Forest Thinning Model

Forests are another important kind of renewable natural resource, and their optimal management is becoming a significant current problem. In Kilkki and Vaisanen (1969), a model is developed for forest growth and thinning in connection with Scotch Pine forests in Finland. Thinning is the process of removing some but not all of the trees prior to clearcutting the forest. Besides yielding a harvest of wood, the thinning process also improves the growth rate and quality of the forest. The solution method employed by Kilkki and Vaisanen was based on dynamic programming. We will use the maximum principle approach to solve the model. For related literature, see Clark (1976) and Bowes and Krutilla (1985).

10.2.1 The Forestry Model

We introduce the following notation:

t_0 = the initial age of the forest,

ρ = the discount rate,

$x(t)$ = the volume of usable timber in the forest at time t,

$u(t)$ = the rate of thinning at time t,
p = the constant price per unit volume of timber,
c = the constant cost per unit volume of thinning,
$f(x)$ = the growth function, which is positive, concave, and has a unique maximum at x^m; we assume $f(0) = 0$,
$g(t)$ = the growth coefficient which is a positive, decreasing function of time.

The specific function form for the forest growth used in Kilkki and Vaisanen (1969) is as follows:

$$f(x) = xe^{-\alpha x}, \;\; 0 \le x \le \frac{1}{\alpha},$$

where α is a positive constant. Note that f is increasing and concave in the relevant range, and it takes it maximum at $1/\alpha$. They use the growth coefficient of the form

$$g(t) = at^{-b},$$

where a and b are positive constants.

The forest growth equation is

$$\dot{x} = g(t)f(x) - u(t), \; x(t_0) = x_0. \tag{10.16}$$

The objective is to maximize the discounted profit

$$J = \int_{t_0}^{\infty} e^{-\rho t}(p - c)udt \tag{10.17}$$

subject to (10.16) and the state and control constraints

$$x(t) \ge 0 \text{ and } u(t) \ge 0. \tag{10.18}$$

The control constraint in (10.18) implies that there is no replanting in the forest. In Sect. 10.2.3 we extend this model to incorporate the successive replantings of the forest each time it is clearcut.

10.2.2 Determination of Optimal Thinning

We solve the forest thinning model by using the maximum principle. The Hamiltonian is

$$H = (p - c)u + \lambda[gf(x) - u] \tag{10.19}$$

with the adjoint equation

$$\dot{\lambda} = \lambda[\rho - gf'(x)]. \tag{10.20}$$

The optimal control is

$$u^* = \text{bang}[0,\ \infty;\ p - c - \lambda]. \tag{10.21}$$

The appearance of ∞ as an upper bound in (10.21) simply means that impulse control is permitted.

We do not use the Lagrangian form of the maximum principle to include constraints (10.18) because, as we will see, the forestry problem has a natural ending at a time T for which $x(T) = 0$.

To get the singular control solution triple $\{\bar{x}, \bar{\lambda}, \bar{u}\}$, we must observe that due to the time dependence of $g(t)$, $\bar{x}$ and $\bar{u}$ will be functions of time. From (10.21), we have

$$\bar{\lambda} = p - c, \tag{10.22}$$

which is a constant so that $\dot{\lambda} = 0$. From (10.20),

$$f'(\bar{x}(t)) = \frac{\rho}{g(t)} \text{ or } \bar{x}(t) = f'^{-1}(\rho/g(t)). \tag{10.23}$$

Then, from (10.14),

$$\bar{u}(t) = g(t)f(\bar{x}(t)) - \dot{\bar{x}}(t) \tag{10.24}$$

gives the singular control.

The solution of (10.23) can be illustrated as in Fig. 10.2. Since $g(t)$ is a decreasing function of time, it is clear from Fig. 10.2 that $\bar{x}(t)$ is a decreasing function of time, and then by (10.24), $\bar{u}(t) \geq 0$. It is also clear from (10.23) that $\bar{x}(\hat{T}) = 0$ at time $\hat{T}$, where $\hat{T}$ is given by

$$\frac{\rho}{g(\hat{T})} = f'(0),$$

which, in view of $f'(0) = 1$, gives

$$\hat{T} = e^{-(1/b)\ln(\rho/a)}. \tag{10.25}$$

In Fig. 10.3 we plot $\bar{x}(t)$ as a function of time t. The figure also contains an optimal control trajectory for the case in which $x_0 < \bar{x}(t_0)$. To determine the switching time $\hat{t}$, we first solve (10.14) with $u = 0$. Let $x(t)$ be the solution. Then, $\hat{t}$ is the time at which the $x(t)$ trajectory intersects the $\bar{x}(t)$ curve; see Fig. 10.3.

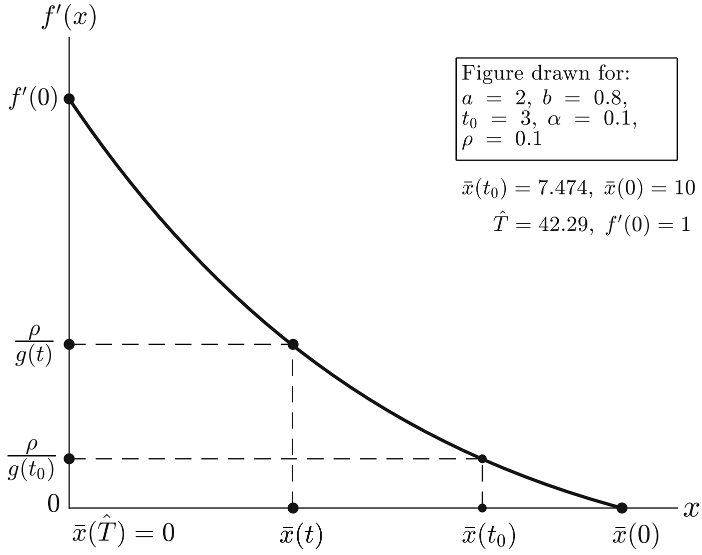

Figure 10.2: Singular usable timber volume $\bar{x}(t)$

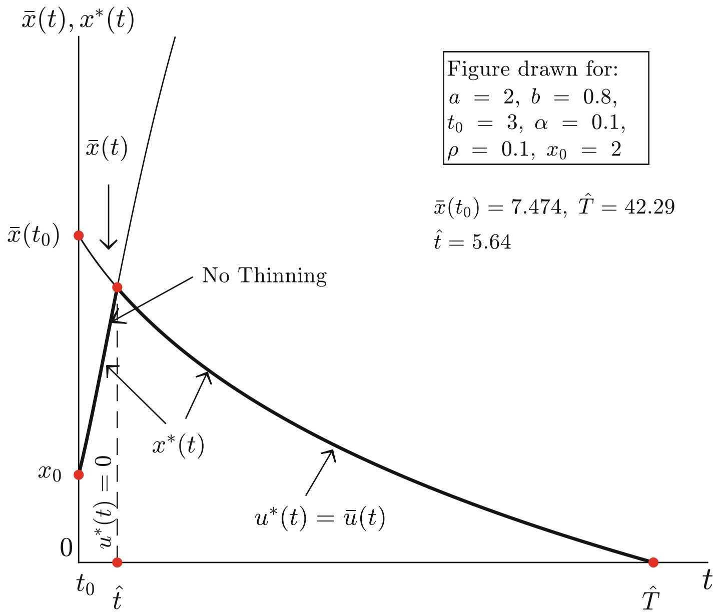

Figure 10.3: Optimal thinning $u^*(t)$ and timber volume $x^*(t)$ for the forest thinning model when $x_0 < \bar{x}(t_0)$

For $x_0 > \bar{x}(t_0)$, the optimal control at t_0 will be the *impulse* cutting to bring the level from x_0 to $\bar{x}(t_0)$ instantaneously. To complete the infinite horizon solution, set $u^*(t) = 0$ for $t \geq \hat{T}$. In Exercise 10.12 you are asked to obtain $\lambda(t)$ for $t \in [0, \infty)$.

10.2.3 A Chain of Forests Model

We now extend the model of Sect. 10.2.1 to incorporate successive replantings of the forest each time it is clearcut. This extension is similar in spirit to the chain of machines model of Sect. 9.3, but with some important differences. We will assume that successive plantings, sometimes called forest rotations, take place at equal intervals. This is similar to what was assumed in the machine replacement problem treated in Sethi (1973b).

Let T be the rotation period, i.e., the time from planting to clearcutting which is to be determined. During the nth rotation, the dynamics of the forest is given by (10.17) with $t \in [(n-1)T, nT]$ and $x[(n-1)T] = 0$. The discounted profit to be maximized is given by

$$\begin{aligned} J(T) &= \sum_{k=1}^{\infty} e^{(k-1)\rho T} \int_0^T e^{-\rho t}(p-c)udt \\ &= \frac{1}{1-e^{-\rho T}} \int_0^T e^{-\rho t}(p-c)udt. \end{aligned} \quad (10.26)$$

From the solution of the model in the previous section, and the assumption that the forest is profitable, it is obvious that $0 \leq T \leq \hat{T}$ as shown in Fig. 10.4. We have two cases to consider, depending on whether $T > \hat{t}$ or $T \leq \hat{t}$.

Case 1: $T > \hat{t}$. From the preceding section it is easy to conclude that the optimal trajectory is as shown in Fig. 10.4. Using the turnpike terminology of Chap. 7, the trajectory from 0 to A is the entry ramp to the turnpike, the trajectory from A to B is on the turnpike, and the trajectory from B to T is the exit ramp. Since $u^*(t) = 0$ on the entry ramp, no timber is collected from time 0 to time $\hat{t}$. Timber is, however, collected by thinning from time $\hat{t}$ to T^- and clearcutting at time T. Note from Fig. 10.4 that $\bar{x}(T)$ is the amount of timber collected from impulse clearcutting $u^*(T) = \text{imp}[\bar{x}(T), 0; T]$ at time T. Thus, we can write the

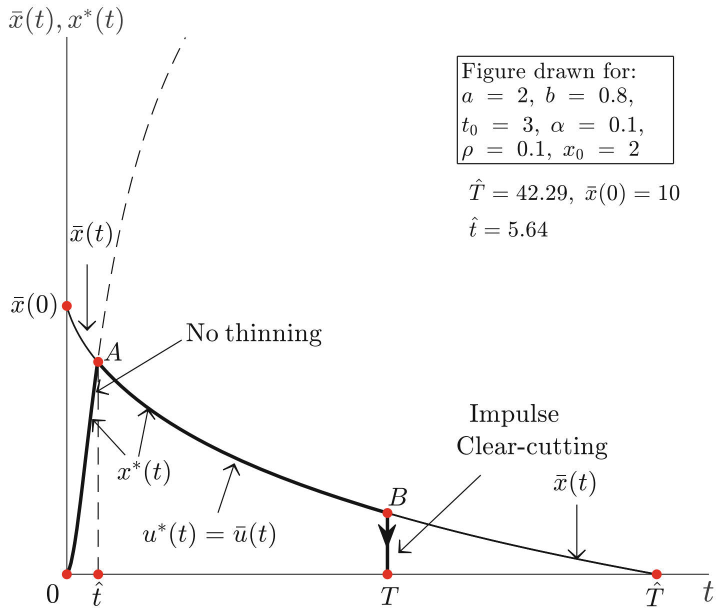

Figure 10.4: Optimal thinning $u^*(t)$ and timber volume $x^*(t)$ for the chain of forests model when $T > \hat{t}$

discounted profit $J^*(T)$ of (10.26) for a given T as

$$J^*(T) \quad = \quad \frac{1}{1-e^{-\rho T}}\left[\int_{\hat{t}}^{T-} e^{-\rho t}(p-c)\bar{u}(t)dt + e^{-\rho T}(p-c)\bar{x}(T)\right]. \tag{10.27}$$

Formally, the second term inside the brackets above represents

$$\int_{T-}^{T} e^{-\rho t}(p-c)\ \mathrm{imp}[\bar{x}(t), 0; t]dt, \tag{10.28}$$

the value of clearcutting at time T. In Exercise 10.13, you are asked to show that this value is precisely the second term.

For finding the optimal value of T in this case, we differentiate (10.27) with respect to T, equate the result to zero, and simplify to obtain (see Exercise 10.14)

$$(1-e^{-\rho T})g(T)f[\bar{x}(T)] - \rho\bar{x}(T) - \rho\int_{\hat{t}}^{T-} e^{-\rho t}\bar{u}(t)dt = 0. \tag{10.29}$$

If the solution T lies in $(\hat{t}, \hat{T}]$, keep it; otherwise set $T = \hat{T}$. Note that (10.29) can also be derived by using the transversality condition (3.15); see Exercise 3.6.

Case 2: $T \le \hat{t}$. The optimal trajectory in this case is as shown in Fig. 10.5. In the Vidale-Wolfe advertising model of Chap. 7, a similar case occurs when T is small; see Fig. 7.10 and compare it with Fig. 10.5. The solution for $x(T)$ is obtained by integrating (10.14) with $u = 0$ and $x_0 = 0$. Let this solution be denoted as $x^*(t)$. Here (10.26) becomes

$$J^*(T) = \frac{e^{-\rho T}}{1 - e^{-\rho T}}(p - c)\bar{x}(T). \tag{10.30}$$

To find the optimal value of T for this case, we differentiate (10.30) with respect to T and equate $dJ^*(T)/dT$ to zero. We obtain (see Exercise 10.14)

$$(1 - e^{-\rho T})g(T)f[\bar{x}(T)] - \rho\bar{x}(T) = 0. \tag{10.31}$$

If the solution lies in the interval $[0, \hat{t}]$ keep it; otherwise set $T = \hat{t}$.

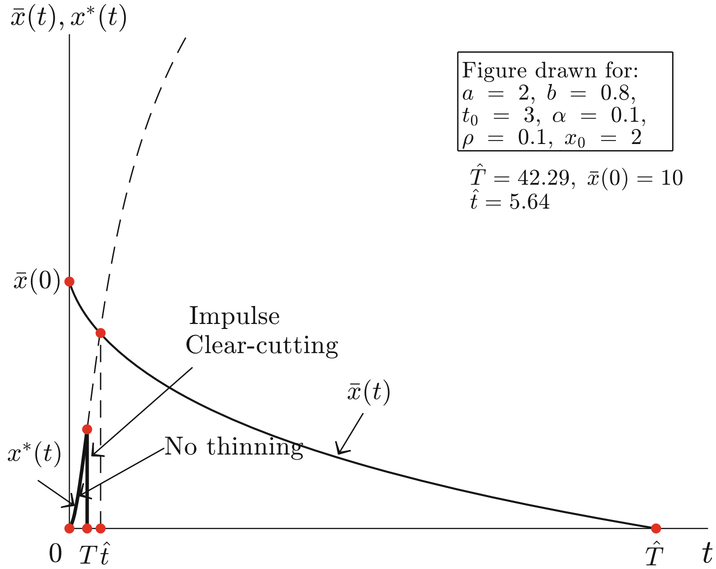

Figure 10.5: Optimal thinning and timber volume $x^*(t)$ for the chain of forests model when $T \le \hat{t}$

The optimal value T^* can be obtained by computing $J^*(T)$ from both cases and selecting whichever is larger; see also Näslund (1969) and Sethi (1973c).

10.3 An Exhaustible Resource Model

In the previous two sections we discussed two renewable resource models. However, many natural resources are nonrenewable or exhaustible. Examples are petroleum, mineral deposits, coal, etc. Given the growing energy shortage, the optimal production and use of these resources is of immense importance to the world. The earliest important work in this area is Hotelling (1931). Since then, a number of studies have been published such as Dasgupta and Heal (1974a), Solow (1974), Weinstein and Zeckhauser (1975), Pindyck (1978a,b), Derzko and Sethi (1981a,b), Amit (1986) and Heal (1993).

In this section, we discuss a simple model taken from a paper by Sethi (1979a). The paper obtains the optimal depletion rate of an exhaustible resource that maximizes a social welfare function involving consumers' surplus and producers' surplus with various weights. Here we treat the special case when these weights are equal.

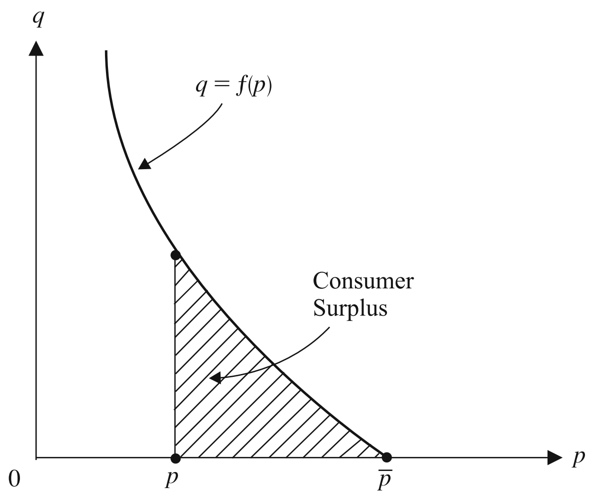

Figure 10.6: The demand function

10.3.1 Formulation of the Model

The model will be developed under the assumption that at a high enough price, say $\bar{p}$, a substitute, preferably renewable, will become available. For example, if the price of fossil fuel becomes sufficiently high, solar energy may become an economic substitute. In the North American context, the resource under consideration could be crude oil and its expensive substitute could be coal and/or tar sands; see, e.g., Fuller and Vickson (1987).

We introduce the following notation:

$p(t)$ = the price of the resource at time t,

q = $f(p)$ is the demand function, i.e., the quantity demanded at price p; $f' \leq 0$, $f(p) > 0$ for $p < \bar{p}$, and $f(p) = 0$ for $p \geq \bar{p}$, where $\bar{p}$ is the price at which the substitute completely replaces the resource. A typical graph of the demand function is shown in Fig. 10.6,

c = $G(q)$ is the cost function; $G(0) = 0$, $G(q) > 0$ for $q > 0$, $G' > 0$ and $G'' \geq 0$ for $q \geq 0$, and $G'(0) < \bar{p}$. The latter assumption makes it possible for the producers to make a positive profit at a price p below $\bar{p}$,

$Q(t)$ = the available stock or reserve of the resource at time t; $Q(0) = Q_0 > 0$,

ρ = the social discount rate; $\rho > 0$,

T = the horizon time, which is the latest time at which the substitute will become available regardless of the price of the natural resource; $T > 0$.

Before stating the optimal control problem, we need the following additional definitions and assumptions. Let

$$c = G[f(p)] = g(p), \tag{10.32}$$

for which it is obvious that $g(p) > 0$ for $p < \bar{p}$ and $g(p) = 0$ for $p \geq \bar{p}$. Let

$$\pi(p) = pf(p) - g(p) \tag{10.33}$$

denote the profit function of the producers, i.e., the *producers' surplus*. Let $\underline{p}$ be the smallest price at which $\pi(p)$ is nonnegative. Assume further that $\pi(p)$ is a concave function in the range $[\underline{p}, \bar{p}]$ as shown in Fig. 10.7. In the figure the point p^m indicates the price which maximizes $\pi(p)$.

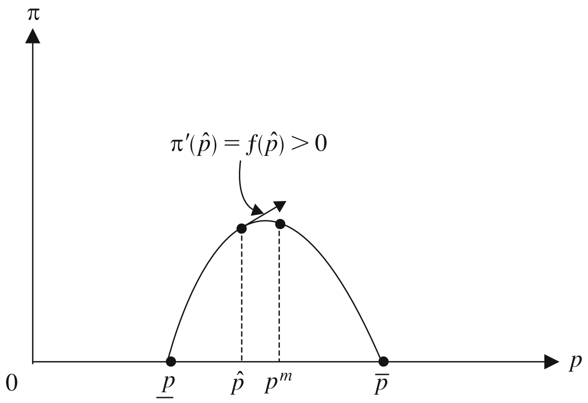

Figure 10.7: The profit function

We also define

$$\psi(p) = \int_p^{\bar{p}} f(y)dy \tag{10.34}$$

as the *consumers' surplus*, i.e., the area shown shaded in Fig. 10.6. This quantity represents the total excess amount that consumers would be willing to pay. In other words, consumers actually pay $pf(p)$, while they would be willing to pay

$$\int_{\bar{p}}^{p} yf'(y)dy = pf(p) + \psi(p).$$

The instantaneous rate of consumers' surplus and producers' surplus is the sum $\psi(p)+\pi(p)$. Let $\hat{p}$ denote the maximum of this sum, i.e., $\hat{p}$ solves

$$\psi'(\hat{p}) + \pi'(\hat{p}) = \hat{p}f'(\hat{p}) - g'(\hat{p}) = 0. \tag{10.35}$$

In Exercise 10.16 you will be asked to show that $\hat{p} < p^m$, as marked in Fig. 10.7. Later we will show that the correct second-order conditions hold at $\hat{p}$.

The optimal control problem is:

$$\max \left\{ J = \int_0^T [\psi(p) + \pi(p)]e^{-\rho t}dt \right\} \tag{10.36}$$

subject to

$$\dot{Q} = -f(p), \;\; Q(0) = Q_0, \tag{10.37}$$

$$Q(T) \geq 0, \tag{10.38}$$

and $p \in \Omega = [\underline{p}, \bar{p}]$. Recall that the sum $\psi(p) + \pi(p)$ is concave in p.

10.3.2 Solution by the Maximum Principle

Form the current-value Hamiltonian

$$H(Q,p,\lambda) = \psi(p) + \pi(p) + \lambda[-f(p)], \tag{10.39}$$

where λ satisfies the relation

$$\dot{\lambda} = \rho\lambda, \;\; \lambda(T) \geq 0, \;\; \lambda(T)Q(T) = 0, \tag{10.40}$$

which implies

$$\lambda(t) = \begin{cases} 0 & \text{if } Q(T) \geq 0 \text{ is not binding,} \\ \lambda(T)e^{\rho(t-T)} & \text{if } Q(T) \geq 0 \text{ is binding.} \end{cases} \tag{10.41}$$

To obtain the optimal control, the Hamiltonian maximizing condition, which is both necessary and sufficient in this case (see Theorem 2.1), is

$$\frac{\partial H}{\partial p} = \psi' + \pi' - \lambda f' = (p-\lambda)f' - g' = 0. \tag{10.42}$$

To show that the solution $s(\lambda)$ for p of (10.42) actually maximizes the Hamiltonian, it is enough to show that the second derivative of the Hamiltonian is negative at $s(\lambda)$. Differentiating (10.42) gives

$$\frac{\partial^2 H}{\partial p^2} = f' - g'' + (p-\lambda)f''.$$

Using (10.42) we have

$$\frac{\partial^2 H}{\partial p^2} = f' - g'' + \frac{g'}{f'}f''. \tag{10.43}$$

From the definition of G in (10.32), we can obtain

$$G'' = \frac{f'g'' - g'f''}{f'^3},$$

which, when substituted into (10.43), gives

$$\frac{\partial^2 H}{\partial p^2} = f' - G''f'^2. \tag{10.44}$$

The right-hand side of (10.44) is strictly negative because $f' < 0$, and $G'' \geq 0$ by assumption. We remark that $\hat{p} = s(0)$ using (10.35) and (10.42), and hence the second-order condition for $\hat{p}$ of (10.35) to give the maximum of H is verified. In Exercise 10.17 you are asked to show that $s(\lambda)$ increases from $\hat{p}$ as λ increases from 0, and that $s(\lambda) = \bar{p}$ when $\lambda = \bar{p} - G'(0)$.

Case 1: The constraint $Q(T) \geq 0$ is not binding. From (10.41), $\lambda(t) \equiv 0$ so that from (10.42) and (10.35),

$$p^* = \hat{p}. \tag{10.45}$$

With this value, the total consumption of the resource is $Tf(\hat{p})$, which must be $\leq Q_0$ so that the constraint $Q(T) \geq 0$ is not binding. Hence,

$$Tf(\hat{p}) \leq Q_0 \tag{10.46}$$

characterizes Case 1 and its solution is given in (10.45).

Case 2: $Tf(\hat{p}) > Q_0$ so that the constraint $Q(T) \geq 0$ is binding. Obtaining the solution requires finding a value of $\lambda(T)$ such that

$$\int_0^{t^*} f(s[\lambda(T)e^{\rho(t-T)}])dt = Q_0, \tag{10.47}$$

where

$$t^* = \min\left\{T, T + \frac{1}{p}\ln\left[\frac{\bar{p} - G'(0)}{\lambda(T)}\right]\right\}. \tag{10.48}$$

The time t^* , if it is less than T, is the time at which $s[\lambda(T)e^{\rho(t^*-T)}] = \bar{p}$. From Exercise 10.17,

$$\lambda(T)e^{\rho(t^*-T)} = \bar{p} - G'(0) \tag{10.49}$$

which, when solved for t^*, gives the second argument of (10.48).

One method to obtain the optimal solution is to define $\bar{T}$ as the longest time horizon during which the resource can be optimally used. Such a $\bar{T}$ must satisfy

$$\lambda(\bar{T}) = \bar{p} - G'(0),$$

and therefore,

$$\int_0^{\bar{T}} f\left(s\left[\{\bar{p} - G'(0)\}e^{\rho(t-\bar{T})}\right]\right)dt = Q_0, \tag{10.50}$$

which is a transcendental equation for $\bar{T}$. We now have two subcases.

Subcase 2a: $T \geq \bar{T}$. The optimal control is

$$p^*(t) = \begin{cases} s\left(\{\bar{p} - G'(0)\}e^{\rho(t-\bar{T})}\right) & \text{for } t \leq \bar{T}, \\ \bar{p} & \text{for } t > \bar{T}. \end{cases} \tag{10.51}$$

Clearly in this subcase, $t^* = \bar{T}$ and

$$\lambda(T) = [\bar{p} - G'(0)]e^{-\rho(\bar{T}-T)}.$$

A sketch of (10.51) is shown in Fig. 10.8.

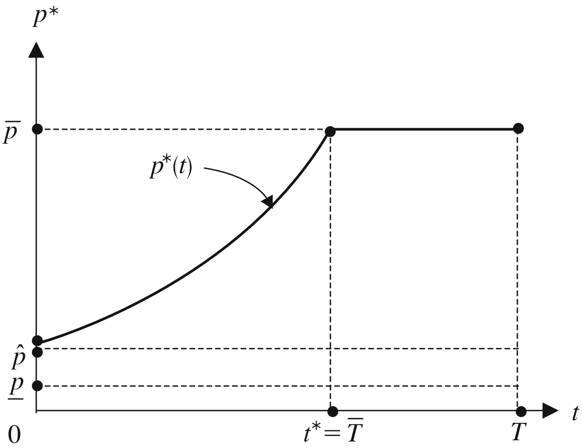

Figure 10.8: Optimal price trajectory for $T \geq \bar{T}$

Subcase 2b: $T < \bar{T}$. Here the optimal price trajectory is

$$p^*(t) = s\left[\lambda(T)e^{\rho(t-T)}\right], \tag{10.52}$$

where $\lambda(T)$ is to be obtained from the transcendental equation

$$\int_0^T f\left(s\left[\lambda(T)e^{\rho(t-T)}\right]\right)dt = Q_0. \tag{10.53}$$

A sketch of (10.52) is shown in Fig. 10.9.

In Exercise 10.18 you are given specific functions for the exhaustible resource model and asked to work out explicit optimal price trajectories for the model.

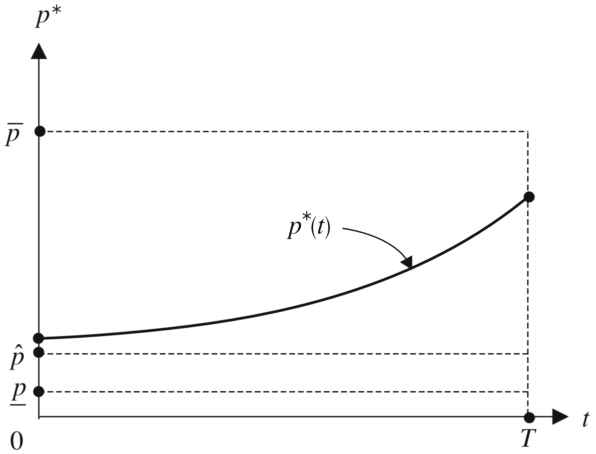

Figure 10.9: Optimal price trajectory for $T < \bar{T}$

Exercises for Chapter 10

E 10.1 As an alternate derivation for the turnpike level $\bar{x}$ of (10.12), use the maximum principle to obtain the optimal long-run stationary equilibrium triple $\{\bar{x}, \bar{u}, \bar{\lambda}\}$.

E 10.2 Prove that $\bar{x} \in (x_b, X)$ and $\bar{u} < U$, where $\bar{x}$ is the solution of (10.12) and x_b is given in (10.4).

E 10.3 Show that $\bar{x}$ obtained from (10.12) decreases as ρ increases. Furthermore, derive the relation (10.15).

E 10.4 Obtain the turnpike level $\bar{x}$ of (10.12) for the special case $g(x) = x(1-x)$, $p = 2$, $c = q = 1$, and $\rho = 0.1$.

E 10.5 Perform the following:

(a) For the Schaefer model with $g(x) = rx(1 - x/X)$ and $q = 1$, derive the formula for the turnpike level $\bar{x}$ of (10.12).

(b) Allen (1973) and Clark (1976) estimated the parameters of the Schaefer model for the Antarctic fin-whale population as follows: $r = 0.08$, $X = 400,000$ whales, and $x_b = 40,000$. Solve for $\bar{x}$ for $\rho = 0, 0.10$, and ∞.

E 10.6 Obtain $\pi'(x)$ from (10.13) and use it in (10.12) to derive (10.14).

E 10.7 Let $\pi(x, u) = [p - c(x)](qux)$ in (10.3), where $c(x)$ is a differentiable, decreasing, and convex function. Derive an expression for $\bar{x}$ satisfying an equation corresponding to (10.12).

E 10.8 Show that extinction is optimal if $\infty > p \geq c(0)$ and $\rho > 2g'(0)$ in Exercise 10.7.

Hint: Use the generalized mean value theorem.

E 10.9 Let the constant price p in Exercise 10.7 be replaced by a time dependent price $p(t)$ which is differentiable with respect to t. Derive the equation $\bar{x}$ corresponding to (10.12) for this nonautonomous problem. Furthermore, find the turnpike level $\bar{x}(t)$ satisfying the derived equation.

E 10.10 Let $\pi(x, u)$ of Exercise 10.7 be

$$\pi(x, u) = [p - c(x)](qux) + V(x),$$

where $V(x)$ with $V'(x) > 0$ is the *conservation value function*, which measures the value to society of having a large fish stock. By deriving the analogue to (10.12), show that the new $\bar{x}$ is larger than the $\bar{x}$ in Exercise 10.7.

E 10.11 When $c(x) = 0$ in Exercise 10.9, show that the analogue to (10.12) reduces to

$$g'(x) = \rho - \frac{\dot{p}}{p}.$$

Give an economic interpretation of this equation.

E 10.12 Find $\lambda(t)$, $t \in [0, \infty)$, for the infinite horizon model of Sect. 10.2.2.

E 10.13 Derive the second term inside the brackets of (10.27) by computing $e^{-\rho T}(p - c)\,\text{imp}[\bar{x}(T), 0; T]$.

E 10.14 Derive (10.29) by using the first-order condition for maximizing $J^*(T)$ of (10.27) with respect to T. Similarly, derive (10.31).

E 10.15 *Forest Fertilization Model* (Näslund 1969). Consider a forestry model in which thinning is not allowed, and the forest is to be clearcut at a fixed time T. Suppose $v(t) \geq 0$ is the rate of fertilization at time t, so that the growth equation is

$$\dot{x} = r(X - x) + f(v, t), \;\; x(0) = x_0,$$

where x is the volume of timber, r and X are positive constants, and f is an increasing, differentiable, concave function of v. The objective is to maximize

$$J = -c\int_0^T e^{-\rho t}v(t)dt + e^{-\rho T}px(T),$$

where p is the price of a unit of timber and c is the unit cost of fertilization.

(a) Show that the optimal control $v^*(t)$ is given by solving the equation

$$\frac{\partial f}{\partial v} = \frac{c}{p}e^{-(\rho+r)(t-T)}.$$

Check that the second order condition for a maximum holds for this $v^*(t)$.

(b) If $f(v) = (1 + t)\ln(1 + v)$, then find explicitly the optimal control $v^*(t)$ under the assumption that $p/c > e^{(\rho+r)T}$. Show further that $v^*(t)$ is increasing and convex in $t \in [0, T]$.

E 10.16 Show that $\hat{p}$ defined in (10.35) satisfies $\underline{p} \leq \hat{p} \leq p^m$.

E 10.17 Show that $s(\lambda)$, the solution of (10.39), increases from $\hat{p}$ as λ increases from 0. Also show that $s(\lambda) = \bar{p}$, when $\lambda = \bar{p} - G'(0)$.

E 10.18 For the model of Sect. 10.3, assume

$$f(p) = \begin{cases} \bar{p} - p & \text{for } p \leq \bar{p}, \\ 0 & \text{for } p > \bar{p}, \end{cases}$$

$$G(q) = q^2.$$

(a) Show that $p^* = 2\bar{p}/3$ if $T \leq 3Q_0/\bar{p}$.

(b) Show that $\bar{T}$ satisfies $\bar{T} + e^{-\rho\bar{T}}/\rho = 1/\rho + 3Q_0/\bar{p}$. Moreover,

$$p^*(t) = \begin{cases} \bar{p}\left(e^{\rho(t-\bar{T})} + 2\right)/3 & \text{if } t \leq \bar{T}, \\ \bar{p} & \text{if } t > \bar{T}, \end{cases}$$

for $T \geq \bar{T}$, and

$$p^*(t) = \frac{2\bar{p}}{3} + \frac{\rho[\bar{p}T - 3Q_0]}{3e^{-\rho t}(e^{\rho T} - 1)}$$

for $T > \bar{T}$.

Chapter 11

Applications to Economics

Optimal control theory has been extensively applied to the solution of economic problems since the early papers that appeared in Shell (1967) and the works of Arrow (1968) and Shell (1969). The field is too vast to be surveyed in detail here, however. Several books in the area are: Arrow and Kurz (1970), Hadley and Kemp (1971), Takayama (1974), Lesourne and Leban (1982), Seierstad and Sydsæter (1987), Feichtinger (1988), Léonard and Long (1992), Van Hilten et al. (1993), Kamien and Schwartz (1992), and Dockner et al. (2000), and Weber (2011). We content ourselves with the discussion of three simple kinds of models.

In Sect. 11.1, two capital accumulation or economic growth models are presented. In Sect. 11.2, we formulate and solve an epidemic control model. Finally, in Sect. 11.3 we discuss a pollution control model.

11.1 Models of Optimal Economic Growth

In this section we develop two simple models of economic growth or capital accumulation. The earliest such model was developed by Ramsey (1928) for an economy having a stationary population; see Exercise 11.7 for one of his models.

The first model treated in Sect. 11.1.1 is a finite horizon fixed-end-point model with a stationary population. The problem is to maximize the present value of the utility of consumption for the society, as well as to accumulate a specified capital stock by the end of the horizon.

The second model incorporates an exogenously and exponentially

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growing population in the infinite horizon setting. A technique known as the method of phase diagrams is used to analyze the model.

For related discussion and extensions of these models, see Arrow and Kurz (1970), Burmeister and Dobell (1970), Intriligator (1971), and Arrow et al. (2007, 2010).

11.1.1 An Optimal Capital Accumulation Model

Consider a one-sector economy in which the stock of capital, denoted by $K(t)$, is the only factor of production. Let $F(K)$ be the output rate of the economy when K is the capital stock. Assume $F(0) = 0,\ F(K) > 0,\ F'(K) > 0$, and $F''(K) < 0$, for $K > 0$. These conditions imply the diminishing marginal productivity of capital as well as the strict concavity of $F(K)$ in K. A part of this output is consumed and the remainder is reinvested for further accumulation of capital stock. Let $C(t)$ be the amount of output allocated to consumption, and let $I(t) = F[K(t)] - C(t)$ be the amount invested. Let δ be the constant rate of depreciation of capital. Then, the capital stock equation is

$$\dot{K} = F(K) - C - \delta K, \quad K(0) = K_0. \tag{11.1}$$

Let $U(C)$ be the society's utility of consumption, where we assume $U'(0) = \infty,\ U'(C) > 0$, and $U''(C) < 0$, for $C \geq 0$. These conditions ensure that $U(C)$ is strictly concave in C. Let ρ denote the social discount rate and T denote the finite horizon. Then, a government which is elected for a term of T years could consider the following problem:

$$\max \left\{ J = \int_0^T e^{-\rho t} U[C(t)] dt \right\} \tag{11.2}$$

subject to (11.1) and the fixed-end-point condition

$$K(T) = K_T, \tag{11.3}$$

where K_T is a given positive constant. It may be noted that replacing (11.3) by $K(T) \geq K_T$ would give the same solution.

11.1.2 Solution by the Maximum Principle

Form the current-value Hamiltonian as

$$H = U(C) + \lambda[F(K) - C - \delta K]. \tag{11.4}$$

The adjoint equation is

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial K} = (\rho + \delta)\lambda - \lambda\frac{\partial F}{\partial K},\ \lambda(T) = \alpha, \tag{11.5}$$

where α is a constant to be determined.

The optimal control is given by

$$\frac{\partial H}{\partial C} = U'(C) - \lambda = 0. \tag{11.6}$$

Since $U'(0) = \infty$, the solution of this condition always gives $C(t) > 0$. An intuitive argument for this result is that a slight increase from a zero consumption rate brings and infinitesimally large marginal utility and therefore optimal consumption will remain strictly positive. Moreover, the capital stock will not be allowed to fall to zero along an optimal path in order to avoid the consumption rate from falling to zero. See Karatzas et al. (1986) for a rigorous demonstration of this result in a related context.

Note that the sufficiency of optimality is easily established here by obtaining the derived Hamiltonian $H^0(K, \lambda)$ by substituting for C from (11.6) in (11.4), and showing that $H^0(K, \lambda)$ is concave in K. This follows easily from the facts that $F(K)$ is concave and $\lambda > 0$ from (11.6) on account of the assumption that $U'(C) > 0$.

The economic interpretation of the Hamiltonian is straightforward. It consists of two terms: the first one gives the utility of current consumption and the second one gives the net investment evaluated by price λ, which, from (11.6), reflects the marginal utility of consumption.

For the economic system to be run optimally, the solution must satisfy the following three conditions:

(a) The static efficiency condition (11.6) which maximizes the value of the Hamiltonian at each instant of time myopically, provided that $\lambda(t)$ is known.

(b) The dynamic efficiency condition (11.5) which forces the price λ of capital to change over time in such a way that the capital stock always yields a net rate of return, which is equal to the social discount rate ρ. That is,

$$d\lambda + \frac{\partial H}{\partial K}dt = \rho\lambda dt.$$

(c) The long-run foresight condition, which establishes the terminal price $\lambda(T)$ of capital in such a way that exactly the terminal capital stock K_T is obtained at T.

Equations (11.1), (11.3), (11.5), and (11.6) form a two-point boundary value problem which can be solved numerically. In Exercise 11.1, you are asked to solve a simple version of the model in which the TPBVP can be solved analytically.

11.1.3 Introduction of a Growing Labor Force

In the preceding sections of this chapter we studied the simplest capital accumulation model in which the population was assumed to be fixed. We now introduce labor as a new factor (treated the same as population, for simplicity), which grows exponentially at a fixed rate g, $0 < g < \rho$. It is now possible to recast the new model in terms of per capita variables so that it is formally similar to the previous model. The introduction of the per capita variables makes it possible to treat the infinite horizon version of the new model.

Let $L(t)$ denote the amount of labor at time t. Since it is growing exponentially at rate g, we have

$$L(t) = L(0)e^{gt}. \tag{11.7}$$

Let $F(K, L)$ be the production function which is assumed to be strictly increasing and concave in both factors of production so that $F_K > 0$, $F_L > 0$, $F_{KK} < 0$, and $F_{LL} < 0$ for $K \geq 0$, $L \geq 0$. Furthermore, it is homogeneous of degree one so that $F(mK, mL) = mF(K, L)$ for $m \geq 0$. We define $k = K/L$ and the per capita production function $f(k)$ as

$$f(k) = \frac{F(K, L)}{L} = F(\frac{K}{L}, 1) = F(k, 1). \tag{11.8}$$

It is clear from the assumptions of F that $f'(k) > 0$ and $f''(k) < 0$ for $k \geq 0$.

To derive the state equation for k, we note that

$$\dot{K} = \dot{k}L + k\dot{L} = \dot{k}L + kgL.$$

Substituting for $\dot{K}$ from (11.1) and defining per capita consumption $c = C/L$, we get

$$\dot{k} = f(k) - c - \gamma k, \;\; k(0) = k_0, \tag{11.9}$$

where $\gamma = g + \delta$.

Let $u(c)$ be the utility of per capita consumption c, where u is assumed to satisfy

$$u'(c) > 0 \text{ and } u''(c) < 0 \text{ for } c \geq 0 \text{ and } u'(0) = \infty. \tag{11.10}$$

As in Sect. 11.1.2, the last condition in (11.10) rules out zero consumption.

According to the position known as total utilitarianism, the society's discounted total utility is $\int_0^\infty e^{-\rho t} L(t) u(c(t)) dt$, which we aim to maximize. In view of (11.7), this is equivalent to maximizing

$$J = \int_0^\infty e^{-rt} u(c) dt, \tag{11.11}$$

where $r = \rho - g > 0$. Note also that $r + \gamma = \rho + \delta$.

Remark 11.1 It is interesting to note that the problem is an infinite version of that in Sect. 11.1.1, if we consider r to be the adjusted discount rate and γ to be the adjusted depreciation rate. This reduction of a model with two factors of production to a one-sector model does not work if we jettison the assumption of an exponentially growing population. Then, the analysis becomes much more complicated. The reader is referred to Arrow et al. (2007, 2010) for economic growth models with non-exponentially and endogenously growing populations.

11.1.4 Solution by the Maximum Principle

The current-value Hamiltonian is

$$H = u(c) + \lambda[f(k) - c - \gamma k]. \tag{11.12}$$

The adjoint equation is

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial k} = (r + \gamma)\lambda - f'(k)\lambda = (\rho + \delta)\lambda - f'(k)\lambda. \tag{11.13}$$

To obtain the optimal control, we differentiate (11.12) with respect to c, set it to zero, and solve

$$u'(c) = \lambda. \tag{11.14}$$

Let $c = h(\lambda) = u'^{-1}(\lambda)$ denote the solution of (11.14). In Exercise 11.3, you are asked to show that $h'(\lambda) < 0$. This can be easily shown by inverting the graph of $u'(c)$ vs. c. Alternatively you can rewrite (11.14) as $u'(h(\lambda)) = \lambda$ and then take its derivative with respect to λ.

To show that the maximum principle is sufficient for optimality, it is enough to show that the derived Hamiltonian

$$H^0(k,\lambda) = u(h(\lambda)) + \lambda[f(k) - h(\lambda) - \gamma k] \tag{11.15}$$

is concave in k for any λ satisfying (11.14). The concavity follows immediately from the facts that λ is positive from (11.10) and (11.14) and $f(k)$ is concave because of the assumptions on $F(K,L)$.

Equations (11.9), (11.13), and (11.14) now constitute a complete autonomous system, since time does not enter explicitly in these equations. Such systems can be analyzed by the phase diagram method, which is used next.

In Fig. 11.1 we have drawn a phase diagram for the two equations

$$\dot{k} = f(k) - h(\lambda) - \gamma k = 0, \tag{11.16}$$
$$\dot{\lambda} = (r+\gamma)\lambda - f'(k)\lambda = 0, \tag{11.17}$$

obtained from (11.9), (11.13), and (11.14). In Exercise 11.2 you are asked to show that the graphs of $\dot{k} = 0$ and $\dot{\lambda} = 0$ are like the dotted curves in Fig. 11.1. Given the nature of these graphs, known as isoclines, it is clear that they have a unique point of intersection denoted as $(\bar{k}, \bar{\lambda})$. In

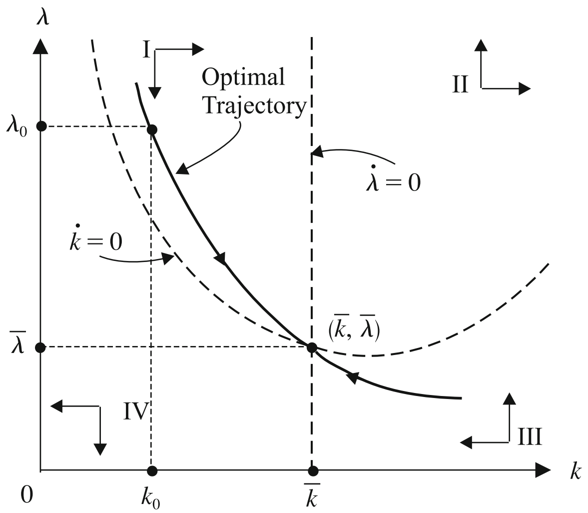

Figure 11.1: Phase diagram for the optimal growth model

other words, $(\bar{k}, \bar{\lambda})$ is the unique solution of the equations

$$f'(\bar{k}) - h(\bar{\lambda}) - \gamma\bar{k} = 0 \text{ and} (r+\gamma) - f'(\bar{k}) = 0. \tag{11.18}$$

The two isoclines divide the plane into four regions, I, II, III, and IV, as marked in Fig. 11.1. To the left of the vertical line $\dot{\lambda} = 0$, we have $k < \bar{k}$ and therefore $r + \gamma < f'(k)$ in view of $f''(k) < 0$. Thus, $\dot{\lambda} < 0$ from (11.13). Therefore, λ is decreasing, which is indicated by the downward pointing arrows in Regions I and IV. On the other hand, to the right of the vertical line, in Regions II and III, the arrows are pointed upward because λ is increasing. In Exercise 11.3, you are asked to show that the horizontal arrows, which indicate the direction of change in k, point to the right above the $\dot{k} = 0$ isocline, i.e., in Regions I and II, and they point to the left in Regions III and IV which are below the $\dot{k} = 0$ isocline.

The point $(\bar{k}, \bar{\lambda})$ represents the optimal long-run stationary equilibrium. The values of $\bar{k}$ and $\bar{\lambda}$ are obtained in Exercise 11.2. The next important thing is to show that there is a unique path starting from any initial capital stock k_0, which satisfies the maximum principle and converges to the steady state $(\bar{k}, \bar{\lambda})$. Clearly such a path cannot start in Regions II and IV, because the directions of the arrows in these areas point away from $(\bar{k}, \bar{\lambda})$. For $k_0 < \bar{k}$, the value of λ_0 (if any) must be selected so that (k_0, λ_0) is in Region I. For $k_0 > \bar{k}$, on the other hand, the point (k_0, λ_0) must be chosen to be in Region III. We analyze the case $k_0 < \bar{k}$ only, and show that there exists a unique λ_0 associated with the given k_0, and that the optimal path, shown as the solid curve in Region I of Fig. 11.1, starts from (k_0, λ_0) and converges to $(\bar{k}, \bar{\lambda})$. It should be obvious that this path also represents the locus of such (k_0, λ_0) for $k_0 \in [0, \bar{k}]$. The analysis of the case $k_0 > \bar{k}$ is left as Exercise 11.4.

In Region I, $\dot{k}(t) > 0$ and $k(t)$ is an increasing function of t as indicated by the horizontal right-directed arrow in Fig. 11.1. Therefore, we can replace the independent variable t by k, and then use (11.16) and (11.17) to obtain

$$\lambda'(k) = \frac{d\lambda}{dk} = \frac{d\lambda}{dt} \Big/ \frac{dk}{dt} = \frac{[f'(k) - (r+\gamma)]\lambda}{h(\lambda) + \gamma k - f(k)}. \tag{11.19}$$

Thus, our task of showing that there exists an optimal path starting from any initial $k_0 < \bar{k}$ is equivalent to showing that there exists a solution of the differential equation (11.19) on the interval $[0, \bar{k}]$, beginning with the boundary condition $\lambda(\bar{k}) = \bar{\lambda}$. For this, we must obtain the value $\lambda'(\bar{k})$. Since both the numerator and the denominator in (11.19) vanish at $k = \bar{k}$, we need to derive $\lambda'(\bar{k})$ by a perturbation argument. To do so, we use (11.19) and (11.18) to obtain

$$\lambda'(k) = \frac{[r + \gamma - f'(k)]\lambda}{f(k) - \gamma k - h(\lambda)} = \frac{[f'(\bar{k}) - f'(k)]\lambda}{f(k) - f(\bar{k}) - \gamma k + \gamma \bar{k} - h(\lambda) + h(\bar{\lambda})}.$$

We use L'Hôpital's rule to take the limit as $k \to \bar{k}$ and obtain

$$\lambda'(\bar{k}) = \frac{-f''(\bar{k})\bar{\lambda}}{f'(\bar{k}) - \gamma - h'(\bar{\lambda})} = \frac{-f''(\bar{k})\bar{\lambda}}{f'(\bar{k}) - \gamma - \lambda'(\bar{k})/u''(h(\bar{\lambda}))}, \tag{11.20}$$

or

$$-\frac{(\lambda'(\bar{k}))^2}{u''(h(\bar{\lambda}))} + \lambda'(\bar{k})[f'(\bar{k}) - \gamma] + \bar{\lambda} f''(\bar{k}) = 0. \tag{11.21}$$

Note that the second equality in (11.20) uses the relation $h'(\bar{\lambda}) = 1/u''(h(\bar{\lambda}))$ obtained by differentiating $u'(c) = u'(h(\lambda)) = \lambda$ of (11.14) with respect to λ at $\lambda = \bar{\lambda}$.

It is easy to see that (11.21) has one positive solution and one negative solution. We take the negative solution for $\lambda'(\bar{k})$ because of the following consideration. With the negative solution, we can prove that the differential equation (11.19) has a smooth solution, such that $\lambda'(k) < 0$. For this, let

$$\pi(k) = f(k) - k\gamma - h(\lambda(k)).$$

Since $k < \bar{k}$, we have $r+\gamma-f'(k) < 0$. Then from (11.19), since $\lambda'(\bar{k}) < 0$, we have $\lambda(\bar{k} - \varepsilon) > \lambda(\bar{k})$. Also since $\bar{\lambda} > 0$ and $f''(\bar{k}) < 0$, Eq. (11.20) with $\lambda'(\bar{k})$ implies

$$\pi'(\bar{k}) = f'(\bar{k}) - \gamma - \frac{\lambda'(\bar{k})}{u''(h(\bar{\lambda}))} < 0,$$

and thus,

$$\pi(\bar{k} - \varepsilon) = f(\bar{k} - \varepsilon) - \gamma(\bar{k} - \varepsilon) - h(\lambda(\bar{k} - \varepsilon)) > 0.$$

Therefore, the derivative at $\bar{k} - \varepsilon$ is well defined and $\lambda'(\bar{k} - \varepsilon) < 0$. We can proceed as long as

$$\pi'(k) = f'(k) - \gamma - \frac{\lambda'(k)}{u''(h(\lambda(k)))} < 0. \tag{11.22}$$

This implies that $f(k) - k\gamma - h(\lambda) > 0$, and also since $r + \gamma - f'(k)$ remains negative for $k < \bar{k}$, we have $\lambda'(k) < 0$.

Suppose now that there is a point $\tilde{k} < \bar{k}$ with $\pi(\tilde{k}) = 0$. Then, since $\pi(\tilde{k} + \varepsilon) > 0$, we have $\pi'(\tilde{k}) \geq 0$. But at $\tilde{k}$, $\pi(\tilde{k}) = 0$ in (11.19) implies $\lambda'(\tilde{k}) = -\infty$, and then from (11.22), we have $\pi'(\tilde{k}) = -\infty$, which is a contradiction with $\pi'(\tilde{k}) \geq 0$. Thus, we can proceed on the whole interval $[0, \bar{k}]$. This indicates that the path $\lambda(k)$ (shown as the solid line in Region I of Fig. 11.1) remains above the curve

$$\dot{k} = f(k) - k\gamma - h(\lambda) = 0,$$

shown as the dotted line in Fig. 11.1 when $k < \bar{k}$. Thus, we can set $\lambda_0 = \lambda(k_0)$ for $0 \leq k_0 \leq \bar{k}$ and have the optimal path starting from (k_0, λ_0) and converging to $(\bar{k}, \bar{\lambda})$.

Similar arguments hold when the initial capital stock $k_0 > \bar{k}$, in order to show that the optimal path (shown as the solid line in Region III of Fig. 11.1) exists in this case. You have already been asked to carry out this analysis in Exercise 11.4.

We should mention that the conclusions derived in this subsection could have been reached by invoking the Global Saddle Point Theorem stated in Appendix D.7, but we have chosen instead to carry out a detailed analysis for illustrating the use of the phase diagram method. The next time we use the phase diagram method will be in Sect. 11.3.3, and there we shall rely on the Global Saddle Point Theorem.

11.2 A Model of Optimal Epidemic Control

Certain infectious epidemic diseases are seasonal in nature. Examples are the common cold, the flu, and certain children's diseases. When it is beneficial to do so, control measures are taken to alleviate the effects of these diseases. Here we discuss a simple control model due to Sethi (1974c) for analyzing an epidemic problem. Related problems have been treated by Sethi and Staats (1978), Sethi (1978c), and Francis (1997). See Wickwire (1977) for a good survey of optimal control theory applied to the control of pest infestations and epidemics, and Swan (1984) for applications to biomedicine.

11.2.1 Formulation of the Model

Let N be the total fixed population. Let $x(t)$ be the number of infectives at time t so that the remaining $N - x(t)$ is the number of susceptibles. To keep the model simple, assume that no immunity is acquired so that when infected people are cured, they become susceptible again. The state equation governing the dynamics of the epidemic spread in the population is

$$\dot{x} = \beta x(N - x) - vx, \quad x(0) = x_0, \tag{11.23}$$

where β is a positive constant termed *infectivity* of the disease, and v is a control variable reflecting the level of medical program effort. Note that $x(t)$ is in $[0, N]$ for all $t > 0$ if x_0 is in that interval.

The objective of the control problem is to minimize the present value of the cost stream up to a horizon time T, which marks the end of the season for that disease. Let h denote the unit social cost per infective, let m denote the cost of control per unit level of program effort, and let Q denote the capability of the health care delivery system providing an upper bound on v. The optimal control problem is:

$$\max \left\{ J = \int_0^T -(hx + mv)e^{-\rho t} dt \right\} \tag{11.24}$$

subject to (11.23), the terminal constraint that

$$x(T) = x_T, \tag{11.25}$$

and the control constraint

$$0 \le v \le Q.$$

11.2.2 Solution by Green's Theorem

Rewriting (11.23) as

$$vdt = [\beta x(N - x)dt - dx]/x$$

and substituting into (11.24) yields the line integral

$$J_\Gamma = \int_\Gamma - \left\{ [hx + m\beta(N - x)]e^{-\rho t} dt - \frac{m}{x} e^{-\rho t} dx \right\}, \tag{11.26}$$

where Γ is a path from x_0 to x_T in the (t, x)-space. Let Γ_1 and Γ_2 be two such paths from x_0 to x_T, and let R be the region enclosed by Γ_1 and Γ_2. By Green's theorem, we can write

$$J_{\Gamma_1 - \Gamma_2} = J_{\Gamma_1} - J_{\Gamma_2} = \int\int_R - \left[\frac{m\rho}{x} - h + m\beta \right] e^{-\rho t} dt dx. \tag{11.27}$$

To obtain the singular control we set the integrand of (11.27) equal to zero, as we did in Sect. 7.2.2. This yields

$$x = \frac{\rho}{h/m - \beta} = \frac{\rho}{\theta}, \tag{11.28}$$

where $\theta = h/m - \beta$. Define the singular state x^s as follows:

$$x^s = \begin{cases} \rho/\theta & \text{if } 0 < \rho/\theta < N, \\ N & \text{otherwise.} \end{cases} \tag{11.29}$$

The corresponding singular control level

$$v^s = \beta(N - x^s) = \begin{cases} \beta(N - \rho/\theta) & \text{if } 0 < \rho/\theta < N, \\ 0 & \text{otherwise.} \end{cases} \tag{11.30}$$

We will show that x^s is the turnpike level of infectives. It is instructive to interpret (11.29) and (11.30) for the various cases. If $\rho/\theta > 0$, then $\theta > 0$ so that $h/m > \beta$. Here the smaller the ratio h/m, the larger the turnpike level x^s, and therefore, the smaller the medical program effort should be. In other words, the smaller the social cost per infective and/or the larger the treatment cost per infective, the smaller the medical program effort should be.

When $\rho/\theta < 0$, you are asked to show in Exercise 11.9 that $x^s = N$ in the case $h/m < \beta$, which means the ratio of the social cost to the treatment cost is smaller than the infectivity coefficient. Therefore, in this case when there is no terminal constraint, the optimal trajectory involves no treatment effort. An example of this case is the common cold where the social cost is low and treatment cost is high.

The optimal control for the fortuitous case when $x_T = x^s$ is

$$v^*(x(t)) = \begin{cases} Q & \text{if } x(t) > x^s, \\ v^s & \text{if } x(t) = x^s, \\ 0 & \text{if } x(t) < x^s. \end{cases} \tag{11.31}$$

When $x_T \neq x^s$, there are two cases to consider. For simplicity of exposition we assume $x_0 > x^s$ and T and Q to be large.

Case 1: $x_T > x^s$. The optimal trajectory is shown in Fig. 11.2. In Exercise 11.8 you are asked to show its optimality by using Green's theorem.

Case 2: $x_T < x^s$. The optimal trajectory is shown in Fig. 11.3. It can be shown that x goes asymptotically to $N - Q/\beta$ if $v = Q$. The level is marked in Fig. 11.3.

The optimal control shown in Figs. 11.2 and 11.3 assumes $0 < x^s < N$. It also assumes that T is large so that the trajectory will spend some time on the turnpike and Q is large so that $x^s \geq N - Q/\beta$. The graphs are drawn for $x_0 > x^s$ and $x^s < N/2$; for all other cases see Sethi (1974c).

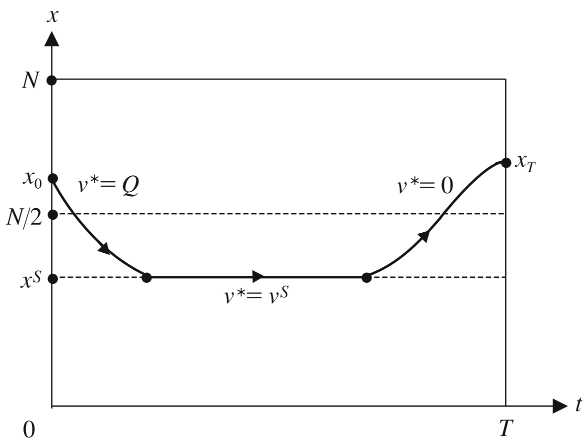

Figure 11.2: Optimal trajectory when $x_T > x^s$

11.3 A Pollution Control Model

In this section we will describe a simple pollution control model due to Keeler et al. (1971). We will describe this model in terms of an economic system in which labor is the only primary factor of production, which is allocated between food production and DDT production. It is assumed that all of the food produced is used for consumption. On the other hand, all of the DDT produced is used as a secondary factor of production which, along with labor, determines the food output. However, when used, DDT causes pollution, which can only be reduced by natural decay. The objective of the society is to maximize the total present value of the utility of food less the disutility of pollution due to the use of DDT.

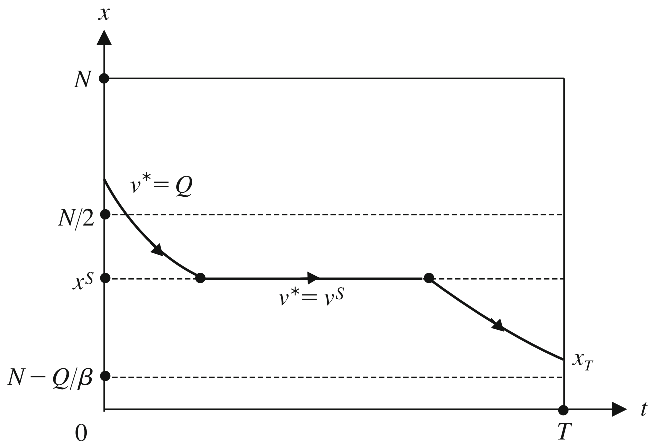

Figure 11.3: Optimal trajectory when $x_T < x^s$

11.3.1 Model Formulation

We introduce the following notation:

L = the total labor force, assumed to be constant for simplicity,

v = the amount of labor used for DDT production,

$L - v$ = the amount of labor used for food production,

P = the stock of DDT pollution at time t,

$a(v)$ = the rate of DDT output; $a(0) = 0,\ a' > 0,\ a'' < 0,$ for $v \geq 0$,

δ = the natural exponential decay rate of DDT pollution,

$C(v)$ = $f[L - v, a(v)]$ = the rate of food output to be consumed; $C(v)$ is concave, $C(0) > 0,\ C(L) = 0;\ C(v)$ attains a unique maximum at $v = V > 0;$ see Fig. 11.4. Note that a sufficient condition for $C(v)$ to be strictly concave is $f_{12} \geq 0$ along with the usual concavity and monotonicity conditions on f (see Exercise 11.10),

$u(C)$ = the utility function of consuming the food output $C \geq 0$; $u'(0) = \infty,\ u'(C) > 0,\ u''(C) < 0,$

$h(P)$ = the disutility function of pollution stock $P \geq 0$; $h'(0) = 0,\ h'(P) > 0,\ h''(P) > 0.$

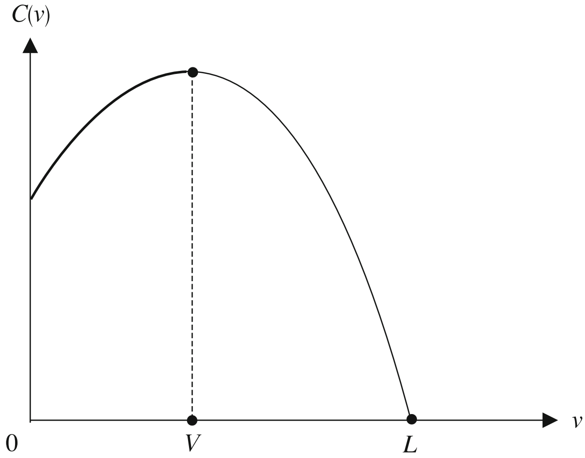

Figure 11.4: Food output function

The optimal control problem is:

$$\max \left\{ J = \int_0^\infty e^{-\rho t}[u(C(v)) - h(P)]dt \right\} \tag{11.32}$$

subject to

$$\dot{P} = a(v) - \delta P, \quad P(0) = P_0, \tag{11.33}$$

$$0 \le v \le L. \tag{11.34}$$

From Fig. 11.4, it is obvious that v is at most V, since the production of DDT beyond that level decreases food production and increases DDT pollution. Hence, (11.34) can be reduced to simply

$$v \ge 0. \tag{11.35}$$

11.3.2 Solution by the Maximum Principle

Form the current-value Lagrangian

$$L(P, v, \lambda, \mu) = u[C(v)] - h(P) + \lambda[a(v) - \delta P] + \mu v \tag{11.36}$$

using (11.32), (11.33) and (11.35), where

$$\dot{\lambda} = (\rho + \delta)\lambda + h'(P), \tag{11.37}$$

and

$$\mu \ge 0 \text{ and } \mu v = 0. \tag{11.38}$$

The optimal solution is given by

$$\frac{\partial L}{\partial v} = u'[C(v)]C'(v) + \lambda a'(v) + \mu = 0. \tag{11.39}$$

Since the derived Hamiltonian is concave, conditions (11.36)–(11.39) together with

$$\lim_{t \to \infty} \lambda(t) = \bar{\lambda} = \text{constant} \tag{11.40}$$

are sufficient for optimality; see Theorem 2.1 and Sect. 2.4. The phase diagram analysis presented below gives $\lambda(t)$ satisfying (11.40).

11.3.3 Phase Diagram Analysis

From the assumptions on $C(v)$ or from Fig. 11.4, we see that $C'(0) > 0$. This means that $du/dv = u'(C(v))C'(v)|_{v=0} > 0$. This along with $h'(0) = 0$ implies that $v > 0$, meaning that it pays to produce some positive amount of DDT in equilibrium. Therefore, the equilibrium value of the Lagrange multiplier is zero, i.e., $\bar{\mu} = 0$. From (11.33), (11.37) and (11.39), we get the equilibrium values $\bar{P}, \bar{\lambda}$, and $\bar{v}$ as follows:

$$\bar{P} = \frac{a(\bar{v})}{\delta}, \tag{11.41}$$

$$\bar{\lambda} = -\frac{h'(\bar{P})}{\rho + \delta} = -\frac{u'[C(\bar{v})]C'(\bar{v})}{a'(\bar{v})}. \tag{11.42}$$

From (11.42) and the assumptions on the derivatives of g, C and a, we know that $\bar{\lambda} < 0$. From this and (11.37), we conclude that $\lambda(t)$ is always negative. The economic interpretation of λ is that $-\lambda$ is the imputed cost of pollution. Let $v = \Phi(\lambda)$ denote the solution of (11.39) with $\mu = 0$. On account of (11.35), define

$$v^* = \max[0, \Phi(\lambda)]. \tag{11.43}$$

We know from the interpretation of λ that when λ increases, the imputed cost of pollution decreases, which can justify an increase in the DDT production to ensure an increased food output. Thus, it is reasonable to assume that

$$\frac{d\Phi}{d\lambda} > 0,$$

and we will make this assumption. It follows that there exists a unique λ^c such that $\Phi(\lambda^c) = 0,\ \Phi(\lambda) < 0$ for $\lambda < \lambda^c$ and $\Phi(\lambda) > 0$ for $\lambda > \lambda^c$.

To construct the phase diagram, we must plot the isoclines $\dot{P} = 0$ and $\dot{\lambda} = 0$. These are, respectively,

$$P = \frac{a(v^*)}{\delta} = \frac{a[\max\{0, \Phi(\lambda)\}]}{\delta}, \tag{11.44}$$

$$h'(P) = -(\rho + \delta)\lambda. \tag{11.45}$$

Observe that the assumption $h'(0) = 0$ implies that the graph of (11.45) passes through the origin. Differentiating these equations with respect to λ and using (11.43), we obtain

$$\frac{dP}{d\lambda}|_{\dot{P}=0} = \frac{a'(v)}{\delta}\frac{dv}{d\lambda} > 0 \tag{11.46}$$

as the slope of the $\dot{P} = 0$ isocline, and

$$\frac{dP}{d\lambda}|_{\dot{\lambda}=0} = -\frac{(\rho + \delta)}{h''(P)} < 0 \tag{11.47}$$

as the slope of the $\dot{\lambda} = 0$ isocline.

Using (11.41), (11.42), (11.46), and (11.47), we can draw (11.44) and (11.45) in the (λ, P)-space as shown in Fig. 11.5. As in Sect. 11.1.4, these isoclines divide the (λ, P) space in four regions. At any point in each of these regions, we have depicted the direction of the movement of the trajectory with v^* in (11.33) and (11.37). It is easy to conclude that we have $\dot{P} < 0$ ($\dot{P} > 0$) above (below) the $\dot{P} = 0$ isocline and $\dot{\lambda} > 0$ ($\dot{\lambda} < 0$) to the right (left) of the $\dot{\lambda} = 0$ isocline.

The intersection point $(\bar{\lambda}, \bar{P})$ of these isoclines denotes the equilibrium levels for the adjoint variable and the pollution stock, respectively. That there exists an optimal path (shown as the solid line in Fig. 11.5) converging to the equilibrium $(\bar{\lambda}, \bar{P})$ follows directly from the Global Saddle Point Theorem stated in Appendix D.7.

Given λ^c as the intersection of the $\dot{P} = 0$ curve and the horizontal axis, the corresponding ordinate P^c on the optimal trajectory is the related pollution stock level. The significance of P^c is that if the existing pollution stock P is larger than P^c, then the optimal control is $v^* = 0$, meaning no DDT is produced.

Given an initial level of pollution P_0, the optimal trajectory curve in Fig. 11.5 provides the initial value λ_0 of the adjoint variable. With these initial values, the optimal trajectory is determined by (11.33), (11.37), and (11.43). If $P_0 > P^c$, as shown in Fig. 11.5, then $v^* = 0$ until such

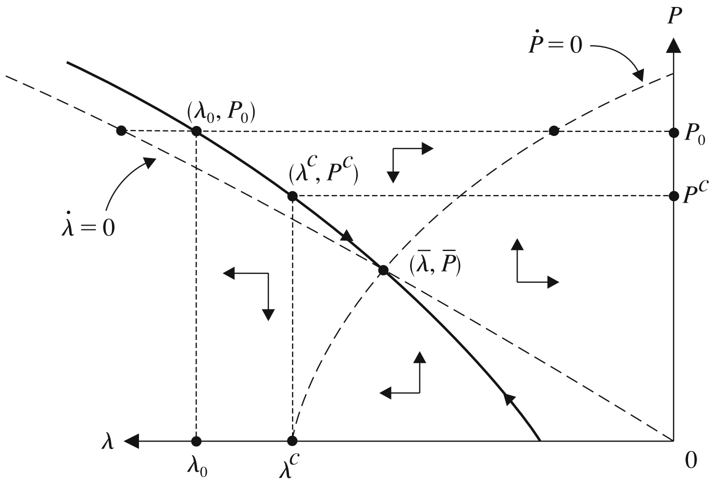

Figure 11.5: Phase diagram for the pollution control model

time that the natural decay of pollution stock has reduced it to P^c. At that time, the adjoint variable has increased to the value λ^c. The optimal control is $v^* = \phi(\lambda)$ from this time on, and the path converges to $(\bar{\lambda}, \bar{P})$.

At equilibrium, $\bar{v} = \Phi(\bar{\lambda}) > 0$, which implies that it is optimal to produce some DDT forever in the long run. The only time when its production is not optimal is at the beginning when the pollution stock is higher than P^c.

It is important to examine the effects of changes in the parameters on the optimal path. In particular, you are asked in Exercise 11.11 to show that an increase in the natural rate of decay of pollution, δ, will increase P^c. That is, when pollution decays at a faster rate, we can increase the threshold level of pollution stock at which to ban the production of the pollutant. For DDT in reality, δ is small so that its complete ban, which has actually occurred, may not be far from the optimal policy.

Here we have presented a very simple model of pollution in which the problem was to choose an optimal production process. Models in which the control variable to determine is the optimal amount to spend in reducing the pollution output of an existing dirty process have also been formulated; see Wright (1974) and Sethi (1977d). For other related models, see Luptacik and Schubert (1982), Hartl and Luptacik (1992), and Hartl and Kort (1996a,b,c, 1997), Xepapadeas and de Zeeuw (1999), and Moser et al. (2014).

11.4 An Adverse Selection Model

In modern contract theory, the term *adverse selection* is used to describe principal-agent models in which an agent has private information before a contract is written. For example, a seller does not know perfectly how much a buyer is willing to pay for a good. A related concept is that of *moral hazard*, when there is present a hidden action not adversely observed by the principal.

In such game situations, clearly the principal would like to know the agent's private information which he cannot learn simply by asking the agent, because it is in the agent's interest to distort the truth. Fortunately, according to the theory of *mechanism design*, the principal can design a game whose rules can influence the agent to act the way he would like. Thanks, particularly to the *revelation principle*, the principal needs only consider games in which the agent truthfully reports her private information.

There is a large literature on contract theory, and we refer the reader to books by Laffont and Mortimort (2001), Bolton and Dewatripont (2005) and Cvitanic and Zhang (2013). For our purposes, we shall next consider a game between a seller and a buyer, where the buyer has private information about her willingness-to-pay for the seller's goods; see Bolton and Dewatripont (2005).

11.4.1 Model Formulation

Consider a transaction between a seller (the principal) and a buyer (the agent) of type $t \in [t_1, t_2]$, $0 \leq t_1 \leq t_2$, represents her willingness-to-pay for seller's goods. We assume in particular that buyer's preferences are represented by the utility function

$$U(q, \phi, t) = ta(q) - \phi, \tag{11.48}$$

where q is the number of units purchased and ϕ is the total amount paid to the seller. We assume $a(0) = 0$, $a' > 0$, and $a'' < 0$.

The seller knows only the distribution $F(t)$, having the density $f(t), t \in [t_1, t_2]$. The seller's unit production cost is $c > 0$, so that his profit from selling q units against a sum of money ϕ is given by

$$\pi = \phi - cq. \tag{11.49}$$

The question of interest here is to obtain a profit-maximizing pair $\{\phi, q\}$ that the seller will be able to induce the buyer of type $\hat{t}$ to choose.

Thanks to the revelation principle, the answer is that the seller can offer a menu of contracts $\{\phi(t), q(t)\}$ which comes from solving the following maximization problem:

$$\max_{q(\cdot),\phi(\cdot)} \int_{t_1}^{t_2} [\phi(t) - cq(t)]f(t)dt \tag{11.50}$$

subject to

$$\text{(IR)}\ \ \hat{t}a(q(\hat{t})) - \phi(\hat{t}) \geq 0,\ \hat{t} \in [t_1, t_2] \tag{11.51}$$

$$\text{(IC)}\ \ \hat{t}a(q(\hat{t})) - \phi(\hat{t}) \geq \hat{t}a(q(t)) - \phi(t),\ t,\ \hat{t} \in [t_1, t_2],\ t \neq \hat{t}. \tag{11.52}$$

The constraints (11.51), called *individual rationality constraints (IR)*, say that the agent of type $\hat{t}$ will participate in the contract. Clearly, given (11.52), we can replace these constraints by a single constraint

$$t_1 a(q(t_1)) - \phi(t_1) \geq 0. \tag{11.53}$$

The left-hand side of the constraints (11.52), called *incentive compatibility constraints (IC)*, is the utility of agent $\hat{t}$ if she chooses the contract intended for her, whereas the right-hand side represents the utility of agent $\hat{t}$ if she chooses the constraint intended for type $t \neq \hat{t}$. The IC constraints, therefore, imply that type $\hat{t}$ agent is better off choosing the contract intended for her than any other contract in the menu.

Clearly, the seller's problem is mathematically difficult as it involves maximizing the seller's profit over a class of functions. So, a way to deal with this problem is to decompose it into an implementation problem (which functions $q(\cdot)$ are incentive compatible?) and an optimization problem (which one is the best implementation function for the seller?)

11.4.2 The Implementation Problem

Given a menu $\{q(\cdot), \phi(\cdot)\}$ that satisfies the seller's problem (11.50)–(11.52), it must be the case in equilibrium that the buyer $\hat{t}$ will choose the contract $\{q(\hat{t}), \phi(\hat{t})\}$. In other words, his utility $\hat{t}a(q(t)) - \phi(t)$ of choosing a contract $\{q(t), \phi(t)\}$ will be maximized at $t = \hat{t}$. Assuming that $q(\cdot)$ and $\phi(\cdot)$ are twice differentiable functions, the first-order and second-order conditions are

$$\hat{t}a'(q(t))\dot{q}(t) - \dot{\phi}(t)|_{t=\hat{t}} = \hat{t}a'(q(\hat{t}))\dot{q}(\hat{t}) - \dot{\phi}(\hat{t}) = 0, \tag{11.54}$$

$$\hat{t}a''(q(t))(\dot{q}(t))^2 + \hat{t}a'(q(t))\ddot{q}(t) - \ddot{\phi}(t)|_{t=\hat{t}} \le 0. \tag{11.55}$$

From (11.54), it follows from replacing $\hat{t}$ by t that

$$ta'(q(t))\dot{q}(t) - \dot{\phi}(t) = 0,\ t \in [t_1, t_2], \tag{11.56}$$

called the *local incentive compatibility condition*, must hold. Differentiating (11.56) gives,

$$ta''(q(t))(\dot{q}(t))^2 + a'(q(t))\dot{q}(t) + ta'(q(t))\ddot{q}(t) - \ddot{\phi}(t) = 0. \tag{11.57}$$

It follows from (11.55), (11.57), and $a' > 0$, that

$$\dot{q}(t) \ge 0. \tag{11.58}$$

This is called the *monotonicity condition*. In Exercise 11.12, you are asked to show that (11.56) and (11.58) are sufficient for (11.52) to hold. Since, these conditions are already necessary, we can say that local incentive compatibility (11.56) and monotonicity (11.58) together are equivalent to the IC condition (11.52).

We can now ready to formulate the seller's optimization problem.

11.4.3 The Optimization Problem

The seller's problem can be written as the following optimal control problem:

$$\max_{u(\cdot)} \int_{t_1}^{t_2} [\phi(t) - cq(t)]f(t)dt \tag{11.59}$$

subject to

$$\dot{q}(t) = u(t), \tag{11.60}$$

$$\dot{\phi}(t) = ta'(q(t))u(t), \tag{11.61}$$

$$t_1 a(q(t_1)) - \phi(t_1) = 0, \tag{11.62}$$

$$u(t) \ge 0. \tag{11.63}$$

Here, $q(t)$ and $\phi(t)$ are state variables and $u(t)$ is a control variable satisfying the control constraint $u(t) \ge 0$. The objective function (11.59) is the expected value of the seller's profit with respect to the density $f(t)$. Equation (11.60) and constraint (11.63) come from the monotonicity condition (11.58). Equation (11.61) with $u(t)$ from (11.60) gives the local incentive compatibility condition (11.56). Finally, (11.62) specifies the IR constraint (11.53) in view of the fact it will be binding for the lowest agent type t_1 at the optimum.

We can now use the sense of the maximum principle (3.12) to write the necessary conditions for optimality. Note that (3.12) is written for problem (3.7) that has specified initial states and some constraints on the terminal state vector $x(T)$ that include the equality constraint $b(x(T),T) = 0$. Our problem, on the other hand, has this type of equality constraint, namely (11.62), on the initial states $q(t_1)$ and $\phi(t_1)$ and no specified terminal states $q(t_2)$ and $\phi(t_2)$. However, since initial time conditions and terminal time conditions can be treated in a symmetric fashion, we can apply the sense of (3.12), as shown in Remark 3.9, to obtain the necessary optimality conditions to problem (11.59)–(11.63). In Exercise 11.13, you are asked to obtain (11.67) and (11.68) by following Remark 3.9 to account for the presence of the equality constraint (11.62) on the initial state variables rather than on the terminal state as in problem (3.7).

To specify the necessary optimality condition, we first define the Hamiltonian.

$$\begin{aligned} H(q,\phi,\lambda,\mu,t) &= [\phi(t) - cq(t)]f(t) + \lambda(t)u(t) + \mu(t)[ta'(q(t)u(t))] \\ &= [\phi(t) - cq(t)]f(t) + [\lambda(t) + \mu(t)ta'(q(t))]u(t) \end{aligned} \tag{11.64}$$

Then for u^* with the corresponding state trajectories q^* and ϕ^* to be optimal, we must have adjoints λ and μ, and a constant β, such that

$$\dot{q}^* = u^*, \dot{\phi}^* = ta'(q^*)u \tag{11.65}$$

$$t_1 a(q^*(t_1)) - \phi^*(t_1) = 0, \tag{11.66}$$

$$\dot{\lambda} = cf - \mu t a''(q^*)u^*,\ \lambda(t_1) = \beta t_1 a'(q^*(t_1)),\ \lambda(t_2) = 0, \tag{11.67}$$

$$\dot{\mu} = -f,\ \mu(t_1) = -\beta,\ \mu(t_2) = 0, \tag{11.68}$$

$$u^*(t) = \text{bang}[0,\infty; \lambda(t) + \mu(t)ta'(q^*(t))]. \tag{11.69}$$

Several remarks are in order at this point. First we see that we have a bang-bang control in (11.69). This means that the $u^*(t)$ can be 0, or greater than 0, or an impulse control. Moreover, in the region when $u^*(t) = 0$, which will occur when $\lambda(t) + \mu(t)ta'(q^*(t)) < 0$, we will have a constant $q^*(t)$, and we will have a singular control $u^*(t) > 0$ if we can keep $\lambda(t) + \mu(t)ta'(q^*(t)) = 0$ by an appropriate choice of $u^*(t)$ along the singular path. An impulse control would occur if the initial $q(t_1)$ were above the singular path. Since in our problem, initial states are not exactly specified, we shall not encounter an impulse control here.

The third remark concerns a numerical way of solving the problem. For this, let us rewrite the boundary conditions in (11.67) and (11.68) and the condition (11.66) as below:

$$t_1 a(q^*(t_1)) - \phi^*(t_1) = 0, \ \lambda(t_1) = -\mu(t_1) t_1 a'(q^*(t_1)) \tag{11.70}$$

$$\lambda(t_2) = \mu(t_2) = 0. \tag{11.71}$$

With (11.71) and a guess of $q(t_2)$ and $\phi(t_2)$, we can solve the differential equation (11.65), (11.67) and (11.68), with $u^*(t)$ in (11.69), backward in time. These will give us the values of $\lambda(t_1)$, $\mu(t_1)$, $q(t_1)$ and $\phi(t_1)$. We can check if these satisfy the two equations in (11.70). If yes, we have arrived at a solution. If not, we change our guess for $q(t_2)$ and $\phi(t_2)$ and start again. As you may have noticed, the procedure is very similar to solving a two-point boundary value problem.

Next we provide an alternative procedure to solve the seller's problem, a procedure used in the theory of mechanism design. This procedure first ignores the nonnegativity constraint (11.60) and solves the relaxed problem given by (11.59)–(11.62). In view of (11.52), let us define

$$u^0(\hat{t}) = \hat{t}a(q(\hat{t})) - \phi(\hat{t}) = \max_t [ta(q(t)) - \phi(t)]. \tag{11.72}$$

By the envelope theorem, we have

$$\frac{du^0(\hat{t})}{d\hat{t}} = \frac{\partial u^0(\hat{t})}{\partial \hat{t}} = a(q(\hat{t})), \tag{11.73}$$

which we can integrate to obtain

$$u^0(t) = \int_{t_1}^{t} a(q(x)dx) + u^0(t_1) = \int_{t_1}^{t} a(q(x))dx, \tag{11.74}$$

since $u^*(t_1) = 0$ at the optimum. Also, since $\phi(t) = ta(q(t)) - u^0(t)$, we can write the seller's profit as

$$\int_{t_1}^{t_2} [ta(q(t)) - \int_{t_1}^{t} a(q(x))dx - cq(t)] f(t)dt. \tag{11.75}$$

Then, integrating by parts, we have

$$\begin{aligned} &\textstyle\int_{t_1}^{t_2} \left[\{ta(q(t)) - cq(t)\} f(t) - a(q(t))(1 - F(t))\right] dt \\ &\quad = \textstyle\int_{t_1}^{t_2} \left[ta(q(t)) - cq(t) - a(q(t))/h(t)\right] f(t)dt, \end{aligned} \tag{11.76}$$

where $h(t) = f(t)/[1 - F(t)]$ is known as the hazard rate. Since we are interested in maximizing the seller's profit with respect to the output schedule $q(\cdot)$, we can maximize the expression under the integral pointwise for each t. The first-order condition for that is

$$\left[t - \frac{1 - F(t)}{f(t)}\right] a'(q(t)) = \left[t - \frac{1}{h(t)}\right] a'(q(t)) = c, \tag{11.77}$$

which gives us the optimal solution of the relaxed problem as

$$\hat{q}(t) = a'^{-1}\left[c\left(t - \frac{1}{h(t)}\right)^{-1}\right]. \tag{11.78}$$

In obtaining (11.78), we had omitted the nonnegativity constraint (11.63) introduced to ensure that $q(t)$ is increasing. Thus, it remains to check if $d\hat{q}(t)/dt \geq 0$. It is straightforward to verify that if the hazard rate $h(t)$ is increasing in t, then $\hat{q}(t)$ is increasing in t. To show this, we differentiate (11.78) to obtain

$$\frac{d\hat{q}(t)}{dt} = -\frac{g(t)a'(\hat{q}(t))}{a''(\hat{q}(t))g(t)},$$

where $g(t) = [t - 1/h(t)]$. Clearly, if $h(t)$ is increasing, then $g(t)$ is increasing, and $d\hat{q}(t)/dt \geq 0$.

In this case, $\hat{q}(t)$ and the corresponding $\hat{\phi}(t)$ obtained from solving the differential equation given by (11.61) and the boundary condition (11.62) give us the optimal menu $\{\hat{\phi}(t), \hat{q}(t)\}$.

What if $h(t)$ is not increasing? In that case, there is a procedure called *bunching and ironing* given by the solution of an optimal control problem to be formulated next. This is because $\hat{q}(t)$ in (11.78) is obtained by solving the relaxed problem that ignores the nonnegativity constraint (11.63), and so it may be that $d\hat{q}/dt$ is strictly negative for some $t \in [\underline{t}, \bar{t}] \subset [t_1, t_2]$ as shown in Fig. 11.6.

Then the seller must choose the optimal $q^*(t)$ to maximize the following constrained optimal control problem:

$$\max_{q(\cdot)} \int_{t_1}^{t_2} \left[ta(q(t)) - cq(t) - \frac{a(q(t))}{h(t)}\right] f(t)dt \tag{11.79}$$

subject to

$$\dot{q}(t) = u(t),\ u(t) \geq 0. \tag{11.80}$$

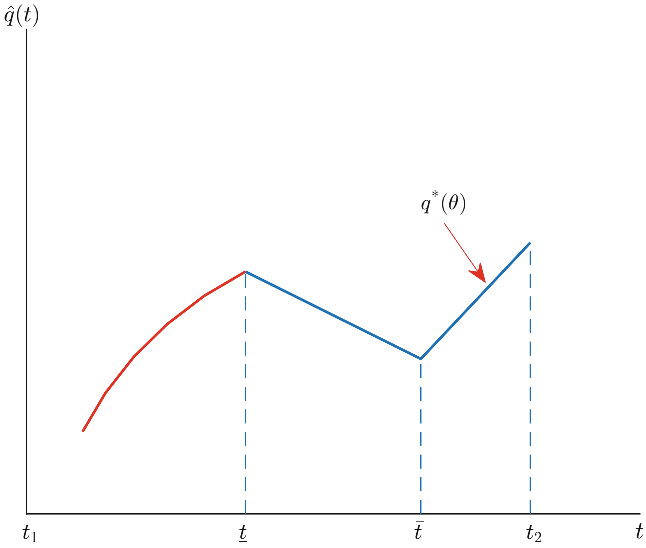

Figure 11.6: Violation of the monotonicity constraint

Now the necessary optimality conditions, with the Hamiltonian defined as

$$H(q, 0, \lambda, t) = (ta(q) - cq - a(q)/h)f + \lambda u, \tag{11.81}$$

are

$$\dot{\lambda} = -\left[(t - 1/h)a'(q) - c\right] f,\ \lambda(t_1) = \lambda(t_2) = 0, \tag{11.82}$$

and

$$u^* = [0, \infty;\ \lambda]. \tag{11.83}$$

We may also note that these conditions are also sufficient since H in (11.81) is concave in q.

Integrating (11.82), we have

$$\lambda(t) = -\int_{t_1}^{t} \left[\left(z - \frac{1}{h(z)}\right) a'(q(z)) - c\right] f(z)dz.$$

Using the transversality conditions in the case when neither the initial nor the terminal state is specified for the state equation (11.80), we obtain

$$0 = \lambda(t_1) = \lambda(t_2) = -\int_{t_1}^{t_2} \left[\left(z - \frac{1}{h(z)}\right) a'(q(z)) - c\right] f(z)dz.$$

Then for $u^*(t) = 0$ on an interval $t \in [\theta_1, \theta_2] \subset [t_1, t_2]$, we must have $\lambda(t) < 0$, $t \in [\theta_1, \theta_2]$. Moreover, when $u^*(t) > 0$, it must be a singular control for which $\lambda(t) = 0$.

But $\lambda(t) = 0$ is the same as the condition (11.77), which means that if $q^*(t)$ is strictly increasing, then it must coincide with $\hat{q}(t)$ in (11.78). It, therefore, only remains to determine the intervals over which $q^*(t)$ is constant. Consider Fig. 11.7

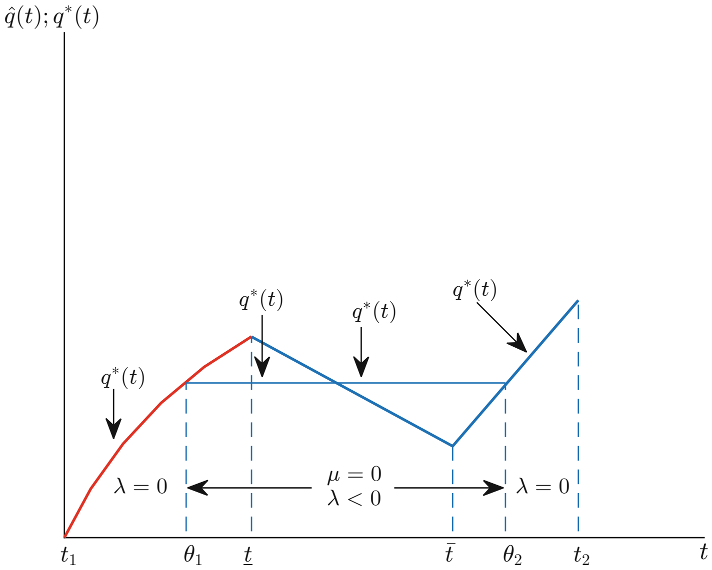

Figure 11.7: Bunching and ironing

By continuity, we must have $\lambda(\theta_1) = \lambda(\theta_2) = 0$, so that

$$\int_{\theta_1}^{\theta_2} \left[\left(z - \frac{1}{h(z)} \right) a'(q^*(z)) - c \right] dz = 0. \tag{11.84}$$

In addition, we must have

$$q^*(\theta_1) = q^*(\theta_2) \tag{11.85}$$

from the continuity of $q^*(\cdot)$. Thus, we have two equations (11.84) and (11.85) and two unknowns, allowing us to obtain the values of θ_1 and θ_2. An interval $[\theta_1, \theta_2]$ over which $q^*(t)$ is constant is known as a *bunching* interval.

Here, we have given a procedure when $\hat{q}(\cdot)$ has only one interval $[\underline{t}, \bar{t}]$ over which it is strictly decreasing. If there are more such intervals, this procedure of ironing and bunching can be extended in an obvious manner.

11.5 Miscellaneous Applications

The number of papers which apply control theory to problems in economics and management science is now so large that it is impossible to cover them in detail within the confines of a single book. We satisfy ourselves by listing selected references with a brief indication of their contents.

For control theory applications to economics, see: Tu (1969) and Southwick and Zionts (1974) for optimal educational investments, Kamien and Schwartz (1971b) for limit pricing and uncertain entry, Treadway (1970) for adjustment costs in the theory of competitive firms, Vousden (1974) for international trade, Harris (1976) for money demand with transaction costs, Raviv (1979) for the design of an optimal insurance policy, Sethi and McGuire (1977) for optimal training and heterogeneous labor, Arthur and McNicoll (1977) for population policy, Brito and Oakland (1977) for optimal income tax, Thompson (1982a,b) for continuous expanding economies, Thépot (1983) for investment and marketing policies in a duopoly, Verheyen (1985) for a theory of firm under government regulations, Hartl and Mehlmann (1986) for renumeration patterns for medical services, Schijndel (1986) for dynamic shareholder behavior under personal taxation, Hartl and Kort (1997) for optimal input substitution in response to environmental constraints, Feichtinger et al. (1998), Behrens et al. (2000, 2002), Tragler et al. (2001), Grass et al. (2008), and Seidl et al. (2016) for optimal control of crime such as illicit drugs and terrorism.

For control theory applications to management science and operations research, see: Nelson (1960) for labor assignments, Fan and Wang (1964), Charnes and Kortanek (1966), Tapiero and Soliman (1972) and Bookbinder and Sethi (1980) for distribution and transportation applications, Nepomiastchy (1970) and Zimin and Ivanilov (1971) for scheduling and network planning problems, Lucas (1971) for research and development, Legey et al. (1973) for city congestion problems, Taylor (1974) for warfare models, Mehra (1975) for national settlement planning, Kalish (1983) for pricing with dynamic demand and production costs, Kalish and Lilien (1983) for optimal price subsidy for accelerating diffusion of innovation, Gaimon (1986c) for optimal acquisition of new technology, Dockner and Jørgensen (1988) and Jedidi et al. (1989) for optimal pricing and/or advertising for monopolistic diffusion models, Hartl and Jørgensen (1985) for manpower planning, Ringbeck (1985) for optimal quality and advertising under asymmetric information, Hartl and Krauth (1989) for optimal production mix, Gaimon (1997) for planning for information technology, Hartl and Kort (2005) for advertising directed to existing and new customers, and Shani et al. (2005) for dynamic irrigation policies.

Finally, we conclude this section by citing a series of rather unusual but humorous applications of optimal control theory that began with the Sethi (1979b) paper on optimal pilfering policies for dynamic continuous thieves. These are: Hartl and Mehlmann (1982, 1983) and Hartl et al. (1992a) on optimal blood consumption by vampires, Hartl and Mehlmann (1986) on renumeration patterns for medical services, Hartl and Jørgensen (1988, 1990) on optimal slidemanship at conferences, Jørgensen (1992) on the dynamics of extramarital affairs, and Feichtinger et al. (1999) on Petrarch's Canzoniere: rational addiction and amorous cycles. See also the monograph by Mehlmann (1997) on unusual and humorous applications of differential games.

Exercises for Chapter 11

E 11.1 For the model formulated in Sect. 11.1.1, assume $F(K) = \gamma K$ and $U(C) = (C - \bar{C})^{1-\theta}/(1-\theta)$, where $0 < \theta < 1$, $\bar{C} > 0$ a constant, and $\gamma - \delta > 0$ a constant satisfying $(\gamma - \delta)(1-\theta) < \rho < \gamma - \delta$. Let $\beta = \gamma - \delta$ and assume $\theta = 1/2$ for simplicity. Also assume that $K_0 e^{\beta T} + \bar{C}(1 - e^{\beta T})/\beta > K_T$ for the problem to be well-posed (note that the left-hand side of this inequality is the amount of capital at T associated with the consumption rate $\bar{C}$). Solve this problem to obtain explicit expressions for the optimal consumption rate and the associated capital and the adjoint trajectories.

E 11.2 Perform the following:

(a) Obtain the value of $\bar{k}$ in Fig. 11.1 from Eq. (11.17).

(b) Show that the graph of $\dot{k} = 0$ starts from $+\infty$ when $k = 0$, decreases to a minimum of $\hat{\lambda}$ at $\hat{k}$, and then increases. Also obtain the expression for $\bar{\lambda}$.

(c) Show that $\bar{k} < \hat{k}$.

E 11.3 Use (11.14) to show that $h'(\lambda) < 0$. Then, conclude that the directions of the horizontal arrows above and below the $\dot{k} = 0$ curve are as drawn in Fig. 11.1.

E 11.4 Show that for any $k_0 > \bar{k}$, there exists a unique optimal path, such as that shown by the solid curve in Region III of Fig. 11.1.

E 11.5 In the formulation of the objective function for the economic growth model in Sect. 11.1.3, we took the position of total utilitarianism. Reformulate and solve the problem if our task is to maximize the present value of the utility of per capita consumption over time.

E 11.6 Use the phase diagram method to solve the advertising model of (7.7) with its objective function replaced by

$$\max_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t}[\pi(G) - c(u)]dt \right\},$$

where $c(u)$ represents an increasing convex advertising cost function with $c(u) \geq 0$, $c'(u) \geq 0$, and $c''(u) > 0$ for $u \geq 0$. This is the model of Gould (1970).

E 11.7 A variation of the optimal capital accumulation model with stationary population, known as Ramsey's model, is:

$$\max \left\{ J = \int_0^\infty [u(c) - B]dt \right\}$$

subject to

$$\dot{k} = f(k) - c - \gamma k, \quad k(0) = k_0,$$

where

$$B = \sup_{c \geq 0} u(c) > 0$$

is the so-called *Bliss point*,

$$\lim_{t \to \infty} u[c(t)] = B$$

so that the integral in the objective function converges, and $\lim_{t \to \infty} u'[c(t)] = 0$; see Ramsey (1928).

(a) Show that the optimal capital stock trajectory satisfies the differential equation

$$u'(f(k) - \gamma k - \dot{k})\dot{k} = B - u(f(k) - \gamma k - \dot{k}).$$

(b) From part (a), derive *Ramsey's rule*

$$\frac{d[u'(c(t))]}{dt} = u'(c(t))[\gamma - f'(k(t))].$$

(c) Assume $u(c) = 2c - c^2/B$ and $f(k) = \alpha k$, where $\alpha - \gamma := \beta > 0$ and $\beta < B/k_0 < 2\beta$. Show that the optimal feedback consumption rule is

$$c^*(k) = 2\beta k - B$$

and the optimal capital trajectory k^* is given by

$$k^*(t) = \frac{1}{\beta}[B - (B - \beta k_0)e^{-\beta t}].$$

E 11.8 Show that the trajectory $x_0 BLx_T$ shown in Fig. 11.2 is optimal for the epidemic model under the stated assumptions. Assume $0 < x^s < N$.

E 11.9 In (11.29), show by using Green's theorem that $x^s = N$ if $\rho/\theta < 0$.

E 11.10 Show that $C(v)$ defined in Sect. 11.3.1 satisfies $C''(v) < 0$ if $f_{12} \geq 0$.

Hint: Note that the usual concavity and monotonicity conditions on the production function f are $f_1 > 0,\ f_2 > 0,\ f_{11} < 0$ and $f_{22} < 0$.

E 11.11 Show that the P^c of Fig. 11.5 increases as δ in Eq. (11.33) increases.

E 11.12 Show that (11.56) and (11.58) imply the (global) IC condition (11.52).

Hint: The proof is by contradiction. First, begin by supposing that (11.52) is violated for some $t > \hat{t}$. Then do the same with $t < \hat{t}$.

E 11.13 In problem (3.7), the terminal equality constraint $b(x(T), T) = 0$ results in the term $\beta b_x(x(T), T)$ in the terminal condition (3.11) on the adjoint variable. In problem (11.59)–(11.63), we have the equality constraint (11.62) on the initial states $q(t_1)$ and $\phi(t_1)$ instead, which we can write as $b((q(t_1), \phi(t_1)), t_1) = t_1 a(q(t_1)) - \phi(t_1) = 0$. Now apply (3.11) in a symmetric fashion to obtain the initial conditions (11.67) and (11.68) on the adjoint variables.

Chapter 12

Stochastic Optimal Control

In previous chapters we assumed that the state variables of the system are known with certainty. When the variables are outcomes of a random phenomenon, the state of the system is modeled as a stochastic process. Specifically, we now face a *stochastic optimal control problem* where the state of the system is represented by a controlled stochastic process. We shall only consider the case when the state equation is perturbed by a Wiener process, which gives rise to the state as a Markov diffusion process. In Appendix D.2 we have defined the Wiener process, also known as Brownian motion. In Sect. 12.1, we will formulate a stochastic optimal control problem governed by stochastic differential equations involving a Wiener process, known as Itô equations. Our goal will be to synthesize optimal feedback controls for systems subject to Itô equations in a way that maximizes the expected value of a given objective function.

In this chapter, we also assume that the state is (fully) observed. On the other hand, when the system is subject to noisy measurements, we face partially observed optimal control problems. In some important special cases, it is possible to separate the problem into two problems: optimal estimation and optimal control. We discuss one such case in Appendix D.4.1. In general, these problems are very difficult and are beyond the scope of this book. Interested readers can consult some references listed in Sect. 12.5.

In Sect. 12.2, we will extend the production planning model of Chap. 6 to allow for some uncertain disturbances. We will obtain an optimal production policy for the stochastic production planning problem thus formulated. In Sect. 12.3, we will solve an optimal stochastic advertising

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problem explicitly. The problem is a modification as well as a stochastic extension of the optimal control problem of the Vidale-Wolfe advertising model treated in Sect. 7.2.4. In Sect. 12.4, we will introduce investment decisions in the consumption model of Example 1.3. We will consider both risk-free and risky investments. Our goal will be to find optimal consumption and investment policies in order to maximize the discounted value of the utility of consumption over time.

In Sect. 12.5, we will conclude the chapter by mentioning other types of stochastic optimal control problems that arise in practice.

12.1 Stochastic Optimal Control

In Appendix D.1 on the Kalman filter, we obtain the optimal state estimation for linear systems with noise and noisy measurements. In Sect. D.4.1, we see that for stochastic linear-quadratic optimal control problems, the separation principle allows us to solve the problem in two steps: to obtain the optimal estimate of the state and to use it in the optimal feedback control formula for deterministic linear-quadratic problems.

In this section we will introduce the possibility of controlling a system governed by Itô stochastic differential equations. In other words, we will introduce control variables into Eq. (D.20). This produces the formulation of a stochastic optimal control problem.

It should be noted that for such problems, the separation principle does not hold in general. Therefore, to simplify the treatment, it is often assumed that the state variables are observable, in the sense that they can be directly measured. Furthermore, most of the literature on these problems uses dynamic programming or the Hamilton-Jacobi-Bellman framework rather than a stochastic maximum principle. In what follows, therefore, we will formulate the stochastic optimal control problem under consideration, and provide a brief, informal development of the Hamilton-Jacobi-Bellman equation for the problem. A detailed analysis of the problem is available in Fleming and Rishel (1975). For problems involving jump disturbances, see Davis (1993) for the methodology of optimal control of piecewise deterministic processes. For stochastic optimal control in discrete time, see Bertsekas and Shreve (1996).

Let us consider the problem of maximizing

$$E\left[\int_0^T F(X_t, U_t, t)dt + S(X_T, T)\right], \tag{12.1}$$

where X_t is the state at time t and U_t is the control at time t, and together they are required to satisfy the Itô stochastic differential equation

$$dX_t = f(X_t, U_t, t)dt + G(X_t, U_t, t)dZ_t, \;\; X_0 = x_0, \tag{12.2}$$

where $Z_t, \; t \in [0, T]$ is a standard Wiener process.

For convenience in exposition we assume the drift coefficient function $F : E^1 \times E^1 \times E^1 \to E^1$, $S : \; E^1 \times E^1 \to E^1$, $f : \; E^1 \times E^1 \times E^1 \to E^1$ and the diffusion coefficient function $G : E^1 \times E^1 \times E^1 \to E^1$, so that (12.2) is a scalar equation. We also assume that the functions F and S are continuous in their arguments and the functions f and G are continuously differentiable in their arguments. For multidimensional extensions of this problem, see Fleming and Rishel (1975).

Since (12.2) is a scalar equation, the subscript t here represents only time t. Thus, writing X_t, U_t, and Z_t in place of writing $X(t), U(t)$, and $Z(t)$, respectively, will not cause any confusion and, at the same time, will eliminate the need for writing many parentheses.

To solve the problem defined by (12.1) and (12.2), let $V(x, t)$, known as the *value function*, be the expected value of the objective function (12.1) from t to T, when an optimal policy is followed from t to T, given $X_t = x$. Then, by the principle of optimality,

$$V(x, t) = \max_U E[F(x, U, t)dt + V(x + dX_t, t + dt)]. \tag{12.3}$$

By Taylor's expansion, we have

$$\begin{aligned} V(x + dX_t, t + dt) = V(x, t) \quad & + V_t dt + V_x dX_t + \tfrac{1}{2} V_{xx}(dX_t)^2 \\ & + \tfrac{1}{2} V_{tt}(dt)^2 + \tfrac{1}{2} V_{xt} dX_t dt \\ & + \text{higher-order terms.} \end{aligned} \tag{12.4}$$

From (12.2), we can formally write

$$(dX_t)^2 = f^2(dt)^2 + G^2(dZ_t)^2 + 2fGdZ_t dt, \tag{12.5}$$

$$dX_t dt = f(dt)^2 + GdZ_t dt. \tag{12.6}$$

The exact meaning of these expressions comes from the theory of stochastic calculus; see Arnold (1974, Chapter 5), Durrett (1996) or Karatzas and Shreve (1997). For our purposes, it is sufficient to know the multiplication rules of stochastic calculus:

$$(dZ_t)^2 = dt, \; dZ_t dt = 0, \; dt^2 = 0. \tag{12.7}$$

Substitute (12.4) into (12.3) and use (12.5), (12.6), (12.7), and the property that $E[dZ_t] = 0$ to obtain

$$V = \max_U \left[F dt + V + V_t dt + V_x f dt + \frac{1}{2} V_{xx} G^2 dt + o(dt) \right]. \tag{12.8}$$

Note that we have suppressed the arguments of the functions involved in (12.8).

Canceling the term V on both sides of (12.8), dividing the remainder by dt, and letting $dt \to 0$, we obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max_U [F + V_t + V_x f + \frac{1}{2} V_{xx} G^2] \tag{12.9}$$

for the value function $V(t, x)$ with the boundary condition

$$V(x, T) = S(x, T). \tag{12.10}$$

Just as we had introduced a current-value formulation of the maximum principle in Sect. 3.3, let us derive a current-value version of the HJB equation here. For this, in a way similar to (3.29), we write the objective function to be maximized as

$$E \int_0^T [\phi(X_t, U_t) e^{-\rho t} + \psi(X_T) e^{-\rho T}]. \tag{12.11}$$

We can relate this to (12.1) by setting

$$F(X_t, U_t, t) = \phi(X_t, U_t) e^{-\rho t} \text{ and } S(X_T, T) = \psi(X_T) e^{-\rho T}. \tag{12.12}$$

It is important to mention that the explicit dependence on time t in (12.11) is only via the discounting term. If it were not the case, there would be no advantage in formulating the current-value version of the HJB equation.

Rather than develop the current-value HJB equation in a manner of developing (12.9), we will derive it from (12.9) itself. For this we define the current-valued value function

$$\tilde{V}(x, t) = V(x, t) e^{\rho t}. \tag{12.13}$$

Then we have

$$V_t = \tilde{V}_t e^{-\rho t} - \rho \tilde{V} e^{-\rho t},\ V_x = \tilde{V}_x e^{-\rho t} \text{ and } V_{xx} = \tilde{V}_{xx} e^{-\rho t}. \tag{12.14}$$

By using these and (12.12) in (12.9), we obtain

$$0 = \max_U [\phi e^{-\rho t} + \tilde{V} e^{-\rho t} - \rho \tilde{V} e^{-\rho t} + V_x f e^{-\rho t} + \frac{1}{2} V_{xx} G^2 e^{-\rho t}].$$

Multiplying by $e^{\rho t}$ and rearranging terms, we get

$$\rho \tilde{V} = \max_U [\phi + \tilde{V}_t + \tilde{V}_x f + \frac{1}{2} \tilde{V}_{xx} G^2]. \tag{12.15}$$

Moreover, from (12.12), (12.13), and (12.10), we can get the boundary condition

$$\tilde{V}(x, T) = \psi(x). \tag{12.16}$$

Thus, we have obtained (12.15) and (12.16) as the current-value HJB equation.

To obtain its infinite-horizon version, it is generally the case that we remove the explicit dependence on t from the function f and G in (12.2), and also assume that $\psi \equiv 0$. With that, the dynamics (12.2) and the objective function (12.11) change, respectively, to

$$dX_t = f(X_t, U_t)dt + G(X_t, U_t)dZ_t,\ X_0 = x_0, \tag{12.17}$$

$$E \int_0^\infty \phi(X_t, U_t) e^{-\rho t} dt. \tag{12.18}$$

It should then be obvious that $\tilde{V}_t = 0$, and we can obtain the infinite-horizon version of (12.15) as

$$\rho \tilde{V} = \max_U [\phi + \tilde{V}_x f + \frac{1}{2} \tilde{V}_{xx} G^2]. \tag{12.19}$$

As for its boundary condition, (12.16) is replaced by a growth condition that is the same, in general, as the growth of the function $\phi(x, U)$ in x. For example, if $\phi(x, U)$ is quadratic in x, we would look for a value function $\tilde{V}(x)$ to be of quadratic growth. See Beyer et al. (2010), Chapter 3, for a related discussion of a polynomial growth case in the discrete time setting.

If we can find a solution of the HJB equation with the given boundary condition (or an appropriate growth condition in the infinite horizon case), then a result called a *verification theorem* suggests that we can construct an optimal feedback control $U^*(x, t)$ (or $U^*(x)$ in the infinite horizon case) by maximizing the right-hand side of the HJB equation with respect U. For further details and extension when the value function is not smooth enough and thus not a classical solution of the HJB equation, see Fleming and Rishel (1975), Yong and Zhou (1999), and Fleming and Soner (1992).

In the next three sections, we will apply this procedure to solve problems in production, marketing and finance.

12.2 A Stochastic Production Inventory Model

In Sect. 6.1.1, we formulated a deterministic production-inventory model. In this section, we extend a simplified version of that model by including a random process. Let us define the following quantities:

I_t = the inventory level at time t (state variable),

P_t = the production rate at time t (control variable),

S = the constant demand rate at time t; $S > 0$,

T = the length of planning period,

I_0 = the initial inventory level,

B = the salvage value per unit of inventory at time T,

Z_t = the standard Wiener process,

σ = the constant diffusion coefficient.

The inventory process evolves according to the stock-flow equation stated as the Itô stochastic differential equation

$$dI_t = (P_t - S)dt + \sigma dZ_t, \quad I_0 \text{ given}, \tag{12.20}$$

where I_0 denotes the initial inventory level. As mentioned in Appendix Sect. D.2, the process dZ_t can be formally expressed as $w(t)dt$, where $w(t)$ is considered to be a white noise process; see Arnold (1974). It can be interpreted as "sales returns," "inventory spoilage," etc., which are random in nature.

The objective function is:

$$\max E\left\{BI_T - \int_0^T (P_t^2 + I_t^2)dt\right\}. \tag{12.21}$$

It can be interpreted as maximization of the terminal salvage value less the cost of production and inventory assumed to be quadratic. In Exercise 12.1, you will be asked to solve the problem with the objective function given by the expected value of the undiscounted version of the integral in (6.2).

As in Sect. 6.1.1 we do not restrict the production rate to be nonnegative. In other words, we permit disposal (i.e., $P_t < 0$). While this is done for mathematical expedience, we will state conditions under which a disposal is not required. Note further that the inventory level is allowed to be negative, i.e., we permit backlogging of demand.

The solution of the above model due to Sethi and Thompson (1981a) will be carried out via the previous development of the HJB equation satisfied by a certain *value function*.

Let $V(x,t)$ denote the expected value of the objective function from time t to the horizon T with $I_t = x$ and using the optimal policy from t to T. The function $V(x,t)$ is referred to as the value function, and it satisfies the HJB equation

$$0 = \max_P[-(P^2 + x^2) + V_t + V_x(P - S) + \frac{1}{2}\sigma^2 V_{xx}] \tag{12.22}$$

with the boundary condition

$$V(x,T) = Bx. \tag{12.23}$$

Note that these are applications of (12.9) and (12.10) to the production planning problem.

It is now possible to maximize the expression inside the bracket of (12.22) with respect to P by taking its derivative with respect to P and setting it to zero. This procedure yields

$$P^*(x,t) = \frac{V_x(x,t)}{2}. \tag{12.24}$$

Substituting (12.24) into (12.22) yields the equation

$$0 = \frac{V_x^2}{4} - x^2 + V_t - SV_x + \frac{1}{2}\sigma^2 V_{xx}, \tag{12.25}$$

which, after the max operation has been performed, is known as the Hamilton-Jacobi equation. This is a partial differential equation which must be satisfied by the value function $V(x,t)$ with the boundary condition (12.23). The solution of (12.25) is considered in the next section.

Remark 12.1 It is important to remark that if the production rate were restricted to be nonnegative, then, as in Remark 6.1, (12.24) would be changed to

$$P^*(x,t) = \max\left[0, \frac{V_x(x,t)}{2}\right]. \tag{12.26}$$

Substituting (12.26) into (12.23) would give us a partial differential equation which must be solved numerically. We will not consider (12.26) further in this chapter.

12.2.1 Solution for the Production Planning Problem

To solve Eq. (12.25) with the boundary condition (12.23) we let

$$V(x,t) = Q(t)x^2 + R(t)x + M(t). \tag{12.27}$$

Then,

$$V_t = \dot{Q}x^2 + \dot{R}x + \dot{M}, \tag{12.28}$$
$$V_x = 2Qx + R, \tag{12.29}$$
$$V_{xx} = 2Q, \tag{12.30}$$

where $\dot{Y}$ denotes dY/dt. Substituting (12.28)–(12.30) in (12.25) and collecting terms gives

$$x^2[\dot{Q}+Q^2-1]+x[\dot{R}+RQ-2SQ]+\dot{M}+\frac{R^2}{2}-RS+\sigma^2 Q = 0. \tag{12.31}$$

Since (12.31) must hold for any value of x, we must have

$$\dot{Q} = 1-Q^2,\ \ Q(T)=0, \tag{12.32}$$
$$\dot{R} = 2SQ-RQ,\ \ R(T)=B, \tag{12.33}$$
$$\dot{M} = RS-\frac{R^2}{4}-\sigma^2 Q,\ \ M(T)=0, \tag{12.34}$$

where the boundary conditions for the system of simultaneous differential equations (12.32), (12.33), and (12.34) are obtained by comparing (12.27) with the boundary condition $V(x,T) = Bx$ of (12.23).

To solve (12.32), we expand $\dot{Q}/(1-Q^2)$ by partial fractions to obtain

$$\frac{\dot{Q}}{2}\left[\frac{1}{1-Q}+\frac{1}{1+Q}\right] = 1,$$

which can be easily integrated. The answer is

$$Q = \frac{y-1}{y+1}, \tag{12.35}$$

where

$$y = e^{2(t-T)}. \tag{12.36}$$

Since S is assumed to be a constant, we can reduce (12.33) to

$$\dot{R}^0 + R^0 Q = 0, \quad R^0(T) = B - 2S$$

by the change of variable defined by $R^0 = R - 2S$. Clearly the solution is given by

$$\log R^0(T) - \log R^0(t) = -\int_t^T Q(\tau)d\tau,$$

which can be simplified further to obtain

$$R = 2S + \frac{2(B-2S)\sqrt{y}}{y+1}. \tag{12.37}$$

Having obtained solutions for R and Q, we can easily express (12.34) as

$$M(t) = -\int_t^T [R(\tau)S - (R(\tau))^2/4 - \sigma^2 Q(\tau)]d\tau. \tag{12.38}$$

The optimal control is defined by (12.24), and the use of (12.35) and (12.37) yields

$$P^*(x,t) = V_x/2 = Qx + R/2 = S + \frac{(y-1)x + (B-2S)\sqrt{y}}{y+1}. \tag{12.39}$$

This means that the optimal production rate for $t \in [0,T]$

$$P_t^* = P^*(I_t^*, t) = S + \frac{(e^{2(t-T)} - 1)I_t^* + (B-2S)e^{(t-T)}}{e^{2(t-T)} + 1}, \tag{12.40}$$

where $I_t^*,\ t \in [0,T]$, is the inventory level observed at time t when using the optimal production rate $P_t^*,\ t \in [0,T]$, according to (12.40).

Remark 12.2 The optimal production rate in (12.39) equals the demand rate plus a correction term which depends on the level of inventory and the distance from the horizon time T. Since $(y-1) < 0$ for $t < T$, it is clear that for lower values of x, the optimal production rate is likely to be positive. However, if x is very high, the correction term will become smaller than $-S$, and the optimal control will be negative. In other words, if inventory level is too high, the factory can save money by disposing a part of the inventory resulting in lower holding costs.

Remark 12.3 If the demand rate S were time-dependent, it would have changed the solution of (12.33). Having computed this new solution in place of (12.37), we can once again obtain the optimal control as $P^*(x,t) = Qx + R/2$.

Remark 12.4 Note that when $T \to \infty$, we have $y \to 0$ and

$$P^*(x,t) \to S - x, \tag{12.41}$$

but the undiscounted objective function value (12.21) in this case becomes $-\infty$. Clearly, any other policy will render the objective function value to be $-\infty$. In a sense, the optimal control problem becomes ill-posed. One way to get out of this difficulty is to impose a nonzero discount rate. You are asked to carry this out in Exercise 12.2.

Remark 12.5 It would help our intuition if we could draw a picture of the path of the inventory level over time. Since the inventory level is a stochastic process, we can only draw a typical sample path. Such a sample path is shown in Fig. 12.1. If the horizon time T is long enough, the optimal control will bring the inventory level to the goal level $\bar{x} = 0$. It will then hover around this level until t is sufficiently close to the horizon T. During the ending phase, the optimal control will try to build up the inventory level in response to a positive valuation B for ending inventory.

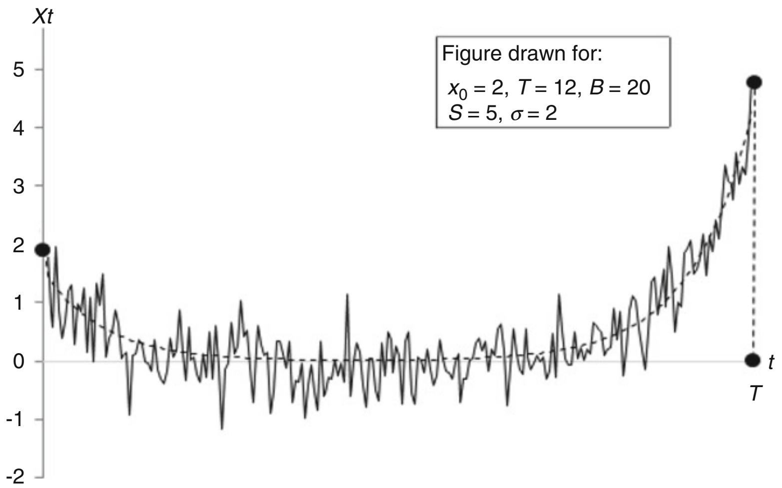

Figure 12.1: A sample path of optimal production rate I_t^* with $I_0 = x_0 > 0$ and $B > 0$

12.3 The Sethi Advertising Model

In this section, we will discuss a stochastic advertising model due to Sethi (1983b). The model is:

$$\begin{cases} \max\ E\left[\int_0^\infty e^{-\rho t}(\pi X_t - U_t^2)dt\right] \\ \text{subject to} \\ dX_t = (rU_t\sqrt{1-X_t} - \delta X_t)dt + \sigma(X_t)dZ_t,\ X_0 = x_0, \\ U_t \geq 0, \end{cases} \tag{12.42}$$

where X_t is the market share and U_t is the rate of advertising at time t, and where, as specified in Sect. 7.2.1, $\rho > 0$ is the discount rate, $\pi > 0$ is the profit margin on sales, $r > 0$ is the advertising effectiveness parameter, and $\delta > 0$ is the sales decay parameter. Furthermore, Z_t is the standard one-dimensional Wiener process and $\sigma(x)$ is the diffusion coefficient function having some properties to be specified shortly. The term in the integrand represents the discounted profit rate at time t. Thus, the integral represents the total value of the discounted profit stream on a sample path. The objective in (12.42) is, therefore, to maximize the expected value of the total discounted profit.

The Sethi model is a modification as well as a stochastic extension of the optimal control formulation of the Vidale-Wolfe advertising model presented in (7.43). The Itô equation in (12.42) modifies the Vidale-Wolfe dynamics (7.25) by replacing the term $rU(1-x)$ by $rU_t\sqrt{1-X_t}$ and adding a diffusion term $\sigma(X_t)dZ_t$ on the right-hand side. Furthermore, the linear cost of advertising U in (7.43) is replaced by a quadratic cost of advertising U_t^2 in (12.42). The control constraint $0 \leq U \leq Q$ in (7.43) is replaced by simply $U_t \geq 0$. The addition of the diffusion term yields a stochastic optimal control problem as expressed in (12.42).

An important consideration in choosing the function $\sigma(x)$ should be that the solution X_t to the Itô equation in (12.42) remains inside the interval $[0, 1]$. Merely requiring that the initial condition $x_0 \in [0, 1]$, as in Sect. 7.2.1, is no longer sufficient in the stochastic case. Additional conditions need to be imposed. It is possible to specify these conditions by using the theory presented by Gihman and Skorohod (1972) for stochastic differential equations on a finite spatial interval. In our case, the conditions boil down to the following, in addition to $x_0 \in (0, 1)$, which has been assumed already in (12.42):

$$\sigma(x) > 0,\ x \in (0,1) \text{ and } \sigma(0) = \sigma(1) = 0. \tag{12.43}$$

It is possible to show that for any feedback control $U(x)$ satisfying

$$U(x) \geq 0,\ x \in (0,1],\ \text{and } U(0) > 0, \tag{12.44}$$

the Itô equation in (12.42) will have a solution X_t such that $0 < X_t < 1$, *almost surely* (i.e., with probability 1). Since our solution for the optimal advertising $U^*(x)$ would turn out to satisfy (12.44), we will have the optimal market share X_t^* lie in the interval $(0,1)$.

Let $V(x)$ denote the value function for the problem, i.e., $V(x)$ is the expected value of the discounted profits from time t to infinity, when $X_t = x$ and an optimal policy U_t^* is followed from time t onwards. Note that since $T = \infty$, the future looks the same from any time t, and therefore the value function does not depend on t. It is for this reason that we have defined the value function as $V(x)$, rather than $V(x,t)$ as in the previous section.

Using now the principle of optimality as in Sect. 12.1, we can write the HJB equation as

$$\rho V(x) = \max_U \left[\pi x - U^2 + V_x(rU\sqrt{1-x} - \delta x) + V_{xx}(\sigma(x))^2/2\right]. \tag{12.45}$$

Maximization of the RHS of (12.45) can be accomplished by taking its derivative with respect to U and setting it to zero. This gives

$$U^*(x) = \frac{rV_x\sqrt{1-x}}{2}. \tag{12.46}$$

Substituting of (12.46) in (12.45) and simplifying the resulting expression yields the HJB equation

$$\rho V(x) = \pi x + \frac{V_x^2 r^2(1-x)}{4} - V_x \delta x + \frac{1}{2}\sigma^2(x)V_{xx}. \tag{12.47}$$

As shown in Sethi (1983b), a solution of (12.47) is

$$V(x) = \bar{\lambda}x + \frac{\bar{\lambda}^2 r^2}{4\rho}, \tag{12.48}$$

where

$$\bar{\lambda} = \frac{\sqrt{(\rho+\delta)^2 + r^2\pi} - (\rho+\delta)}{r^2/2}, \tag{12.49}$$

as derived in Exercise 7.37. In Exercise 12.3, you are asked verify that (12.48) and (12.49) solve the HJB equation (12.47).

We can now obtain the explicit formula for the optimal feedback control as

$$U^*(x) = \frac{r\bar{\lambda}\sqrt{1-x}}{2}. \tag{12.50}$$

Note that $U^*(x)$ satisfies the conditions in (12.44).

As in Exercise 7.37, it is easy to characterize (12.50) as

$$U_t^* = U^*(X_t) = \begin{cases} > \bar{U} & \text{if } X_t < \bar{X}, \\ = \bar{U} & \text{if } X_t = \bar{X}, \\ < \bar{U} & \text{if } X_t > \bar{X}, \end{cases} \tag{12.51}$$

where

$$\bar{X} = \frac{r^2\bar{\lambda}/2}{r^2\bar{\lambda}/2 + \delta} \tag{12.52}$$

and

$$\bar{U} = \frac{r\bar{\lambda}\sqrt{1-\bar{x}}}{2}, \tag{12.53}$$

as given in (7.51).

The market share trajectory for X_t is no longer monotone because of the random variations caused by the diffusion term $\sigma(X_t)dZ_t$ in the Itô equation in (12.42). Eventually, however, the market share process hovers around the equilibrium level $\bar{x}$. It is, in this sense and as in the previous section, also a turnpike result in a stochastic environment.

12.4 An Optimal Consumption-Investment Problem

In Example 1.3 in Chap. 1, we had formulated a problem faced by Rich Rentier who wants to consume his wealth in a way that will maximize his total utility of consumption and bequest. In that example, Rich Rentier kept his money in a savings plan earning interest at a fixed rate of $r > 0$.

In this section, we will offer Rich the possibility of investing a part of his wealth in a risky security or stock that earns an expected rate of return that equals $\alpha > r$. Rich, now known as Rich Investor, must optimally allocate his wealth between the risk-free savings account and the risky stock over time and consume over time so as to maximize his total utility of consumption. We will assume an infinite horizon problem in lieu of the bequest, for convenience in exposition. One could, however, argue that Rich's bequest would be optimally invested and consumed by his heir, who in turn would leave a bequest that would be optimally invested and consumed by a succeeding heir and so on. Thus, if Rich considers the utility accrued to all his heirs as his own, then he can justify solving an infinite horizon problem without a bequest.

In order to formulate the stochastic optimal control problem of Rich Investor, we must first model his investments. The savings account is easy to model. If S_0 is the initial deposit in the savings account earning an interest at the rate $r > 0$, then we can write the accumulated amount S_t at time t as

$$S_t = S_0 e^{rt}.$$

This can be expressed as a differential equation, $dS_t/dt = rS_t$, which we will rewrite as

$$dS_t = rS_t dt,\ S_0 \geq 0. \tag{12.54}$$

Modeling the stock is much more complicated. Merton (1971) and Black and Scholes (1973) have proposed that the stock price P_t can be modeled by an Itô equation, namely,

$$\frac{dP_t}{P_t} = \alpha dt + \sigma dZ_t,\ P_0 > 0, \tag{12.55}$$

or simply,

$$dP_t = \alpha P_t dt + \sigma P_t dZ_t,\ P_0 > 0, \tag{12.56}$$

where $P_0 > 0$ is the given initial stock price, α is the average rate of return on stock, σ is the standard deviation associated with the return, and Z_t is a standard Wiener process.

Remark 12.6 The LHS in (12.55) can be written also as $d\ln P_t$. Another name for the process Z_t is *Brownian Motion*. Because of these, the price process P_t given by (12.55) is often referred to as a *logarithmic Brownian Motion*. It is important to note from (12.56) that P_t remains nonnegative at any $t > 0$ on account of the fact that the price process has almost surely continuous sample paths (see Sect. D.2). This property nicely captures the limited liability that is incurred in owning a share of stock.

In order to complete the formulation of Rich's stochastic optimal control problem, we need the following additional notation:

W_t = the wealth at time t,

$$
\begin{aligned}
C_t &= \text{the consumption rate at time } t, \\
Q_t &= \text{the fraction of the wealth invested in stock at time } t, \\
1 - Q_t &= \text{the fraction of the wealth kept in the savings account at time } t, \\
U(C) &= \text{the utility of consumption when consumption is at the rate } C\text{; the function } U(C) \text{ is assumed to be increasing and concave}, \\
\rho &= \text{the rate of discount applied to consumption utility}, \\
B &= \text{the bankruptcy parameter, to be explained later.}
\end{aligned}
$$

Next we develop the dynamics of the wealth process. Since the investment decision Q is unconstrained, it means Rich is allowed to buy stock as well as to sell it short. Moreover, Rich can deposit in, as well as borrow money from, the savings account at the rate r.

While it is possible to rigorously obtain the equation for the wealth process involving an intermediate variable, namely, the number N_t of shares of stock owned at time t, we will not do so. Instead, we will write the wealth equation informally as

$$
\begin{aligned}
dW_t &= Q_t W_t \alpha dt + Q_t W_t \sigma dZ_t + (1 - Q_t) W_t r dt - C_t dt \\
&= (\alpha - r) Q_t W_t dt + (r W_t - C_t) dt + \sigma Q_t W_t dZ_t, \; W_0 \text{ given},
\end{aligned}
\tag{12.57}
$$

and provide an intuitive explanation for it. The term $Q_t W_t \alpha dt$ represents the expected return from the risky investment of $Q_t W_t$ dollars during the period from t to $t+dt$. The term $Q_t W_t \sigma dZ_t$ represents the risk involved in investing $Q_t W_t$ dollars in stock. The term $(1 - Q_t) W_t r dt$ is the amount of interest earned on the balance of $(1 - Q_t) W_t$ dollars in the savings account. Finally, $C_t dt$ represents the amount of consumption during the interval from t to $t + dt$.

In deriving (12.57), we have assumed that Rich can trade continuously in time without incurring any broker's commission. Thus, the change in wealth dW_t from time t to time $t+dt$ is due to consumption as well as the change in share price. For a rigorous development of (12.57) from (12.54) and (12.55), see Harrison and Pliska (1981).

Since Rich can borrow an unlimited amount and invest it in stock, his wealth could fall to zero at some time T. We will say that Rich goes bankrupt at time T, when his wealth falls zero at that time. It is clear that T is a random variable defined as

$$
T = \inf\{t \geq 0 | W_t = 0\}. \tag{12.58}
$$

This special type of random variable is called a *stopping time*, since it is observed exactly at the instant of time when wealth falls to zero.

We can now specify Rich's objective function. It is:

$$\max \left\{ J = E\left[\int_0^T e^{-\rho t} U(C_t)dt + e^{-\rho T}B\right]\right\}, \tag{12.59}$$

where we have assumed that Rich experiences a payoff of B, in the units of utility, at the time of bankruptcy. B can be positive if there is a social welfare system in place, or B can be negative if there is remorse associated with bankruptcy. See Sethi (1997a) for a detailed discussion of the bankruptcy parameter B.

Let us recapitulate the optimal control problem of Rich Investor:

$$\begin{cases} \max \left\{ J = E\left[\int_0^T e^{-\rho t} U(C_t)dt + e^{-\rho T}B\right]\right\} \\ \text{subject to} \\ dW_t = (\alpha - r)Q_t W_t dt + (rW_t - C_t)dt + \sigma Q_t W_t dZ_t,\ W_0 \text{ given}, \\ C_t \geq 0. \end{cases} \tag{12.60}$$

As in the infinite horizon problem of Sect. 12.2, here also the value function is stationary with respect to time t. This is because T is a stopping time of bankruptcy, and the future evolution of wealth, investment, and consumption processes from any starting time t depends only on the wealth at time t and *not* on time t itself. Therefore, let $V(x)$ be the value function associated with an optimal policy beginning with wealth $W_t = x$ at time t. Using the principle of optimality as in Sect. 12.1, the HJB equation satisfied by the value function $V(x)$ for problem (12.60) can be written as

$$\begin{cases} \rho V(x) = \max\limits_{C \geq 0, Q} \quad [(\alpha - r)QxV_x + (rx - C)V_x \\ \qquad\qquad\qquad\qquad +(1/2)Q^2\sigma^2 x^2 V_{xx} + U(C)], \\ V(0) = B. \end{cases} \tag{12.61}$$

This problem and a number of its generalizations are solved explicitly in Sethi (1997a). Here we shall confine ourselves in solving a simpler problem resulting from the following considerations.

It is shown in Karatzas et al. (1986), reproduced as Chapter 2 in Sethi (1997a), that when $B \leq U(0)/\rho$, no bankruptcy will occur. This should be intuitively obvious because if Rich goes bankrupt at any time $T > 0$, he receives B at that time, whereas by not going bankrupt at that time he reaps the utility of strictly more than $U(0)/\rho$ on account of consumption from time T onward. It is shown furthermore that if $U'(0) = \infty$, then the optimal consumption rate will be strictly positive. This is because even an infinitesimally small positive consumption rate results in a proportionally large amount of utility on account of the infinite marginal utility at zero consumption level. A popular utility function used in the literature is

$$U(C) = \ln C, \tag{12.62}$$

which was also used in Example 1.3. This function gives an infinite marginal utility at zero consumption, i.e.,

$$U'(0) = 1/C|_{C=0} = \infty. \tag{12.63}$$

We also assume $B = U(0)/\rho = -\infty$. These assumptions imply a strictly positive consumption level at all times and no bankruptcy.

Since Q is already unconstrained, having no bankruptcy and only positive (i.e., interior) consumption level allows us to obtain the form of the optimal consumption and investment policy simply by differentiating the RHS of (12.61) with respect to Q and C and equating the resulting expressions to zero. Thus,

$$(\alpha - r)xV_x + Q\sigma^2 x^2 V_{xx} = 0,$$

i.e.,

$$Q^*(x) = -\frac{(\alpha - r)V_x}{x\sigma^2 V_{xx}}, \tag{12.64}$$

and

$$C^*(x) = \frac{1}{V_x}. \tag{12.65}$$

Substituting (12.64) and (12.65) in (12.61) allows us to remove the max operator from (12.61), and provides us with the equation

$$\rho V(x) = -\frac{\gamma (V_x)^2}{V_{xx}} + \left(rx - \frac{1}{V_x}\right) V_x - \ln V_x, \tag{12.66}$$

where

$$\gamma = \frac{(\alpha - r)^2}{2\sigma^2}. \tag{12.67}$$

This is a nonlinear ordinary differential equation that appears to be quite difficult to solve. However, Karatzas et al. (1986) used a change of variable that transforms (12.66) into a second-order, linear, ordinary differential equation, which has a known solution. For our purposes, we will simply guess that the value function is of the form

$$V(x) = A \ln x + B, \tag{12.68}$$

where A and B are constants, and obtain the values of A and B by substitution in (12.66). Using (12.68) in (12.66), we see that

$$\begin{aligned}\rho A \ln x + \rho B &= \gamma A + \left(rx - \frac{x}{A}\right)\frac{A}{x} - \ln\left(\frac{A}{x}\right) \\ &= \gamma A + rA - 1 - \ln A + \ln x.\end{aligned}$$

By comparing the coefficients of $\ln x$ and the constants on both sides, we get $A = 1/\rho$ and $B = (r - \rho + \gamma)/\rho^2 + \ln\rho/\rho$. By substituting these values in (12.68), we obtain

$$V(x) = \frac{1}{\rho}\ln(\rho x) + \frac{r - \rho + \gamma}{\rho^2},\ x \geq 0. \tag{12.69}$$

In Exercise 12.4, you are asked by a direct substitution in (12.66) to verify that (12.69) is indeed a solution of (12.66). Moreover, $V(x)$ defined in (12.69) is strictly concave, so that our concavity assumption made earlier is justified.

From (12.69), it is easy to show that (12.64) and (12.65) yield the following feedback policies:

$$Q^*(x) = \frac{\alpha - r}{\sigma^2}, \tag{12.70}$$

$$C^*(x) = \rho x. \tag{12.71}$$

The investment policy (12.70) says that the optimal fraction of the wealth invested in the risky stock is $(\alpha - r)/\sigma^2$, i.e.,

$$Q_t^* = Q^*(W_t) = \frac{\alpha - r}{\sigma^2},\ t \geq 0, \tag{12.72}$$

which is a constant over time. The optimal consumption policy is to consume a constant fraction ρ of the current wealth, i.e.,

$$C_t^* = C^*(W_t) = \rho W_t,\ t \geq 0. \tag{12.73}$$

This problem and its many extensions have been studied in great detail. See, e.g., Sethi (1997a).

12.5 Concluding Remarks

In this chapter, we have considered stochastic optimal control problems subject to Itô differential equations. For impulse stochastic control, see Bensoussan and Lions (1984). For stochastic control problems with jump Markov processes or, more generally, martingale problems, see Fleming and Soner (1992), Davis (1993), and Karatzas and Shreve (1998). For problems with incomplete information or partial observation, see Bensoussan (2004, 2018), Elliott et al. (1995), and Bensoussan et al. (2010).

For applications of stochastic optimal control to manufacturing problems, see Sethi and Zhang (1994a), Yin and Zhang (1997), Sethi et al. (2005), Bensoussan (2011), and Bensoussan et al. (2007b,c,d, 2008a,b, 2009a,b,c). For applications to problems in finance, see Sethi (1997a), Karatzas and Shreve (1998), and Bensoussan et al. (2009d). For applications in marketing, see Tapiero (1988), Raman (1990), and Sethi and Zhang (1995b). For applications of stochastic optimal control to economics including economics of natural resources, see, e.g., Pindyck (1978a,b), Rausser and Hochman (1979), Arrow and Chang (1980), Derzko and Sethi (1981a), Bensoussan and Lesourne (1980, 1981), Malliaris and Brock (1982), and Brekke and Øksendal (1994).

Exercises for Chapter 12

E 12.1 Solve the production-inventory problem with the state equation (12.20) and the objective function

$$\min\left\{J = E\int_0^T [\frac{h}{2}(I-\hat{I})^2 + \frac{c}{2}(P-\hat{P})^2]dt\right\},$$

where $h > 0, c > 0, \hat{I} \geq 0$ and $\hat{P} \geq 0$; see the objective function (6.2) for the interpretation of these parameters.

E 12.2 Formulate and solve the discounted infinite-horizon version of the stochastic production planning model of Sect. 12.2. Specifically, assume $B = 0$ and replace the objective function in (12.21) by

$$\max E\left\{\int_0^\infty -e^{-\rho t}(P_t^2 + I_t^2)dt\right\}.$$

E 12.3 Verify by direct substitution that the value function defined by (12.48) and (12.49) solves the HJB equation (12.47).

E 12.4 Verify by direct substitution that the value function in (12.69) solves the HJB equation (12.66).

E 12.5 Solve the consumption-investment problem (12.60) with the utility function $U(C) = C^\beta$, $0 < \beta < 1$, and $B = 0$.

E 12.6 Solve Exercise 12.5 when $U(C) = -C^\beta$ with $\beta < 0$ and $B = -\infty$.

E 12.7 Solve the optimal consumption-investment problem:

$$V(x) = \max \left\{ J = E \left[\int_0^\infty e^{-\rho t} \ln(C_t - s) dt \right] \right\}$$

subject to

$$\begin{aligned} dW_t &= (\alpha - r)Q_t W_t dt + (rW_t - C_t)dt + \sigma Q_t W_t dZ_t, \; W_0 = x, \\ C_t &\geq s. \end{aligned}$$

Here $s > 0$ denotes a minimal subsistence consumption, and we assume $0 < \rho < 1$. Note that the value function $V(s/r) = -\infty$. Guess a solution of the form

$$V(x) = A \ln(x - s/r) + B.$$

Find the constants A, B, and the optimal feedback consumption and investment allocation policies $C^*(x)$ and $Q^*(x)$, respectively. Characterize these policies in words.

E 12.8 Solve the consumption-investment problem:

$$V(x) = \max \left\{ J = E \left[\int_0^\infty e^{-\rho t} (C_t - s)^\beta dt \right] \right\}$$

subject to

$$\begin{aligned} dW_t &= (\alpha - r)Q_t W_t dt + (rW_t - C_t)dt + \sigma Q_t W_t dZ_t, \; W_0 = x, \\ C_t &\geq s. \end{aligned}$$

Here $s > 0$ denotes a minimal subsistence consumption and we assume $0 < \rho < 1$ and $0 < \beta < 1$. Note that the value function $V(s/r) = 0$. Therefore, guess a solution of the form

$$V(x) = A(x - s/r)^\beta.$$

Find the constant A and the optimal feedback consumption and investment allocation policies $C^*(x)$ and $Q^*(x)$, respectively. Characterize these policies in words.

Chapter 13

Differential Games

In previous chapters, we were mainly concerned with the optimal control problems formulated by a single objective function (or a single decision maker). However, there are situations when there may be more than one decision maker, each having one's own objective function that each is trying to maximize, subject to a set of differential equations. This extension of optimal control theory is referred to as the *theory of differential games.*

The study of differential games was initiated by Isaacs (1965). After the development of Pontryagin's maximum principle, it became clear that there was a connection between differential games and optimal control theory. In fact, differential game problems represent a generalization of optimal control problems in cases where there is more than one controller or player. However, differential games are conceptually far more complex than optimal control problems in the sense that it is no longer obvious what constitutes a solution; see Starr and Ho (1969), Ho (1970), Varaiya (1970), Friedman (1971), Leitmann (1974), Case (1979), Selten (1975), Mehlmann (1988), Berkovitz (1994), Basar and Olsder (1999), Dockner et al. (2000), and Basar et al. (2010). Indeed, there are a number of different types of solutions such as minimax, Nash, Stackelberg, along with possibilities of cooperation and bargaining; see, e.g., Tolwinski (1982) and Haurie et al. (1983). We will discuss minimax solutions for zero-sum differential games in Sect. 13.1, Nash solutions for nonzero-sum games in Sect. 13.2, and Stackelberg differential games in Sect. 13.3.

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13.1 Two-Person Zero-Sum Differential Games

Consider the state equation

$$\dot{x} = f(x, u, v, t),\ x(0) = x_0, \tag{13.1}$$

where we may assume all variables to be scalar for the time being. Extension to the vector case simply requires appropriate reinterpretations of each of the variables and the equations. In this equation, we let u and v denote the controls applied by players 1 and 2, respectively. We assume that

$$u(t) \in U,\ v(t) \in V,\ t \in [0, T],$$

where U and V are convex sets in E^1. Consider further the objective function

$$J(u, v) = S[x(T)] + \int_0^T F(x, u, v, t)dt, \tag{13.2}$$

which player 1 wants to maximize and player 2 wants to minimize. Since the gain of player 1 represents a loss to player 2, such games are appropriately termed *zero-sum games*. Clearly, we are looking for admissible control trajectories u^* and v^* such that

$$J(u^*, v) \geq J(u^*, v^*) \geq J(u, v^*). \tag{13.3}$$

The solution (u^*, v^*) is known as the *minimax* solution. Here u^* and v^* stand for $u^*(t),\ t \in [0, T]$, and $v^*(t),\ t \in [0, T]$, respectively.

The necessary conditions for u^* and v^* to satisfy (13.3) are given by an extension of the maximum principle. To obtain these conditions, we form the Hamiltonian

$$H = F + \lambda f \tag{13.4}$$

with the adjoint variable λ satisfying the equation

$$\dot{\lambda} = -H_x,\ \lambda(T) = S_x[x(T)]. \tag{13.5}$$

The necessary condition for trajectories u^* and v^* to be a minimax solution is that for $t \in [0, T]$,

$$H(x^*(t), u^*(t), v^*(t), \lambda(t), t) = \min_{v \in V} \max_{u \in U} H(x^*(t), u, v, \lambda(t), t), \tag{13.6}$$

which can also be stated, with suppression of (t), as

$$H(x^*, u^*, v, \lambda, t) \geq H(x^*, u^*, v^*, \lambda, t) \geq H(x^*, u, v^*, \lambda, t) \tag{13.7}$$

for $u \in U$ and $v \in V$. Note that (u^*, v^*) is a saddle point of the Hamiltonian function H.

Note also that if u and v are unconstrained, i.e., when, $U = V = E^1$, condition (13.6) reduces to the first-order necessary conditions

$$H_u = 0 \text{ and } H_v = 0, \tag{13.8}$$

and the second-order conditions are

$$H_{uu} \leq 0 \text{ and } H_{vv} \geq 0. \tag{13.9}$$

We now turn to the treatment of nonzero-sum differential games.

13.2 Nash Differential Games

In this section, let us assume that we have N players where $N \geq 2$. Let $u^i \in U^i,\ i = 1, 2, \ldots, N$, represent the control variable for the ith player, where U^i is the set of controls from which the ith player can choose. Let the state equation be defined as

$$\dot{x} = f(x, u^1, u^2, \ldots, u^N, t). \tag{13.10}$$

Let J^i, defined by

$$J^i = S^i[x(T)] + \int_0^T F^i(x, u^1, u^2, \ldots, u^N, t)dt, \tag{13.11}$$

denote the objective function which the ith player wants to maximize. In this case, a *Nash solution* is defined by a set of N admissible trajectories

$$\{u^{1*}, u^{2*}, \ldots, u^{N*}\}, \tag{13.12}$$

which have the property that

$$J^i(u^{1*}, u^{2*}, \ldots, u^{N*}) = \max_{u^i \in U^i} J^i(u^{1*}, u^{2*}, \ldots, u^{(i-1)*}, u^i, \ldots, u^{(i+1)*}, \ldots, u^{N*}) \tag{13.13}$$

for $i = 1, 2, \ldots, N$.

To obtain the necessary conditions for a Nash solution for nonzero-sum differential games, we must make a distinction between open-loop and closed-loop controls.

13.2.1 Open-Loop Nash Solution

The open-loop Nash solution is defined when the set of trajectories in (13.12) are given as functions of time satisfying (13.13). To obtain the maximum principle type conditions for such solutions to be a Nash solution, let us define the Hamiltonian functions

$$H^i(x, u^1, u^2, \ldots, u^N, \lambda^i) = F^i + \lambda^i f \tag{13.14}$$

for $i = 1, 2, \ldots, N$, with λ^i satisfying

$$\dot{\lambda}^i = -H^i_x,\ \lambda^i(T) = S^i_x[x(T)]. \tag{13.15}$$

The Nash control u^{i*} for the ith player is obtained by maximizing the ith Hamiltonian H^i with respect to u^i, i.e., u^{i*} must satisfy

$$\begin{aligned} &H^i(x^*, u^{1*}, \ldots, u^{(i-1)*}, u^{i*}, u^{(i+1)*}, \ldots, u^{N*}, \lambda, t) \geq \\ &\qquad H^i(x^*, u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, \lambda, t),\ t \in [0, T], \end{aligned} \tag{13.16}$$

for all $u^i \in U^i,\ i = 1, 2, \ldots, N$.

Deal et al. (1979) formulated and solved an advertising game with two players and obtained the open-loop Nash solution by solving a two-point boundary value problem. In Exercise 13.1, you are asked to obtain their boundary value problem. See also Deal (1979).

13.2.2 Feedback Nash Solution

A feedback Nash solution is obtained when (13.12) is defined in terms of the current state of the system. To avoid confusion, we let

$$u^{i*}(x, t) = \phi^i(x, t),\ i = 1, 2, \ldots, N. \tag{13.17}$$

For these controls to represent a feedback Nash strategy, we must recognize the dependence of the other players' actions on the state variable x. Therefore, we need to replace the adjoint equation (13.15) by

$$\dot{\lambda}^i = -H^i_x - \sum_{j=1}^{N} H^i_{u^j}\phi^j_x = -H^i_x - \sum_{j=1, j\neq i}^{N} H^i_{u^j}\phi^j_x. \tag{13.18}$$

The presence of the summation term in (13.18) makes the necessary condition for the feedback solution virtually useless for deriving computational algorithms; see Starr and Ho (1969). It is, however, possible to use a dynamic programming approach for solving extremely simple nonzero-sum games, which require the solution of a partial differential equation. We will use this approach in Sect. 13.3.

The troublesome summation term in (13.18) is absent in three important cases: (a) in optimal control problems ($N = 1$) since $H_u u_x = 0$, (b) in two-person zero-sum games because $H^1 = -H^2$ so that $H^1_{u^2} u^2_x = -H^2_{u^2} u^2_x = 0$ and $H^2_{u^1} u^1_x = -H^1_{u^1} u^1_x = 0$, and (c) in open-loop nonzero-sum games because $u^j_x = 0$. It certainly is to be expected, therefore, that the feedback and open-loop Nash solutions are going to be different, in general. This can be shown explicitly for the linear-quadratic case.

We conclude this section by providing an interpretation to the adjoint variable λ^i. It is the sensitivity of the ith player's profit to a perturbation in the state vector. If the other players are using closed-loop strategies, any perturbation δx in the state vector causes them to revise their controls by the amount $\phi^j_x \delta x$. If the ith Hamiltonian H^i were maximized with respect to u^j, $j \neq i$, this would not affect the ith player's profit; but since $\partial H^i / \partial u^j \neq 0$ for $i \neq j$, the reactions of the other players to the perturbation influence the ith player's profit, and the ith player must account for this effect in considering variations of the trajectory.

13.2.3 An Application to Common-Property Fishery Resources

Consider extending the fishery model of Sect. 10.1 by assuming that there are two producers having unrestricted rights to exploit the fish stock in competition with each other. This gives rise to a nonzero-sum differential game analyzed by Clark (1976).

Equation (10.2) is modified by

$$\dot{x} = g(x) - q^1 u^1 x - q^2 u^2 x, \; x(0) = x_0, \tag{13.19}$$

where $u^i(t)$ represents the rate of fishing effort and $q^i u^i x$ is the rate of catch for the ith producer, $i = 1, 2$. The control constraints are

$$0 \leq u^i(t) \leq U^i, \; i = 1, 2, \tag{13.20}$$

the state constraints are

$$x(t) \geq 0, \tag{13.21}$$

and the objective function for the ith producer is the total present value of his profits, namely,

$$J^i = \int_0^\infty (p^i q^i x - c^i) u^i e^{-\rho t} dt, \; i = 1, 2. \tag{13.22}$$

To find the feedback Nash solution for this model, we let $\bar{x}^i$ denote the turnpike (or optimal biomass) level given by (10.12) on the assumption that the ith producer is the sole-owner of the fishery. Let the bionomic equilibrium x_b^i and the corresponding control u_b^i associated with producer i be defined by (10.4), i.e.,

$$x_b^i = \frac{c^i}{p^i q^i} \text{ and } u_b^i = \frac{g(x_b^i)p^i}{c^i}. \tag{13.23}$$

As shown in Exercise 10.2, $x_b^i < \bar{x}^i$, and we assume U^i to be sufficiently large so that $u_b^i \leq U^i$. We also assume that

$$x_b^1 < x_b^2, \tag{13.24}$$

which means that producer 1 is more efficient than producer 2, i.e., producer 1 can make a positive profit at any level in the interval $(x_b^1, x_b^2]$, while producer 2 loses money in the same interval, except at x_b^2, where he breaks even. For $x > x_b^2$, both producers make positive profits.

Since $U^1 \geq u_b^1$ by assumption, producer 1 has the capability of driving the fish stock level down to at least x_b^1 which, by (13.24), is less than x_b^2. This implies that producer 2 cannot operate at a sustained level above x_b^2; and at a sustained level below x_b^2, he cannot make a profit. Hence, his optimal feedback policy is bang-bang:

$$u^{2*}(x) = \begin{cases} U^2 & \text{if } x > x_b^2, \\ 0 & \text{if } x \leq x_b^2. \end{cases} \tag{13.25}$$

As far as producer 1 is concerned, he wants to attain his turnpike level $\bar{x}^1$ if $\bar{x}^1 \leq x_b^2$. If $\bar{x}^1 > x_b^2$ and $x_0 \geq \bar{x}^1$, then from (13.25) producer 2 will fish at his maximum rate until the fish stock is driven to x_b^2. At this level, it is optimal for producer 1 to fish at a rate which maintains the fish stock at level x_b^2 in order to keep producer 2 from fishing. Thus, the optimal feedback policy for producer 1 can be stated as

$$u^{1*}(x) = \left\{ \begin{array}{ll} U^1 & \text{if } x > \bar{x}^1 \\ \bar{u}^1 = \frac{g(\bar{x}^1)}{q^1 \bar{x}^1} & \text{if } x = \bar{x}^1 \\ 0 & \text{if } x < \bar{x}^1 \end{array} \right\}, \text{ if } \bar{x}^1 < x_b^2, \tag{13.26}$$

$$u^{1*}(x) = \left\{ \begin{array}{ll} U^1 & \text{if } x > x_b^2 \\ \frac{g(x_b^2)}{q^1 x_b^2} & \text{if } x = x_b^2 \\ 0 & \text{if } x < x_b^2 \end{array} \right\}, \quad \text{if } \bar{x}^1 \geq x_b^2. \tag{13.27}$$

The formal proof that policies (13.25)–(13.27) give a Nash solution requires direct verification using the result of Sect. 10.1.2. The Nash solution for this case means that for all feasible paths u^1 and u^2,

$$J^1(u^{1*}, u^{2*}) \geq J^1(u^1, u^{2*}), \tag{13.28}$$

and

$$J^2(u^{1*}, u^{2*}) \geq J^2(u^{1*}, u^2). \tag{13.29}$$

The direct verification involves defining a modified growth function

$$g^1(x) = \begin{cases} g(x) - q^2 U^2 x & \text{if } x > x_b^2, \\ g(x) & \text{if } x \leq x_b^2, \end{cases}$$

and using the Green's theorem results of Sect. 10.1.2. Since $U^2 \geq u_b^2$ by assumption, we have $g^1(x) \leq 0$ for $x > x_b^2$. From (10.12) with g replaced by g^1, it can be shown that the new turnpike level for producer 1 is $\min(\bar{x}^1, x_b^2)$, which defines the optimal policy (13.26)–(13.27) for producer 1. The optimality of (13.25) for producer 2 follows easily.

To interpret the results of the model, suppose that producer 1 originally has sole possession of the fishery, but anticipates a rival entry. Producer 1 will switch from his own optimal sustained yield $\bar{u}_1$ to a more intensive exploitation policy *prior* to the anticipated entry.

We can now guess the results in situations involving N producers. The fishery will see the progressive elimination of inefficient producers as the stock of fish decreases. Only the most efficient producers will survive. If, ultimately, two or more maximally efficient producers exist, the fishery will converge to a classical bionomic equilibrium, with zero sustained economic rent.

We have now seen that a feedback Nash solution involving $N \geq 2$ competing producers results in the long-run erosion of economic rents. This conclusion depends on the assumption that producers face an infinitely elastic supply of all factors of production going into the fishing effort, but typically the methods of licensing entrants to regulated fisheries make some attempt to also control the factors of production such as permitting the licensee to operate only a single vessel of specific size.

In order to develop a model for the licensing of fishermen, we let the control variable v^i denote the capital stock of the ith producer and let the concave function $f(v^i)$, with $f(0) = 0$, denote the *fishing mortality function* for $i = 1, 2, \ldots, N$. This requires the replacement of $q^i u^i$ in the previous model by $f(v^i)$. The extended model becomes nonlinear in control variables. You are asked in Exercise 13.3 to formulate this new model and develop necessary conditions for a feedback Nash solution for this game involving N producers. The reader is referred to Clark (1976) for further details. For other papers on applications of differential games to fishery management, see Hämäläinen et al. (1984, 1985, 1986, 1990).

13.3 A Feedback Nash Stochastic Differential Game in Advertising

In this section, we will study a competitive extension of the Sethi advertising model discussed in Sect. 12.3. This will give us a stochastic differential game, for which we aim to obtain a feedback Nash equilibrium by using a dynamic programming approach developed in Sect. 12.1. We should note that this approach can also be used to obtain feedback Nash equilibria in deterministic differential games as an alternative to the maximum principle approach developed in Sect. 13.2.2.

Specifically, we consider a duopoly market in a mature product category where total sales are distributed between two firms, labeled as Firm 1 and Firm 2, which compete for market share through advertising expenditures. We let X_t denote the market share of Firm 1 at time t, so that the market share of Firm 2 is $(1 - X_t)$. Let U_{1t} and U_{2t} denote the advertising effort rates of Firms 1 and 2, respectively, at time t. Using the subscript $i \in \{1, 2\}$ to reference the two firms, let $r_i > 0$ denote the advertising effectiveness parameter, $\pi_i > 0$ denote the sales margin, $\rho_i > 0$ denote the discount rate, and $c_i > 0$ denote the cost parameter so that the cost of advertising effort u by Firm i is $c_i u^2$. Further, let $\delta > 0$ be the churn parameter, Z_t be the standard one-dimensional Wiener process, and $\sigma(x)$ be the diffusion coefficient function as defined in Sect. 12.3. Then, in view of the competition between the firms, Prasad and Sethi (2004) extend the Sethi model dynamics in (12.42) as the Itô stochastic differential equation

$$\begin{aligned} dX_t &= [r_1 U_{1t}\sqrt{1-X_t} - \delta X_t - r_2 U_{2t}\sqrt{X_t} + \delta(1-X_t)]dt + \sigma(X_t)dZ_t, \\ & \qquad X(0) = x_0 \in [0,1]. \end{aligned} \tag{13.30}$$

We formulate the optimal control problem faced by the two firms as

$$\max_{U_1 \geq 0} \left\{ V^1(x_0) = E\int_0^\infty e^{-\rho_1 t}\left[\pi_1 X_t - c_1 U_{1t}^2\right] dt \right\}, \tag{13.31}$$

$$\max_{U_2 \geq 0} \left\{ V^2(x_0) = E\int_0^\infty e^{-\rho_2 t}\left[\pi_2 (1-X_t) - c_2 U_{2t}^2\right] dt \right\}, \tag{13.32}$$

subject to (13.30). Thus, each firm seeks to maximize its expected, discounted profit stream subject to the market share dynamics.

To find the feedback Nash equilibrium solution, we form the Hamilton-Jacobi-Bellman (HJB) equations for the value functions $V^1(x)$ and $V^2(x)$:

$$\begin{aligned} \rho_1 V^1 &= \max_{U_1 \geq 0}\{H^1(x, U_1, U_2, V_x^1) + (\sigma(x))^2 V_{xx}^1/2\} \\ &= \max_{U_1 \geq 0}\{\pi_1 x - c_1 U_1^2 + V_x^1[r_1 U_1 \sqrt{1-x} - r_2 U_2 \sqrt{x} - \delta(2x-1)] \\ &\quad + (\sigma(x))^2 V_{xx}^1/2\}, \end{aligned} \tag{13.33}$$

$$\begin{aligned} \rho_2 V^2 &= \max_{U_2 \geq 0}\{H^2(x, U_1, U_2, V_x^2) + (\sigma(x))^2 V_{xx}^2/2\} \\ &= \max_{U_2 \geq 0}\{\pi_2(1-x) - c_2 U_2^2 \\ &\quad + V_x^2[r_1 U_1 \sqrt{1-x} - r_2 U_2 \sqrt{x} - \delta(2x-1)] \\ &\quad + (\sigma(x))^2 V_{xx}^2/2\}, \end{aligned} \tag{13.34}$$

where the Hamiltonians are as defined in (13.14). We use the first-order conditions for Hamiltonian maximization to obtain the optimal feedback advertising decisions

$$U_1^*(x) = V_x^1(x) r_1 \sqrt{1-x}/2c_1 \text{ and } U_2^*(x) = -V_x^2(x) r_2 \sqrt{x}/2c_2. \tag{13.35}$$

Since it is reasonable to expect that $V_x^1 \geq 0$ and $V_x^2 \leq 0$, these controls will turn out to be nonnegative as we will see later.

Substituting (13.35) in (13.33) and (13.34), we obtain the Hamilton-Jacobi equations

$$\begin{aligned}\rho_1 V^1 &= \pi_1 x + (V_x^1)^2 r_1^2 (1-x)/4c_1 + V_x^1 V_x^2 r_2^2 x/2c_2 \\ &\quad -V_x^1 \delta(2x-1) + (\sigma(x))^2 V_{xx}^1/2, \qquad (13.36)\end{aligned}$$

$$\begin{aligned}\rho_2 V^2 &= \pi_2 (1-x) + (V_x^2)^2 r_2^2 x/4c_2 + V_x^1 V_x^2 r_1^2 (1-x)/2c_1 \\ &\quad -V_x^2 \delta(2x-1) + (\sigma(x))^2 V_{xx}^2/2. \qquad (13.37)\end{aligned}$$

As in Sect. 12.3, we look for the following forms for the value functions

$$V^1 = \alpha_1 + \beta_1 x \text{ and } V^2 = \alpha_2 + \beta_2 (1-x). \qquad (13.38)$$

These are inserted into (13.36) and (13.37) to determine the unknown coefficients $\alpha_1, \beta_1, \alpha_2$, and β_2. Equating the coefficients of x and the constants on both sides of (13.36) and the coefficients of $(1-x)$ and the constants on both sides of (13.37), the following four equations emerge, which can be solved for the unknowns $\alpha_1, \beta_1, \alpha_2$, and β_2:

$$\rho_1 \alpha_1 = \beta_1^2 r_1^2/4c_1 + \beta_1 \delta, \qquad (13.39)$$

$$\rho_1 \beta_1 = \pi_1 - \beta_1^2 r_1^2/4c_1 - \beta_1 \beta_2 r_2^2/2c_2 - 2\beta_1 \delta, \qquad (13.40)$$

$$\rho_2 \alpha_2 = \beta_2^2 r_2^2/4c_2 + \beta_2 \delta, \qquad (13.41)$$

$$\rho_2 \beta_2 = \pi_2 - \beta_2^2 r_2^2/4c_2 - \beta_1 \beta_2 r_1^2/2c_1 - 2\beta_2 \delta. \qquad (13.42)$$

Let us first consider the special case of symmetric firms, i.e., when $\pi = \pi_1 = \pi_2$, $c = c_1 = c_2$, $r = r_1 = r_2$, and $\rho = \rho_1 = \rho_2$, and therefore $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$. The four equations in ((13.39)–(13.42)) reduce to the following two:

$$\rho\alpha = \beta^2 r^2/4c + \beta\delta \text{ and } \rho\beta = \pi - 3\beta^2 r^2/4c - 2\beta\delta. \qquad (13.43)$$

There are two solutions for β. One is negative, which clearly makes no sense. Thus, the remaining positive solution is the correct one. This also allows us to obtain the corresponding α. The solution is

$$\alpha = [(\rho-\delta)(W - \sqrt{W^2 + 12R\pi}) + 6R\pi]/18R\rho, \qquad (13.44)$$

$$\beta = (\sqrt{W^2 + 12R\pi} - W)/6R, \qquad (13.45)$$

where $R = r^2/4c$ and $W = \rho + 2\delta$. With this the value functions in (13.38) are defined, and the controls in (13.35) for the case of symmetric firms can be written as

$$u_1^*(x) = \frac{\beta_1 r_1 \sqrt{1-x}}{2c_1} = \frac{\beta r \sqrt{1-x}}{2c} \text{ and } u_2^*(x) = \frac{\beta_2 r_2 \sqrt{x}}{2c_2} = \frac{\beta r \sqrt{x}}{2c},$$

which are clearly nonnegative as required.

We return now to the general case of asymmetric firms. For this, we re-express equations ((13.39)–(13.42)) in terms of a single variable β_1, which is determined by solving the quartic equation

$$\begin{aligned} 3R_1^2\beta_1^4 + 2R_1(W_1+W_2)\beta_1^3 &+ (4R_2\pi_2 - 2R_1\pi_1 - W_1^2 + 2W_1W_2)\beta_1^2 \\ &+ 2\pi_1(W_1 - W_2)\beta_1 - \pi_1^2 = 0. \end{aligned} \tag{13.46}$$

This equation can be solved explicitly to give four roots. We will find that only one of these is positive, and select it as our value of β_1. With that, other coefficients can be obtained by solving for α_1 and β_2 and then, in turn, α_2, as follows:

$$\alpha_1 = \beta_1(\beta_1 R_1 + \delta)/\rho_1, \tag{13.47}$$

$$\beta_2 = (\pi_1 - \beta_1^2 R_1 - \beta_1 W_1)/2\beta_1 R_2, \tag{13.48}$$

$$\alpha_2 = \beta_2(\beta_2 R_2 + \delta)/\rho_2, \tag{13.49}$$

where $R_1 = r_1^2/4c_1,\ R_2 = r_2^2/4c_2,\ W_1 = \rho_1 + 2\delta,$ and $W_2 = \rho_2 + 2\delta.$

It is worthwhile to mention that firm i's advertising effectiveness parameter r_i and advertising cost parameter c_i manifest themselves through $R_i = {r_i}^2/4c_i$. This would suggest that R_i is a measure of firm i's *advertising power*. This can be seen more clearly in Exercise 13.6 involving two firms that are identical in all other aspects except that $R_2 > R_1$. Specifically in that exercise, you are asked to use *Mathematica* or another suitable software program to solve (13.46) to obtain β_1 and then obtain the coefficients α_1, α_2, and β_2 by using (13.47)–(13.49), when $\rho_1 = \rho_2 = 0.05,\ \pi_1 = \pi_2 = 1,\ \delta = 0.01,\ R_1 = 1,\ R_2 = 4,\ x_0 = 0.5,$ and $\sigma(x) = \sqrt{0.5x(1-x)}$. Figure 13.1 represents a sample path of the market share of the two firms with this data.

It is noteworthy to see that both firms are identical except in their advertising powers R_1 and R_2. With $R_2 > R_1$, firm 2 is more powerful and we see that this results in its capture of an increasing share of the market average over time beginning with exactly one half of the market at time 0.

13.4 A Feedback Stackelberg Stochastic Differential Game of Cooperative Advertising

The preceding sections in this chapter dealt with differential games in which all players make their decisions simultaneously. We now discuss

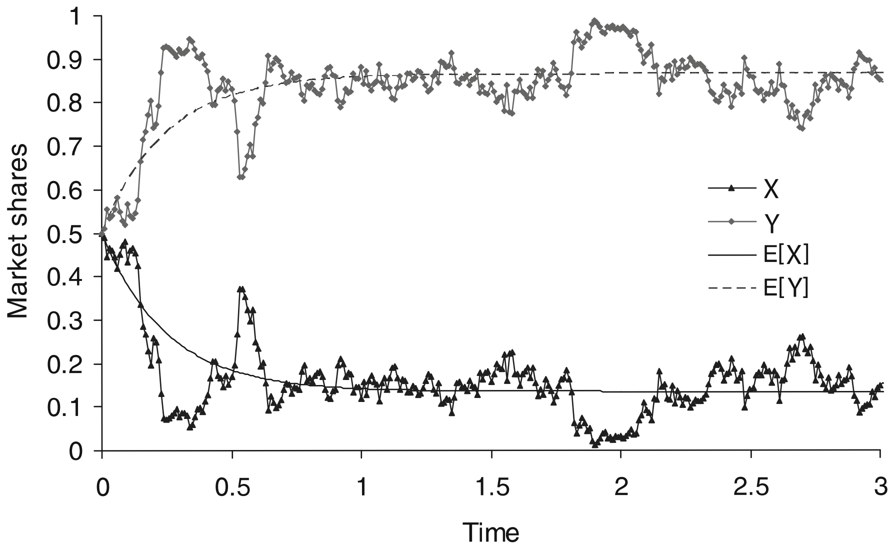

Figure 13.1: A sample path of optimal market share trajectories

a differential game in which two players make their decisions in a hierarchical manner. The player having the right to move first is called the leader and the other player is called the follower. If there are two or more leaders, they play Nash, and the same goes for the followers.

In terms of solutions of Stackelberg differential games, we have open-loop and feedback solutions. An open-loop Stackelberg equilibrium specifies, at the initial time (say, $t = 0$), the decisions over the entire horizon. As in Sect. 13.1, there is a maximum principle for open-loop solutions. Typically, open-loop solutions are not time consistent in the sense that at any time $t > 0$, the remaining decision may no longer be optimal; see Exercise 13.2. A *feedback* or *Markovian Stackelberg equilibrium*, on the other hand, consists of decisions expressed as functions of the current state and time. Such a solution is time consistent.

In this section, we will not develop the general theory, for which we refer the reader to Basar and Olsder (1999), Dockner et al. (2000), and Bensoussan et al. (2014, 2015a, 2018). Instead, we will formulate a Stackelberg differential game of cooperative advertising between a manufacturer as the leader and a retailer as the follower, and obtain a feedback Stackelberg solution. This formulation is due to He et al. (2009). A verification theorem that applies to this problem can be found in Bensoussan et al. (2018).

The manufacturer sells a product to end users through the retailer. The product is in a mature category where sales, expressed as a fraction of the potential market, is influenced through advertising expenditures. The manufacturer as the leader decides on an advertising support scheme via a *subsidy rate*, i.e., he will contribute a certain percentage of the advertising expenditure by the retailer. Specifically, the manufacturer decides on a subsidy rate $W_t,\ 0 \le W_t \le 1$, and the retailer as the follower decides on the advertising effort level $U_t \ge 0,\ t \ge 0$.

As in Sect. 12.3, the cost of advertising is quadratic in the advertising effort U_t. Then, with the advertising effort U_t and the subsidy rate W_t, the manufacturer's and the retailer's advertising expenditures are $W_t U_t^2$ and $(1 - W_t)U_t^2$, respectively. The market share dynamics is given by the Sethi model

$$dX_t = (rU_t\sqrt{1 - X_t} - \delta X_t)dt + \sigma(X_t)dZ_t,\ X_0 = x_0. \tag{13.50}$$

The corresponding expected profits of the retailer and the manufacturer are, respectively, as follows:

$$J_R = E\left[\int_0^\infty e^{-\rho t}(\pi X_t - (1 - W_t)U_t^2)dt\right], \tag{13.51}$$

$$J_M = E\left[\int_0^\infty e^{-\rho t}\left(\pi_M X_t - W_t U_t^2\right)dt\right]. \tag{13.52}$$

A solution of this Stackelberg differential game depends on the available information structure. We shall assume that at each time t, both players know the current system state and the follower knows the action of the leader. The concept of equilibrium that applies in this case is that of feedback Stackelberg equilibrium. For this and other information structures and equilibrium concepts, see Bensoussan et al. (2015a).

Next we define the rules, governing the sequence of actions, by which this game will be played over time. To be specific, the sequence of plays at any time $t \ge 0$ is as follows. First, the manufacturer observes the market share X_t at time t and selects the subsidy rate W_t. Then, the retailer observes this action W_t and, knowing also the market share X_t at time t, sets the advertising effort rate U_t as his response to W_t. The system evolves over time as this game is played in continuous time beginning at time $t = 0$. One could visualize this game as being played at times $0, \delta t, 2\delta t, \ldots$, and then let $\delta t \to 0$.

Next, we will address the question of how players choose their actions at any given t. Specifically, we are interested in deriving an equilibrium menu $W(x)$ for the leader representing his decision when the state is x

at time t, and a menu $U(x, W)$ for the follower representing his decision when he observes the leader's decision to be W in addition to the state x at time t. For this, let us first define a feedback Stackelberg equilibrium, and then develop a procedure to obtain it.

We begin with specifying the admissible strategy spaces for the manufacturer and the retailer, respectively:

$$\begin{aligned}\mathcal{W} &= \{W | W : [0,1] \to [0,1] \\ &\quad \text{and } W(x) \text{ is Lipschitz continuous in } x\} \\ \mathcal{U} &= \{U | U : [0,1] \times [0,1] \to [0,\infty) \\ &\quad \text{and } U(x, W) \text{ is Lipschitz continuous in } (x, W)\}.\end{aligned}$$

For a pair of strategies $(W, U) \subset \mathcal{W} \times \mathcal{U}$, let $Y_s,\ s \geq t$, denote the solution of the state equation

$$dY_s = (rU(Y_s, W_s)\sqrt{1 - Y_s} - \delta Y_s)ds + \sigma(Y_s)dZ_s,\ Y_t = x. \tag{13.53}$$

We should note that Y_s here stands for $Y_s(t, x; W, U)$, as the solution depends on the specified arguments. Then $J_M^{t,x}(W(\cdot), U(\cdot, W(\cdot)))$ and $J_R^{t,x}(W(\cdot), U(\cdot, W(\cdot)))$ representing the current-value profits of the manufacturer and retailer at time t are, respectively,

$$\begin{aligned}&J_M^{t,x}(W(\cdot), U(\cdot, W(\cdot))) \\ &= E \int_t^\infty e^{-\rho(s-t)}[\pi_M Y_s - W(Y_s)\{U(Y_s, W(Y_s))\}^2],\end{aligned} \tag{13.54}$$

$$\begin{aligned}&J_R^{t,x}(W(\cdot), U(\cdot, W(\cdot))) \\ &= E \int_t^\infty e^{-\rho(s-t)}[\pi Y_s - (1 - W(Y_s))\{U(Y_s, W(Y_s))\}^2],\end{aligned} \tag{13.55}$$

where we should stress that $W(\cdot), U(\cdot, W(\cdot))$ evaluated at any state ζ are $W(\zeta), U(\zeta, W(\zeta))$. We can now define our equilibrium concept.

A pair of strategies $(W^*, U^*) \in \mathcal{W} \times \mathcal{U}$ is called a feedback Stackelberg equilibrium if

$$\begin{aligned}&J_M^{t,x}(W^*(\cdot), U^*(\cdot, W^*(\cdot))) \\ &\geq J_M^{t,x}(W(\cdot), U^*(\cdot, W(\cdot))),\ W \in \mathcal{W}, x \in [0,1],\ t \geq 0,\end{aligned} \tag{13.56}$$

and

$$\begin{aligned}&J_R^{t,x}(W^*(\cdot), U^*(\cdot, W^*(\cdot))) \\ &\geq J_R^{t,x}(W^*(\cdot), U(\cdot, W^*(\cdot))),\ U \in \mathcal{U}, x \in [0,1],\ t \geq 0.\end{aligned} \tag{13.57}$$

It has been shown in Bensoussan et al. (2014) that this equilibrium is obtained by solving a pair of Hamilton-Jacobi-Bellman equations where a static Stackelberg game is played at the Hamiltonian level at each t, and where

$$H^M(x, W, U, \lambda^M) = \pi_M x - WU^2 + \lambda^M(rU\sqrt{1-x} - \delta x) \tag{13.58}$$

$$H^R(x, W, U, \lambda^R) = \pi x - (1-W)U^2 + \lambda^R(rU\sqrt{1-x} - \delta x) \tag{13.59}$$

are the Hamiltonians for the manufacturer and the retailer, respectively. To solve this Hamiltonian level game, we first maximize H^R with respect to U in terms of x and W. The first order condition gives

$$U^*(x, W) = \frac{\lambda^R r\sqrt{1-x}}{2(1-W)}, \tag{13.60}$$

as the optimal response of the follower for any decision W by the leader. We then substitute this for U in H^M to obtain

$$\begin{aligned} H^M(x, W, U^*(x, W), \lambda^M) &= \pi_M x - \frac{W(\lambda^R r)^2(1-x)}{4(1-W)^2} \\ &\quad + \lambda^M\left(\frac{\lambda^R r^2(1-x)}{2(1-W)} - \delta x\right). \end{aligned} \tag{13.61}$$

The first-order condition of maximizing H^M with respect to W gives us

$$W(x) = \frac{2\lambda^M - \lambda^R}{2\lambda^M + \lambda^R}. \tag{13.62}$$

Clearly $W(x) \geq 1$ makes no intuitive sense because it would induce the retailer to spend an infinite amount on advertising, and that would not be optimal for the leader. Moreover, λ^M and λ^R, the marginal valuations of the market share of the leader and the follower, respectively, are expected to be positive, and therefore it follows from (13.62) that $W(x) < 1$. Thus, we set,

$$W^*(x) = \max\left\{0, \frac{2\lambda^M - \lambda^R}{2\lambda^M + \lambda^R}\right\}. \tag{13.63}$$

We can now write the HJB equations as

$$\begin{aligned} \rho V^R &= H^R(x, W^*(x), U^*(x, W^*(x)), V_x^R) + (\sigma(x))^2 V_{xx}^R/2 \\ &= \pi x + \frac{(V_x^R r)^2(1-x)}{4(1-W^*(x))} - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2} \end{aligned} \tag{13.64}$$

$$\begin{aligned}\rho V^M &= H^M(x, W^*(x), U^*(x, W^*(x)), V_x^M) + (\sigma(x))^2 V_{xx}^M/2 \\ &= \pi_M x - \frac{(V_x^R r)^2(1-x)W^*(x)}{4(1-W^*(x))^2} + \frac{V_x^M V_x^R r^2(1-x)}{2(1-W^*(x))} \\ &\quad -V_x^M \delta x + (\sigma(x))^2 V_{xx}^M/2 \qquad (13.65)\end{aligned}$$

The solution of these equations will yield the value functions $V^M(x)$ and $V^R(x)$. With these in hand, we can give the equilibrium menu of actions to the manufacturer and the retailer to guide their decisions at each t. These menus are

$$W^*(x) = \max\left\{0, \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}\right\} \text{ and } U^*(x, W) = \frac{V_x^R r\sqrt{1-x}}{2(1-W)}. \qquad (13.66)$$

To solve for the value function, we next investigate the two cases where the subsidy rate is (a) zero and (b) positive, and determine the condition required for no subsidy to be optimal.

Case (a): No Co-op Advertising ($W^* = 0$). Inserting $W^*(x) = 0$ into (13.66) gives

$$U^*(x, 0) = \frac{rV_x^R\sqrt{1-x}}{2}. \qquad (13.67)$$

Inserting $W^*(x) = 0$ into (13.65) and (13.64), we have

$$\rho V^M = \pi_M x + \frac{V_x^M V_x^R r^2(1-x)}{2} - V_x^M \delta x + \frac{(\sigma(x))^2 V_{xx}^M}{2}, \qquad (13.68)$$

$$\rho V^R = \pi x + \frac{(V_x^R)^2 r^2(1-x)}{4} - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2}. \qquad (13.69)$$

Let $V^M(x) = \alpha_M + \beta_M x$ and $V^R(x) = \alpha + \beta x$. Then, $V_x^M = \beta_M$ and $V_x^R = \beta$. Substituting these into (13.68) and (13.69) and equating like powers of x, we can express all of the unknowns in terms of β, which itself can be explicitly solved. That is, we obtain

$$\beta = \frac{2\pi}{\sqrt{(\rho+\delta)^2 + r^2\pi} + (\rho+\delta)},\ \beta_M = \frac{2\pi_M}{2(\rho+\delta) + \beta r^2}, \qquad (13.70)$$

$$\alpha = \frac{\beta^2 r^2}{4\rho},\ \alpha_M = \frac{\beta\beta_M r^2}{2\rho}. \qquad (13.71)$$

Using (13.71) in (13.67), we can write $U^*(x) = \sqrt{\rho\alpha(1-x)}$. Finally, we can derive the required condition from the right-hand side of $W^*(x)$ in (13.66), which is $2V_x^M \le V_x^R$, for no co-op advertising ($W^* = 0$) in the equilibrium. This is given by $2\beta^M \le \beta$, or

$$\frac{4\pi_M}{2(\rho+\delta) + \frac{2\pi r^2}{\sqrt{(\rho+\delta)^2+r^2\pi}+(\rho+\delta)}} \le \frac{2\pi}{\sqrt{(\rho+\delta)^2+r^2\pi}+(\rho+\delta)}. \tag{13.72}$$

After a few steps of algebra, this yields the required condition

$$\theta := \frac{\pi_M}{\sqrt{(\rho+\delta)^2+r^2\pi}} - \frac{\pi}{\sqrt{(\rho+\delta)^2+r^2\pi}+(\rho+\delta)} \le 0. \tag{13.73}$$

Next, we obtain the solution when $\theta > 0$.

Case (b): Co-op Advertising ($W^* > 0$). Then, $W^*(x)$ in (13.66) reduces to

$$W^*(x) = \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}. \tag{13.74}$$

Inserting this for $W^*(x)$ into (13.65) and (13.64), we have

$$\begin{aligned} \rho V^M &= \pi_M x - \frac{r^2(1-x)[4(V_x^M)^2 - (V_x^R)^2]}{16} \\ &\quad + \frac{V_x^M r^2(1-x)[2V_x^M + V_x^R]}{4} \\ &\quad - V_x^M \delta x + \frac{(\sigma(x))^2 V_{xx}^M}{2}, \end{aligned} \tag{13.75}$$

$$\rho V^R = \pi x + \left[\frac{(V_x^R)^2 r^2(1-x)}{4}\right]\left[\frac{2V_x^M + V_x^R}{2V_x^R}\right] - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2}. \tag{13.76}$$

Once again, $V^M(x) = \alpha_M + \beta_M x$, $V^R = \alpha + \beta x$, $V_x^M = \beta_M$, $V_x^R = \beta$. Substituting these into (13.75) and (13.76) and equating like powers of x, we have

$$\alpha = \frac{\beta(\beta + 2\beta_M)r^2}{8\rho}, \tag{13.77}$$

$$(\rho+\delta)\beta = \pi - \frac{\beta(\beta+2\beta_M)r^2}{8}, \tag{13.78}$$

$$\alpha_M = \frac{(\beta+2\beta_M)^2 r^2}{16\rho}, \tag{13.79}$$

$$(\rho+\delta)\beta_M = \pi_M - \frac{(\beta+2\beta_M)^2 r^2}{16}. \tag{13.80}$$

Using (13.66), (13.74), and (13.79), we can write $U^*(x, W^*(x))$, with a slight abuse of notation, as

$$U^*(x) = \frac{r(V_x^R + 2V_x^M)\sqrt{1-x}}{4} = \sqrt{\rho\alpha_M(1-x)}. \tag{13.81}$$

The four equations (13.77)–(13.80) determine the solutions for the four unknowns, α, β, α_M, and β_M. From (13.78) and (13.80), we can obtain

$$\beta^3 + \frac{2\pi_M}{\rho+\delta}\beta^2 + \frac{8\pi}{r^2}\beta - \frac{8\pi^2}{(\rho+\delta)r^2} = 0. \tag{13.82}$$

If we denote

$$a_1 = \frac{2\pi_M}{\rho+\delta},\ a_2 = \frac{8\pi}{r^2},\ \text{and}\ a_3 = \frac{-8\pi^2}{(\rho+\delta)r^2},$$

then $a_1 > 0,\ a_2 > 0$, and $a_3 < 0$. From Descarte's Rule of Signs, there exists a unique, positive real root. The two remaining roots may be both imaginary or both real and negative. Since this is a cubic equation, a complete solution can be obtained. Using *Mathematica* or following Spiegel et al. (2008), we can write down the three roots as

$$\begin{aligned}
\beta(1) &= S + T - \frac{1}{3}a_1,\\
\beta(2) &= -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{\sqrt{3}}{2}i(S-T),\\
\beta(3) &= -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{\sqrt{3}}{2}i(S-T),
\end{aligned}$$

with

$$S = \sqrt[3]{R+\sqrt{Q^3+R^2}},\ T = \sqrt[3]{R-\sqrt{Q^3+R^2}},\ i = \sqrt{-1},$$

where

$$Q = \frac{3a_2 - a_1^2}{9},\ R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}.$$

Next, we identify the positive root in each of the following three cases:

Case 1 ($Q > 0$)**:** We have $S > 0 > T$ and $Q^3 + R^2 > 0$. There is one positive root and two imaginary roots. The positive root is $\beta = S + T - (1/3)a_1$.

Table 13.1: Optimal feedback Stackelberg solution

	(a) if $\theta \le 0$ No co-op equilibrium	(b) if $\theta > 0$ Co-op equilibrium
Retailer's profit V^R	$V^R(x) = \alpha + \beta x$	$V^R(x) = \alpha + \beta x$
Manufacturer's profit V^M	$V^M(x) = \alpha_M + \beta_M x$	$V^M(x) = \alpha_M + \beta_M x$
Coefficients of profit functions, $\alpha, \beta, \alpha_M, \beta_M$ obtained from:	$\beta = \frac{2\pi}{\sqrt{(\rho+\delta)^2+r\pi}+(\rho+\delta)}$ $\beta_M = \frac{2\pi_M}{2(\rho+\delta)+\beta r^2}$ $\alpha = \frac{\beta^2 r^2}{4\rho}$ $\alpha_M = \frac{\beta\beta_M r^2}{2\rho}$	$\beta = \frac{\pi}{\rho+\delta} - \frac{\beta(\beta+2\beta_M)r^2}{8(\rho+\delta)}$ $\beta_M = \frac{\pi_M}{\rho+\delta} - \frac{(\beta+2\beta_M)^2 r^2}{16(\rho+\delta)}$ $\alpha = \frac{\beta(\beta+2\beta_M)r^2}{8\rho}$ $\alpha_M = \frac{(\beta+2\beta_M)^2 r^2}{16\rho}$
Subsidy rate $W^*(x) =$	0	$\frac{2\beta_M-\beta}{2\beta_M+\beta} = 1 - \frac{\alpha}{\alpha_M}$
Advertising effort $U^*(x) =$	$\frac{r\beta\sqrt{1-x}}{2} = \sqrt{\rho\alpha(1-x)}$	$\frac{r(\beta+2\beta_M)\sqrt{1-x}}{4} = \sqrt{\rho\alpha_M(1-x)}$

Case 2 ($Q < 0$ and $Q^3 + R^2 > 0$): There are three real roots with one positive root, which is $\beta = S + T - (1/3)a_1$.

Case 3 ($Q < 0$ and $Q^3 + R^2 < 0$): S and T are both imaginary. We have three real roots with one positive root. While subcases can be given to identify the positive root, for our purposes, it is enough to identify it numerically.

Finally, we can conclude that $2\beta_M - \beta > 0$ so that $W^* > 0$, since if this were not the case, then W^* would be zero, and we would once again be in Case (a).

We can now summarize the optimal feedback Stackelberg equilibrium in Table 13.1. In Exercises 13.7–13.10, you are asked to further explore the model of this section when the parameters $\pi = 0.25$, $\pi_M = 0.5$, $r =$

2, $\rho = 0.05$, $\delta = 1$, and $\sigma(x) = 0.25\sqrt{x(1-x)}$. For this case, He et al. (2009) obtain the comparative statics as shown in Fig. 13.2.

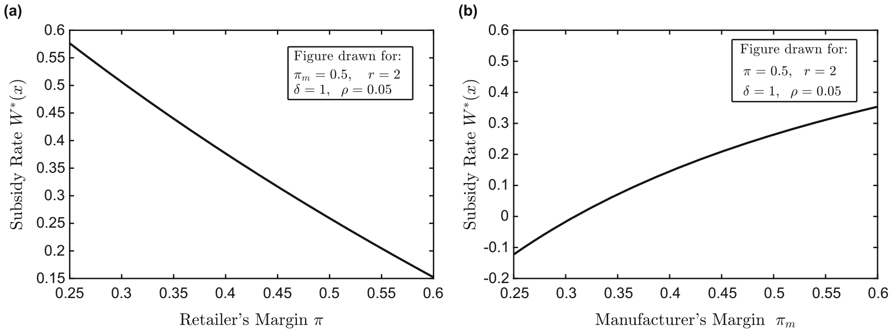

Figure 13.2: Optimal subsidy rate vs. (**a**) Retailer's margin and (**b**) Manufacturer's margin

There have been many applications of differential games in marketing in general and optimal advertising in particular. Some references are Bensoussan et al. (1978), Deal et al. (1979), Deal (1979), Jørgensen (1982a), Rao (1984, 1990), Dockner and Jørgensen (1986, 1992), Chintagunta and Vilcassim (1992), Chintagunta and Jain (1994, 1995), Fruchter (1999a), Jarrar et al. (2004), Martín-Herrán et al. (2005), Breton et al. (2006), Jørgensen and Zaccour (2007), He and Sethi (2008), Naik et al. (2008), Zaccour (2008a), Jørgensen et al. (2009), Prasad and Sethi (2009). The literature on advertising differential games is surveyed by Jørgensen (1982a) and the literature on management applications of Stackelberg differential games is reviewed by He et al. (2007). Monographs are written by Erickson (2003) and Jørgensen and Zaccour (2004). For applications of differential games to economics and management science in general, see the book by Dockner et al. (2000).

Exercises for Chapter 13

E 13.1 *A Bilinear Quadratic Advertising Model* (Deal et al. 1979). Let x_i be the market share of firm i and u_i be its advertising rate, $i = 1, 2$.

The state equations are

$$\begin{aligned}\dot{x}_1 &= b_1u_1(1-x_1-x_2)+e_1(u_1-u_2)(x_1+x_2)-a_1x_1\\ &\quad x_1(0)=x_{10},\\ \dot{x}_2 &= b_2u_2(1-x_1-x_2)+e_2(u_2-u_1)(x_1+x_2)-a_2x_2\\ &\quad x_2(0)=x_{20},\end{aligned}$$

where $b_i,\ e_i,$ and a_i are given positive constants. Firm i wants to maximize

$$J_i = w_i e^{-\rho T}x_i(T)+\int_0^T (c_i x_i - u_i^2)e^{-\rho t}dt,$$

where $w_i,\ c_i$, and ρ are positive constants. Derive the necessary conditions for the open-loop Nash solution, and formulate the resulting boundary value problem. In a related paper, Deal (1979) provides a numerical solution to this problem with $e_1 = e_2 = 0$.

E 13.2 Let $x(t)$ denote the stock of pollution at time $t \in [0,T]$ that affects the welfare of two countries, one of which is the leader and the other the follower. The state dynamics is

$$\dot{x} = u + v,\ \ x(0) = x_0,$$

where u and v are emission rates of the leader and the follower, respectively. Let their instantaneous utility functions be

$$u-(u^2+x^2)/2 \text{ and } v-(v^2+x^2)/2,$$

respectively. Obtain the open-loop Stackelberg solution. By re-solving this problem at time $\tau,\ 0<\tau<T$, show that the first solution obtained is time inconsistent.

Hint: Apply first the maximum principle to the follower's problem for any given leader's decision u. Let λ^F denote the adjoint variable associated with the state x; Clearly $\lambda^F(T) = 0$. Then apply the maximum principle to the leader's problem, treating the follower's adjoint equation as a "state" equation in addition to the state equation for x. Let the adjoint variables associated with x and λ^F be λ^L and μ, respectively. Clearly $\lambda^L(T) = 0$. However, the transversality condition for μ will be $\mu(0) = 0$ in view of Remark 3.9. See Basar and Olsder (1999) and Dockner et al. (2000) for further details.

E 13.3 Develop the nonlinear model for licensing of fisherman described toward the end of Sect. 13.2.3 by rewriting (13.19) and (13.22) for the model. Derive the adjoint equation for λ^i for the ith producer, and show that the feedback Nash policy for producer i is given by

$$f'(v^{i*}) = \frac{c^i}{(p^i - \lambda^i)x}.$$

E 13.4 Consider an N-firm oligopoly. Let $S_i(t)$ denote the cumulative sales by time t of firm $i \in \{1, 2, ..., N\}$ and define $S(t) = \sum_{i=1}^N S_i(t)$. Let $A_i(t)$ denote firm i's advertising rate. With positive constants a, b, and d, assume that the differential game has the diffusion dynamics

$$\dot{S}_i(t) = [a + b \log A_i(t) + dS(t)][M - S(t)], \quad S_i(0) = S_{i0} \geq 0,$$

which means that a firm can stimulate its sales through advertising (but subject to decreasing returns) and that demand learning effects (imitation) are industry-wide. (If these effects were firm-specific we would have S_i instead of S in the brackets on the right-hand side of the dynamics.) Payoffs are given by

$$J^i = \int_0^T [(p_i - c_i)\dot{S}_i(t) - A_i(t)]dt,$$

in which prices and unit costs are constant. Since $\dot{S}_i(t)$ in the expression for J^i is stated in terms of the state variable $S(t)$ and the control variables $A_i(t),\ i \in \{1, 2, \ldots, N\}$, formulate the differential game problem with $S(t)$ as the state variable. In the open-loop Nash equilibrium, show that the advertising rates are monotonically decreasing over time.

Hint: Assume $\partial^2 H^i / \partial S^2 \leq 0$ so that H^i is concave in S. Use this condition to prove the monotone property.

E 13.5 Solve (13.43) to obtain the solution for α and β given in (13.44) and (13.45).

E 13.6 Use *Mathematica* or another suitable software program to solve the quartic equation (13.46). Show that for $\rho_1 = \rho_2 = 0.05$, $\pi_1 = \pi_2 = 1$, $\delta = 0.01$, $R_1 = 1$, $R_2 = 4$, the only positive solution for β_1 is 0.264545. Figure 13.1 gives a sample path of the optimal market shares of the two firms for this problem. Draw another sample path.

E 13.7 In the Stackelberg differential game of Sect. 13.4 let $\pi = 0.25$, $\pi_M = 0.5$, $r = 2$, $\rho = 0.05$, and $\delta = 1$. Obtain the coefficients $\alpha, \beta, \alpha_M, \beta_M$, and show that $W^* = 0.58$. Graph the value functions $V^M(x) = \alpha_M + \beta_M x$, $V(x) = \alpha + \beta x$, and their sum $V^M(x) + V(x)$, as the functions of the market share x.

E 13.8 Suppose the manufacturer in Exercise 13.7 does not behave optimally and decides instead to offer no cooperative advertising. Obtain the value functions of the manufacturer and the retailer. Compare the manufacturer's value function in this case with $V_M(x)$ in Exercise 13.7. Furthermore, when $x_0 = 0.5$, obtain the manufacturer's loss in expected profit when compared to the optimal expected profit $V_M(x_0)$ in Exercise 13.7.

E 13.9 Suppose that the manufacturer and the retailer in the problem of Sect. 13.4 are integrated into a single firm. Then, formulate the stochastic optimal control problem of the integrated firm. Also, using the data in Exercise 13.7, obtain the value function $V^I(x) = \alpha_I + \beta_I x$ of the integrated firm, and compare it to $V^M(x) + V(x)$ obtained in Exercise 13.7.

E 13.10 Let $\sigma(x) = 0.25\sqrt{x(1-x)}$ and the initial market share $x_0 = 0.1$. Use the optimal feedback advertising effort $U^*(x)$ in (13.50) to determine the optimal market share X_t^* over time. You may use MATLAB or another suitable software to graph a sample path of $X_t^*, t \geq 0$.

Appendix A

Solutions of Linear Differential Equations

A.1 First-Order Linear Equations

Consider the equation

$$\dot{x} + ax = b(t),\ x(0) = x_0, \tag{A.1}$$

where a is a constant real number and $b(t)$ is a given function of t. If we multiply both sides of this equation by the integrating factor e^{at}, we get

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at},$$

which can be written at any time τ as

$$d(x(\tau)e^{a\tau}) = b(\tau)e^{a\tau}d\tau.$$

Integrating from 0 to t and then multiplying throughout by e^{-at}, we get the solution of (A.1) as

$$x(t) = e^{-at}x_0 + \int_0^t e^{-a(t-\tau)}b(\tau)d\tau. \tag{A.2}$$

If we generalize (A.1) by replacing the constant a by a function $a(t)$, we get

$$\dot{x}(t) + a(t)x(t) = b(t),\ x(0) = x_0. \tag{A.3}$$

S. P. Sethi, *Optimal Control Theory*,
https://doi.org/10.1007/978-3-319-98237-3

We can then use the integrating factor $e^{\int_0^t a(s)ds}$, and with that you are asked to show in Exercise A.1 by employing a procedure similar to that for the solution of (A.3) that

$$x(t) = x_0 e^{-\int_0^t a(s)ds} + \int_0^t b(\tau) e^{-\int_\tau^t a(s)ds} d\tau. \tag{A.4}$$

A.2 Second-Order Linear Equations with Constant Coefficients

Consider the equation

$$\ddot{x} + a_1 \dot{x} + ax = b(t), \tag{A.5}$$

where a and a_1 are constants and $b(t)$ is a function of t. This equation requires two boundary conditions to be completely specified. These, for example, could be the values of $x(t)$ at two points in time or the values of $x(0)$ and $\dot{x}(0)$.

A general solution of (A.5) has the form

$$x(t) = x_n(t) + x_p(t), \tag{A.6}$$

where $x_n(t)$ is a homogeneous solution, defined to be a solution of (A.5) with $b(t)$ set at 0, and $x_p(t)$ is the particular solution. Clearly $\ddot{x}_n + a_1 \dot{x}_n + ax_n = 0$ and $\ddot{x}_p + a_1 \dot{x}_p + ax_p = b(t)$.

To obtain a homogeneous solution, let m_1 and m_2 be the roots of the auxiliary equation

$$m^2 + a_1 m + a = 0.$$

Then there are 3 cases shown in Table A.1.

Next we provide the particular solution of Eq. (A.5). Since this solution depends on the function $b(t)$, we will provide this in Table A.2.

It is easy to extend Row 3 and Row 5 of Table A.2 for a polynomial $P(t)$ of degree n. See Zwillinger (2003) for details.

For solutions of higher order linear differential equations with constant coefficients and many other differential equations, the reader is referred to Zwillinger (2003) and Polyanin and Zaitsev (2003).

A.3 System of First-Order Linear Equations

In vector form, a system of first-order linear equations reads

$$\dot{x} + Ax = b(t), \; x(0) = x_0, \tag{A.7}$$

Table A.1: Homogeneous solution forms for Eq. (A.5)

Root	General solution form
$m_1 \neq m_2$, real	$x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$
$m_1 = m_2 = m$, real	$x(t) = (C_1 + C_2 t)e^{mt}$
$m_1 = p + qi$, $m_2 = p - qi$	$x(t) = e^{pt}(C_1 \sin\ qt + C_2 \cos\ qt)$

Table A.2: Particular solutions for Eq. (A.5)

	$b(t)$	The particular solution of (A.5)
(1)	e^{rt}	$e^{rt}/(r^2 + a_1 r + a)$
(2)	$\sin\theta t$	$\frac{(a-\theta^2)\sin\theta t - a_1\cos\theta t}{(a-\theta^2)^2+(a_1\theta)^2}$
(3)	$P(t) = \alpha + \beta t + \gamma t^2$	$\frac{1}{a}[P(t) - \frac{a_1}{a}P'(t) + \frac{a_1^2 - a}{a^2}P''(t)]$
(4)	$e^{rt}\sin\theta$	Multiply row 2 by e^{rt} Replace a_1 by $a_1 + 2r$ Replace a by $a + a_1 r + r^2$
(5)	$P(t)e^{rt}$	Multiply row 3 by e^{rt} Replace a_1 by $a_1 + 2r$ Replace a by $a + a_1 r + r^2$

where x is an n-column vector, A is an $n \times n$ matrix of constants, and b is a function of t. We will present two ways of solving the first-order system (A.7).

The first method involves the matrix exponential function e^{tA} defined by the power series

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \cdots = \sum_0^\infty \frac{(tA)^k}{k!}. \tag{A.8}$$

It can be shown that this series converges (component by component) for all values of t. Also it is differentiable (component by component) for all values of t and satisfies

$$\frac{d}{dt}(e^{tA}) = Ae^{tA} = (e^{tA})A. \tag{A.9}$$

By analogy with (A.2), we can write the solution of (A.7) as

$$x(t) = e^{-tA}x_0 + \int_0^t e^{-(t-\tau)A}b(\tau)d\tau. \tag{A.10}$$

Although (A.10) represents a formal expression for the solution of (A.7), it does not provide a computationally convenient way of getting explicit solutions.

For the second method we assume that the matrix A is diagonalizable, i.e., that there exists a nonsingular square matrix P such that

$$P^{-1}AP = \Lambda. \tag{A.11}$$

Here Λ is the diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \tag{A.12}$$

where the diagonal elements, $\lambda_1, \ldots, \lambda_n$, are eigenvalues of A. The ith column of P is the column eigenvector associated with the eigenvalue λ_i (to see this multiply both sides of (A.11) by P on the left). By looking at (A.8) it is easy to see that

$$P^{-1}e^{tA}P = e^{t\Lambda} \text{ and } Pe^{t\Lambda}P^{-1} = e^{tA}, \tag{A.13}$$

where

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{t\lambda_n} \end{bmatrix}. \tag{A.14}$$

Using (A.13) into (A.10), we can write the solution to (A.7) as

$$x(t) = (Pe^{-t\Lambda}P^{-1})x_0 + \int_0^t Pe^{-(t-\tau)\Lambda}P^{-1}b(\tau)d\tau. \tag{A.15}$$

Since well-known algorithms are available for finding eigenvalues and eigenvectors of a matrix, the solution (A.15) can be computed in a straightforward manner.

A.4 Solution of Linear Two-Point Boundary Value Problems

In linear-quadratic control problems with linear salvage values (e.g., the production-inventory problem in Sect. 6.1) we require the solution of linear two-point boundary value problems of the form

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{A.16}$$

with boundary conditions

$$x(0) = x_0 \quad \text{and} \quad \lambda(T) = \lambda_T. \tag{A.17}$$

The solution of this system will be of the form (A.15), which can be restated as

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} + \begin{bmatrix} R_1(t) \\ R_2(t) \end{bmatrix}, \tag{A.18}$$

where the $\lambda(0)$ is a vector of unknowns. They can be determined by setting

$$\lambda_T = Q_{21}(T)x(0) + Q_{22}(T)\lambda(0) + R_2(T), \tag{A.19}$$

which is a system of linear equations for the variables $\lambda(0)$.

A.5 Solutions of Finite Difference Equations

In this book we will have uses for finite difference equations only in Chaps. 8 and 9. For that reason we will give only a brief introduction to solution techniques for them. Readers who wish more details can consult one of several texts on difference equations; see, e.g., Goldberg (1986) or Spiegel (1971).

If $f(k)$ is a real function of time, then the *difference operator* applied to f is defined as

$$\Delta f(k) = f(k+1) - f(k). \tag{A.20}$$

The *factorial power* of k is defined as

$$k^{(n)} = k(k-1)(k-2)\ldots(k-(n-1)). \tag{A.21}$$

It is easy to show that

$$\Delta k^{(n)} = nk^{(n-1)}. \tag{A.22}$$

Because this formula is similar to the corresponding formula for the derivative $d(k^n)/dk$, the factorial powers of k play an analogous role for finite differences that the ordinary powers of k play for differential calculus.

If $f(k)$ is a real function of time, then the *anti-difference operator* Δ^{-1} applied to f is defined as another function $g = \Delta^{-1}f(k)$ with the property that

$$\Delta g = f(k). \tag{A.23}$$

One can easily show that

$$\Delta^{-1}k^{(n)} = (1/(n+1))k^{(n+1)} + c, \tag{A.24}$$

where c is an arbitrary constant. Equation (A.24) corresponds to the integration formula for powers of k in calculus.

Note that formulas (A.22) and (A.24) are similar to, respectively, differentiation and integration of the power function k^n in calculus. By analogy with calculus, therefore, we can solve difference equations involving polynomials in ordinary powers of k by first rewriting them as polynomials involving factorial powers of k so that (A.22) and (A.24) can be used. We show next how to do this.

A.5.1 Changing Polynomials in Powers of k into Factorial Powers of k

We first give an abbreviated list of formulas that show how to change powers of k into factorial powers of k:

$$\begin{aligned}
k^0 &= k^{(0)} = 1 \quad \text{(by definition)}, \\
k^1 &= k^{(1)}, \\
k^2 &= k^{(1)} + k^{(2)}, \\
k^3 &= k^{(1)} + 3k^{(2)} + k^{(3)}, \\
k^4 &= k^{(1)} + 7k^{(2)} + 6k^{(3)} + k^{(4)}, \\
k^5 &= k^{(1)} + 15k^{(2)} + 25k^{(3)} + 10k^{(4)} + k^{(5)}.
\end{aligned}$$

The coefficients of the factorial powers on the right-hand sides of these equations are called *Stirling numbers of the second kind*, after the person who first derived them. This list can be extended by using a more complete table of these numbers, which can be found in books on difference equations cited earlier.

Example A.1 Express $k^4 - 3k + 4$ in terms of factorial powers.

Solution Using the equations above we have

$$k^4 = k^{(1)} + 7k^{(2)} + 6k^{(3)} + k^{(4)},\ -3k = -3k^{(1)},\ 4 = 4,$$

so that

$$k^4 - 3k + 4 = k^{(4)} + 6k^{(3)} + 7k^{(2)} - 2k^{(1)} + 4.$$

Example A.2 Solve the following difference equation in Example 8.7:

$$\Delta\lambda^k = -k + 5, \lambda^6 = 0.$$

Solution We first change the right-hand side into factorial powers so that it becomes

$$\Delta\lambda^k = -k^{(1)} + 5.$$

Applying (A.24), we obtain

$$\lambda^k = -(1/2)k^{(2)} + 5k^{(1)} + c,$$

where c is a constant. Applying the condition $\lambda^6 = 0$, we find that $c = -15$, so that the solution is

$$\lambda^k = -(1/2)k^{(2)} + 5k^{(1)} - 15. \tag{A.25}$$

However, we would like the answer to be in ordinary powers of k. The way to do that is discussed in the next section.

A.5.2 Changing Factorial Powers of k into Ordinary Powers of k

In order to change factorial powers of k into ordinary powers of k, we make use of the following formulas:

$$\begin{aligned}
k^{(1)} &= k, \\
k^{(2)} &= -k + k^2, \\
k^{(3)} &= 2k - 3k^2 + k^3, \\
k^{(4)} &= -6k + 11k^2 - 6k^3 + k^4, \\
k^{(5)} &= 24k - 50k^2 + 35k^3 - 10k^4 + k^5.
\end{aligned}$$

The coefficients of the factorial powers on the right-hand sides of these equations are called *Stirling numbers of the first kind.* This list can also be extended by using a more complete table of these numbers, which can be found in books on difference equations.

Solution of Example A.2 Continued By substituting the first two of the above formulas into (A.25), we see that the desired answer is

$$\lambda^k = -(1/2)k^2 + (11/2)k - 15, \tag{A.26}$$

which is the solution needed for Example 8.7.

Exercises for Appendix A

E A.1 Show that the solution of Eq. (A.3) is given by (A.4).

E A.2 If $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, show that $\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Use (A.15) to solve (A.7) for this data, given that $z(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

E A.3 If $A = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$, show that $\Lambda = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$.

Use (A.15) to solve (A.7) for this data, given that $z(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$.

E A.4 After you have read Sect. 6.1, re-solve the production-inventory example stated in Eqs. (6.1) and (6.2), (ignoring the control constraint ($P \geq 0$) by the method of Sect. A.4. The linear two-point boundary value problem is stated in Eqs. (6.6) and (6.7).

Appendix B

Calculus of Variations and Optimal Control Theory

Here we introduce the subject of the calculus of variations by analogy with the classical topic of maximization and minimization in calculus; see Gelfand and Fomin (1963), Young (1969), and Leitmann (1981) for rigorous treatments of the subject. The problem of the calculus of variations is that of determining a function that maximizes a given functional, the objective function. An analogous problem in calculus is that of determining a point at which a specific function, the objective function, is maximum. This, of course, is done by taking the first derivative of the function and equating it to zero. This is what is called the first-order condition for a maximum. A similar procedure will be employed to derive the first-order condition for the variational problem. The analogy with classical optimization extends also to the second-order maximization condition of calculus. Finally, we will show the relationship between the maximum principle of optimal control theory and the necessary conditions of the calculus of variations. It is noted that this relationship is similar to the one between the Kuhn-Tucker conditions in mathematical programming and the first-order conditions in classical optimization.

We start with the "simplest" variational problem in the next section.

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https://doi.org/10.1007/978-3-319-98237-3

B.1 The Simplest Variational Problem

Assume a function $x : C^1[0,T] \to E^1$, where $C^1[0,T]$ is a class of functions defined over the interval $[0,T]$ with continuous first derivatives. For simplicity in exposition, we are assuming $x(t)$ to be a scalar function of $t \in [0,T]$, and the extension to a vector function is straightforward. Here t simply denotes the independent variable which need not be time. Assume further that a function in this class is termed *admissible* if it satisfies the terminal conditions

$$x(0) = x_0 \quad \text{and} \quad x(T) = x_T. \tag{B.1}$$

We are thus dealing with a fixed-end-point problem. Examples of admissible functions for the problem are shown in Fig. B.1; see Chapters 2 and 3 of Gelfand and Fomin (1963) for problems other than the simplest problem, i.e., the problems with other kinds of conditions for the end points.

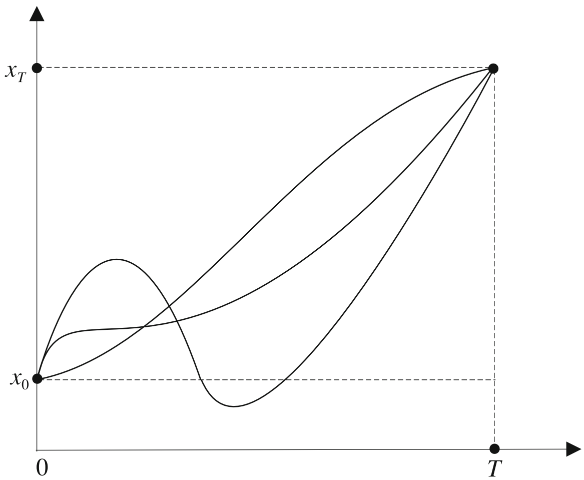

Figure B.1: Examples of admissible functions for the problem

The problem under consideration is to obtain the admissible function x^* for which the functional

$$J(x) = \int_0^T F(x, \dot{x}, t)dt \tag{B.2}$$

has a relative maximum. We will assume that all first and second partial derivatives of the function $F : E^1 \times E^1 \times E^1 \to E^1$ are continuous.

B.2 The Euler-Lagrange Equation

The necessary first-order conditions in classical optimization were obtained by considering small changes about the solution point. For the variational problem, we consider small variations about the solution function. Let $x(t)$ be the solution and let

$$y(t) = x(t) + \varepsilon\eta(t),$$

where $\eta(t) : C^1[0, T] \to E^1$ is an arbitrary continuously differentiable function satisfying

$$\eta(0) = \eta(T) = 0, \tag{B.3}$$

and $\varepsilon \geq 0$ is a small number. A sketch of these functions is shown in Fig. B.2.

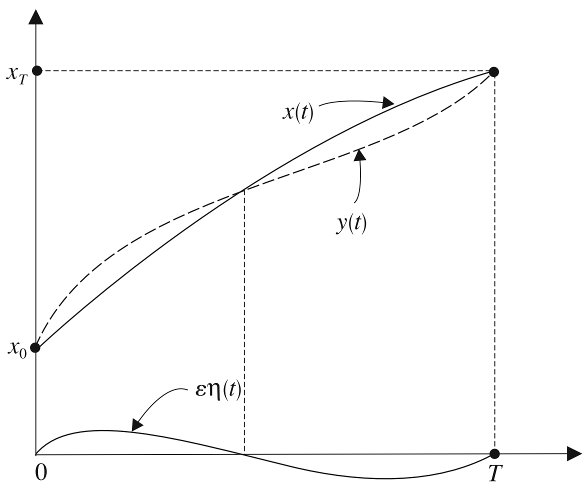

Figure B.2: Variation about the solution function

The value of the objective functional associated with $y(t)$ can be considered a function of ε, i.e.,

$$V(\varepsilon) = J(y) = \int_0^T F(x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta}, , t)dt.$$

However, $x(t)$ is a solution and therefore $V(\varepsilon)$ must have a maximum at $\varepsilon = 0$. This means

$$\delta J \stackrel{\triangle}{=} \left.\frac{dV}{d\varepsilon}\right|_{\varepsilon=0} = 0,$$

where δJ is known as the variation δJ in J. Differentiating $V(\varepsilon)$ with respect to ε and setting $\varepsilon = 0$ yields

$$\delta J = \left.\frac{dV}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T (F_x \eta + F_{\dot{x}} \dot{\eta}) dt = 0,$$

which after integrating the second term by parts provides

$$\delta J = \left.\frac{dV}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T F_x \eta dt + (F_{\dot{x}} \eta)|_0^T - \int_0^T \frac{d}{dt}(F_{\dot{x}}) \eta dt = 0. \tag{B.4}$$

Because of the end conditions on η, the expression simplifies to

$$\delta J = \left.\frac{dV}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^T [F_x - \frac{d}{dt} F_{\dot{x}}] \eta dt = 0.$$

We now use the *fundamental lemma* of the calculus of variations which states that if h is a continuous function and $\int_0^T h(t)\eta(t)dt = 0$ for every continuous function $\eta(t)$, then $h(t) = 0$ for all $t \in [0, T]$. The reason that this lemma holds, without going into details of a rigorous proof which is available in Gelfand and Fomin (1963), is as follows. Suppose that $h(t) \neq 0$ for some $t \in [0, T]$. Since $h(t)$ is continuous, there is, therefore, an interval $(t_1, t_2) \subset [0, T]$ over which h is nonzero and has the same sign. Now selecting $\eta(t)$ such

$$\eta(t) \text{ is } \begin{cases} > 0, & t \in (t_1, t_2) \\ 0, & \text{otherwise,} \end{cases}$$

it is possible to make the integral $\int_0^T h(t)\eta(t)dt \neq 0$. Thus, by contradiction, $h(t)$ must be identically zero over the entire interval $[0, T]$.

By using the fundamental lemma, we have the necessary condition

$$F_x - \frac{d}{dt} F_{\dot{x}} = 0 \tag{B.5}$$

known as the *Euler-Lagrange equation*, or simply the *Euler equation*, which must be satisfied by a maximal solution x^*. In other words, the solution $x^*(t)$ must satisfy

$$F_x(x^*, \dot{x}^*, t) - \frac{d}{dt} F_{\dot{x}}(x^*, \dot{x}^*, t) = 0. \tag{B.6}$$

We note that the Euler equation is a second-order ordinary differential equation. This can be seen by taking the total time derivative of $F_{\dot{x}}$ in (B.5) to obtain

$$F_x - F_{\dot{x}t} - (F_{\dot{x}x}\dot{x})' - (F_{\dot{x}\dot{x}}\ddot{x})' = 0. \tag{B.7}$$

The boundary conditions for this equation are obviously the end-point conditions $x(0) = x_0$ and $x(T) = x_T$.

Special Case (i): When F does not depend explicitly on $\dot{x}$.

In this case, the Euler equation (B.5) reduces to

$$F_x = 0,$$

which is nothing but the first-order condition of classical optimization. In this case, the dynamic problem is a succession of static classical optimization problems.

Special Case (ii): When F does not depend explicitly on x.

The Euler equation reduces to

$$\frac{d}{dt}F_{\dot{x}} = 0, \tag{B.8}$$

which we can integrate as

$$F_{\dot{x}} = C, \tag{B.9}$$

where C is a constant.

Special Case (iii): When F does not depend explicitly on t.

In this important special case, the Euler equation (B.7) reduces to

$$F_x - (F_{\dot{x}x}\dot{x})' - (F_{\dot{x}\dot{x}}\ddot{x})' = 0. \tag{B.10}$$

On multiplying the left hand side of (B.10) by $\dot{x}$ on the right, and adding and subtracting the term $F_{\dot{x}}\ddot{x}$, transforms (B.10) to

$$\frac{d}{dt}(F - F_{\dot{x}}\dot{x}) = 0. \tag{B.11}$$

We can solve the above equation as

$$F - F_{\dot{x}}\dot{x} = C, \tag{B.12}$$

where C is a constant.

B.3 The Shortest Distance Between Two Points on the Plane

The problem is to show that the straight line passing through two points on a plane is the shortest distance between the two points. The problem can be stated as follows:

$$\begin{cases} \min \displaystyle\int_0^T \sqrt{1+\dot{x}^2}dt \\ \text{subject to} \\ x(0) = x_0 \text{ and } x(T) = x_T. \end{cases}$$

Here t refers to distance rather than time. Since $F = -\sqrt{1+\dot{x}^2}$ does not depend explicitly on x, we are in the second special case and the first integral (B.9) of the Euler equation is

$$F_{\dot{x}} = -\dot{x}(1+\dot{x}^2)^{-\frac{1}{2}} = C.$$

This implies that $\dot{x}$ is a constant, which results in the solution

$$x^*(t) = C_1 t + C_2,$$

where C_1 and C_2 are constants. These can be evaluated by imposing boundary conditions which give $C_1 = (x_T - x_0)/T$ and $C_2 = x_0$. Thus,

$$x^*(t) = \left[\frac{x_T - x_0}{T}\right] t + x_0,$$

which is the straight line passing through x_0 and x_T.

B.4 The Brachistochrone Problem

The problem arises from the search for the shape of a wire along which a bead will slide, without friction, in the least time from a given point A to another point B, under the influence of gravity; see Fig. 1.1.

Let t denote the horizontal axis, x denote the vertical axis (measured vertically down), and let the (t, x) coordinates of A and B be $(0, 0)$ and (T, b), respectively. Thus, $x(0) = 0$ and $x(T) = b$. It is reasonable to assume $b \geq 0$, so that point B is not higher than point A.

The time τ_{AB} required for the bead to slide from point A to point B along a wire formed in the shape of a curve $x(t)$ is given as

$$\tau_{AB} = \int_0^{s_T} \frac{ds}{v},$$

where v represents velocity and s_T is the final displacement measured along the curve. Since $ds^2 = dx^2 + dt^2$, we can write

$$ds = \sqrt{1 + \dot{x}^2}dt,$$

where $\dot{x} = dx/dt$ (note that t does not denote time here). From elementary physics, it is known that if $v(t = 0) = 0$ and a denotes the acceleration due to gravity, then

$$v(t) = \sqrt{2ax(t)}, \quad t \in [0, T].$$

Then,

$$\tau_{\mathrm{AB}} = \int_0^T \sqrt{\frac{1 + \dot{x}^2}{2ax}} dt. \tag{B.13}$$

The purpose of the Brachistochrone problem is to find $x(t), t \in [0, T]$, so as to minimize the time τ_{AB}. This is a variational problem, which in view of a being a constant, can be stated as follows:

$$\min \left\{ J(x) = \int_0^T F(x, \dot{x}, t) dt = \int_0^T \sqrt{\frac{1 + \dot{x}^2}{x}} dt \right\}. \tag{B.14}$$

As we can see, the integral F in the above problem does not depend explicitly on t, and the problem (B.14) belongs to the third special case. Using the first integral (B.11) of the Euler equation for this case, we have

$$\sqrt{\frac{1 + \dot{x}^2}{x}} - \dot{x}^2 \sqrt{\frac{1}{x(1 + \dot{x}^2)}} = \frac{1}{k} \quad \text{(a constant)}.$$

We can reduce this to

$$\frac{dx}{dt} = \sqrt{\frac{k^2 - x}{x}},$$

which we rewrite as

$$\frac{dx}{\sqrt{\frac{k^2 - x}{x}}} = dt. \tag{B.15}$$

By performing a change of variable according to

$$x = k^2 \sin^2 \theta = k^2 \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right) \tag{B.16}$$

and recognizing that $x(t = 0) = 0$ corresponds to $\theta = 0$, we can integrate (B.15) to obtain

$$\int_0^\theta 2k^2 \sin^2 \theta d\theta = k^2(\theta - \frac{1}{2} \sin 2\theta) = \int_0^t dt = t. \tag{B.17}$$

By setting $2\theta = \phi$ in (B.16) and (B.17), we can write the solution parametrically as

$$\left. \begin{array}{rcl} t & = & k^2(\phi - \sin\phi)/2 \\ x & = & k^2(1 - \cos\phi)/2 \end{array} \right\}, \tag{B.18}$$

which are known to be equations representing a cycloid, as depicted in Fig. 1.1 in Chap. 1. Furthermore, since the initial condition $x(0) = 0$ is already incorporated in performing the integration in (B.17), we must use the terminal condition $x(T) = b$ for determining the constant k. Clearly, if we let ϕ_1 be defined by the relation

$$\frac{b}{T} = \frac{1 - \cos\phi_1}{\phi_1 - \sin\phi_1}, \tag{B.19}$$

then we can write

$$k^2 = \frac{2b}{(1 - \cos\phi_1)} = \frac{2T}{(\phi_1 - \sin\phi_1)}. \tag{B.20}$$

The value of ϕ_1 can be easily obtained numerically for any given values of $b > 0$ and $T > 0$.

With these, the optimal solution $x^*(t)$ is the cycloid given parametrically as

$$\left. \begin{array}{rcl} t & = & T\left(\frac{\phi - \sin\phi}{\phi_1 - \sin\phi_1}\right) \\ x^* & = & b\left(\frac{1 - \cos\phi}{1 - \cos\phi_1}\right) \end{array} \right\}. \tag{B.21}$$

Furthermore, the minimum time τ^*_{AB} can be obtained as

$$\tau^*_{\mathrm{AB}} = \sqrt{2a} J(x^*) = \int_0^T \sqrt{\frac{1 + (\dot{x}^*(t))^2}{x^*(t)}} dt. \tag{B.22}$$

In Exercise B.1, you are asked to obtain ϕ_1 for $T = b = 1\,\mathrm{m}$, and then obtain the minimum time τ^*_{AB}.

B.5 The Weierstrass-Erdmann Corner Conditions

So far we have only considered functionals defined for smooth curves. This is, however, a restricted class of curves which qualify as solutions, since it is easy to give examples of variational problems which have no solution in this class. Consider, for example, the objective functional

$$\min \left\{ J(x) = \int_{-1}^{1} x^2(1-\dot{x})^2 dt \right\}, \quad x(-1)=0, \quad x(1)=1.$$

The greatest lower bound for $J(x)$ for smooth $x = x(t)$ satisfying the boundary conditions is obviously zero. Yet there is no $x \in C^1[-1,1]$ with $x(-1) = 0$ and $x(1) = 1$, which achieves this value of $J(x)$. In fact, the minimum is achieved for the curve

$$x^*(t) = \begin{cases} 0, & -1 \le t \le 0, \\ t, & 0 < t \le 1, \end{cases}$$

which has a *corner* (i.e., a discontinuous first derivative) at $t = 0$. Such a piecewise smooth extremal with corners is called a *broken extremal.*

We now enlarge the class of admissible functions by relaxing the requirement that they be smooth everywhere. The larger class is the class of piecewise continuous functions which are continuously differentiable almost everywhere in $[0,T]$, i.e., except at some points in $[0,T]$.

Let x, defined on the interval $[0,T]$, have a corner at $\tau \in [0,T]$. Let us decompose $J(x)$ as

$$\begin{aligned} J(x) &= \int_0^T F(x,\dot{x},t)dt = \int_0^\tau F(x,\dot{x},t)dt + \int_\tau^T F(x,\dot{x},t)dt \\ &= J_1(x) + J_2(x). \end{aligned}$$

It is clear that on each of the intervals $[0,\tau)$ and $(\tau,T]$, the Euler equation must hold.

To compute variations δJ_1 and δJ_2, we must recognize that the two 'pieces' of x are not fixed-end-point problems. We must require that the two pieces of x join continuously at $t = \tau$; the point $t = \tau$ can, however, move freely as shown in Fig. B.3.

This will require a slightly modified version of formula (B.4) for writing out the variations; see pp. 55–56 in Gelfand and Fomin (1963).

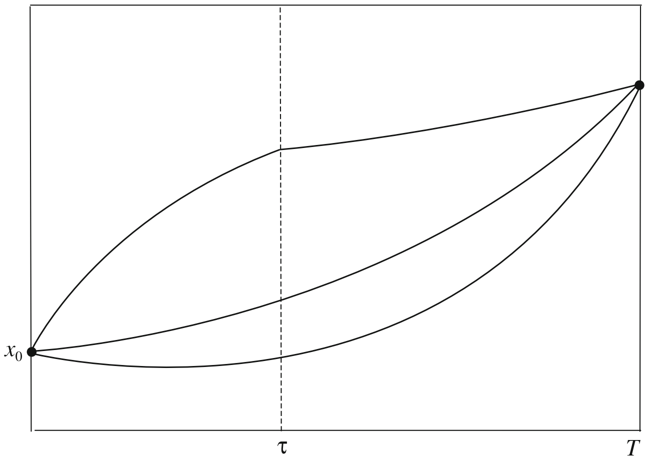

Figure B.3: A broken extremal with corner at τ

Equating the sum of variations

$$\delta J = \delta J_1 + \delta J_2 = 0$$

for x^* to be an extremal and using the fact that it must be continuous at $t = \tau$ implies

$$F_{\dot{x}}|_{\tau^-} = F_{\dot{x}}|_{\tau^+}\ , \tag{B.23}$$

$$[F - F_{\dot{x}}\dot{x}]_{\tau^-} = [F - F_{\dot{x}}\dot{x}]_{\tau^+}. \tag{B.24}$$

These conditions are called Weierstrass-Erdmann corner conditions, which must hold at the point τ where the extremal has a corner.

In each of the interval $[0, \tau)$ and $(\tau, t]$, the extremal x must satisfy the Euler equation (B.5). Solving these two equations will provide us with four constants of integration since the Euler equations are second-order differential equations. These constants can be found from the end-point conditions (B.1) and Weierstrass-Erdmann conditions (B.23) and (B.24).

B.6 Legendre's Conditions: The Second Variation

The Euler equation is a necessary conditions analogous to the first-order condition for a maximum (or minimum) in the classical optimization problems of calculus. The condition analogous to the second-order necessary condition for a maximum x^* is the Legendre condition

$$F_{\dot{x}\dot{x}} \leq 0. \tag{B.25}$$

To obtain this condition, we use the second-order condition of classical optimization on function $V(\varepsilon)$ to be a maximum at $\varepsilon = 0$, i.e.,

$$\left.\frac{d^2V(\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_0^T (F_{xx}\eta^2 + 2F_{x\dot{x}}\eta\dot{\eta} + F_{\dot{x}\dot{x}}\dot{\eta}^2)dt \leq 0. \tag{B.26}$$

Integrating the middle term by parts and using (B.3), we can transform (B.26) into a more convenient form

$$\int_0^T (Q\eta^2 + P\dot{\eta}^2)dt \leq 0, \tag{B.27}$$

where

$$Q = Q(t) = F_{x\dot{x}} - \frac{d}{dt}F_{x\dot{x}} \quad \text{and} \quad P = P(t) = F_{\dot{x}\dot{x}}.$$

While it is possible to rigorously obtain (B.25) from (B.27), we will only provide a qualitative argument for this. If we consider the quadratic functional (B.27) for functions $\eta(t)$ satisfying $\eta(0) = 0$, then $\eta(t)$ will be small in $[0, T]$ if $\dot{\eta}(t)$ is small in $[0, T]$. The converse is not true, however, since it is easy to construct $\eta(t)$ which is small but has a large derivative $\dot{\eta}(t)$ in $[0, T]$. Thus, $P\dot{\eta}^2$ plays the dominant role in (B.27); i.e., $P\dot{\eta}^2$ can be much larger than $Q\eta^2$ but it cannot be much smaller (provided $P \neq 0$). Therefore, it might be expected that the sign of the functional in (B.8) is determined by the sign of the coefficient $P(t)$, i.e., (B.27) implies (B.25). For a rigorous proof, see Gelfand and Fomin (1963).

We note that the strengthened Legendre condition (i.e., with a strict inequality in (B.25)), the Euler equation, and one other condition called strengthened Jacobi condition are sufficient for a maximum. The reader can consult Chapter 5 of Gelfand and Fomin (1963) for details.

B.7 Necessary Condition for a Strong Maximum

So far we have discussed necessary conditions for a *weak maximum*. By weak maximum we mean that the candidate extremals are smooth or piecewise smooth functions. The concept of a *strong maximum* on the other hand requires that the candidate extremal need only be continuous functions. Without going into details, which are available in Gelfand and Fomin (1963), we state a necessary condition for a strong maximum. This is called the *Weierstrass necessary condition.* The condition is analogous to the one in the static case that the objective function be concave. It states that if the functional (B.2) has a strong maximum for the extremal x^* satisfying (B.1), then

$$E(x^*, \dot{x}^*, t, u) \le 0 \tag{B.28}$$

for every finite u, where E is the *Weierstrass Excess Function* defined as

$$E(x, \dot{x}, t, u) = F(x, u, t) - F(x, \dot{x}, t) - F_{\dot{x}}(x, \dot{x}, t)(u - \dot{x}). \tag{B.29}$$

Note that this condition is always met if $F(x, \dot{x}, t)$ is concave in $\dot{x}$.

The proof of (B.28) is by contradiction. Suppose there exists a $\tau \in [0, T]$ and a vector q such that

$$E(\tau, x^*(\tau), \dot{x}^*(\tau), q) > 0.$$

It is then possible to suitably modify x^* to y, which is close to x^* in $C^1[0, T]$, such that

$$\triangle J = \int F(y, \dot{y}, t)dt - \int F(x^*, \dot{x}^*, t)dt > 0,$$

contradicting the hypothesis that $J(x)$ has a strong maximum at x^*.

B.8 Relation to Optimal Control Theory

It is possible to derive the necessary conditions of the calculus of variations from the maximum principle. This is strongly reminiscent of the relationship between the first-order conditions of classical optimization and the Kuhn-Tucker conditions of mathematical programming.

First, we note that the calculus of variations problem can be stated as an optimal control problem as follows:

$$\begin{cases} \max\left\{J = \int_0^T F(x,u,t)dt\right\} \\ \text{subject to} \\ \dot{x} = u,\ x(0) = x_0,\ x(T) = x_T, \\ u \in \Omega = E^n. \end{cases} \tag{B.30}$$

The Hamiltonian is

$$H(x,u,\lambda,t) = F(x,u,t) + \lambda u \tag{B.31}$$

with the adjoint variable λ satisfying

$$\dot{\lambda} = -H_x = -F_x. \tag{B.32}$$

Maximizing the Hamiltonian with respect to u yields

$$H_u = F_{\dot{x}} + \lambda = 0, \tag{B.33}$$

from which we obtain

$$\lambda = -F_{\dot{x}}. \tag{B.34}$$

Differentiating (B.34) with respect to time gives

$$\dot{\lambda} = -\frac{d}{dt}F_{\dot{x}}.$$

This equation with (B.32) implies the Euler equation

$$F_x - \frac{d}{dt}F_{\dot{x}} = 0.$$

From (B.30) and (B.32), the second-order condition $H_{uu} \leq 0$ for the maximization of the Hamiltonian leads to

$$F_{\dot{x}\dot{x}} \leq 0,$$

known as the *Legendre condition.*

By the maximum principle, if u^* is an optimal control with x^* denoting the corresponding trajectory, then for each $t \in [0,T]$,

$$H(x^*,u^*,\lambda,t) \geq H(x^*,u,\lambda,t),$$

where u is any other control. By the definition of the Hamiltonian (B.31), $\dot{x}^* = u^*$ from (B.32), and Eq. (B.33), we have

$$F(x^*, \dot{x}^*, t) - F_{\dot{x}}(x^*, \dot{x}^*, t)\dot{x}^* \geq F(x^*, u, t) - F_{\dot{x}}(x^*, \dot{x}^*, t)u,$$

which by transposition of the terms yields the *Weierstrass necessary condition*

$$E(x^*, \dot{x}^*, t, u) = F(x^*, u, t) - F(x^*, \dot{x}^*, t) - F_{\dot{x}}(x^*, \dot{x}^*, t)(u - \dot{x}^*) \leq 0.$$

We have just proved the equivalence of the maximum principle and the Weierstrass necessary condition in the case where Ω is open. In cases when Ω is closed and when the optimal control is on the boundary of Ω, the Weierstrass necessary condition holds no longer in general. The maximum principle still applies, however.

Finally, according to the maximum principle, both λ and H are continuous functions of time. That is,

$$\begin{aligned} \lambda(\tau^-) &= \lambda(\tau^+), \\ H(x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau) &= H(x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau). \end{aligned}$$

However,

$$\lambda = -F_{\dot{x}} \quad \text{and} \quad H = F - F_{\dot{x}}\dot{x},$$

which means that the right-hand sides must be continuous with respect to time, i.e., even across corners. These are precisely the Weierstrass-Erdmann corner conditions.

Exercises for Appendix B

E B.1 Solve (B.19) numerically to obtain ϕ_1 for $T = b = 1\,\text{m}$. Then, use the formulas (B.21) and (B.22) to compute the minimum time τ^*_{AB}. Note that the gravitational acceleration rate $a = 9.81\,\text{m/s}^2$.

Appendix C

An Alternative Derivation of the Maximum Principle

Recall that in the derivation of the maximum principle in Chap. 2, we assumed the twice differentiability of the value function $V(x,t)$ with respect to the state variable x. Looking at (2.31), we can observe that the smoothness assumptions on the value function do not arise in the statement of the maximum principle. Also since it is not an exogenously given function, there is no a priori reason to assume the twice differentiability. Moreover, there arise cases in which the value function $V(x,t)$ is not even differentiable in x.

In what follows, we will give an alternate derivation. This proof follows the course pointed out by Pontryagin et al. (1962) but with certain simplifications. It appears in Fel'dbaum (1965) and, in our opinion, it is one of the simplest proofs for the maximum principle which is not related to dynamic programming and thus permits the elimination of assumptions about the differentiability of the return function $V(t,x)$.

We select the Mayer form of the problem (2.5) for deriving the maximum principle in this section. It will be convenient to reproduce (2.5) here as (C.1):

$$\begin{cases} \max\limits_{u(t)\in\Omega(t)} \{J = cx(T)\} \\ \text{subject to} \\ \dot{x} = f(x,u,t), \quad x(0) = x_0. \end{cases} \tag{C.1}$$

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C.1 Needle-Shaped Variation

Let $u^*(t)$ be an optimal control with corresponding state trajectory $x^*(t)$. We sketch $u^*(t)$ in Fig. C.1 and $x^*(t)$ in Fig. C.2 in a scalar case. Note that the kink in $x^*(t)$ at $t = \theta$ corresponds to the discontinuity in $u^*(t)$ at $t = \theta$.

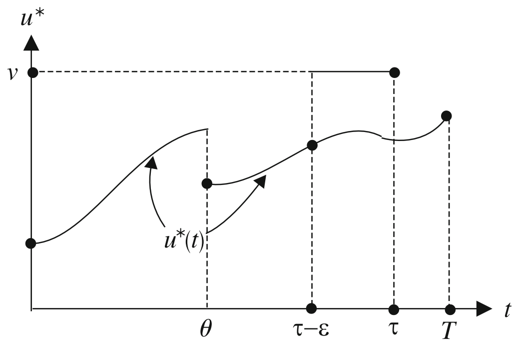

Figure C.1: Needle-shaped variation

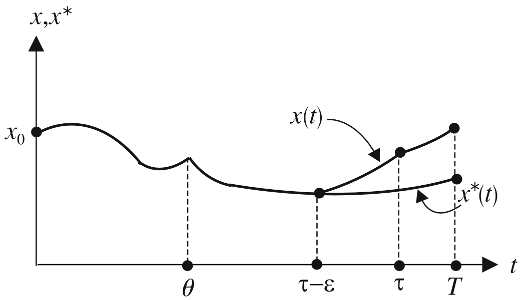

Figure C.2: Trajectories $x^*(t)$ and $x(t)$ in a one-dimensional case

Let τ denote any time in the open interval $(0, T)$. We select a sufficiently small ε to insure that $\tau - \varepsilon > 0$ and concentrate our attention on this small interval $(\tau - \varepsilon, \tau]$. We vary the control on this interval while keeping the control on the remaining intervals $[0, \tau - \varepsilon]$ and $(\tau, T]$ fixed.

Specifically, the modified control is

$$u(t) = \begin{cases} v \in \Omega, & t \in (\tau - \varepsilon, \tau], \\ u^*(t), & \text{otherwise.} \end{cases} \tag{C.2}$$

This is called a *needle-shaped* variation as shown in Fig. C.1. It is a jump function and is different from variations in the calculus of variations; see Appendix B. Also the difference $v - u^*$ is finite and need not be small. However, since the variation is on a small time interval, its influence on the subsequent state trajectory can be proved to be 'small'. This is done in the following.

Let the subsequent motion be denoted by $x(t) \neq x^*(t)$ for $t > \tau - \varepsilon$. In Fig. C.2, we have sketched $x(t)$ corresponding to $u(t)$.

Let

$$\delta x(t) = x(t) - x^*(t), \quad t \geq \tau - \varepsilon,$$

denote the change in the state variables. Obviously $\delta x(\tau - \varepsilon) = 0$. Clearly,

$$\delta x(\tau) \approx \varepsilon[\dot{x}(s) - \dot{x}^*(s)], \tag{C.3}$$

where s denotes some intermediate time in the interval $(\tau - \varepsilon, \tau]$. In particular, we can write (C.3) as

$$\begin{aligned} \delta x(\tau) &= \varepsilon[\dot{x}(\tau) - \dot{x}^*(\tau)] + o(\varepsilon) \\ &= \varepsilon[f(x(\tau), v, \tau) - f(x^*(\tau), u^*(\tau), \tau] + o(\varepsilon). \end{aligned} \tag{C.4}$$

But $\delta x(\tau)$ is small since f is assumed to be bounded. Furthermore, since f is continuous and the difference $\delta x(\tau) = x(\tau) - x^*(\tau)$ is small, we can rewrite (C.4) as

$$\delta x(t) \approx \varepsilon[f(x^*(\tau), v, \tau) - f(x^*(\tau), u^*(\tau), \tau)]. \tag{C.5}$$

Since the initial difference $\delta x(\tau)$ is small and since $u^*(\tau)$ does not change from $t > \tau$ on, we may conclude that $\delta x(t)$ will be small for all $t > \tau$. Being small, the law of variation of $\delta x(t)$ can be found from linear equations for small changes in the state variables. These are called *variational equations*. From the state equation in (C.1), we have

$$\frac{d(x^* + \delta x)}{dt} = f(x^* + \delta x, u^*, t) \tag{C.6}$$

or,

$$\frac{dx^*}{dt} + \frac{d(\delta x)}{dt} \approx f(x^*, u^*, t) + f_x \delta x \tag{C.7}$$

or using (C.1),

$$\frac{d}{dt}(\delta x) \approx f_x(x^*, u^*, t)\delta x, \quad \text{for } t \geq \tau, \tag{C.8}$$

with the initial condition $\delta x(\tau)$ given by (C.5).

The basic idea in deriving the maximum principle is that equations (C.8) are linear variational equations and result in an extraordinary simplification. We next obtain the adjoint equations.

C.2 Derivation of the Adjoint Equation and the Maximum Principle

For this derivation, we employ two methods. The direct method, similar to that of Hartberger (1973), is the consequence of directly integrating (C.8). The indirect method avoids this integration by a trick which is instructive.

Direct Method. Integrating (C.8) we get

$$\delta x(T) = \delta x(\tau) + \int_\tau^T f_x[x^*(t), u^*(t), t]\delta x(t)dt, \tag{C.9}$$

where the initial condition $\delta x(\tau)$ is given in (C.5).

Since $\delta x(T)$ is the change in the terminal state from the optimal state $x^*(T)$, the change in the objective function δJ must be negative. Thus,

$$\delta J = c\delta x(T) = c\delta x(\tau) + \int_\tau^T cf_x[x^*(t), u^*(t), t]\delta x(t)dt \le 0. \tag{C.10}$$

Furthermore, since (C.8) is a linear homogeneous differential equation, we can write its general solution as

$$\delta x(t) = \Phi(t, \tau)\delta x(\tau), \tag{C.11}$$

where the fundamental solution matrix or the transition matrix $\Phi(t, \tau) \in E^{n\times n}$ obeys

$$\frac{d}{dt}\Phi(t, \tau) = f_x[x^*(t), u^*(t)t]\Phi(t, \tau), \qquad \Phi(\tau, \tau) = I, \tag{C.12}$$

where I is an $n \times n$ identity matrix; see Appendix A.

Substituting for $\delta x(t)$ from (C.11) into (C.10), we have

$$\delta J = c\delta x(\tau) + \int_\tau^T cf_x[x^*(t), u^*(t), t]\Phi(t, \tau)\delta x(\tau)dt \le 0. \tag{C.13}$$

This induces the definition

$$\lambda^*(t) = \int_\tau^T cf_x[x^*(t), u^*(t), t]\Phi(t,\tau)dt + c, \tag{C.14}$$

which when substituted into (C.13), yields

$$\delta J = \lambda^*(\tau)\delta x(\tau) \le 0. \tag{C.15}$$

But $\delta x(\tau)$ is supplied in (C.5). Noting that $\varepsilon > 0$, we can rewrite (C.15) as

$$\lambda^*(\tau)f[x^*(\tau), v, \tau] - \lambda^*(\tau)f[x^*(\tau), u^*(\tau), \tau] \le 0. \tag{C.16}$$

Defining the Hamiltonian for the Mayer form as

$$H[x, u, \lambda, t] \overset{\triangle}{=} \lambda f(x, u, t), \tag{C.17}$$

we can rewrite (C.16) as

$$H[x^*(\tau), u^*(\tau), \lambda(\tau), \tau] \ge H[x^*(\tau), v, \lambda(\tau), \tau]. \tag{C.18}$$

Since this can be done for almost every τ, we have the required Hamiltonian maximizing condition.

The differential equation form of the adjoint equation (C.14) can be obtained by taking its derivative with respect to τ. Thus,

$$\begin{aligned}\frac{d\lambda(\tau)}{d\tau} &= \int_\tau^T cf_x[x^*(t), u^*(t), t]\frac{d\Phi(t,\tau)}{d\tau}dt \\ &\quad -cf_x[x^*(\tau), u^*(\tau), \tau].\end{aligned} \tag{C.19}$$

It is also known that the transition matrix has the property:

$$\frac{d\Phi(t,\tau)}{d\tau} = -\Phi(t,\tau)f_x[x^*(\tau), u^*(\tau), \tau],$$

which can be used in (C.19) to obtain

$$\begin{aligned}\frac{d\lambda(\tau)}{d\tau} = -\int_\tau^T cf_x[x^*(t), u^*(t), t]\Phi(t,\tau)f_x[x^*(\tau), u^*(\tau), \tau]dt \\ -cf_x[x^*(\tau), u^*(\tau), \tau].\end{aligned} \tag{C.20}$$

Using the definition (C.14) of $\lambda(\tau)$ in (C.20), we have

$$\frac{d\lambda(\tau)}{d\tau} = -\lambda(\tau)f_x[x^*(\tau), u^*(\tau), \tau]$$

with $\lambda(T) = c$, or using (C.17) and noting that τ is arbitrary, we have

$$\dot{\lambda} = -\lambda f_x[x^*, u^*, t] = -H_x[x^*, u^*, \lambda, t), \quad \lambda(T) = c. \tag{C.21}$$

This completes the derivation of the maximum principle along with the adjoint equation using the direct method.

Indirect Method. The indirect method employs a trick which simplifies considerably the derivation. Instead of integrating (C.8) explicitly, we now assume that the result of this integration yields $c\delta x(T)$ as the change in the state at the terminal time. As in (C.10), we have

$$\delta J = c\delta x(T) \leq 0. \tag{C.22}$$

First, we define

$$\lambda(T) \stackrel{\triangle}{=} c, \tag{C.23}$$

which makes it possible to write (C.22) as

$$\delta J = c\delta x(T) = \lambda(T)\delta x(T) \leq 0. \tag{C.24}$$

Note parenthetically that if the objective function $J = S(x(T))$, we must define $\lambda(T) = \partial S[x(T)]/\partial x(T)$ giving us

$$\delta J = \frac{\partial S[x(T)]}{\partial x(T)}\delta x(T) = \lambda(T)\delta x(T).$$

Now, $\lambda(T)\delta x(T)$ is the change in the objective function due to a change $\delta x(T)$ at the terminal time T. That is, $\lambda(T)$ is the *marginal return* or the marginal change in the objective function per unit change in the state at time T. But $\delta x(T)$ cannot be known without integrating (C.8). We do know, however, the value of the change $\delta x(\tau)$ at time τ which caused the terminal change $\delta x(T)$ via (C.8).

We would therefore like to pose the problem of obtaining the change δJ in the objective function in terms of the known value $\delta x(\tau)$; see Fel'dbaum (1965). Simply stated, we would like to obtain the marginal return $\lambda(\tau)$ per unit change in state at time τ. Thus,

$$\lambda(\tau)\delta x(\tau) = \delta J = \lambda(T)\delta x(T) \leq 0. \tag{C.25}$$

Obviously, knowing $\lambda(\tau)$ will make it possible to make an inference about δJ, which is directly related to the needle-shaped variation applied in the small interval $(\tau - \varepsilon, \tau]$.

However, since τ is arbitrary, our problem of finding $\lambda(\tau)$ can be translated to one of finding $\lambda(t)$, $t \in [0, T]$, such that

$$\lambda(t)\delta x(t) = \lambda(T)\delta x(T), \quad t \in [0, T], \tag{C.26}$$

or in other words,

$$\lambda(t)\delta x(t) = \text{constant}, \quad \lambda(T) = c. \tag{C.27}$$

It turns out that the differential equation which $\lambda(t)$ must satisfy can be easily found. From (C.27),

$$\frac{d}{dt}[\lambda(t)\delta x(t)] = \lambda \frac{\delta x}{dt} + \dot{\lambda}\delta x = 0, \tag{C.28}$$

which after substituting for $d\delta x/dt$ from (C.8) becomes

$$\lambda f_x \delta x + \dot{\lambda}\delta x = (\lambda f_x + \dot{\lambda})\delta x = 0. \tag{C.29}$$

Since (C.29) is true for arbitrary δx, we have

$$\dot{\lambda} = -\lambda f_x = -H_x \tag{C.30}$$

using the definition (C.17) for the Hamiltonian.

The Hamiltonian maximizing condition can be obtained by substituting for $\delta x(\tau)$ from (C.5) into (C.25). This is the same as what we did in (C.15) through (C.18).

The purpose of the alternative proof was to demonstrate the validity of the maximum principle for a simple problem without knowledge of any return function. For more complex problems, one needs complicated mathematical analysis to rigorously prove the maximum principle without making use of return functions. A part of mathematical rigor is in proving the existence of an optimal solution without which necessary conditions are meaningless; see Young (1969).

Appendix D

Special Topics in Optimal Control

In this appendix we will discuss a number of specialized topics in seven sections. These are the Kalman and Kalman-Bucy filters, the Weiner Process, Itô's Lemma, linear-quadratic problems, second-order variations, singular control, and the Sethi-Skiba points. These topics are referred to but not discussed in the main body of the text. While we will not be able to go into great detail, we will provide an adequate description of these topics for our purposes. For further details, the reader can consult the references cited in the respective sections dealing with these topics.

D.1 The Kalman Filter

So far in this book, we have assumed that the values of the state variables can be measured with certainty. In many cases the assumption that the value of a state variable can be directly measured and exactly determined may not be realistic.

There are two types of random disturbances present. The first kind, termed *measurement noise*, arises because of imprecise measurement instruments, inaccurate recording systems, etc. In many cases the measurement technique involves observations of functions of state variables, from which the values of some or all of the state variables are inferred; e.g., measuring the inventory of a natural gas reservoir involves pressure

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measurements together with physical laws relating pressure and volume.

The second kind can be termed *system noise*, in which the system itself is subjected to random disturbances. For instance, sales may follow a stochastic process, which affects the system equation (6.1) relating inventory, production, and sales. In the cash balance example, the demand for cash as well as the interest rates in (5.1) and (5.2) can be represented by stochastic processes.

In analyzing systems in which one or both of these kinds of noises are present, it is important to be able to make good *estimates* of the values of the state variables. We discuss the Kalman and Kalman-Bucy filters devoted to optimal estimation of current values of state variables given past measurements. The Kalman filter will be described in this section, for which further details can be obtained from references such as Kalman (1960a,b), Bryson and Ho (1975), Anderson and Moore (1979), and Kumar and Varaiya (1986). The Kalman-Bucy filter for continuous-time linear systems will be described briefly in Sect. D.3 and the readers can refer to Fleming and Rishel (1975) and Arnold (1974) for further details.

Consider a dynamic stochastic system in discrete time described by the difference equation

$$x^{t+1} - x^t = A_t x^t + G_t w^t,\ t = 0, 1, ..., N-1, \tag{D.1}$$

or

$$x^{t+1} = (A_t + I)x^t + G_t w^t,\ t = 0, 1, ..., N-1, \tag{D.2}$$

where x^t is an n-component (column) state vector, w^t is a k-component (column) system noise vector, A_t is an $n \times n$ matrix, and G_t is an $n \times k$ matrix. The initial state x_0 is assumed to be a Gaussian (normal) random variable with mean and $n \times n$ covariance matrix given by

$$E[x^0] = \bar{x}^0 \text{ and } E[(x^0 - \bar{x}^0)(x^0 - \bar{x}^0)'] = \Sigma_0. \tag{D.3}$$

Without loss of generality, we confine ourselves to the case when w^t is a standard Gaussian purely random sequence with

$$E[w^t] = 0 \text{ and } E[w^t(w^\tau)'] = I\delta_{t\tau}, \tag{D.4}$$

where for $t = 0, 1, ..., N,\ \tau = 0, 1, ..., N,$

$$\delta_{t\tau} = \begin{cases} 0 & \text{if } t \neq \tau, \\ 1 & \text{if } t = \tau. \end{cases} \tag{D.5}$$

Thus, the random vectors w^t and w^τ are independent standard normal variables for $t \neq \tau$. We also assume that the sequence w^t is independent of the initial condition x^0, i.e., the $k \times n$ matrix

$$E[w^t(x^0 - \bar{x}^0)'] = 0,\ t = 0, 1, ..., N. \tag{D.6}$$

The process of measurement of the state variables x^t yields a r-dimensional vector y^t which is related to x^t by the transformation

$$y^t = H_t x^t + v^t,\ t = 0, 1, ..., N, \tag{D.7}$$

where H_t is the state-to-measurement transformation matrix of dimension $r \times n$, and v^t is a Gaussian purely random sequence of r-dimensional measurement noise vectors having the following properties:

$$E[v^t] = 0,\ \ E[v^t(v^\tau)'] = R_t \delta_{t\tau}, \tag{D.8}$$

$$E[w^t(v^\tau)'] = 0,\ \ E[(x^0 - \bar{x}^0)(\bar{v}^t)'] = 0. \tag{D.9}$$

In (D.8) the matrix R_t is the $r \times r$ covariance matrix for the random variable v^t, and it is therefore positive semidefinite, symmetric, and nonsingular. The requirements in (D.9) mean that the additive measurement noise is independent of the system noise as well as the initial state.

Given a sequence of observations $y^0, y^1, y^2, \ldots, y^i$ up to time i, we would like to obtain the maximum likelihood estimate of the state x^i, or equivalently, to find the weighted least squares estimate. In order to derive the estimate $\hat{x}^i$ of x^i, we require the use of the Bayes theorem and an application of calculus to find the unconstrained minimum of a quadratic form. This derivation is straightforward but lengthy. It yields the following recursive procedure for finding the estimate $\hat{x}^t,\ t = 0, 1, ..., i,\ i \leq N$:

$$\begin{aligned}
\hat{x}^t &= \bar{x}^t + K_t(y^t - H_t \bar{x}^t), && \text{(D.10)}\\
\bar{x}^{t+1} &= (A_t + I)\hat{x}^t,\ \ \bar{x}^0 \text{ given}, && \text{(D.11)}\\
K_t &= P_t {H_t}' R_t^{-1}, && \text{(D.12)}\\
P_t &= (\Sigma_t^{-1} + {H_t}' R_t^{-1} H_t)^{-1}, && \text{(D.13)}\\
\Sigma_{t+1} &= (A_t + I)P_t(I + A_t)' + G_t G_t',\ \Sigma_0 \text{ given}. && \text{(D.14)}
\end{aligned}$$

The procedure in expressions (D.10)–(D.14) is known as the *Kalman filter* for linear discrete-time processes.

The interpretation of (D.10) is that the estimate $\hat{x}^t$ is equal to the mean value $\bar{x}^t$ plus a correction term which is proportional to the difference between the actual measurement y^t and the predicted measurement $H_t\bar{x}^t$. Also,

$$\Sigma_t = E[(x^t - \bar{x}^t)(x^t - \bar{x}^t)'],$$

the error covariance before the measurement at time t, and

$$P_t = E[(x^t - \hat{x}^t)(x^t - \hat{x}^t)'],$$

the error covariance matrix after the measurement at time t. In other words, Σ_t and P_t are measures of uncertainties in the state before and after the measurement at time t, respectively. Thus, the proportionality matrix K_t can be interpreted as the ratio between the uncertainty P_t in the state and the measurement uncertainty R_t. Because of this property of K_t, it is called the *Kalman gain* in the engineering literature.

It is important to note that the propagation of P_t given by (D.13) and (D.14) is independent of the measurements. Thus, it can be computed offline and stored. The computation of updated estimates by (D.10) and (D.11) involves only the current measurement and error covariance, and can therefore be done in real time. Finally, prediction of the state beyond the period up to which measurements are available can be done as

$$\hat{x}^{t+1} = \bar{x}^{t+1} = (A_t + I)\hat{x}^t + G_t\bar{w}^t, \quad t \geq i, \quad i \in N, \tag{D.15}$$

with $\hat{x}^i$ obtained from the filter (D.10)–(D.14).

D.2 Wiener Process and Stochastic Calculus

A continuous 1-dimensional process Z is a (standard) *Wiener process* on an interval $[0, T]$ if

1. Z has independent increments;
2. The increment $Z_t - Z_\tau$ is Gaussian with mean 0 and variance $|t - \tau|$ for any $t, \tau, \in [0, T]$;
3. Z_0 is Gaussian with mean 0.

This definition easily generalizes to define a k-dimensional Wiener process.

A Wiener process is also called a Brownian motion, as it models the motion of a particle in a fluid. It has been shown that a Wiener process is nowhere differentiable; a Brownian particle does not possess a velocity at any instant. Furthermore, it is a process with unbounded variation, i.e., its length in any finite interval is infinite. The Wiener process is difficult to draw, although Fig. D.1 is an attempt to sketch a continuous sample path that, at the same time, conveys the flavor of its "wild" nature.

Nevertheless, the formal time derivative of a Wiener process is termed *white noise* in the engineering literature. Thus, $w_t = dZ/dt$ can be regarded as a stationary process in which the random variables w_t and w_τ, $t \neq \tau$, are independent with $Ew_t = Ew_\tau = 0$ and the covariance $E[w_t w_s] = \delta_{t\tau}$. One can see that w_t is a continuous time analogue of the discrete-time process w^t defined in the previous section.

Next we wish to define an integral $\int_s^t G(\tau)dZ_\tau$ for a rather wide class of processes G. Specifically, it will be the class M_0 of all real-valued, stochastic processes G on $[0,T]$ such that $\int_0^T |G(\tau)|^2 d\tau < \infty$ with probability 1. Given the wild nature of the Wiener process, the integral cannot be defined in the sense of Reimann-Steiltjes for every function in M_0.

Therefore, we resort to the concept of a *stochastic integral in the Itô sense*. For this, let us define the subclass $M \subset M_0$ such that any $G \in M$ satisfies $E\int_0^T |G(\tau)|^2 d\tau < \infty$. Let $G_j \in M$ be a step process on $[0,t]$ in the sense that there is a partition consisting of points $\tau_0, \tau_1, \ldots, \tau_m$ with $0 < \tau_0 < \tau_1 < \ldots < \tau_m = t$. For this step process, the integral equals the Riemann-Steiltjes sum

$$\int_0^t G_j(\tau)dZ_\tau = \sum_{k=1}^{m} G_j(\tau_{k-1})[Z_{\tau_k} - Z_{\tau_{k-1}}]. \tag{D.16}$$

We then define the stochastic integral for any $G \in M_0$ by taking a sequence of step processes $G_j, j = 1, 2, \ldots$, such that $\int_0^t |G_j(\tau) - G(\tau)|^2 d\tau$ converges to zero in probability as $j \to \infty$. Then, the sequence of random variables defined in (D.16) converges, as $j \to \infty$, to a limit in probability, which is defined as $\int_0^t G(\tau)dZ_\tau$, written simply as $\int_0^t GdZ$. It can be shown that the limit does not depend on the approximating sequence G_j with probability 1 for each t.

It is important to note the following important properties of Itô's stochastic integral. The integral $\int_0^t GdZ$ can be defined simultaneously for all $t \in [0,T]$, so that it is continuous on $[0,T]$. Furthermore, for any $H, G \in M_0$, we have

$$E\int_0^t G(\tau)dZ_\tau = 0, \quad E\int_0^t H(\tau)dZ_\tau = 0,$$

and

$$E\left[\left(\int_0^t G(\tau)dZ_\tau\right)\left(\int_0^t H(\tau)dZ_\tau\right)\right] = E\int_0^t G(\tau)H(\tau)d\tau. \tag{D.17}$$

Equation (D.17) serves as motivation for the frequently used symbolic notation

$$(dZ_t)^2 = dt. \tag{D.18}$$

Now that we have defined the stochastic integral, it remains to specify the stochastic differential rule. Let f, G, and X be one-dimensional stochastic processes such that $E\int_0^T |f|dt < \infty$, $G \in M_0$, X is continuous, and

$$X_t - X_0 = \int_0^t f(\tau)d\tau + \int_0^t G(\tau)dZ_\tau, \;\; 0 \le t \le T. \tag{D.19}$$

This equation is a stochastic integral equation, for which it is customary to use the suggestive notation

$$dX_t = f(t)dt + G(t)dZ_t, \;\; X_0 \text{ given},$$

or simply

$$dX = fdt + GdZ, \;\; X_0 \text{ given}. \tag{D.20}$$

Now let the one-dimensional process $Y_t = \psi(X_t, t),\ t \in [0, T]$, where the function $\psi(x, t)$ is continuously differentiable in t and twice continuously differentiable in x. Then, it possesses the stochastic differential

$$\begin{aligned} dY_t &= \psi_t(X_t,t) + \psi_X(X_t,t)dx_t + \frac{1}{2}\psi_{XX}(X_t,t)G^2(t)dt \\ &= [\psi_t(X_t,t) + \psi_X(X_t,t)f(t) + \frac{1}{2}\psi_{XX}(X_t,t)G^2(t)]dt \\ &\quad +\psi_X(X_t,t)G(t)dZ_t,\ Y_0 = \psi(X_0,0). \end{aligned} \tag{D.21}$$

Equation (D.21) is to be interpreted in the sense that its integral form from 0 to t holds with probability 1, i.e.,

$$\begin{aligned} Y(x_t,t) &= Y(x_0,0) \\ &\quad + \int_0^t [\psi_s(x_s,s) + \psi_x(x_s,s)f(s) + \frac{1}{2}\psi_{xx}(x_s,s)G^2(s)]ds \\ &\quad + \int_0^t \psi(x_s,s)G(s)dZ_s,\ w.p.1. \end{aligned} \tag{D.22}$$

It is worth pointing out that the term $\frac{1}{2}\psi_{xx}G^2dt$ does not appear in the differential rule of elementary calculus. This is an important difference as seen in Chap. 12, where we discuss stochastic optimal control problems. Also, a multi-dimensional generalization of (D.16)–(D.22) is straightforward.

D.3 The Kalman-Bucy Filter

The continuous-time analogue of the Kalman filter is known as the *Kalman-Bucy filter*. Here, the difference equation (D.2) is replaced by the linear stochastic differential equation

$$dX_t = A(t)X_t dt + G(t)dZ_t, \;\; 0 \le t \le T, \tag{D.23}$$

which is a special case of the Itô stochastic differential equation (D.20) introduced in Chap. 12. In this equation, X_t is an n-component (column) state vector, Z_t is the value at time t of a standard k-component (column) Wiener process Z, and the matrices $A(t)$ and $G(t)$ of dimensions $n \times n$ and $n \times k$, respectively, are continuous in t. Furthermore,

$$E(X_0) = \bar{X}_0, \text{ and } E[(X_0 - \bar{X}_0)(X_0 - \bar{X}_0)'] = \Sigma_0. \tag{D.24}$$

The measurement process (D.7) is replaced by

$$dY_t = H(t)X_t + \sigma(t)d\xi_t, \;\; Y_0 = 0, \tag{D.25}$$

where ξ is a standard r-dimensional Weiner process and the $k \times r$ matrix $\sigma(t)$ is such that the $k \times k$ matrix $R(t) := \sigma(t)\sigma'(t)$ is positive definite. Note that the term $\sigma(t)d\xi_t$ in (D.25) represents the noise term, which corresponds to v^t in (D.7). Thus, the term $R(t)$ corresponds to the covariance matrix R_t in Sect. D.1 on the Kalman filter.

The filtering problem is to find the weighted least square estimate of X_t given the measurements up to time t. It can be shown that the optimal estimate is the conditional expectation

$$\hat{X}_t = E[X_t | Y_s, \; 0 \le s \le t]. \tag{D.26}$$

Furthermore, it can be obtained recursively by the following Kalman-Bucy filter:

$$d\hat{X}_t = A(t)\hat{X}_t dt + K(t)[dY_t - H(t)\hat{X}_t dt], \; \hat{X}_0 = \bar{X}_0, \tag{D.27}$$

$$K(t) = P(t)H'(t)R^{-1}(t), \tag{D.28}$$

$$\begin{aligned}\dot{P}(t) &= (A(t)P(t) + P(t)A'(t) - K(t)H(t)P(t) \\ &\quad + G(t)G'(t)), \ P(0) = \Sigma_0, \end{aligned} \tag{D.29}$$

where $H'(t)$ denotes the transpose $(H(t))'$ and $R^{-1}(t)$ means the inverse $(R(t))^{-1}$, as the notational convention defined in Chap. 1. The interpretations of $P(t)$ and $K(t)$ are the same as in the previous section.

The filter (D.27)–(D.29) is the *Kalman-Bucy filter* (Kalman and Bucy 1961) for linear systems in continuous time. Equation (D.29) is called the matrix Riccati equation. Besides engineering applications, the Kalman filter and its extensions are very useful in econometric and financial modeling; see Buchanan and Norton (1971), Chow (1975), Aoki (1976), Naik et al. (1998), and Bhar (2010).

D.4 Linear-Quadratic Problems

An important problem in systems theory, especially engineering sciences, is to synthesize feedback controllers. These controllers provide optimal control as a function of the state of the system. A usual method of obtaining these controllers is to solve the Hamilton-Jacobi-Bellman partial differential equation (2.19). This equation is nonlinear in general, which makes it very difficult to solve in closed form. Thus, it is not possible in most cases to obtain optimal feedback control schemes explicitly.

It is, however, feasible in many cases to obtain perturbation feedback control, which refers to control in the vicinity of an optimal path. These perturbation schemes require the approximation of the problem by a linear-quadratic problem in the vicinity of an optimal path (see Sect. D.5), and feedback control for the approximating problem is easy to obtain.

A linear-quadratic control problem is a problem with linear dynamics and a quadratic objective function. First, we treat a special case called the *Regulator Problem*:

$$\min_u \left\{ x'(T)S_T x(T) + \int_0^T (x'Cx + u'Du)dt \right\} \tag{D.30}$$

subject to

$$\dot{x} = Ax + Bu, \quad x(0) = x_0. \tag{D.31}$$

Here $x \in E^n$, $u \in E^m$, and the appropriate dimensional matrices $C, D, A,$ and B, when time-dependent, are assumed to be continuous in time t. Furthermore, we shall assume the matrices C and S_T to be positive semidefinite and, without loss of generality, symmetric, and matrix D to be symmetric and positive definite.

To solve the regulator problem for an explicit feedback controller, we rewrite it as that of maximizing

$$J = \int_0^T -(x'Cx + u'Du)dt - x'(T)S_T x(T)$$

subject to (D.31). Clearly, this is a special case of the optimal control problem (2.4) and we can apply (2.15) and (2.16) to obtain the Hamilton-Jacobi-Bellman equation

$$0 = \max_u \left\{ -(x'Cx + u'Du) + V_x[Ax + Bu] + V_t \right\} \tag{D.32}$$

with the terminal boundary condition

$$V(x, T) = -x'(T)S_T x(T). \tag{D.33}$$

By checking that $V(\gamma x, t) = \gamma^2 V(x, t)$ and $V(x, t) + V(y, t) = \frac{1}{2}[V(x+y, t) + V(x - y, t)$, we can establish that the value function $V(x, t)$ is of a quadratic form. Thus, let

$$V(x(t), t) = -x'(t)S(t)x(t) \tag{D.34}$$

for some matrix $S(t)$, symmetric without loss of generality. Then $V_t = -x'\dot{S}x$ and $V_x = -2(Sx)' = -2x'S$. Using these relations in (D.32), we get

$$\begin{aligned} x'\dot{S}x &= \max_u \left\{ -x'Cx - u'Du - 2x'SAx - 2x'SBu \right\} \\ &= -\min_u \left\{ x'Cx + u'Du + 2x'SAx + 2x'SBu \right\}. \end{aligned} \tag{D.35}$$

To find the minimum of the expression on the right-hand side of (D.35), we observe the following identity obtained by completing the square:

$$\begin{aligned} x'Cx + u'Du + 2x'SAx + 2x'SBu &= (u + D^{-1}B'Sx)'D(u + D^{-1}B'Sx) \\ &\quad + x'(C - SBD^{-1}B'S + SA + A'S)x. \end{aligned}$$

Because matrix D is positive definite, it follows that the minimum is achieved in (D.35) by the control

$$u^* = -D^{-1}B'Sx. \tag{D.36}$$

Then from (D.35) and (D.36), we obtain,

$$x'\dot{S}x = -x'[C - SBD^{-1}B'S + SA + A'S]x. \tag{D.37}$$

Since this equation holds for all x, we have the matrix differential equation

$$\dot{S} = -SA - A'S + SBD^{-1}B'S - C, \tag{D.38}$$

called a *matrix Riccati equation*, with the terminal condition

$$S(T) = S_T \tag{D.39}$$

obtained from (D.33), where S_T is specified in (D.30).

A solution procedure for Riccati equations appears in Bryson and Ho (1975) or Anderson and Moore (1990). With the solution S of (D.38) and (D.39), we have the optimal feedback control as in (D.36).

To see that the optimal control u^* in (D.36) maximizes the Hamiltonian $H = -x'Cx - u'Du + V_x[Ax + Bu]$, let us use (D.32) to obtain

$$2(Du^*)' = 2u^{*\prime}D' = -2x'SB(D')^{-1}D' = -2x'SB = V_xB,$$

which is precisely the first-order condition for the maximum of the right-hand side of (D.32). Moreover, the first-order condition yields a global maximum of the Hamiltonian, which is concave since the matrix D is positive definite.

A generalization of (D.30) to include a cross-product term to allow for interactions between the state x and control u, which would be useful in the next section on the second variation, is to set

$$J = -x'(T)S_Tx(T) - \int_0^T (x', u') \begin{bmatrix} C & N \\ N' & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt, \tag{D.40}$$

and the problem is to maximize J subject to the state equation (D.31). It is easy to see that the integrand in (D.40) can be rewritten as $x'Cx+u'Du+2x'Nu$. Furthermore, with the definition $\tilde{u} = u+D^{-1}N'x$, the generalized problem defined by (D.40) and (D.31) can be reduced to the standard regulator problem of maximizing $\int_0^T -[\tilde{u}'D^{-1}\tilde{u} + x'(C - ND^{-1}N')x] - x'(T)S_Tx(T)$ subject to $\dot{x} = (A - BD^{-1}N')x + B\tilde{u}$, provided that the matrix $C - ND^{-1}N'$ is positive semidefinite. We can then use formulas (D.36), (D.38), and (D.39), to obtain the solution of the transformed problem and then use the definition of $\tilde{u}$ to write the feedback control of the generalized problem as

$$u^*(x) = -D^{-1}[N' + B'S]x, \tag{D.41}$$

where

$$\begin{aligned}\dot{S} &= -S(A - BD^{-1}N') - (A' - ND^{-1}B')S \\ &\quad + SBD^{-1}B'S + ND^{-1}N' - C \\ &= -SA - A'S + (SB + N)D^{-1}(B'S + N') - C \end{aligned} \tag{D.42}$$

with

$$S(T) = S_T. \tag{D.43}$$

D.4.1 Certainty Equivalence or Separation Principle

Suppose Eq. (D.31) is changed by the presence of the stochastic term $G(t)dZ_t$ as defined in (D.23) so that we have the Itô equation

$$dX_t = (A(t)X_t + B(t)U_t)dt + G(t)dZ_t,$$

and X_0 is a normal random variable with

$$E[X_0] = 0, \quad E[X_0X_0'] = \Sigma_0.$$

Because of the presence of uncertainty in the system equation, we modify the objective function in (D.40) as follows:

$$\max \left\{ J = E\left[-X_T'S_TX_T - \int_0^T (X_t', U_t') \begin{pmatrix} C_t & N_t \\ N_t' & D_t \end{pmatrix} \begin{pmatrix} X_t \\ U_t \end{pmatrix} dt \right] \right\}.$$

Assume further that X_t cannot be directly measured and the measurement process is given by (D.25), i.e.,

$$dY_t = H(t)X_t + \sigma(t)d\xi_t, \; Y_0 = 0.$$

The optimal control U_t^* for this linear-quadratic stochastic optimal control problem can be shown to be given by (D.41) with X_t replaced by its estimate $\hat{X}_t$; see Arnold (1974). Thus,

$$U_t^* = -D(t)^{-1}[N'(t) + B'(t)S(t)]\hat{X}_t,$$

where $S(t)$ is given by (D.42) and (D.43), and $\hat{X}_t$ is given by the *Kalman-Bucy filter*:

$$\begin{aligned} d\hat{X}_t &= [A(t)\hat{X}_t + B(t)U_t^*]dt + K(t)(dY_t - H(t)\hat{X}_t dt), \quad \hat{X}(0) = 0, \\ K(t) &= P(t)H'(t)R^{-1}(t), \\ \dot{P}(t) &= A(t)P(t) + P(t)A'(t) - K(t)H(t)P(t) + G(t)G'(t), \quad P(0) = \Sigma_0. \end{aligned}$$

The above procedure has received two different names in the literature. In economics it is called the *certainty equivalence principle*; see Simon (1956). In engineering and mathematics literature it is called the *separation principle*; see Fleming and Rishel (1975). When we call it the certainty equivalence principle, we are emphasizing the fact that $\hat{X}_t$ can be used for the purposes of optimal feedback control as if it were the certain value of the state variable X_t. Whereas the term separation principle emphasizes the fact that the process of determining the optimal control can be broken down into two steps: first, estimate X_t by using the optimal filter; second, use that estimate in the optimal feedback control formula for the deterministic problem.

D.5 Second-Order Variations

Second-order variations in optimal control theory are analogous to the second-order conditions in the classical optimization problem of calculus. To discuss the second-order variational condition is difficult when the control variable u is constrained to be in the control set Ω. So we make the simplifying assumption that $\Omega = R^m$, and thus the control u is unconstrained. As a result, we are now dealing with the problem:

$$\max_u \left\{ J = \int_0^T F(x,u,t)dt + \Phi[x(T)] \right\} \tag{D.44}$$

subject to

$$\dot{x} = f(x,u,t), \quad x(0) = x_0. \tag{D.45}$$

From Chap. 2, we know that the first-order necessary conditions for this problem are given by

$$\dot{\lambda} = -H_x, \quad \lambda(T) = 0, \tag{D.46}$$

$$H_u = 0, \tag{D.47}$$

where the Hamiltonian H is given by

$$H = F + \lambda f. \tag{D.48}$$

Since u is unconstrained, these conditions may be easily derived by the method of calculus of variations. To see this, we write the augmented objective functional as

$$\bar{J} = \Phi[x(T)] + \int_0^T [H(x, u, \lambda, t) - \lambda \dot{x}] dt. \tag{D.49}$$

Consider small perturbation from the extremal path given by (D.45)–(D.48) as a result of small perturbations $\delta x(0)$ in the initial state. Define the resulting perturbations in state, adjoint, and control variables by $\delta x(t)$, $\delta\lambda(t)$, and $\delta u(t)$, respectively. These, of course, will be obtained by linearizing ((D.45)–(D.47)) around the external path:

$$\frac{d\delta x}{dt} = f_x \delta x + f_u \delta u, \quad \delta x(0)\text{specified}, \tag{D.50}$$

$$\frac{d\delta\lambda}{dt} = -(H_{xx}\delta x)^T - \delta\lambda f^T - (H_{xu}\delta u), \tag{D.51}$$

$$\begin{aligned} \delta H_u &= (H_{ux}\delta x)^T + \delta\lambda(H_u\lambda)^T + (H_{uu}\delta u)^T \\ &= (H_{uu}\delta x)^T + \delta\lambda f_u + (H_{uu}\delta u)^T = 0. \end{aligned} \tag{D.52}$$

Alternatively, we may consider an expansion of the objective function and the state equation to second order since the first-order terms vanish about a trajectory which satisfies ((D.44)–(D.47)). From Bryson and Ho (1975), this may be accomplished by expanding (D.49) to second order and all the constraints to first order. Thus, we have

$$\delta^2 \bar{J} = \frac{1}{2}(\delta x^T(T)\Phi_{xx}\delta x(T)) + \frac{1}{2}\int_0^T (\delta x, \delta u) \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt \tag{D.53}$$

subject to

$$\frac{d\delta x}{dt} = f_x \delta x + f_u \delta u, \quad \delta x(0) \text{ specified}. \tag{D.54}$$

Since we are interested in a neighboring extremal path, we must determine $\delta u(t)$ so as to maximize $\delta^2 \bar{J}$ subject to (D.54). This problem is a linear-quadratic problem discussed in the previous section. For this problem, the optimal control $\delta u^*(t)$ is given by the formula (D.42), provided $H_{uu}(t)$ is nonsingular for $0 \leq t \leq T$. The case when $H_{uu}(t)$ is singular for a finite time interval is treated in Sect. D.6. Thus, recognizing that $G = \Phi_{xx}$, $C = H_{xx}$, $N = H_{xu}$, $D = H_{uu}$, $A = f_x$, and $B = f_u$, we have

$$\delta u^*(t) = H_{uu}^{-1}[H_{ux} + f_u^T S(t)]\delta x(t), \tag{D.55}$$

where

$$\dot{S} + S f_x + f_u^T S - (S f_u + H_{xu}) H_{uu}^{-1} (f_u^T S + H_{ux}) + H_{xx} = 0, \qquad S(T) = \Phi_{xx}. \tag{D.56}$$

While a number of second-order conditions can be obtained by proceeding further from this manner, we will be interested only in the concavity condition (or *strengthened Legendre-Clebsch condition*). It is possible to show that neighboring stationary paths exist (in a weak sense; i.e., δx and δu are small) if

$$H_{uu}(t) < 0 \quad \text{for} \quad 0 \leq t \leq T, \tag{D.57}$$

or in other words, $H_{uu}(t)$ is negative semidefinite. First-order conditions, conditions (D.57), and the condition that $S(t)$ is finite for $0 \leq t \leq T$ represent sufficient conditions for a trajectory to be a local maximum. We are not being specific here because in this book we would be relying mostly on the sufficiency conditions developed in Chaps. 2–4, which are based on certain concavity requirements. We are stating (D.57) because of its similarity to the second-order condition for a local maximum in the classical maximization problem.

We must note that

$$H_u = 0 \quad \text{and} \quad H_{uu} \leq 0 \tag{D.58}$$

form necessary conditions for a trajectory to be a local maximum.

D.6 Singular Control

In some optimization problems including some problems treated in this text, extremal arcs satisfying $H_u = 0$ occur on which the matrix H_{uu} is singular. Such arcs are called *singular arcs*. Note that these arcs satisfy (D.58) but not the strengthened condition (D.57). While no general sufficiency conditions are available for singular arcs, some additional necessary conditions known as the generalized Legendre-Clebsch conditions have been developed. A good reference on singular control is Bell and Jacobson (1975).

We will only discuss the case in which the Hamiltonian is linear in one or more of the control variables. For these systems, $H_u = 0$ implies that the coefficient of the linear control term in the Hamiltonian vanishes identically along a singular arc. Thus, the control is not determined in terms of x and λ by the Hamiltonian maximizing condition $H_u = 0$. Instead, the control is determined by the requirement that the coefficient of these linear terms remain zero on the singular arc. That is, the time derivatives of H_u must be zero. Having obtained the control by setting $dH_u/dt = 0$ (or by setting higher time derivatives to equal zero) along the singular arc, we must check additional necessary conditions analogous to the second-order condition (D.57). For a maximization problem with a single control variable, these conditions turn out to be

$$(-1)^k \frac{\partial}{\partial u}\left[\frac{d^{2k} H_u}{dt^{2k}}\right] \leq 0, \qquad k = 0, 1, 2, ... \tag{D.59}$$

The conditions (D.59) are called the *generalized Legendre-Clebsch conditions.*

For applications of these conditions to problems in production and finance, see e.g., Maurer et al. (2005) and Davis and Elzinga (1971). The Davis-Elzinga model is covered in Exercise 5.17 in Chap. 5. For numerical solutions of singular control problems, see Maurer (1976).

Example D.1 We present an example treated by Johnson and Gibson (1963):

$$\max\left\{J = -\frac{1}{2}\int_0^T x_1^2 dt\right\} \tag{D.60}$$

subject to

$$\dot{x}_1 = x_2 + u, \qquad x_1(0) = a, \tag{D.61}$$

$$\dot{x}_2 = -u, \qquad x(0) = b, \tag{D.62}$$

$$x_1(T) = x_2(T) = 0. \tag{D.63}$$

Solution We form the Hamiltonian

$$H = -\frac{1}{2}x_1^2 + \lambda_1(x_2 + u) + \lambda_2(-u), \tag{D.64}$$

where the adjoint equations are

$$\dot{\lambda}_1 = x_1, \quad \dot{\lambda}_2 = -\lambda_1. \tag{D.65}$$

The optimal control is bang-bang plus singular. Singular arcs must satisfy

$$H_u = \lambda_1 - \lambda_2 = 0 \tag{D.66}$$

for a finite time interval. The optimal control can, therefore, be obtained by

$$\frac{dH_u}{dt} = \dot{\lambda}_1 - \dot{\lambda}_2 = x_1 + \lambda_1 = 0. \tag{D.67}$$

Differentiating once more with respect to time t, we obtain

$$\frac{d^2 H_u}{dt^2} = \dot{x}_1 + \dot{\lambda}_1 = x_2 + u + x_1 = 0,$$

which implies

$$u = -(x_1 + x_2) \tag{D.68}$$

along the singular arc. We now verify for the example, the generalized Legendre-Clebsch condition (D.59) for $k = 1$:

$$-\frac{\partial}{\partial u}\left[\frac{d^2 H_u}{dt^2}\right] = -1 \leq 0. \tag{D.69}$$

D.7 Global Saddle Point Theorem

In this section, we provide an important result for a class of stationary infinite-horizon optimal control problems such as those treated in Chap. 11. In particular, we are concerned here with the one-dimensional state problem defined in (3.97) without the mixed constraint and the terminal inequality constraints, i.e.,

$$\max\left\{J = \int_0^\infty \phi(x,u)e^{-\rho t}dt\right\}, \tag{D.70}$$

$$\dot{x} = f(x,u), \ x(0) = x_0. \tag{D.71}$$

An application of the maximum principle results in an adjoint equation

$$\dot{\lambda} = \rho\lambda - \phi_x - \lambda f_x \tag{D.72}$$

and a Hamiltonian maximizing control $u^*(x, \lambda)$. Substituting this for u in (D.71) and (D.72) gives rise to a canonical system of differential equations

$$\dot{x} = f^*(x, \lambda) \text{ and } \dot{\lambda} = \psi^*(x, \lambda). \tag{D.73}$$

A saddle point $(\bar{x}, \bar{\lambda})$ of the system (D.73) satisfies

$$f^*(\bar{x}, \bar{\lambda}) = 0 \text{ and } \psi^*(\bar{x}, \bar{\lambda}) = 0. \tag{D.74}$$

The important issue for this problem is the existence and uniqueness of an optimal path that steers the system from an initial value x_0 to the steady state $\bar{x}$. This is equivalent to finding a value λ_0 so that the system (D.73) starting from (x_0, λ_0) moves asymptotically to $(\bar{x}, \bar{\lambda})$. A sufficient condition for this to happen is given in the following theorem.

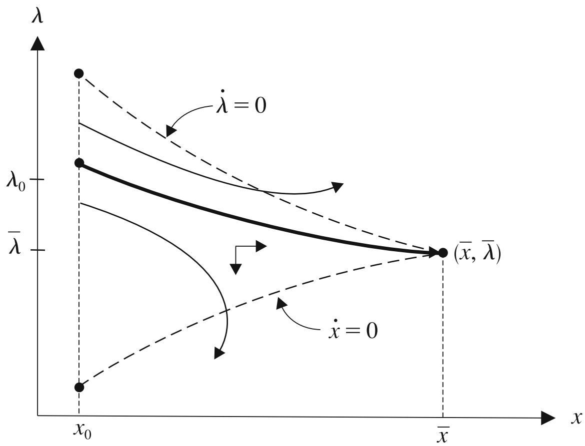

Figure D.1: Phase diagram for system (D.73)

Theorem D.1 (Global Saddle Point Theorem) *Let* $(\bar{x}, \bar{\lambda})$ *be a unique saddle point of the canonical system* (D.73) *of the differential equations and let* x_0 *be a given initial state for which the vertical line* $x = x_0$ *(see Fig. D.1) intersects both isoclines* $\dot{x} = f^*(x, \lambda) = 0$ *and* $\dot{\lambda} = \psi^*(x, \lambda) = 0$. *Assume further that the region bounded by the isoclines and the line* $x = x_0$ *has a triangular shape as in Fig. D.1 (i.e., the isoclines themselves do not intersect in the open interval between* x_0 *and* $\bar{x}$*). Then, there exists a unique saddle point path starting for* $x = x_0$ *and leading to the saddle point* $(\bar{x}, \bar{\lambda})$.

The proof of this theorem, based on Theorem 1.2 and Corollaries 1.1 and 1.2 from Hartman (1982), can be found in Feichtinger and Hartl (1986).

D.8 The Sethi-Skiba Points

In Exercise 2.9, we defined autonomous optimal control problems. Here, we limit the discussion to autonomous systems that are discounted infinite-horizon optimal control problems with one-dimensional state, defined as follows:

$$\max_{u(t)\in\Omega}\left\{J=\int_0^\infty e^{-\rho t}\phi(x(t),u(t))dt\right\}$$

subject to

$$\dot{x}(t)=f(x(t),u(t)),\ \ x(0)\text{ given},$$

with $\rho>0$ as the discount rate. In addition to assuming that the function ϕ and f are continuously differentiable, we assume that the integral in the objective function J converges for any admissible solution $x(t),u(t),t\geq 0$. In such problems, there may arise multiple equilibria depending on the initial condition. Suppose x_0 is an initial value for $x(0)$, such that the system starting from it exhibits multiple optimal solutions or equilibria. Thus, at least in the neighborhood of x_0, the system moves to one equilibrium if $x(0)>x_0$ and to another if $x(0)<x_0$. In other words, x_0 is an indifference point from which the system could move to either of two equilibria. Such points were originally identified by Sethi (1977b, 1979c). Subsequently, Skiba (1978) and Dechert and Nishimura (1983) explored these indifference points for one-sector optimal economic growth models with nonconvex production functions, in contrast to concave production functions treated in Sect. 11.1. These points are also referred to as the *DNSS points*, where the acronym DNSS stands for Dechert, Nishimura, Sethi, and Skiba. Before it became known that Sethi (1977b) had already identified them prior to Skiba (1978), these points were also called Skiba points.

Below we present a simple example that exhibits a Sethi-Skiba point at $x_0=0$. For further discussion on these points, see Grass et al. (2008), Zeiler et al. (2010), Kiseleva and Wagener (2010), and Caulkins et al. (2015a).

Example D.2 Solve the problem:

$$\max\left\{J=\int_0^\infty e^{-\rho t}x(t)u(t)dt\right\}$$

subject to

$$\dot{x}(t)=-x(t)+u(t),\ x(0)=x_0, \tag{D.75}$$
$$u(t)\in[-1,+1],\ t\geq 0.$$

Let us first solve this problem for $x_0<0$. We form the Hamiltonian

$$H=x(t)u(t)+\lambda(t)(-x(t)+u(t)) \tag{D.76}$$

with

$$\dot{\lambda}(t)=(1+\rho)\lambda(t)-u(t). \tag{D.77}$$

Since H is linear in u, the optimal policy is

$$u^*(t)=\text{bang}[-1,1;x(t)+\lambda(t)]. \tag{D.78}$$

For $x_0<0$, the state equation reveals that $u^*(t)=-1$ will give the largest decrease of $x(t)$ and keep $x(t)<0,\ t\geq 0$. Thus, it will maximize the product $x(t)u(t)$ for each $t>0$. We also note that the long-run stationary equilibrium in this case is $(\bar{x},\bar{u},\bar{\lambda})=(-1,-1,-1/(1+\rho))$. It is also easy to verify that the solution $u^*(t)=-1,\ x^*(t)=-1+e^{-t}(x_0+1)$, and $\lambda(t)=-1/(1+\rho),\ t\geq 0$, satisfies (D.75), (D.77) along with the sufficiency transversality condition (3.99), and maximizes the Hamiltonian in (D.76).

Similarly, we can argue that for $x_0>0$, the optimal solution is $u^*(t)=+1,\ x^*(t)=1+e^{-t}(x_0-1)>0$, and $\lambda(t)=1/(1+\rho),\ t\geq 0$. The long-run stationary equilibrium in this case is $(\bar{x},\bar{u},\bar{\lambda})=(1,1,1/(1+\rho)$.

Then by symmetry, we can conclude that if $x_0=0$, both $u^*(t)=-1$ and $u^*(t)=+1,\ t\geq 0$, yield the same objective function, and hence both are optimal. Thus, $x_0=0$ is a Sethi-Skiba point for this example.

Clearly, at this point, the choice between using $u^*(0)=-1$ and $u^*(0)=+1$ will determine the equilibrium the system approaches. Notice that once the system has moved away from $x_0=0$, there is no more choice left in choosing the control.

It is possible that at a Sethi-Skiba point, a decision maker can influence the equilibrium that the system would move to, by choosing a control from the set of possible optimal controls. This may have important implications. In a model of controlling illicit drugs, Grass et al. (2008) derive a Sethi-Skiba point, signifying a critical number of addicts, such that if there are fewer addicts than the critical number, it is optimal to use an eradication strategy that uses massive treatment spending that drive the number of addicts down to zero. On the other hand, if there are more than the critical number of addicts, then it is optimal to use an accommodation strategy that uses a moderate level of treatment spending that balances the social cost of drug use and the cost of treatment.

This is a case of a classic Sethi-Skiba point acting as a "tipping point" between the two strikingly different equilibria, one of which may be more socially or politically favored than the other, and the social planner can use an optimal control to move to the more favored equilibrium.

We conclude this subsection by mentioning that the Sethi-Skiba points are exhibited in the production management context by Feichtinger and Steindl (2006) and Moser et al. (2014), in the open-source software context by Caulkins et al. (2013a), and in other contexts by Caulkins et al. (2011, 2013b, 2015a).

D.9 Distributed Parameter Systems

Thus far, our efforts have been directed to the study of the control of systems governed by systems of ordinary differential or difference equations. Such systems are often called *lumped parameter systems*. It is possible to generalize these to systems in which the state and control variables are defined in terms of space as well as time dimensions. These are called *distributed parameter systems* and are described by a set of partial differential or difference equations.

For example, in the lumped parameter advertising models of the type treated in Chap. 7, we solved for the optimal rate of advertising expenditure at each instant of time. However, in the analogous distributed parameter advertising models, we must obtain the optimal advertising expenditure rate at every geographic location of interest at each instant of time; see Seidman et al. (1987) and Marinelli and Savin (2008). In other economic problems, the spatial coordinates might be income, quality, age, etc. Derzko et al. (1980), for example, discuss a cattle-ranching model in which the spatial dimension measures the age of a cow.

Let y denote a one dimensional spatial coordinate, let t denote time, and let $x(t, y)$ be a one dimensional state variable. Let $u(t, y)$ denote a control at (t, y) and let the state equation be

$$\frac{\partial x}{\partial t} = g(t, y, x, \frac{\partial x}{\partial y}, u) \tag{D.79}$$

for $t \in [0, T]$ and $y \in [0, h]$. We denote the region $[0, T] \times [0, h]$ by D, and we let its boundary ∂D be split into two parts Γ_1 and Γ_2 as shown in Fig. D.2. The initial conditions will be stated on the part Γ_1 of the boundary ∂D as

$$x(0, y) = x_0(y) \tag{D.80}$$

and

$$x(t, 0) = v(t). \tag{D.81}$$

In Fig. D.2, (D.80) is the initial condition on the vertical portion of Γ_1, whereas (D.81) is that on the horizontal portion of Γ_1. More specifically, in (D.80) the function $x_0(y)$ gives the starting distribution of x with respect to the spatial coordinate y. The function $v(t)$ in (D.81) is an exogenous *breeding function* of x at time t when $y = 0$, which in the cattle ranching model mentioned above, measures the number of newly born calves at time t. To be consistent we make the obvious assumption that

$$x(0, 0) = x_0(0) = v(0). \tag{D.82}$$

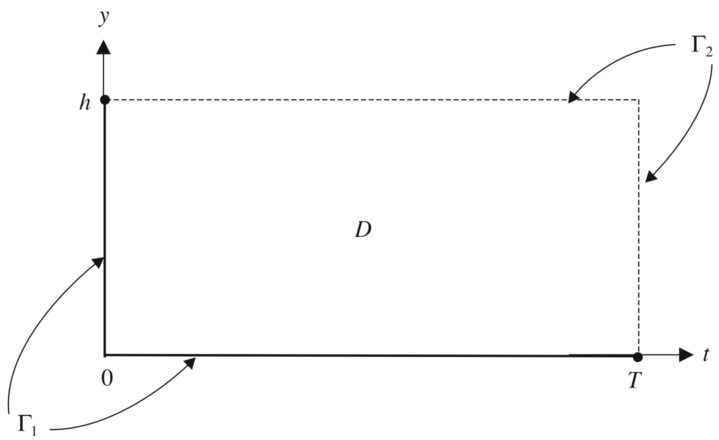

Figure D.2: Region D with boundaries Γ_1 and Γ_2

Let $F(t, y, x, u)$ denote the profit rate when $x(t, y) = x$ and $u(t, y) = u$ at a point (t, y) in D. Let $Q(t)$ be the price of one unit of $x(t, h)$ at time t and let $S(y)$ be the salvage value of one unit of $x(T,y)$ at time T. Then the objective function is:

$$\max_{u(t,y)\in\Omega}\left\{J=\int_0^T\int_0^h F(t,y,x(t,y),u(t,y))dydt\right.$$
$$\left.+\int_0^T Q(t)x(t,h)dt+\int_0^h S(y)x(T,y)dy\right\}, \tag{D.83}$$

where Ω is the set of allowable controls.

We will formulate, without giving proofs, a procedure for solving the problem in (D.79)–(D.83) by a distributed parameter maximum principle, which is analogous to the ordinary one. A more complete treatment of this topic can be found in Sage (1968), Butkowskiy (1969), Ahmed and Teo (1981), Tzafestas (1982b), Derzko et al. (1984), Brokate (1985), and Veliov (2008).

In order to obtain necessary conditions for a maximum, we introduce the Hamiltonian

$$H=F+\lambda f, \tag{D.84}$$

where the spatial adjoint function $\lambda(t,y)$ satisfies

$$\frac{\partial\lambda}{\partial t}=-\frac{\partial H}{\partial x}+\frac{\partial}{\partial t}\left[\frac{\partial H}{\partial x_t}\right]+\frac{\partial}{\partial y}\left[\frac{\partial H}{\partial x_y}\right], \tag{D.85}$$

where $x_t=\partial x/\partial t$ and $x_y=\partial x/\partial y$. The boundary conditions on λ are stated for the Γ_2 part of the boundary of D (see Fig. D.2) as follows:

$$\lambda(t,h)=Q(t) \tag{D.86}$$

and

$$\lambda(T,y)=S(y). \tag{D.87}$$

Once again we need a consistency requirement similar to (D.82). It is

$$\lambda(T,h)=Q(T)=S(h), \tag{D.88}$$

which gives the consistency requirement in the sense that the price and the salvage value of a unit $x(T,h)$ must agree.

We let $u^*(t,y)$ denote the optimal control at (t,y). Then the distributed parameter maximum principle requires that

$$H(t,y,x^*,x_t^*,x_y^*,u^*,\lambda)\geq H(t,y,x^*,x_t^*,x_y^*,u,\lambda) \tag{D.89}$$

for all $(t,y)\in D$ and all $u\in\Omega$.

We have stated only a simple form of the distributed parameter maximum principle which is sufficient for most applications in management science and economics, such as Derzko et al. (1980), Haurie et al. (1984), Feichtinger et al. (2006a), and Kuhn et al. (2015). More general forms of the maximum principle are available in the references cited earlier. Among other things, these general forms allow for the function F in (D.83) to contain arguments such as $\partial x/\partial y$, $\partial^2 x/\partial y^2$, etc. It is also possible to consider controls on the boundary. In this case $v(t)$ in (D.81) will become a control variable.

Exercises for Appendix D

E D.1 Consider the discrete-time dynamics

$$\begin{cases} x^{t+1} - x^t &= ax^t + w^t, \\ y^t &= hx^t + v^t, \end{cases} \tag{D.90}$$

where w^t and v^t are Gaussian purely random sequences with

$$E[w^t] = E[v^t] = 0, \;\; E[w^t w^\tau] = q\delta_{t\tau},$$

$$E[v^t v^\tau] = r\delta_{t\tau},$$

where h, q, and r are constants. The initial condition x^0 is a Gaussian random variable with mean μ and variance Σ_0. Use the Kalman filter (D.10)–(D.14) to obtain the recursive equations

$$\hat{x}^{t+1} - \hat{x^t} = a\hat{x}^t + \frac{P_{t+1}h}{r}(y^{t+1} - h(a+1)\hat{x}^t), \;\; \hat{x}_0 = \mu$$

and

$$P_{t+1} = \frac{r[(a+1)^2 P_t + q]}{r + h^2[(a+1)^2 P_t + q]}, \;\; p_0 = r\Sigma_0/(r + \Sigma_0 h^2).$$

E D.2 Consider the continuous-time dynamics of the simplest nontrivial filter

$$\begin{cases} dX_t &= \sqrt{q}\, dZ_t, x_0 \text{ given}, \\ dY_t &= X_t + \sqrt{r}\, d\xi_t, Y_0 = 0, \end{cases} \tag{D.91}$$

where Z and ξ are standard Brownian motions, q and σ are positive constants, and X_0 is a Gaussian random variable with mean 0 and variance Σ_0. Show that the Kalman-Bucy filter is given by

$$d\hat{X}_t = \frac{P(t)}{r}(dY_t - \hat{X}_t dt), \quad \hat{X}_0 = 0,$$

and

$$P(t) = \sqrt{rq}\,\frac{1 + be^{-2\alpha t}}{1 - be^{-2\alpha t}},$$

where

$$\alpha = \sqrt{q/r} \quad \text{and} \quad b = \frac{\Sigma_0 - \sqrt{rq}}{\Sigma_0 + \sqrt{rq}}.$$

Hint: In solving the Riccati equation for $P(t)$, you will need the formula

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} ln \left(\frac{u-a}{u+a} \right).$$

E D.3 Let $w(u) = u$ in Exercise 7.39. Analyze the various cases that may arise in this problem from the viewpoint of obtaining the Sethi-Skiba points.

E D.4 The economic growth model of Sect. 11.1.3 exhibits a Sethi-Skiba point if we assume the production function $f(k)$ to be convex initially and then concave, i.e., $f''(k) > 0$ for $k < k^s$ and $f''(k) = 0$ at $k = k^s$ and $f''(k) < 0$ for $k > k^s$ for some $k^s \in (0, \infty)$. Analyze this problem with the additional mixed constraints $0 \leq c \leq f(k)$. See Skiba (1978) and Feichtinger and Hartl (1986).

Appendix E

Answers to Selected Exercises

Completely worked solutions to all exercises in this book are contained in a forthcoming *Teachers' Manual*, which will be made available to instructors by the publisher when it is ready.

Chapter 1

1.1 (a) Feasible. $J = -333{,}333$.

1.3 $J = 36$.

1.5 (a) $C = \$157{,}861/\text{year}$.
(b) $J = 103.41$ utils.
(c) \$15,000/year.

1.6 (b) $W(20) = 985{,}648$; $J = 104.34$.

1.14 $\text{imp}(G_1, G_2; t) = (G_1 - G_2)e^{-\rho t}$.

Chapter 2

2.4 The optimal control is

$$u^*(t) = \begin{cases} 2 & \text{if } 0 \le t \le 2 - \ln 2.5, \\ \text{undefined} & \text{if } t = 2 - \ln 2.5, \\ 0 & \text{if } t > 2 - \ln 2.5. \end{cases}$$

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2.17 $u^* = \text{bang}(0, 1; \lambda_1 - \lambda_2)$, where $\lambda(t) = (8e^{-2(t-18)}, 4e^{-2(t-18)})$.

2.18 (a) $x(100) = 30 - 20e^{-10} \approx 30$.

(b) $u^* = 3$ for $t \in [0, 100]$.

(c) $u^*(t) = \begin{cases} 3 & \text{for } t \in [0,\ 100 - 10\ln 2], \\ 0 & \text{otherwise.} \end{cases}$

2.19 $\dot{x} = f(x) + b(x)u,\ x(0) = x_0,\ x(T) = 0.$

$$\dot{u} = [b(x)^2 g'(x) - 2cu\{b(x)f'(x) - b'(x)f(x)\}]/[2cb(x)].$$

2.22 (a) $u^* = \text{bang}[0, 1; (g_1K_1 + g_2K_2)(\lambda_1 - \lambda_2)]$.

(c) $\hat{t} = T - (1/g_2)\ln[(g_2b_1 - g_1b_2)/(g_2 - g_1)b_2]$.

2.29 (a) $C^*(t) = \rho W_0 e^{(r-\rho)t}/(1 - e^{-\rho T})$.

(b) $\dot{C}^*(t) = K(r - \rho)$.

2.30 (a) $\dot{\lambda} = x + 3\lambda x^2,\ \lambda(1) = 0$, and $\dot{x} = -x^3 + \lambda,\ x(0) = 1$.

Chapter 3

3.1 $x - u_1 \geq 0,\ u_1 - u_2 \geq 0,\ u_1 \geq 0,\ 1 + u_2 \geq 0.$

3.2 $X = [-1,\ 5]$.

3.9 $L = F(x, u) + \lambda f(x, u, t) + \mu g(x, u, t),$

$$\dot{\lambda} = -(\dot{\alpha}/\alpha)\lambda - \frac{\partial L}{\partial x},\ \mu \geq 0,\ \mu g = 0.$$

3.13 (a) $\lambda(t) = 10\left[1 - e^{0.1(t-100)}\right],$

$$\mu = \begin{cases} 0 & \text{if } K = 300, \\ -10\left[1 - e^{0.1(K/3-100)}\right] & \text{if } K < 300, \end{cases}$$

$u^*(t) = \text{bang}[0, 3; \lambda + \mu]$.

The problem is infeasible for $K > 300$.

(b) $t^{**} = \min[0,\ 100 - K/3]$,

$$u^*(t) = \begin{cases} 0 & \text{for } t \le t^{**}, \\ 3 & \text{for } t > t^{**}. \end{cases}$$

3.17 $\lambda(t) = t - 1$.

3.23 11.87 min.

3.25 $u^* = -1,\ T^* = 5$.

3.26 $u^* = -2,\ T^* = 5/2$.

3.43 (a) $\{\bar{I}, \bar{P}, \bar{\lambda}\} = \{I_1 - \rho(S - P_1),\ S,\ 2(S - P_1)\}$.
(b) $I = I_1$.

Chapter 4

4.2 $u^*(t) = -1,\ \mu_1 = -\lambda = 1/2 - t,\ \mu_2 = \eta = 0$.

4.3 One solution appears in Fig. 3.1. Another solution is $u(t) = 1/2$ for $t \in [0, 2]$. There are many others.

4.5 (a) $u^* = 0$.

(c) $$u^* = \begin{cases} 1, & 0 \le t \le 1 - T, \\ 0, & 1 - T < t \le T. \end{cases}$$

(e) $J = -(1/8 + 1/8K)$.

(f) $J = -1/8$.

Chapter 5

5.1 (a) $$u^*(t) = \begin{cases} 5, & t \le 1 + 6 \ln 0.99 \approx 0.94, \\ 0, & t > 0.094. \end{cases}$$

(b) $$\lambda_2(t)/\lambda_1(t) = e^{3(t^2-4t+1)/12},\ u^*(t) = \begin{cases} -5, & 0 \le t \le 0.28, \\ 0, & 0.28 < t \le 0.4, \\ 5, & 0.4 < t \le 0.93, \\ 0, & 0.93 < t \le 1.0. \end{cases}$$

5.4 (b) $f(t^*) = t^* - 10\ln(1 - 0.3e^{0.1t^*})$.

(c) $t^* = 1.969327,\ J(t^*) = 19.037$.

5.8 $u^* = v^* = 0$ for all t.

5.10 $u^* = 0,\ v^* = 4/5$ for $t \in [0, 49]$,

$u^* = 0,\ v^* = 0$ for $t \in [49, 60]$,

$J^* = 34{,}420$.

Chapter 6

6.4 $Q(t) = t^4 - 160t^3 + 1740t^2 - 7360t + 9639$.

6.9 $v^* = \text{sat}[-V_2, V_1; (\lambda_2 - \lambda_1 p)2\beta\lambda_1]$.

6.10 $v^*(t) \approx 3e^{-3t},\ y^*(t) \approx 1 - 3e^{-3t}$.

6.12 $J^* = 10.56653$.

6.14
$$u^*(t) = \begin{cases} 0, & 0 \le t \le 7/3, \\ 2, & 7/3 < t < 3, \\ -1, & 3 \le t < 13/3, \\ 0, & 13/3 \le t \le 6. \end{cases}$$

6.15
$$\mu_1 = \begin{cases} -\frac{5}{2}t + \frac{5}{2}, & t \in [0, 1], \\ 0, & t \in (0, 3]. \end{cases}$$

$$\mu_2 = \begin{cases} 0, & t \in [0, 1.8), \\ -\frac{1}{2}t + \frac{3}{2}, & t \in [1.8, 3]. \end{cases}$$

$$\eta = \begin{cases} 0, & t \in [0, 1) \bigcup (1.8, 3], \\ -\frac{5}{2}t + \frac{5}{2}, & t \in [1, 1.8). \end{cases}$$

6.16 (a) $v^*(t) = \begin{cases} -1 & \text{for } t \in [0, 1.8), \\ 1 & \text{for } t \in (1.8, 3]. \end{cases}$

(b) $v^*(t) = 1$ for $t \in [0, 10]$.

6.18 $$v^*(t) = \begin{cases} -1 & \text{for } t \in [0, 1/2], \\ 0 & \text{for } t \in (1/2, 23/12], \\ +1 & \text{for } t \in (23/12, 29/12], \\ 0 & \text{for } t \in (29/12, 4]. \end{cases}$$

6.19 $$u^*(t) = \begin{cases} 0, & \text{for } 0 \le t \le t_1, \\ h(t - t_1)/c, & \text{for } t_1 < t \le T, \end{cases}$$

where $t_1 = T - \sqrt{2BC/h}$.

Chapter 7

7.1 $p^* = 102.5 + 0.2G$.

7.7 $(\bar{u})/(pS) = (\delta\beta)/(\eta(\rho + \delta))$.

7.15 The reachable set is $[x_0 e^{-\delta T}, (x_0 - \bar{x})e^{-(\delta + rQ)T} + \bar{x}]$, where $\bar{x} = rQ/(W + rQ)$.

7.20 (b)

$$t_1 = \frac{1}{rQ + \delta} \ln \frac{x_0}{x^s}, \qquad t_2 = \frac{1}{rQ + \delta} \ln \frac{\bar{x} - x^s}{\bar{x} - x_T}.$$

7.21

$$T \ge \frac{1}{rQ + \delta} \ln \frac{rQ(1 - x_0) - \delta x_0}{rQ(1 - x^s) - \delta x^s} + \frac{1}{\delta} \ln \frac{x_s}{x_T}.$$

7.28 $\text{imp}(A, B; t) = -\frac{1}{r} \ln \left[\frac{1-A}{1-B} \right]$.

7.29 (b) $J = 0.6325$.

7.35 The equations corresponding to (6.28) and (6.29) can be obtained by replacing ρ by $\rho + \dot{r}/r$. The form of (6.30) remains unchanged.

Chapter 8

8.1 (a) $y = 1$, $z = 3$.

(b) $y = 2$, $z = 10$.

8.2 (a) (1,3) is a relative maximum.

(b) (2,10) is a relative maximum.

8.3 $x = 50$; $x = 80$.

8.6 (a) $x = 4$ is a local maximum.

(b) $x = 8$ is a local maximum and $x = 20$ is a local and a global maximum.

8.7 (a) $(0, 0)$ is the nearest point.

(b) $(1/2, 1/2)$ is the nearest point.

8.8 $(1/\sqrt{5}, 2/\sqrt{5})$ is the closest point.

8.9 (a) $(2\sqrt{2}, 0)$.

(b) $(0, 2)$.

(c) $(0, 2)$.

8.13 $\lambda_i^T = \partial F/\partial x_i^T$ for $i = 1, 2, \ldots, n$; $\lambda_{n+1}^T = 1$. Note that here T denotes the terminal time, and *not* the transpose operation.

8.17 $$u^{k*} = \begin{cases} +1 & \text{if } \lambda^{k+1}b > 1, \\ -1 & \text{if } \lambda^{k+1}b < -1 \\ 0 & \text{if } |\lambda^{k+1}b| < 1. \end{cases} \text{, where } \lambda^k = (I + A)^{T-k}\lambda^T,$$

Chapter 9

9.2 $t^s = 5.25,\ T = 11.$

9.4 $T = t^s = 2.47.$

9.5 $t^s = 0,\ T = 30.$

9.7 $u^*(t) = \text{sat}[0, 1; u^0(t)]$, where $u^0(t) = \left[2 - e^{0.05(t-34.8)}\right]^2/(1+t)$, $t_1 \approx 3$; $t_2 - T = 34.8.$

Chapter 10

10.4 $\bar{x} = 0.734.$

10.5 (a)

$$\bar{x} = \frac{X}{4}\left[\left(1 - \frac{\rho}{r} + \frac{c}{Xp}\right) + \sqrt{\left(1 - \frac{\rho}{r} + \frac{c}{Xp}\right)^2 + \frac{8c\rho}{prX}}\right].$$

(b) For $\rho = 0$, $\bar{x} = 220{,}000$. For $\rho = 0.1$, $\bar{x} = 86{,}000$. For $\rho = \infty$, $\bar{x} = 40{,}000$.

10.7 $[g'(x) - \rho][p - c(x)] - c'(x)g(x) = 0.$

10.9 $[g'(x) - \rho][p - c(x)] - c'(x)g(x) + \dot{p} = 0.$

Chapter 11

11.1 $\lambda(t) = \lambda_0 e^{(\rho-\beta)t}$, where

$$\lambda_0 = \frac{[K_0 e^{\beta T} + \bar{C}(1 - e^{\beta T})/\beta - K_T](2\rho - \beta)}{e^{\beta T} - e^{2(\beta-\rho)T}},$$

$$K(t) = K_0 e^{\beta t} + \frac{\bar{C}}{\beta}(1 - e^{\beta t}) - \frac{\lambda_0}{\beta - 2\rho}(e^{2(\beta-\rho)t} - e^{\beta t}).$$

Chapter 12

12.5 $q^*(x) = \frac{\alpha - r}{(1-\beta)\sigma^2},\ c^*(x) = \frac{1}{1-\beta}\left(\rho - r\beta - \frac{\gamma\beta}{1-\beta}\right)x,$

$V(x) = [\frac{1-\beta}{\rho - r\beta - \gamma\beta/(1-\beta)}]^{1-\beta} x^\beta,\ x \geq 0.$

Chapter 13

13.2 $u^*(t) = 1 + \lambda^L(t)$, $v^*(t) = 1 + \lambda^F(t)$, where λ^L and λ^F are the solution of the linear differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda}^F \\ \dot{\lambda}^L \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda^F \\ \lambda^L \\ \mu \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with the boundary conditions

$x(0) = x_0,\ \lambda^F(T) = 0,\ \lambda^L(T) = 0$, and $\mu(0) = 0$.

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