Well-Posedness and Asymptotic Behavior for a Nonlinear Wave Equation

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Dedicated to Prof. Enrique Fernández-Cara on the occasion of his 60th birthday.

Abstract We consider the initial boundary value problem for nonlinear damped wave equations of the form $u'' + M(\int_\Omega |(-\Delta)^s u|^2 \, dx)\Delta u + (-\Delta)^\alpha u' = f$, with Neumann boundary conditions. We prove global existence of solutions, when $s \in [1/2, 1]$ and $\alpha \in (0, 1]$, and we show that the energy of these ones decays exponentially, as $t \to \infty$. The uniqueness of solutions is also obtained when $\alpha \in [1/2, 1]$.

Keywords Wave equation · Well-posedness · Asymptotic behavior

AMS Subject Classifications 35L70, 35B40, 74K10

1 Introduction

Problem on vibrations of the elastic bodies has been extensively studied in the last decades. We will look at the following nonlinear model for small deformations of an elastic membrane:

$$u'' + M\left(\int_\Omega \left|(-\Delta)^s u\right|^2 dx\right) \Delta u = f, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is the region occupied by the membrane. In Eq. (1.1), the prime $'$ stands for temporal derivative, $M = M(\lambda)$ is a positive real function defined for all $\lambda \geq 0$ and connected with the initial tension and with the characteristic of the material of the membrane, and Δ is the Laplace operator. The unknown $u = u(x, t)$ represents the vertical displacement of a point x of the membrane at time t, and $f = f(x, t)$ is an external force. Equation (1.1) was derived by Kirchhoff [9] for the case $s = 1/2$ and by Carrier [4] for the case $s = 0$.

Equation (1.1) with different boundary conditions was studied by several authors. For the Kirchhoff equation ((1.1) with $s = 1/2$) we can mention the existence results of Bernstein [2] in the one dimensional case with some restrictions on Fourier series of the data, and the results by Lions [11] and Pohozhaev [24] which considered the data in a special class of analytic functions. Medeiros-Milla Miranda in [14] studied local well-posedness for (1.1) under very weak hypothesis on the data. The general case, when $s \in [0, 1]$, was analyzed by Cousin et al. in [6], where the authors obtained existence of global solution in classes of Pohozhaev.

By adding a dissipative mechanism in Eq. (1.1), i.e.,

$$u'' + M\left(\int_\Omega \left|(-\Delta)^s u\right|^2 dx\right) \Delta u + (-\Delta)^\alpha u' = f, \tag{1.2}$$

we can cite several works which have obtained some decay rate of the solutions. For example, for the Kirchhoff equation ($s = 1/2$) with Dirichlet boundary conditions and $\alpha = 0$, we mention Brito [3], Nishihara-Yamada [20], Ono [22], and Yamada [26] which have obtained well-posedness and stability (as $t \to \infty$) results by considering data $(u_0, u_1, f) \in D(-\Delta) \times D((-\Delta)^{1/2}) \times L^2(0, T; D((-\Delta)^{1/2}))$ and satisfying a certain smallness conditions. Here and in what follows, $D(A)$ represents the domain of the operator A. Considering the case $\alpha = 1$ (strong dissipation) we cite the works of Matos-Pereira [13], Mimoni et al. [18], Nishihara [19], Ono [21], and Vasconcelos-Teixeira [25] which contain results of global solvability and exponential decay (as $t \to \infty$) of solutions. Still with Dirichlet boundary conditions and data $(u_0, u_1, f) \in H_0^1 \cap D((-\Delta)^\alpha) \times L^2 \times L^2(0, T; L^2)$, Medeiros and Milla Miranda in [15] obtained global existence and exponential decay (as $t \to \infty$) of solutions of (1.2) when $\alpha \in (0, 1]$. The uniqueness has been proved when $\alpha \in [1/2, 1]$. Considering Neumann boundary condition and $\alpha = 1$, Aassila in [1] studied the global existence and asymptotic behaviour (as $t \to \infty$) of solutions of the Kirchhoff equation. Relative to Carrier equation ($s = 0$) the literature is not so extensive, even so, we can mention Cousin et al. [7], Frota-Goldstein [8], Larkin [10], and Park et al. [23] which analyzed existence of global solutions and energy decay for this one with a nonlinear dissipative term. Besides all the previously mentioned works, we still indicated for the interested readers to consult the works by Medeiros et al. [16, 17] which contain an extensive list of results obtained for Kirchhoff-Carrier equation.

In this work, we consider a problem associated to (1.2) with Neumann boundary conditions, i.e.,

$$\begin{cases} u'' + M\left(\int_\Omega |(-\Delta)^s u|^2\,dx\right)\Delta u + (-\Delta)^\alpha u' = f \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \\ u(\cdot,0) = u_0, \quad u'(\cdot,0) = u_1 \quad \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary Γ, ν is the unit outward normal to Γ. The purpose of the present paper is to analyze the well-posedness and asymptotic behavior (as $t \to \infty$) of solutions for the problem (1.3) under the following conditions:

$$M(\lambda) \geq m_0 > 0, \quad \forall\lambda \geq 0, \tag{1.4}$$

$$0 < \alpha \leq 1 \quad \text{and} \quad 1/2 \leq s \leq 1, \tag{1.5}$$

and data $(u_0, u_1, f) \in D((-\Delta)^\alpha) \cap D((-\Delta)^{\alpha+s-\frac{1}{2}}) \times D((-\Delta)^{s-\frac{1}{2}}) \times L^2(0,T; D((-\Delta)^{s-\frac{1}{2}}))$. It is important to point out that, as in [15], the initial motivation was to obtain information as $\alpha \to 0$, but to our best knowledge, the existence of global solution of this system with $\alpha = 0$ and no restriction on the data is still an open (and seems to be difficult) problem. Returning to our results, to obtain the existence of solutions for (1.3), we need to construct a complete orthonormal system in a closed subspace of L^2 and to project the problem in this closed subspace. Thus, we can decompose the solutions of problem in two parts: one belonging to the kernel and another in the range of an operator, which corresponds to solutions of the projected problem. We also show that the projected solution decays in an exponential rate. The uniqueness of this solutions is obtained when $\alpha \in [1/2, 1]$. For $\alpha \in (0, 1/2)$ the uniqueness is still an open problem.

The paper is organized as follows. Section 2 contains some notations and essential results which we will apply in this work. In Sect. 3 we prove existence of global solution for (1.3) employing the Faedo-Galerkin method. The key point of the proof is to obtain the complete orthonormal system before mentioned. Concerning to uniqueness we will use energy method with a special regularization. Finally, in Sect. 4 we prove the exponential decay for the energy associated to projected solutions of the problem (1.3) making use of the perturbed energy method as in Zuazua [27].

2 Some Notations and Results

In this section we establish some important results that help us in the development of our work. Also we give some notations and we define the spaces and operators that we will use during the paper. We define the linear operator A_0 in $L^2(\Omega)$ as follows:

$$\left| \begin{array}{l} D(A_0) = \left\{ u \in H^2(\Omega);\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma = \partial\Omega \right\}, \\ A_0 u = -\Delta u, \quad \forall u \in D(A_0). \end{array} \right. \tag{2.1}$$

It is well know that the operator A_0 is nonnegative, selfadjoint and the resolvent $(I + \lambda A_0)^{-1}$ is compact for all $\lambda > 0$.

We recall a result from [1] that will be needed in the sequel.

Lemma 2.1 *Let $\mathcal{H}$ be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$. Consider $\mathcal{A} : D(\mathcal{A}) \subsetneq \mathcal{H} \to \mathcal{H}$ a nonnegative selfadjoint operator with domain $D(\mathcal{A})$ and range $R(\mathcal{A})$ in H. Suppose that $(I + \mathcal{A})^{-1}$ is a compact operator. Then*

- *(i) $R(\mathcal{A})$ is closed and $\mathcal{H} = N(\mathcal{A}) \oplus R(\mathcal{A})$,*
- *(ii) The operator $\left[\mathcal{A}|_{D(\mathcal{A})\cap R(\mathcal{A})}\right]^{-1} : R(\mathcal{A}) \to R(\mathcal{A})$ is compact, where $\mathcal{A}|_{D(\mathcal{A})\cap R(\mathcal{A})}$ is the restriction of $\mathcal{A}$ to $D(\mathcal{A}) \cap R(\mathcal{A})$.*

According to Lemma 2.1, we can guarantee, for the operator A_0 defined in (2.1), that

$$R(A_0) = \left\{ v \in L^2(\Omega);\ \textstyle\int_\Omega v(x)dx = 0 \right\} \text{ is closed in } L^2(\Omega),$$

$$L^2(\Omega) = N(A_0) \oplus R(A_0), \text{ with } N(A_0) = \{v(x) = \text{constant a.e. in } \Omega\},$$

and

$$\left[A_0|_{D(A_0)\cap R(A_0)}\right]^{-1} : R(A_0) \to R(A_0) \text{ is compact.}$$

Let $P : L^2(\Omega) \to R(A_0)$ be the orthogonal projection of $L^2(\Omega)$ onto $R(A_0)$. Then

$$Pu(x) = u(x) - \overline{u}, \quad \forall u \in L^2(\Omega),$$

where $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx$ and $|\Omega|$ is the measure of Ω.

Let us denote by $(\cdot, \cdot)$ and $|\cdot|$ the inner product and norm in $L^2(\Omega)$, respectively. We consider the system

$$\begin{cases} u'' + M\left(\left|A_0^s u\right|^2\right) A_0 u + A_0^\alpha u' = f \quad \text{in} \quad L^2(\Omega), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.2}$$

If $u(t)$ is a solution to (2.2), then by Lemma 2.1 we have $u(t) = u_1(t) + u_2(t)$, where $u_1(t) \in N(A_0)$ and $u_2(t) \in D(A_0) \cap R(A_0)$. Furthermore, we deduce that

$$\begin{cases} u_1'' + u_2'' + M\left(\left|A_0^s u_2\right|^2\right) A_0 u_2 + A_0^\alpha u_2' = f \quad \text{in} \quad L^2(\Omega), \\ u(0) = u_1(0) + u_2(0) = u_{01} + u_{02}, \\ u'(0) = u_1'(0) + u_2'(0) = u_{11} + u_{12}, \end{cases}$$

where we have used the fact that $A_0^s u(t) = A_0^s u_2(t)$ and $A_0^\alpha u'(t) = A_0^\alpha u_2'(t)$. In this way, we can decompose the last system as follows:

$$\begin{cases} u_1''(t) = 0 \quad \text{in} \quad N(A_0), \\ u_1(0) = u_{01}, \quad u_1'(0) = u_{11}, \end{cases} \tag{2.3}$$

and

$$\begin{cases} u_2'' + M\left(|A^s u_2|^2\right) A u_2 + A^\alpha u_2' = f \quad \text{in} \quad R(A), \\ u_2(0) = u_{02}, \quad u_2'(0) = u_{12}, \end{cases} \tag{2.4}$$

where $A = A_0|_{D(A_0)\cap R(A_0)}$. If we can solve (2.3) and (2.4), we will get the solution $u(t) = u_1(t) + u_2(t)$ for (2.2).

For (2.3), we obtain the following explicit solution:

$$u_1(t) = u_{01} + u_{11}t.$$

The analysis of the well-posedness of global (weak) solutions of (2.4), when

$$u_0 \in V := D(A^s) \cap D(A^{\alpha+s-\frac{1}{2}}), \quad u_1 \in H := D(A^{s-\frac{1}{2}}), \quad \text{and} \quad f \in L^2(0,T;H),$$

(α and s as in (1.5)) and the their asymptotic behavior, as $t \to \infty$, are our objectives in this paper. This will be done in the next two sections.

3 Well-Posedness

This section is devoted to show the well-posedness for the system (2.4). The following result holds.

Theorem 3.1 *Let us suppose $M \in C^0([0,\infty[,\mathbb{R})$, s, and α satisfying (1.4) and (1.5), and let us consider data $(u_0, u_1, f) \in V \times H \times L^2(0,T;H)$. Then there exists at least a function $u : \Omega \times [0,T] \to \mathbb{R}$ verifying the following conditions:*

$$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A^{\frac{\alpha}{2}+s})), \tag{3.1}$$

$$u' \in L^\infty(0,T;H) \cap L^2(0,T;D(A^{\frac{\alpha}{2}+s-\frac{1}{2}})), \tag{3.2}$$

$$u'' + M\left(\left|A^s u\right|^2\right) Au + A^\alpha u' = f \quad \textit{in} \quad L^2(0,T;D(A^{\frac{\alpha}{2}+s-1}) \cap D(A^{-\frac{\alpha}{2}+s-\frac{1}{2}})), \tag{3.3}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \textit{in} \quad \Omega. \tag{3.4}$$

Furthermore, if $M \in C^1([0,\infty[,\mathbb{R})$ and $\alpha \geq 1/2$, the function u satisfying (3.1)–(3.4) is unique.

Proof To prove the existence of solutions, we will use the Faedo-Galerkin method. For this, we consider $\{w_\nu\}_{\nu\in\mathbb{N}}$ a special basis in $R(A)$ formed by eigenvectors of A, whose eigenvalues $\{\lambda_\nu\}_{\nu\in\mathbb{N}}$ are such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_\nu \leq \ldots$ with $\lim_{\nu\to\infty}(\lambda_\nu) = \infty$. We denote by $W_m = [w_1, w_2, \ldots, w_m]$ the subspace of V generated by the first m vectors of $\{w_\nu\}_{\nu\in\mathbb{N}}$. Let us find an approximate solution $u_m = u_m(t) \in W_m$ defined by $u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j$, where $g_{jm}(t)$ are found as solutions of the following initial value problem for the system of ordinary differential equations:

$$\begin{cases} (u_m''(t), v) + M\left(|A^s u_m|^2\right)(Au_m(t), v) + (A^\alpha u_m'(t), v) = (f(t), v), \ \forall v \in W_m, \\ u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad V, \\ u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad H. \end{cases} \tag{3.5}$$

System (3.5) has solutions u_m defined on a certain interval $[0, t_m]$, for $t_m < T$ (see, for example, [5, Th. 1.1, p. 43]). Moreover, the functions u_m and u_m' are absolutely continuous in this interval. Thus, we can guarantee the existence of u_m'' almost everywhere in $[0, t_m]$. This solution can be extended to whole interval $[0, T]$ by using the first estimate that we shall prove in the next step.

Estimate I Taking $v = 2A^{2s-1}u_m'(t)$ in $(3.5)_1$, we have

$$\begin{aligned}&(u_m''(t), 2A^{2s-1}u_m'(t)) + M\left(|A^s u_m(t)|^2\right)(Au_m(t), 2A^{2s-1}u_m'(t)) \\ &+(A^\alpha u_m'(t), 2A^{2s-1}u_m'(t)) = (f(t), 2A^{2s-1}u_m'(t)).\end{aligned}$$

So

$$\begin{aligned}&\frac{d}{dt}\left\{\left|A^{s-\frac{1}{2}}u_m'(t)\right|^2 + \widehat{M}\left(|A^s u_m(t)|^2\right)\right\} + 2\left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}u_m'(t)\right|^2 \\ &\leq 2\left|A^{s-\frac{1}{2}}f(t)\right|\left|A^{s-\frac{1}{2}}u_m'(t)\right| \leq \left|A^{s-\frac{1}{2}}f(t)\right|^2 + \left|A^{s-\frac{1}{2}}u_m'(t)\right|^2,\end{aligned}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(t)dt$. Integrating from 0 to t, $0 \leq t \leq t_m$, we obtain

$$\begin{aligned}&\left|A^{s-\frac{1}{2}}u_m'(t)\right|^2 + \widehat{M}\left(|A^s u_m(t)|^2\right) + 2\int_0^t \left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}u_m'(\xi)\right|^2 d\xi \\ &\leq \left|A^{s-\frac{1}{2}}u_{1m}\right|^2 + \widehat{M}\left(|A^s u_{0m}|^2\right) + 2\int_0^t \left|A^{s-\frac{1}{2}}f(\xi)\right|^2 d\xi \\ &+\int_0^t \left|A^{s-\frac{1}{2}}u_m'(\xi)\right|^2 d\xi.\end{aligned}$$

In this way, by $(3.5)_2$, $(3.5)_3$, and since $f \in L^2(0,T;H)$, the above inequality implies that

$$\begin{aligned}&\left|A^{s-\frac{1}{2}}u_m'(t)\right|^2 + m_0\left|A^s u_m(t)\right|^2 + 2\int_0^t \left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}u_m'(\xi)\right|^2 d\xi\\ &\le C + \int_0^t \left|A^{s-\frac{1}{2}}u_m'(\xi)\right|^2 d\xi,\end{aligned}$$

where $C > 0$ is a constant independent of m and t. Thus, by Gronwall's Lemma, we obtain

$$\left|A^{s-\frac{1}{2}}u_m'(t)\right|^2 + m_0\left|A^s u_m(t)\right|^2 + 2\int_0^t \left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}u_m'(\xi)\right|^2 d\xi \le C,$$

where $C > 0$ is a constant independent of m and t. Therefore

$$\left|\begin{aligned}&(u_m) \text{ is bounded in } L^\infty(0,T;D(A^s)),\\ &(u_m') \text{ is bounded in } L^\infty(0,T;H)\cap L^2(0,T;D(A^{\frac{\alpha}{2}+s-\frac{1}{2}}))\end{aligned}\right. \tag{3.6}$$

and, consequently, we can prolong the approximate solution $u_m(t)$ for all t in $[0,T]$.

Estimate II Let us consider $v = A^{2s+\alpha-1}u_m(t)$ in $(3.5)_1$, then

$$\begin{aligned}&\frac{d}{dt}(u_m'(t), A^{2s+\alpha-1}u_m(t)) - \left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}u_m'(t)\right|^2 + M\left(\left|A^s(u_m(t))\right|^2\right)\left|A^{s+\frac{\alpha}{2}}u_m\right|^2\\ &+\frac{1}{2}\frac{d}{dt}\left|A^{s+\alpha-\frac{1}{2}}u_m(t)\right|^2 = (f(t), A^{2s+\alpha-1}u_m(t)).\end{aligned}$$

Integrating this identity from 0 to t, $t \in [0,T]$, we get

$$\begin{aligned}&\frac{1}{2}\left|A^{s+\alpha-\frac{1}{2}}u_m(t)\right|^2 + \int_0^t M\left(\left|A^s u_m(\xi)\right|^2\right)\left|A^{s+\frac{\alpha}{2}}u_m(\xi)\right|^2 d\xi\\ &= -(A^{s-\frac{1}{2}}u_m'(t), A^{s+\alpha-\frac{1}{2}}u_m(t))\\ &+(A^{s-\frac{1}{2}}u_{1m}, A^{s+\alpha-\frac{1}{2}}u_{0m}) + \frac{1}{2}\left|A^{s+\alpha-\frac{1}{2}}u_{0m}\right|^2\\ &+\int_0^t \left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}u_m'(\xi)\right|^2 d\xi + \int_0^t (A^{s-\frac{1}{2}}f(\xi), A^{s+\alpha-\frac{1}{2}}u_m(\xi))d\xi.\end{aligned}$$

Using the Young's inequality and $(3.6)_2$ we obtain

$$\begin{aligned}&\frac{1}{4}\left|A^{s+\alpha-\frac{1}{2}}u_m(t)\right|^2 + m_0\int_0^t \left|A^{s+\frac{\alpha}{2}}u_m(\xi)\right|^2 d\xi \le C + 4\left|A^{s-\frac{1}{2}}u_m'(t)\right|^2\\ &+\frac{1}{2}\int_0^T \left|A^{s-\frac{1}{2}}f(t)\right|^2 dt + \frac{1}{2}\int_0^t \left|A^{s+\alpha-\frac{1}{2}}u_m(\xi)\right|^2 d\xi \le C + \int_0^t \left|A^{s+\alpha-\frac{1}{2}}u_m(\xi)\right|^2 d\xi,\end{aligned}$$

where $C > 0$ is a constant independent of m and t, $t \in [0, T]$. So, applying again the Gronwall's Lemma, we can conclude that

$$\frac{1}{4}\left|A^{s+\alpha-\frac{1}{2}}u_m(t)\right|^2 + m_0\int_0^t \left|A^{s+\frac{\alpha}{2}}u_m(\xi)\right|^2 d\xi \le C,$$

where $C > 0$ is constant independent of m and t, $t \in [0, T]$. Therefore

$$(u_m) \text{ is bounded in } L^\infty(0, T; D(A^{s+\alpha-\frac{1}{2}})) \cap L^2(0, T; D(A^{s+\frac{\alpha}{2}})). \tag{3.7}$$

Passage to the Limit From estimates (3.6) and (3.7), there exists a subsequence of (u_m), still denoted in the same form, such that

$$\left| \begin{array}{llll} u_m \to u & \text{weak} - * & \text{in} & L^\infty(0, T; D(A^s) \cap D(A^{s+\alpha-\frac{1}{2}})), \\ u_m \to u & \text{weakly} & \text{in} & L^2(0, T; D(A^{s+\frac{\alpha}{2}})), \\ u'_m \to u' & \text{weak} - * & \text{in} & L^\infty(0, T; H), \\ u'_m \to u' & \text{weakly} & \text{in} & L^2(0, T; D(A^{\frac{\alpha}{2}+s-\frac{1}{2}})). \end{array} \right. \tag{3.8}$$

To treat the convergence of the nonlinear term, we observe that, since the injections $D(A^{s+\frac{\alpha}{2}}) \subset D(A^s) \subset H$ are continuous and the embedding of $D(A^{s+\frac{\alpha}{2}})$ into $D(A^s)$ is compact, it follows by (3.6), (3.7), and Aubin-Lions' Compactness Theorem that exists a subsequence of (u_m), which we still denote by (u_m), and a function $u : [0, T] \to \mathbb{R}$, such that

$$u_m \to u \quad \text{strongly in} \quad L^2(0, T; D(A^s)).$$

Then there exists a subsequence of (u_m), which we still denote by (u_m), such that

$$\left|A^s u_m(t)\right|^2 \to \left|A^s u(t)\right|^2 \quad \text{a.e. in} \quad (0, T).$$

By the continuity of M, we have

$$M\left(\left|A^s u_m(t)\right|^2\right) \to M\left(\left|A^s u(t)\right|^2\right) \quad \text{a.e. in} \quad (0, T),$$

and

$$M\left(\left|A^s u_m(t)\right|^2\right) \le C \quad \text{a.e. in} \quad (0, T).$$

Thus, by the Lebesgue's Dominated Convergence Theorem, we get

$$M\left(\left|A^s u_m(t)\right|^2\right) \to M\left(\left|A^s u(t)\right|^2\right) \quad \text{strongly in} \quad L^2(0, T). \tag{3.9}$$

The convergences (3.8) and (3.9) are sufficient to pass to the limit in $(3.5)_1$ and to obtain the function u satisfying (3.1)–(3.3). By standard arguments, we can verify the initial conditions (3.4).

Before proving the uniqueness, we consider the following lemma.

Lemma 3.1 *If* $\frac{1}{2} \leq \alpha \leq 1$, $u \in L^2(0,T; D(A^{s+\frac{\alpha}{2}})$, *and* $u' \in L^2(0,T; D(A^{s+\frac{\alpha}{2}-\frac{1}{2}}))$, *then*

$$\frac{d}{dt}\left|A^s u\right|^2 = 2\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right). \tag{3.10}$$

Proof *We consider the space* $W(0,T)$ *defined by*

$$W(0,T) = \left\{v;\ v \in L^2\left(0,T; D(A^{s+\frac{\alpha}{2}})\right),\ v' \in L^2(0,T; D(A^{s+\frac{\alpha}{2}-\frac{1}{2}}))\right\}$$

equipped with the norm

$$\|v\|^2_{W(0,T)} = \|v\|^2_{L^2(0,T;D(A^{s+\frac{\alpha}{2}}))} + \|v\|^2_{L^2(0,T;D(A^{s+\frac{\alpha}{2}-\frac{1}{2}}))}.$$

By Lions-Magenes [12, p. 13], we have that $\mathcal{D}([0,T]; D(A^{s+\frac{\alpha}{2}}))$ *is dense in* $W(0,T)$. *Taking* $\varphi \in \mathcal{D}([0,T]; D(A^{s+\frac{\alpha}{2}}))$, *it follows that* $\varphi' \in \mathcal{D}\left([0,T]; D(A^{s+\frac{\alpha}{2}})\right)$. *We also have* $D(A^{s+\frac{\alpha}{2}}) \subset D(A^{s-\frac{\alpha}{2}+\frac{1}{2}})$ *with continuous injections, because* $s+\frac{\alpha}{2} \geq s-\frac{\alpha}{2}+\frac{1}{2}$. *In this way, we can assert that* $A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi' \in D\left(A^{1-\alpha}\right)$ *and*

$$\begin{aligned}
\frac{d}{dt}\left|A^s\varphi\right|^2 &= \frac{d}{dt}\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi\right)\\
&= \left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi',\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi\right) + \left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi'\right)\\
&= \left(A^{1-\alpha}\left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi'\right),\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi\right) + \left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi'\right)\\
&= \left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi',\, A^{1-\alpha}\left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi\right)\right) + \left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi'\right)\\
&= \left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi',\, A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi\right) + \left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi'\right)\\
&= 2\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}\varphi,\, A^{s+\frac{\alpha}{2}-\frac{1}{2}}\varphi'\right),
\end{aligned}$$

for all $\varphi \in \mathcal{D}\left([0,T]; D(A^{s+\frac{\alpha}{2}})\right)$. *In this way, using density arguments, we get (3.10) and this proves the lemma.* ■

Returning to uniqueness of solution, to prove it, we will make use of the energy method with a special regularization. In fact, firstly we observe that we can not multiply Eq. (3.3) by $A^{2s-1}u'$ directly because $A^{2s-1}u' \in L^\infty(0,T;D(A^{s-\frac{1}{2}})^*) \cap L^2(0,T,D(A^{s-\frac{\alpha}{2}-\frac{1}{2}})^*)$ and $u'' \in L^2(0,T;D(A^{\frac{\alpha}{2}})^*)$ and therefore the duality $\langle u'', A^{2s-1}u'\rangle$ does not make sense. To overcome this difficulty, let us consider the function u defined over $\mathbb{R}$ with the properties analogous with the properties of u over $[0,T]$ (which is possible by reflection). Let us consider a sequence of mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$, that is, a sequence of functions $\rho_\varepsilon \geq 0$ on $\mathbb{R}$ such that

$$\rho_\varepsilon \in C_c^\infty(\mathbb{R}), \quad \operatorname{supp}\rho_\varepsilon \subset [-\varepsilon,\varepsilon], \quad \int_{-\infty}^{\infty} \rho_\varepsilon(s)\,ds = 1.$$

Taking

$$u_\varepsilon(x,t) = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s)u(x,s)ds,$$

we can see that $u''_\varepsilon \in L^2(0,T,D(A^{s-\frac{1}{2}}))$ and so the duality $\left\langle u''_\varepsilon, A^{2s-1}u'_\varepsilon\right\rangle$ makes sense.

Let us suppose that u and v are two solutions in the conditions of Theorem 3.1. Defining $w_\varepsilon = u_\varepsilon - v_\varepsilon$ and $w = u - v$, we have that

$$w''_\varepsilon + \rho_\varepsilon * \left[M\left(\left|A^s u\right|^2\right)Au - M\left(\left|A^s v\right|^2\right)Av\right] + A^\alpha w'_\varepsilon = 0. \tag{3.11}$$

Making the duality between (3.11) and $A^{2s-1}w'_\varepsilon$, one has

$$\begin{aligned}&\left\langle w''_\varepsilon, A^{2s-1}w'_\varepsilon\right\rangle + \left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)Aw, A^{2s-1}w'_\varepsilon\right\rangle \\ &+ \left\langle \rho_\varepsilon * \left[M\left(|A^s u|^2\right) - M\left(|A^s v|^2\right)\right]Av, A^{2s-1}w'_\varepsilon\right\rangle + \left\langle A^\alpha w'_\varepsilon, A^{2s-1}w'_\varepsilon\right\rangle = 0.\end{aligned} \tag{3.12}$$

Notice that

$$\begin{aligned}\left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)Aw, A^{2s-1}w'_\varepsilon\right\rangle &= \left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'_\varepsilon\right\rangle \\ &= \left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'_\varepsilon - A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle \\ &+ \left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle.\end{aligned} \tag{3.13}$$

By Lemma 3.1, we have

$$\begin{aligned}\left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle &= \frac{1}{2}\frac{d}{dt}\left\langle \rho_\varepsilon * M\left(|A^s u|^2\right)A^s w, A^s w\right\rangle \\ &- \left\langle \rho_\varepsilon * [M'(|A^s u|^2)\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right)]A^s w, A^s w\right\rangle.\end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left|A^{s-\frac{1}{2}}w_\varepsilon'\right|^2+\left\langle\rho_\varepsilon * M(|A^s u|^2)A^s w, A^s w\right\rangle\right)+\left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right|^2\\&=\left\langle\rho_\varepsilon*\left[M\left(|A^s v|^2\right)-M\left(|A^s u|^2\right)\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right\rangle\\&+\left\langle\rho_\varepsilon*\left[M'\left(|A^s u|^2\right)\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right)\right]A^s w, A^s w\right\rangle\\&+\left\langle\rho_\varepsilon*M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'-A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right\rangle.\end{aligned}\tag{3.15}$$

Integrating (3.15) from 0 to $t\le T$ we have

$$\begin{aligned}&\frac{1}{2}\left(\left|A^{s-\frac{1}{2}}w_\varepsilon'\right|^2+\left\langle\rho_\varepsilon*M\left(|A^s u|^2\right)A^s w, A^s w\right\rangle\right)+\int_0^t\left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right|^2 ds\\&=\int_0^t\left\langle\rho_\varepsilon*\left[M\left(|A^s v|^2\right)-M\left(|A^s u|^2\right)\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right\rangle ds\\&+\int_0^t\left\langle\rho_\varepsilon*\left[M'\left(|A^s u|^2\right)\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right)\right]A^s w, A^s w\right\rangle ds\\&+\int_0^t\left\langle\rho_\varepsilon*M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'-A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right\rangle ds.\end{aligned}$$

We can rewrite the above equality as follows

$$\begin{aligned}&\frac{1}{2}\left(\left|A^{s-\frac{1}{2}}w_\varepsilon'\right|^2+\left\langle\rho_\varepsilon*M\left(|A^s u|^2\right)A^s w, A^s w\right\rangle\right)+\int_0^t\left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right|^2 ds\\&=\int_0^t\left\langle\rho_\varepsilon*\left[M\left(|A^s v|^2\right)-M\left(|A^s u|^2\right)\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle ds\\&+\int_0^t\left\langle\rho_\varepsilon*\left[M\left(|A^s v|^2\right)-M\,|A^s u|^2\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'-A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle ds\\&+\int_0^t\left\langle\rho_\varepsilon*\left[M'\left(|A^s u|^2\right)\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right)\right]A^s w, A^s w\right\rangle ds\\&+\int_0^t\left\langle\rho_\varepsilon*M\left(|A^s u|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'-A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right\rangle ds.\end{aligned}\tag{3.16}$$

Notice that, as $\varepsilon\to 0$, we have

$$\int_0^t\left\langle\rho_\varepsilon*\left[M\left(|A^s v|^2\right)-M\left(|A^s u|^2\right)\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'-A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle ds\to 0\tag{3.17}$$

and

$$\int_0^t\left\langle\rho_\varepsilon*M\left(\left|A^s u\right|^2\right)A^{s-\frac{\alpha}{2}+\frac{1}{2}}w, A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'-A^{\frac{\alpha}{2}+s-\frac{1}{2}}w_\varepsilon'\right\rangle ds\to 0.\tag{3.18}$$

Thus, passing (3.16) to the limit, as $\varepsilon \to 0$, and taking into account the convergences in (3.17) and (3.18), we get

$$\begin{aligned}&\frac{1}{2}\left|A^{s-\frac{1}{2}}w'\right|^2+\left\langle M\left(|A^s u|^2\right)A^s w,\,A^s w\right\rangle+\int_0^t\left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right|^2ds\\&=\int_0^t\left\langle\left[M\left(|A^s v|^2\right)-M\left(|A^s u|^2\right)\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v,\,A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle ds\\&+2\int_0^t\left\langle M'\left(|A^s u|^2\right)\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u,\,A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right)A^s w,\,A^s w\right\rangle ds.\end{aligned}\tag{3.19}$$

Using (1.4) and the fact that $s+\frac{\alpha}{2}\geq s+\frac{1}{2}-\frac{\alpha}{2}$, we have that there exists $\xi=\xi\,(t)$ between $|A^s u\,(t)|^2$ and $|A^s v\,(t)|^2$ such that

$$\begin{aligned}&\left\langle\left[M\left(\left|A^s v\right|^2\right)-M\left(\left|A^s u\right|^2\right)\right]A^{s-\frac{\alpha}{2}+\frac{1}{2}}v,\,A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle\\&\leq\left|M'(\xi)\right|\left(\left|A^s u\right|+\left|A^s v\right|\right)\left|\left|A^s u\right|-\left|A^s v\right|\right|\left|\left\langle A^{s-\frac{\alpha}{2}+\frac{1}{2}}v,\,A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right\rangle\right|\\&\leq C\left|A^s w\right|\left|\left\langle A^{s+\frac{1}{2}-\frac{\alpha}{2}}v,\,A^{s-\frac{1}{2}+\frac{\alpha}{2}}w'\right\rangle\right|\\&\leq C\left|A^s w\right|\left|A^{s+\frac{1}{2}-\frac{\alpha}{2}}v\right|\left|A^{s-\frac{1}{2}+\frac{\alpha}{2}}w'\right|\leq C\left|A^{s+\frac{1}{2}-\frac{\alpha}{2}}v\right|^2\left|A^s w\,(t)\right|^2+\frac{1}{2}\left|A^{s-\frac{1}{2}+\frac{\alpha}{2}}w'\right|^2\\&\leq C\left|A^{s+\frac{\alpha}{2}}v\right|^2\left|A^s w\right|^2+\frac{1}{2}\left|A^{s-\frac{1}{2}+\frac{\alpha}{2}}w'\right|^2.\end{aligned}\tag{3.20}$$

We can also note by (1.4) that

$$\left\langle M'\left(\left|A^s u\right|^2\right)\left(A^{s-\frac{\alpha}{2}+\frac{1}{2}}u,\,A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right)A^s w,\,A^s w\right\rangle\leq C\left(\left|A^{s+\frac{\alpha}{2}}u\right|^2+\left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\right|^2\right)\left|A^s w\right|^2.\tag{3.21}$$

Combining (3.19)–(3.21), it follows that

$$m_0\left|A^s w\right|^2+\frac{1}{2}\int_0^t\left|A^{\frac{\alpha}{2}+s-\frac{1}{2}}w'\right|^2ds\leq C\int_0^t h(s)\left|A^s w(s)\right|^2ds,\tag{3.22}$$

with $h\,(t)=\left|A^{s+\frac{\alpha}{2}}u\,(t)\right|^2+\left|A^{s+\frac{\alpha}{2}}v\,(t)\right|^2+\left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'\,(t)\right|^2\in L^1\,(0,T)$. Applying the Gronwall's Lemma in (3.22), we conclude that $w(t)=0$, for all $t\in[0,T]$, and this gives the uniqueness. ■

Remark 3.1 As an immediate consequence of the estimates to obtain existence of solutions in the proof of Theorem 3.1, we have that if $f(x,\cdot)$ is defined in the interval $(0,\infty)$, then (3.1)–(3.3) hold when we consider $T=\infty$.

4 Asymptotic Behavior

The aim of this section is to study the asymptotic behavior, as $t \to \infty$, of the energy $E(t)$ associated to solution of the problem (2.4) (with $f = 0$). This energy is given by

$$E(t) = \frac{1}{2}\left|A^{s-\frac{1}{2}}u'(t)\right|^2 + \frac{1}{2}\widehat{M}\left(\left|A^s u(t)\right|^2\right), \quad \forall t \geq 0. \tag{4.1}$$

Recall that $\widehat{M}(\lambda) = \int_0^\lambda M(t)dt$. The main result of this section is the following.

Theorem 4.1 *Under the assumptions of Theorem 3.1 with $f = 0$, there exist positive constants C and γ such that the energy (4.1) satisfies*

$$E(t) \leq CE(0)e^{-\gamma t}, \quad \forall t \geq 0. \tag{4.2}$$

Proof A simple computation gives

$$E_m'(t) = -\left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m'\right|^2 \leq -\lambda_1^\alpha \left|A^{s-\frac{1}{2}}u_m'\right|^2, \tag{4.3}$$

where $E_m(t)$ is the energy similar to (4.1) associated to the approximated system (3.5) and λ_1 is the first eigenvalue of A. From (4.3), we see that $E_m(t)$ is non-increasing function.

For an arbitrary $\varepsilon > 0$, we define the perturbed energy

$$E_{m\varepsilon}(t) = (1 + \varepsilon c)\, E_m(t) + \varepsilon F(t), \tag{4.4}$$

with $c > 0$ being a constant to be determined later and

$$F(t) = \left(A^{s-\frac{1}{2}}u_m(t),\, A^{s-\frac{1}{2}}u_m'(t)\right).$$

Notice that

$$|F(t)| \leq C_1 E_m(t), \tag{4.5}$$

where $C_1 = \max\left\{C_0^2/m_0, c, 1\right\}$ and $C_0 > 0$ is the immersion constant of $D(A^s)$ into $D(A^{s-\frac{1}{2}})$. By (4.4) and (4.5)

$$|E_{m\varepsilon}(t) - (1 + \varepsilon c)\, E_m(t)| \leq \varepsilon C_1 E_m(t)$$

or

$$[1 + \varepsilon(c - C_1)]\, E_m(t) \leq E_{m\varepsilon}(t) \leq [1 + \varepsilon(c + C_1)]\, E_m(t).$$

Taking $0 < \varepsilon < \min\{1/2(C_1 - c), 1/(C_1 + c)\}$, we get

$$\frac{1}{2}E_m(t) \le E_{m\varepsilon}(t) \le 2E_m(t). \tag{4.6}$$

Considering the derivative of the function $F(t)$ and using (3.5)$_1$ (with $f = 0$), we obtain

$$\begin{aligned} F'(t) &= \left|A^{s-\frac{1}{2}}u'_m\right|^2 + \left(A^{s-\frac{1}{2}}u_m, A^{s-\frac{1}{2}}u''_m\right) = \left|A^{s-\frac{1}{2}}u'_m\right|^2 + \left(A^{2s-1}u_m, u''_m\right) \\ &= \left|A^{s-\frac{1}{2}}u'_m\right|^2 - \left(A^{2s-1}u_m, M\left(|A^s u_m|^2\right)Au_m\right) - \left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'_m\right) \\ &= \left|A^{s-\frac{1}{2}}u'_m\right|^2 - M\left(|A^s u_m|^2\right)|A^s u_m|^2 - \left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'_m\right). \end{aligned} \tag{4.7}$$

By (4.3) and (4.7) one has

$$\begin{aligned} E'_\varepsilon(t) + \varepsilon F'(t) \le& -\lambda_1^\alpha \left|A^{s-\frac{1}{2}}u'_m\right|^2 + \varepsilon\left|A^{s-\frac{1}{2}}u'_m\right|^2 - \varepsilon M\left(\left|A^s u_m\right|^2\right)\left|A^s u_m\right|^2 \\ &- \varepsilon\left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'_m\right). \end{aligned} \tag{4.8}$$

Notice that

$$\begin{aligned} \left|\left(A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m, A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'_m\right)\right| &\le \frac{\delta}{2}\left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u'_m\right|^2 + \frac{1}{2\delta}\left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m\right|^2 \\ &\le -\frac{\delta}{2}E'_m(t) + \frac{1}{2\delta}\left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m\right|^2, \end{aligned} \tag{4.9}$$

with $\delta > 0$ being a constant to be chosen, and

$$\begin{aligned} \left|A^{s+\frac{\alpha}{2}-\frac{1}{2}}u_m\right|^2 &= \sum_{0<\lambda_\nu\le 1}\lambda_\nu^{2s+\alpha-1}\left|(u_m, w_\nu)\right|^2 + \sum_{\lambda_\nu\ge 1}\lambda_\nu^{2s+\alpha-1}\left|(u_m, w_\nu)\right|^2 \\ &\le |u_m|^2 + \left|A^s u_m\right|^2 \le \left(\frac{1+\lambda_1}{m_0}\right)\widehat{M}\left(\left|A^s u_m\right|^2\right). \end{aligned} \tag{4.10}$$

We also have

$$-M\left(\left|A^s u_m\right|^2\right)\left|A^s u_m\right|^2 \le -\frac{m_0}{\tau}\widehat{M}\left(\left|A^s u_m(t)\right|^2\right), \tag{4.11}$$

where $\tau = \max\left\{M(s);\ 0 \le s \le \frac{2E(0)}{m_0}\right\}$. Combining (4.8)–(4.11), it follows that

$$E'_m(t) + \varepsilon\frac{\delta}{2}E'_m(t) + \varepsilon F'(t) \le -\left(\lambda_1^\alpha - \varepsilon\right)\left|A^{s-\frac{1}{2}}u'_m\right|^2 + \left[\frac{\varepsilon}{2\delta}\left(\frac{1+\lambda_1}{m_0}\right) - \frac{\varepsilon m_0}{\tau}\right]\widehat{M}\left(\left|A^s u_m\right|^2\right). \quad (4.12)$$

Choosing $\delta = \frac{\tau(1+\lambda_1)}{m_0^2}$ and $c = \delta/2$, we obtain by (4.4) and (4.12) that

$$E'_{m\varepsilon}(t) \le -\left(\lambda_1^\alpha - \varepsilon\right)\left|A^{s-\frac{1}{2}}u'_m\right|^2 - \frac{\varepsilon m_0}{2\tau}\widehat{M}\left(\left|A^s u_m\right|^2\right). \quad (4.13)$$

Taking $\delta_0 = \min\left\{2(\lambda_1^\alpha - \varepsilon), \frac{\varepsilon m_0}{\tau}\right\}$, we can conclude by (4.6) and (4.13) that

$$E'_{m\varepsilon}(t) \le -\frac{\delta_0}{C_3}E_{m\varepsilon}(t), \quad \forall t \ge 0,$$

which implies

$$E_{m\varepsilon}(t) \le E_{m\varepsilon}(0)e^{-\frac{\delta_0}{C_3}t}, \quad \forall t \ge 0. \quad (4.14)$$

Combining (4.6) and (4.14) we get

$$E_m(t) \le \frac{C_3}{C_2}E_m(0)e^{-\frac{\delta_0}{C_3}t}, \quad \forall t \ge 0. \quad (4.15)$$

Taking the $\liminf_{m\to\infty}$ in both sides of (4.15) and according the convergences $(3.5)_2$, $(3.5)_3$, (3.8), and (3.9), we deduce the inequality (4.2) and Theorem 4.1 is proved. ■

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