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Modelling of Output Flows in Queuing Systems and Networks

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Abstract. A simplification of Burke theorem proof [1] and its generalizations for queuing systems and networks are considered. The proof simplification is based on the fact that points in output flow take place in moments when Markov process of customers number in queuing system has jumps down. First steps in this direction were made in [2]. But here we improved proves of main results and consider queuing systems in random environment. In such way it is possible to obtain a property of the mutual independence of the flow into disjoint periods of time and to calculate intensity of output flow. In this case Poisson input flow with randomly varying intensity may be represented as Poisson flow with average intensity also. If this flow is independent with service process then it is possible to simplify significantly consideration of queuing systems in random environment. These assumptions may be applied to a consideration of multiphase type networks [3] which are convenient in analysis of queuing models with retrial queues [4–8].

Keywords: An output Poisson flow · The Jackson network
A random environment · A directed graph · A non-return set of nodes

1 Introduction

This paper is devoted to analysis of output flows in queuing systems and networks. In first part of the paper we consider simplification of Burke theorem proof [1] and its generalizations. The proof simplification is based on the fact that points in output flow take place in moments when Markov process of customers number in queuing system has jumps down. In such way it is possible to obtain a property of the mutual independence of the flow into disjoint periods of time. Then it is possible knowing the process of customers number distribution to calculate intensities of such jumps down and so to calculate intensity of output flow. This approach allows to obtain different corollaries for output flows in open and close queuing networks.

Such consideration may be applied not only to output flows but to input flows also. In this paper it is shown that for some stochastic models Poisson input flow with randomly varying intensity coincides by distribution with Poisson flow with

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average intensity. This fact allows to analyse and to calculate distributions of processes in open queuing network with finite number of nodes, infinite number of servers in nodes, exponential distributions of service times and Poisson input flow with randomly varying intensity. A presence of infinite number of servers in the network nodes [4–6] together with the statement that the cardinality of counting set of counting sets is counting set also allows to transform initial queuing network into queuing network of multiphase type [3] so that in each node a customer may be served no more than once. A transformation of the Jackson network into the multiphase type network is closely connected with models of retrial queues [7,8].

It is proved that all flows of so transformed network in stationary regime are Poisson. Synergetic effects in this network are analysed using a replacement of infinite number of servers by finite number of them. Synergetic effect means that if number of servers in nodes and intensity of input flow increase in $n \to \infty$ times then probability of queues existence on finite time interval tends to zero.

This investigation is based on the Burke theorem [1] that in stationary regime output flow of multiserver system $M|M|n|\infty$ is Poisson.

2 New Proof of Burke Theorem and Its Corollaries

In [1] the following statement is proved: in queung system $M|M|n|\infty$ in stationary state, the output flow has the same distribution as the input flow. Recently, however, interest in the study of flows in queuing systems is increased. Now it is necessary to give a more compact and convenient for generalizations proof of this theorem.

A random sequence of points will be called a Poisson flow with continuously differentiable intensity $\lambda(t),\ t \geq 0$, if the following conditions are satisfied [9, p. 12, 13], [10, p. 20, 35 – 37]:

(a) the probability of the existence of the point of flow on the time interval $[t, t+h)$ does not depend on the location of the points of the flow up to the time t (this property is called lack of follow-through and expresses the mutual independence of the flow into disjoint periods of time);
(b) the probability that a flow point appears in the semi-interval $[t, t+h)$ is $\lambda(t)h + o(h),\ h \to 0$;
(c) the probability of occurrence of two or more flow points in the range $[t, t+h)$ is $o(h),\ h \to 0$.

Let the system $A_n = M|M|n|\infty$ of the Poisson input flow has an intensity $\lambda > 0$, and the service time has an exponential distribution with the parameter $\mu > 0,\ 1 \leq n < \infty$. Denote $P_{k,n}(t),\ k \geq 0$, distribution of the number of customers in the system at the time t.

Theorem 1. *The output flow in queuing system A_n is Poisson with intensity*

$$a(t) = \sum_{0<k} \mu P_{k,n}(t) \min(k, n).$$

Proof. Let the output flow $T_n = \{0 \le t_1 < t_2 < \ldots\}$ be A_n described by a random function $y_n(t)$ equal to the number of points of this flow on the segment $[0, t)$. Denote $x_n(t)$ the number of customers in the system A_n at the time t. It is known that a random process $x_n(t)$ is Markov process (of death and birth of [10, Chap. II, Sect. 1]), with each point of the T_n flow corresponding to the time of the jump down process $x_n(t)$. Therefore, the output flow T_n satisfies the condition (a). In turn, the condition (b) follows from the equalities:

$$P(y_n(t+h) = y_n(t)+1) = \sum_{k=1}^{n} P(y_n(t+h) = y_n(t)+1/x_n(t)=k)P_{k,n}(t)$$

$$+P(y_n(t+h = y_n(t)+1/x_n(t) > n)\sum_{k>n} P_{k,n}(t)$$

$$= \sum_{k=1}^{n} P_{k,n}(t)(k\mu h + o_k(h)) + \sum_{k>n} P_{k,n}(t)(n\mu h + o_0(h)) = a(t)\mu h + o(h),$$

where for $h \to 0$ we have $\dfrac{o_k(h)}{h} \to 0,\ k = 0, \ldots, n,$

$$o(h) = \sum_{k=1}^{n} P_{k,n}(t)o_k(h) + \sum_{k>n} P_{k,n}(t)o_0(h),\ o_0(h)/h \to 0.$$

Thus, the output flow T_n satisfies the condition (b). Check of condition (c) is quite obvious.

Theorem 2. *In queuing system A_n, when the ergodicity condition $\lambda < \mu$ is satisfied and the process $x_n(t)$ is stationary, the output flow is Poisson with intensity λ.*

Proof. Denote $P_{k,n},\ k = 0, 1, \ldots,$ stationary probabilities of ergodic process $x_n(t)$. The system of Kolmogorov-Chapman equalities for $P_{k,n},\ k = 0, 1, \ldots,$ is following:

$$0 = -P_{0,n}\lambda + P_{1,n}\mu_1,\ 0 = -P_{k,n}(\lambda + \mu_k) + P_{k-1,n}\lambda + P_{k+1,n}\mu_{k+1}, \quad (1)$$

with $\mu_k = \min(k, n)\mu,\ k = 1, 2, \ldots$ Prove by an induction that from Formulas (1) we have

$$0 = -P_{k,n}\lambda + P_{k+1,n}\mu_{k+1},\ k = 0, 1, \ldots \quad (2)$$

Indeed for $k = 0$ this statement is a corollary of the first equation in Formulas (1). Assume that the equality (2) is true for $k = i$, then from equations in (1) and induction asumption we have for $k = i+1$:

$$0 = [-P_{i+1,n}(\lambda + \mu_{i+1}) + P_{i,n}\lambda + P_{i+2,n}\mu_{i+2}] + [-P_{i,n}\lambda + P_{i+1,n}\mu_{i+1}]$$

$$= -P_{i+1,n}\lambda + P_{i+2,n}\mu_{i+2}.$$

Consequently the equations (2) are true for all $k = 0, 1, \dots$ Summarize equalities (2) by $k = 0, 1, \dots,$ we obtain

$$\lambda = \sum_{k \geq 0} P_{k+1,n} \mu_{k+1}. \tag{3}$$

So from Theorem 1 we have the statement of Theorem 2.

Remark 1. Using the scheme of the proof of Theorem 2, it is possible to extend the results to output flows of systems with limited queue, with priority service, with unreliable servers [10, Sect. 7].

3 Poisson Flows in Stationary Queuing Networks

Consider an open queuing network (Jackson network [11]) S with a Poisson input flow of intensity λ_0, consisting of a finite number of nodes $k = 0, 1, \dots, m$ with exponentially distributed service times. The dynamics of the movement of customers in the network is set by the route matrix $\Theta = ||\theta_{i,j}||_{i,j=0}^{m}$, where $\theta_{i,j}$ is the probability of customer transition after service in the i-th node to j-th node, $\theta_{0,0} = 0$, where the node 0 is an external source and at the same time a drain for customers leaving the network. The i node contains $l_i < \infty$ servers, the service time of which has an exponential distribution with the parameter $\mu_i,\ i = 1, \dots, m$.

Assume that route matrix $\Theta = ||\theta_{i,j}||_{i,j=0}^{m}$ is indecomposable, i.e.

$$\forall\, i,\ j \in \{0, \dots, m\}\ \exists\, i_1, \dots, i_r \in \{0, \dots, m\} :\ \theta_{i,i_1} > 0, \theta_{i_1,i_2} > 0, \dots, \theta_{i_r,j} > 0.$$

Then for a fixed $\lambda_0 > 0$, the system of linear algebraic equations for intensities of fluxes coming from nodes of S

$$\lambda_k = \lambda_0 \theta_{0,k} + \sum_{t=1}^{m} \lambda_t \theta_{t,k},\ k = 1, \dots, m \tag{4}$$

has the only solution $(\lambda_1, \dots, \lambda_m)$ $\lambda_1 > 0, \dots, \lambda_m > 0$, [12, p. 13].

The system (4) is called the system of balance relations and plays an important role in the formulation and the proof of the product Jackson theorem [11], widely used in queuing theory. If

$$\lambda_i < l_i \mu_i,\ i = 1, \dots, m,$$

then the discrete Markov process $(n_1(t), \dots, n_m(t)),\ t \geq 0$, describing the number of customers in the network nodes has a limiting distribution $P_S(k_1, \dots, k_m)$, independent of initial conditions and representable in the form

$$P_S(k_1, \dots, k_m) = \prod_{i=1}^{m} P_i(k_i),$$

where $P_i(k_i)$ is the limiting distribution of the number of customers in a stand-alone l_i - channel queuing system with Poisson input flow of intensity $\lambda_i,\ i = 1, \dots, m$.

In [13] network S is mapped to a directed graph G with edges corresponding to positive elements of the route matrix. Let's call the vertex set $U \subseteq \{0, 1, \dots, m\}$ irrevocable if from any node not included in U, there is no edge to the node belonging to U. Then all flows passing through the edges from the node set U to the node set $\{0, 1, \dots, m\} \setminus U$, are independent and Poisson.

Theorem 3. *Flow $T_S^i,\ i = 1, \dots, m$, coming out of node i of open queuing network S, with stationary process $(n_1(t), \dots, n_m(t)),\ t \geq 0$, is Poisson with intensity λ_i.*

Proof. Indeed, the points of the flow T_s^i, exiting the i, node are the moments of jumps down the $n_i(t)$ component of the discrete Markov process $(n_1(t), \dots, n_m(t))$, $t \geq 0$. Hence the flow T_S^i satisfies the condition (a). Conditions (b), (c) are checked similarly to the proof of Theorem 1. Note that the limit probability that the i node contains k_i of customers is $P_i(k_i)$, and the flow rate T_S^i is $\lambda_i,\ i = 1, \dots, m$.

Theorem 4. *Flows $T_S^i,\ i = 1, \dots, m$, are independent.*

Proof. From Theorem 3 and independence of stationary random variables $n_j(t)$, $j = 1, \dots, m$, it follows that the union

$$T_S = \bigcup_{j=1}^{m} t_S^j$$

of flows leaving the nodes of open queuing network S is also Poisson flow with intensity $\lambda_\Sigma = \sum_{j=1}^{m} \lambda_j$. And each point of the combined flow T_S belongs to the flow T_S^i with probability $\dfrac{\lambda_i}{\lambda_\Sigma}$. Hence the flows $T_S^i,\ i = 1, \dots, m$, are independent.

Remark 2. Theorems 3, 4 enhance the results of the article [13], removing restrictions on the independent Poisson flows considered in it.

Consider now a closed queueing network $\overline{S}$, consisting of a finite number of nodes $i = 1, \dots, m$. The i node contains $l_i < \infty$ servers, the service time on which has an exponential distribution with the parameter $\mu_i,\ i = 1, \dots, m$. A finite number N of customers move along network $\overline{S}$. The dynamics of the customers movement in the network is specified by the matrix $\overline{\Theta} = ||\overline{\theta}_{i,j}||_{i,j=1}^m$, where $\overline{\theta}_{i,j}$ is the probability of transition after service of customer in the ith node to j-th one.

Let the route matrix $\overline{\Theta}$ be indecomposable, i.e.

$$\forall\, i,\ j \in \{1, \dots, m\}\ \exists\, i_1, \dots, i_r \in \{1, \dots, m\} :\ \overline{\theta}_{i,i_1} > 0, \overline{\theta}_{i_1,i_2} > 0, \dots, \overline{\theta}_{i_r,j} > 0.$$

Then for a fixed $\lambda_1 > 0$, the system of linear algebraic equations

$$\lambda_k = \sum_{t=1}^{m} \lambda_t \overline{\theta}_{t,k},\ k = 1, \ldots, m \tag{5}$$

has a unique solution of $(\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 > 0, \ldots, \lambda_m > 0$, [12, p. 13].

For a closed queueing network $\overline{S}$ with N customers discrete Markov process $(\overline{n}_1(t), \ldots, \overline{n}_m(t)),\ t \geq 0$, describing the number of customers in the network nodes has a limit distribution of $P_{\overline{S}}(k_1, \ldots, k_m)$, independent of the initial conditions and presented in the form

$$P_{\overline{S}}(k_1, \ldots, k_m) = \frac{\prod_{i=1}^{m} P_i(k_i)}{\sum_{k_1,\ldots,k_m:\ k_1+\ldots+k_m=N} \prod_{i=1}^{m} P_i(k_i)},\ k_1 + \ldots + k_m = N.$$

Hence, the stationary probability $\pi_i(k_i)$ that in a node i of the network $\overline{S}$ there is k_i customers satisfies the equality

$$\pi_i(k_i) = \sum_{k_j,\ 1\leq j\neq i\leq m,\ \sum_{1\leq j\neq i\leq m} k_j = N-k_i} P_{\overline{S}}(k_1, \ldots, k_m),\ k_i = 0, \ldots, N.$$

Theorem 5. *The flow $T^i_{\overline{S}}$, leaving the i node of the closed queueing network $\overline{S}$ with the total number of customers N, being in a stationary state, is Poisson with intensity $\sum_{k_i=1}^{N} \min(k_i, l_i)\mu_i \pi_i(k_i),\ i = 1, \ldots, m.$*

Proof. Indeed, the points of the flow $T^i_{\overline{S}}$, exiting the node i, are the moments of jumps down the component $\overline{n}_i(t)$ of the discrete Markov process $(\overline{n}_1(t), \ldots, \overline{n}_m(t))$, $t \geq 0$. Consequently, the flow $T^i_{\overline{S}}$ satisfies condition (a). Conditions (b), (c) are proved similarly to the proof of Theorem 1.

4 Queuing System $M|M|1|\infty$ with Random Intensities of Input Flow and Service

Consider queuing system $A_1 = M|M|1|\infty$ with a service intensity of $\mu(t)$ and a Poisson input flow Λ with an intensity of $\lambda(t)$, which are randomly changed by the following rules. Let the time axis $t \geq 0$ be split into half-intervals

$$[T_0 = 0, T_1 = T_0 + \xi_1),\ [T_1, T_2 = T_1 + \xi_2), \ldots,$$

where $\xi_1, \xi_2, \ldots$ are independent random variables with distribution

$$P(\xi_k > t) = \exp(-\sigma t),\ t \geq 0,\ k = 1, 2, \ldots$$

with parameter $\sigma > 0$.

We introduce a discrete Markov chain $n_l,\ l=1,\dots,$ with a set of states $\{1,\dots,N\}$ and an irreducible transition matrix $||\theta_{i,j}||_{i,j=1}^N$. Markov chain $n_l,\ l=1,\dots,$ has a unique (with positive components) solution $(\psi_1,\dots,\psi_N)$ of the system of Kolmogorov-Chapman stationary equations

$$\psi_i=\sum_{j=1}^N \psi_j\theta_{j,i},\ l=1,\dots,N. \tag{6}$$

We now introduce the Markov process $n(t),\ t\geq 0$, such that $n(t)=n_l,\ t\in[T_{l-1},T_l),\ l=1,\dots$ It is obvious that the stationary distribution $(\psi_1,\dots,\psi_N)$ of the Markov chain $n_l,\ l=1,\dots,$ is a stationary distribution of the Markov process $n(t),\ t\geq 0$. Indeed denote

$$\psi_i(t)=p(n(t)=i),\ i=1,\dots,N, \tag{7}$$

then the Kolmogorov-Chapman system of equations for Markov process $n(t)$ has the form

$$\dot{\psi}_i(t)=-\sigma\psi_i(t)+\sigma\sum_{j=1}^N \psi_j(t)\theta_{j,i},\ i=1,\dots,n,$$

so the system of Kolmogorov-Chapman stationary equations coincides with (6). Let's call such a queuing system as $M|M|1|\infty$ in a random environment.

Suppose that on each half-interval $[T_{k-1},T_k)$ the input flow to the $M|M|1|\infty$ system is Poisson with intensity $\lambda(t)=\lambda_{n_l},\ t\in[T_{l-1},T_l),\ l=1,2,\dots,$ and the service intensity satisfies the relations $\mu(t)=\mu_{n_l},\ t\in[T_{l-1},T_l),\ l=1,2,\dots,$ where $\lambda_1,\dots,\lambda_N,\ \mu_1,\dots,\mu_N$ are some positive numbers. It is worthy to remark that in this system the input flow and the process of service (random sequence of service times) are dependent random objects.

Theorem 6. *The stationary output flow in the system $M|M|1|\infty$ in a random environment is Poisson with an average intensity $a=\sum_{j=1}^N \psi_j\lambda_j$.*

Proof. Consider Markov random process $(x(t),n(t)),\ t\geq 0$, whose first component sets the number of customers in the system $M|M|1|\infty$ and write for its stationary probabilities $p_{i,j}$ Kolmogorov-Chapman equations:

$$\lambda_i p_{0,i}=-\sigma p_{0,i}+\mu_i p_{1,i}+\sigma\sum_{j=1}^N p_{0,j}\theta_{j,i},\ i=1,...,N,$$

$$(\lambda_i+\mu_i+\sigma)p_{k,i}=\lambda_i p_{k-1,i}+\mu_i p_{k+1,i}+\sigma\sum_{j=1}^N p_{k,j}\theta_{j,i},\ i=1,...,N,\ k=1,2,... \tag{8}$$

We introduce the following notation at $i=1,\dots,N$:

$A_{0,i}=-\lambda_i p_{0,i}+\mu_i p_{1,i},\ A_{k,i}=-(\lambda_i+\mu_i)p_{k,i}+\lambda_i p_{k-1,i}+\mu_i p_{k+1,I},\ k=1,2,\dots,$

$$B_{k,i} = -\sigma p_{k,i} + \sigma \sum_{j=1}^{N} p_{k,j}\theta_{j,i},\ k = 0, 1, \ldots$$

Then the equations (8) may be rewritten as

$$0 = A_{k,i} + B_{k,i},\ i = 1, \ldots, N,\ k = 0, 1, \ldots \tag{9}$$

Denote $C_{k,i} = \sum_{r=0}^{k} A_{r,i},\ D_{k,i} = \sum_{r=0}^{k} B_{r,i}$, then by Formulas (9) we have

$$0 = C_{k,i} + D_{k,i},\ i = 1, \ldots, N,\ k = 0, 1, \ldots \tag{10}$$

Obviously, the following relations are fulfilled:

$$\frac{1}{\sigma}\sum_{i=1}^{N} B_{k,i} = -\sum_{i=1}^{N} p_{k,i} + \sum_{i=1}^{N}\sum_{j=1}^{N} p_{k,j}\theta_{j,i} = -\sum_{i=1}^{N} p_{k,i} + \sum_{j=1}^{N}\sum_{i=1}^{N} p_{k,j}\theta_{j,i} = 0.$$

and consequently

$$\sum_{i=1}^{N} D_{k,i} = 0. \tag{11}$$

By induction of k we can obtain equalities by analogy with Theorem 2 proof:

$$C_{k,i} = -\lambda_i p_{k,i} + \mu_i p_{k+1,i},\ i = 1, \ldots, N,\ k = 0, 1, \ldots \tag{12}$$

Summing up the equations (10) by $i = 1, \ldots, N,\ k = 0, 1, \ldots$, and using Formulas (11), (12), we obtain:

$$0 = -\sum_{i=1}^{N} \lambda_i \sum_{k=0}^{\infty} p_{k,i} + \sum_{i=1}^{N} \mu_i \sum_{k=0}^{\infty} p_{k+1,i}. \tag{13}$$

The second term in Formula (13) is the intensity of a of the output Poisson flow in a given queuing system. In turn, by virtue of formulas (7), (13) we obtain that the intensity

$$a = \sum_{j=1}^{N} \psi_j \lambda_j. \tag{14}$$

Remark 3. By methods of Theorem 1 proof it is easy to obtain that the flow Λ is Poisson with intensity $a = \sum_{j=1}^{N} \psi_j \lambda_j$. Indeed, let us consider the Markov process $(y(t), n(t)),\ t \geq 0$, where $y(t)$ is the number of customers of the input flow that came to the system up to t. This process has the following transient intensities: the transition intensity $(m, i) \rightarrow (m, j)$ equals $\sigma\theta_{i,j}$, the intensity of the transition $(m,i) \to (m+1,i)$ equals to $\lambda_i,\ i,j = 1,\dots,N,\ m = 0,1,\dots$ So the jump intensity $y(t) \to y(t)+1$ equals

$$\sum_{1\le j\le N,\ 0\le m} p(y(t)=m,\ n(t)=j)\lambda_j = \sum_{j=1}^{N} p(n(t)=j)\lambda_j = \sum_{j=1}^{N} \psi_j\lambda_j = a.$$

Thus, the random flow Λ by distribution coincides with the Poisson flow of average intensity a.

Remark 4. The statement of Remark 3 allows to obtain criterion's of ergodicity, to derive formulas for stationary distributions, to analyse output flows for manifold queuing systems with independent input flow Λ and sequences of service times: open queuing network of Jackson type, queuing systems with failures, queuing systems with feedbacks [4].

5 Transformation of Open Queuing Network into Multiphase Type Queuing Network

Following [3] demonstrate how to transform open queuing network into multiphase type queuing network. Consider open queuing network S with finite number of nodes $U = \{0,1,\dots,m\}$ and input flow Λ. As the flow Λ and service times of customers in different nodes are independent then it is convenient to consider the flow Λ as Poisson flow with average intensity $\lambda_0 = a$. Paths of customers in the network S are defined by the route matrix $\Theta = ||\theta_{i,j}||_{i,j=0}^{m},\ \theta_{0,0} = 0$, consisting of probabilities $\theta_{i,j}$ of customers transitions from the node i to the node j after a service in the node i. The node 0 is a source of customers arriving the network and a container of customers departing the network. Here $\theta_{0,i}$ is the probability that input flow customer moves to the node i and $\theta_{i,0}$ is the probability that customer departs network after service in the node i. In the node k of the network S there is infinite number of identical servers with service times which has the distribution

$$F_k(t) = 1 - \exp(-\mu_k t),\ t \ge 0,\ \mu_k,\ 0 < \mu_k < \infty,\ k = 1,\dots,m.$$

Transform the network S into the following network S^*. Each node $k,\ 0 \le k \le m$, is divided into infinite number of nodes $(k,j),\ 1 \le j$. Here nodes with $1 \le k \le m$ are nodes with infinite numbers of servers and nodes with $k = 0$ absorb customers departing the network. A customer arriving the network with the probability $\theta_{0,k}$ moves to the node $(k,1)$. The node $(0,1)$ is sham because $\theta_{0.0} = 0$ and so customers do not visit it. Then after a service in the node $(p,j),\ 1 \le p \le m,\ 1 \le j$, customer with the probability $\theta_{p,q}$ moves to the node $(q,j+1)$ and with the probability $\theta_{p,0}$ moves to the node $(0,j+1)$ - departs the network, $1 \le p,q \le m,\ 1 \le j$. Consequently initial network S is transformed into the network S^* with the nodes set $U^* = \{(k,j),\ 1 \le j,\ 0 \le k \le m\}$. Graphically the network S^* is represented in Fig. 1.

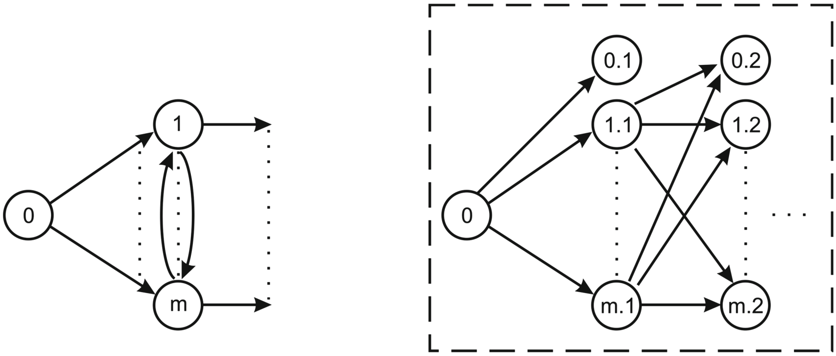

Fig. 1. Transformation of Jackson network (leftward) into multiphase type network (rightward).

The network S^* is constructed similar to retrial queues systems [4–8]. Transformation of the network S into the network S^* does not change paths and service times of customers.

In the network S^* a system of balance equations for stationary intensities of flows arriving the nodes of the set U^* may be solved by recurrent relations

$$\lambda_{k,1} = \lambda_0 \theta_{0,k},\ \lambda_{k,j+1} = \sum_{p=1}^{m} \lambda_{p,j} \theta_{p,k},\ 0 \leq k \leq m,\ 1 \leq j, \tag{15}$$

and its synergetic effects may be analysed in suggestion that each node with infinite number of servers in multiphase type network is replaced by node with large by finite number of servers.

6 Conclusion

It is worthy to devote special attention to an application of Remark 4 to queuing systems and networks with retrial queues. Such systems appear in manifold modern applied problems [4–8]. In this section we connect a representation of the input flow Λ as Poisson flow with average intensity and a consideration of networks with infinite number of servers in their nodes [4–6]. For this purpose we use a transformation of such networks into multiphase type networks [3]. In multiphase type networks it is possible to assume that each customer may be serviced a fixed number of times also not arbitrary ones. This suggestion together with the representation of the input flow Λ as Poisson flow with average intensity and with an assumption that the flow Λ and service process are independent allow to consider models more close to applications.

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References

1. Burke, P.J.: The output of a queuing system. Oper. Res. **4**, 699–704 (1956)
2. Tsitsiashvili, G.S., Osipova, M.A.: Generalization and extension of Burke theorem. Reliab.: Theor. Appl. **13**(1), 59–62 (2018)
3. Tsitsiashvili, G.S., Osipova, M.A., Losev, A.S., Kharchenko, Y.N.: Jackson network as network of multiphase type. Int. Math. Forum **12**(7), 303–310 (2017)
4. Moiseeva, S., Zadiranova, L.: Feedback in infinite-server queuing systems. In: Vishnevsky, V., Kozyrev, D. (eds.) DCCN 2015. CCIS, vol. 601, pp. 370–377. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-30843-2_38
5. Moiseeva, S.P., Zakhorol'naya, I.A.: Athematical model of parallel retrial queueing of multiple requests. Optoelectron. Instrum. Data Process. **47**(6), 567–572 (2011)
6. Moiseev, A., Nazarov, A.: Queueing network $MAP/(GI/\infty)^K$ with high-rate arrivals. Eur. J. Oper. Res. **254**, 161–168 (2016)
7. Nobel, R.D., Tijms, H.C.: Optimal control for an MX/G/1 queue with two service modes. Eur. J. Oper. Res. **113**(3), 610–619 (1999)
8. Nobel, R.D., Tijms, H.C.: Waiting-time probabilities in the "M/G/1" retrial queue. Stat. Neerl. **60**(1), 73–78 (2006)
9. Khinchin, A.Y.: Works on the Mathematical Queuing Theory. Physmatlit, Moscow (1963)
10. Ivchenko, G.I., Kashtanov, V.A., Kovalenko, I.N.: Queuing Theory. Vishaya Shkola, Moscow (1982)
11. Jackson, J.R.: Networks of Waiting Lines. Oper. Res. **5**(4), 518–521 (1957)
12. Basharin, G.P., Tolmachev, A.L.: Queuing network theory and its applications to the analysis of information-computing networks. Itogi Nauki Tech. Ser. Teor. Veroiatn. Mat. Stat. Teor. Kibern. **21**, 3–119 (1983). (in Russian)
13. Beutler, F.J., Melamed, B.: Decomposition and customer streams of feedback networks of queues in equilibrium. Oper. Res. **26**(6), 1059–1072 (1978)