A Retrial Queueing System with Orbital Search of Customers Lost from an Offer Zone

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Abstract. A tandem retrial queueing system with orbital search in which two self-service stations namely, the main station and the offer zone and an orbit for passive customers lost from the offer zone without joining the main station is considered. The main service station is of infinite capacity while the offer zone which works in a random environment and the orbit for passive customers are of finite capacities. Two types of customers arrive to the service stations according to a Marked Markovian Arrival Process (MMAP) with representation (D_0, D_1, D_2). The service times in both stations are exponentially distributed. A virtual search mechanism associated with the main station will be working when the number of customers in the main station is below a pre-assigned level L. The duration of search is exponentially distributed. The condition for system stability is established. The system state distribution in the steady state is obtained. Several system performance characteristics are derived. An associated optimization problem is investigated.

Keywords: Retrial queue · Tandem queue · Main station
Offer zone · Random environment · Passive customers · Orbit

1 Introduction

Tandem queues form an important class of the queueing networks and it serves as a link between the theory of queues and queueing networks. A bibliography of articles on queueing networks with finite capacity service stations can be found in [22]. Most of the literature in this regard assume that the service stations in the tandem network are of finite capacity and the time between successive arrivals to the system are exponentially distributed. [17] gives an algorithm for solving exponential tandem queues with blocking. In [11–13] multi-stage queueing networks with correlated arrivals are considered. In Krishnamoorthy et al. [16] considered a tandem queueing model with two service stations and one of which namely, the offer zone works in a random environment. Artalejo [1,2] gives a detailed bibliography of retrial queues. The monograph by Falin and Templeton [10] gives an introduction to the theory of retrial queues and it describes how

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the theory of retrial queues can well be applied in the analysis of problems which are more realistic as well as practically important. The present paper generalizes the model described in one of our papers [16] to a retrial set-up by setting up an orbit for holding the customers who leave the system after completing their service at the offer zone.

In the present paper, we consider a tandem retrial queueing network with two service stations and an orbit for those customers who discontinued their service after a trial service. This model is the mathematical formulation of a set of real life problems persisting in the field of telecommunication. In the field of telecommunication, various service providers compete for attracting the maximum number of customers to their paid service. Some customers directly enter the paid service while some others would like to make a previous trial of service before subscribing to paid service. Service providers announce various types of offers, incentives and free trials to make the maximum number of customers to continue with their service. They try to minimize the loss of customers from the subscription of their service. Not all the customers, who utilized the offers and free trials, move on to paid service. Some may continue with the same service provider as paid customers, while some others discontinue the service temporarily after the free-trial. But the service providers have a data-base consisting of those customers who discontinued service after the free trial and that may be considered as an orbit. Having discontinued withe service for a short time, a few may have a tendency to come back to paid service, which may be considered as retrials and those retrial rates may be small when compared to direct arrivals to the paid service. So the customers in the orbit may be designated as passive customers. The service providers need a minimum number of customers in paid service for the proper functioning of their system. So whenever the number of customers in the paid service drops down to this pre-assigned value, the service providers try to bring some more customers to the paid service by means of orbital search. This search can be any of the activities like contacting those passive customers over the telephone, sending e-mails, additional cash-back offers etc. Search may result in an additional increment in the number of paid customers and whenever it reaches the optimum level, no more search has been done. Since there is some cost associated with the search, an optimum of this level to switch on the search mechanism is to be found. This problem is modelled mathematically as a tandem retrial queueing system with orbital search in which two service stations, namely the main station and the offer zone are functioning. The main service station is of infinite capacity while the offer zone is of finite capacity. The most important feature of the finite capacity offer zone in our model is that it works in a finite number of random environments, each of which lasts for a time interval whose distribution is Phase Type. In [13] and [12] servers in the same station are independent and identical. In our model the servers of the same station are identical, but the rate at which service is offered at the offer zone depends on the current environmental status of the offer zone. In addition to the retrials from the orbit, search for customers start functioning when the number of customers in the main station drops down to a preassigned level.

In classical queueing models Neuts and Ramalhoto [19] introduced the concept of search of orbital customers by the server at the end of a service completion epoch. In the case of M/G/1 retrial queues, search of orbital customers was introduced by Artalejo et al. [3]. Analysis of multi server queues with orbital search was done by Chakravarthy et al. in [5]. Krishnamoorthy et. al [14] investigated $M/G/1$ Retrial queues with non persistent customers and orbital search. More literature related to orbital search can be found in [6,7,15]. We also assume that the arrivals to the main station and the offer zone is according to a Marked Markovian Arrival Process [MMAP]. In Krishnamoorthy et al. [15] considered a queueing system with MMAP arrivals. Steady state probabilities are computed using Neuts' Matrix Geometric methods [20]. The rate matrix is computed using Logarithmic reduction Algorithm [18]. Various methods for the calculation of the equilibrium distribution of LDQBD's can be found in the papers by Neuts and Rao [21], Bright and Taylor [4] and Ramaswamy [18]. The stability condition is established and the steady state distribution is computed. Several performance measures of the system that influences the efficiency are derived. The cost functions for optimizing the level at which the search mechanism is to be switched off is derived. The control problems that optimizes the maximum capacity of the offer zone as well as the orbit are analyzed.

2 Description of the Model

We consider a tandem retrial queueing system in which, there are two self-service stations namely, the main station and the offer zone. The main station and the offer zone provides the same kind of service. But the service at the offer zone is restricted, for example, some trial service and it can not be continued for as long as they like. But after completing their service at the offer zone, the customers can decide whether to continue their service at the main station or not. There are some restrictions on the period of time they can stay in service at the offer zone. There are two types of customers in this system, say Type A and Type B. Type A customers are those customers who directly enter the main station for service and they do not try to take a trial service. Type B customers are those customers who wish to have a trial service by entering the offer zone and after their service completion at the offer zone, they can decide whether to continue their service at the main station or to leave the system. The offer zone works in a random environment and the environments at the offer zone are designed in such a way to attract the maximum number of type B customers from the offer zone to the main station and to make them get served at the main station. The service at the main station contributes a revenue to the system, while the offer zone has some kind of establishment as well as holding cost associated with it for the proper functioning. After the service completion at the offer zone, Type B customers are assumed to continue their service at the main station with probability η and with its complimentary probability $(1-\eta)$, joins an orbit of passive customers who temporarily discontinued service but retries for service after being idle for sometime. The customers in this orbit are referred to be passive in the sense that their retrial rates are very low compared to the arrival rates to both the stations. Let ν be the rate at which retrials from the offer zone to the main station occur and it is assumed to be lower than the fundamental rates of arrivals to both the stations. For the proper functioning of the system a minimum of L customers are to be ensured at the main station and so whenever the number of customers in service at the main station is below this level L, a virtual search mechanism associated with the main station, starts working and it go in search of customers from the orbit of passive customers. This may be by providing some additional incentives or cash back policies or some other strategies. As a result of this search, customers arrive to the main station at an exponential rate ν^*. The main station is of infinite capacity while both the offer zone and the orbit of passive customers are of finite capacities, say N and M respectively. As a result, when the offer zone is full, Type B customers directly enter the main station with probability γ and leaves the system with probability $(1-\gamma)$. Customers arriving to the orbit when it is full, is lost from the system for ever. Non persistent customers leave the orbit at an exponential rate ζ.

In the present model both type A and type B customers arrive according to a Marked Markovian Arrival Process (MMAP) with representation (D_0, D_1, D_2) where $D_1 = pD^*$ and $D_2 = (1-p)D^*$ for $0 \leq p \leq 1$. MMAP may be viewed as a special case of Markovian Arrival Processes or MAPs which is a more general class of point processes which takes in to account the correlation between inter-event times. It includes both Renewal as well as non-Renewal point processes. Many of the processes which we use in modelling of stochastic processes such as Poisson Processes, PH Renewal Processes, Markov Modulated Poisson Processes (MMPP) come under the class of MAP's. The MMAP governing the arrival of type A and type B customers in the present model is described as follows: Let the underlying Markov chain $\{\nu_t, t \geq 0\}$ be irreducible and let D be the generator of this Markov chain with state space $\{1, 2, 3, \ldots, m\}$. At the end of a sojourn time in state i, which is exponentially distributed with a positive finite parameter λ^i, one of the following events could occur: with probability $p_{ij}(0)$ it can move to state j where $j \neq i$ without an arrival, with probability $p_{ij}(1)$ it can move to state j with an arrival of a type A customer and with probability $p_{ij}(2)$ it can move to state j with an arrival of a type B customer. Let $D_0 = d_{ij}(0)$ be the rate matrix corresponding to those transitions without an arrival. Let $D_1 = d_{ij}(1)$ be the rate matrix corresponding to the arrival of type A customer and let $D_2 = d_{ij}(2)$ be the rate matrix corresponding to the arrival of type B customer. Then the MMAP under consideration is well be described by the parameter matrices (D_0, D_1, D_2) where $D_1 = pD^*$ and $D_2 = (1-p)D^*$ for $0 \leq p \leq 1$. $D = D_0 + D_1 + D_2$ is the infinitesimal generator of the Markov chain corresponding to the MMAP. All the off-diagonal elements of D_0 and all the elements of D_1 and D_2 are non negative. To completely specify a $MMAP(D_0, D_1, D_2)$, the initial probability vector in the Markov chain needs to be specified and we assume that the initial probability vector is the same as the stationary probability vector. That is our MMAP is a stationary MMAP. The average total arrival intensity λ is defined by $\lambda = \boldsymbol{\theta} D_1 \mathbf{e}$, where $\boldsymbol{\theta}$ is the invariant vector of the stationary distribution of the Markov chain $\{\nu_t, t \geq 0\}$. The vector $\boldsymbol{\theta}$ is the unique solution of the system of equations $\boldsymbol{\theta} D = \mathbf{0}, \boldsymbol{\theta}\mathbf{e} = 1$. where $\mathbf{e}$ denotes a column vector of $1'^s$ and $\mathbf{0}$ is a row vector of $0'^s$. The average arrival intensity λ_A and λ_B of type A and type B customers respectively are defined by $\lambda_A = \theta D_1 \mathbf{e}$ and $\lambda_B = \theta D_2 \mathbf{e}$. The squared integral (without differentiating the types of customers) coefficient of variation of intervals between successive arrivals is $c_{var} = 2\lambda\boldsymbol{\theta}(-D_0)^{-1}\mathbf{e} - 1$. The squared coefficient of variation of inter-arrival times of type A customers is $c_{var(A)} = 2\lambda_A\boldsymbol{\theta}[-D_0 - D_2]^{-1}\mathbf{e} - 1$ where as that of inter-arrival times of type B customers is $c_{var(A)} = 2\lambda_B\boldsymbol{\theta}[-D_0 - D_1]^{-1}\mathbf{e} - 1$. The integral coefficient of correlation of two successive intervals between arrivals is given as $c_{cor} = [\lambda\boldsymbol{\theta}(-D_0)^{-1}(D - D_0)(-D_0)^{-1}\mathbf{e} - 1]/c_{var}$.

The main station and the offer zone offers the same service but of the offer zone works in a random environment. We assume that there are a finite number of environments whose duration follows Phase Type distribution and the generator matrix of the Markov process leading to the PH distribution depends on the current environment of the offer zone. Let p_i where $\{i = 1, 2, 3, \ldots, n\}$ is the probability that the offer zone is at environment i. Each environment of the offer zone consists of one or more offers. Let $\{1, 2, \ldots, n\}$ denote the n environments of the offer zone and the duration of time the environment i works follow Phase type distribution with irreducible representation $PH(\beta_i, S_i)$ with M_i phases. The vector S_i^0 is given by $S_i^0 = -S_i\mathbf{e}$. We assume that all the customers in the offer zone are getting served in the same environment and so the offers given to those customers in service at the offer zone change with the change in the environment in which the offer zone works. After service completion at the offer zone type B customers enter the main station with probability η and enter the orbit with probability $(1 - \eta)$ provided it is not fully occupied.

3 Matrix Analytic Solution

We introduce the necessary random variables as follows: Let $N_1(t)$ denote the number of customers in the main station, $N_2(t)$ the number of customers in the offer zone, $N_3(t)$ the number of customers in the orbit, $E(t)$ the environment of the offer zone, $S(t)$ the phase of the environment of the offer zone and $A(t)$ the phase of the arrival process. $E(t)$ can take any of the values $\{1, 2, \ldots, n\}$ depending on the ongoing environment of the offer zone. Then $\{N_1(t), N_2(t), N_3(t), E(t), S(t), A(t)\}$ is a Markov process and it describes the process under consideration. This model can be considered as a Level dependent Quasi-Birth-Death (LDQBD) process and a solution is obtained by Matrix Analytic Method. We define the state space of the QBD under consideration and analyze the structure of its infinitesimal generator.

The state space Ω consists of all elements of the form (i, j, k, r, s, t) where

$$i \geq 0; 0 \leq j \leq N; 0 \leq k \leq M; t = 1, 2, \ldots, m; r = 1, 2, 3 \ldots, n$$

For a fixed value of $r, s = 1, 2, \ldots, M_r$.

Let the ordering of the elements of Ω be lexicographical. The infinitesimal generator Q of the LDQBD describing the model under consideration is of the form

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & & & & & \\ A_2^1 & A_1^1 & A_0^1 & & & & \\ & A_2^2 & A_1^2 & A_0^2 & & & \\ & & A_2^3 & A_1^3 & A_0^3 & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & \cdots & \cdots & \cdots \end{pmatrix}$$

where A_0^i, A_1^i, A_2^i are all square matrices whose entries are block matrices of appropriate dimensions.

A_0^i represents the rate matrix corresponding to the arrival of a customer to the main station; that is transition from level $i \rightarrow i+1$ where $i \geq 0$.

A_2^i represents the rate matrix corresponding to the departure of a customer after service completion at the main station when there are i customers in the main station; that is from level $i \rightarrow i-1$, for $i = 1, 2, \ldots,$ and

A_1^i describes all transitions in which the level does not change (transitions within levels i).

In the following sequel $\otimes$ and $\oplus$ represent the Kronecker Sum and Kronecker product respectively. Let $\mathbf{e}$ denote all one vector of appropriate order and I_M denote an identity matrix of order M.

The structure of the A_1^i for $i \geq 0$ are as follows:

$$A_1^i = \begin{pmatrix} E_1 & E_0 & & & & \\ E_2^1 & E_1 & E_0 & & & \\ & E_2^2 & E_1 & E_0 & & \\ & & \cdots\cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & E_2^{(N-1)} & E_1 & E_0 \\ & & & & E_2^N & E_1^N \end{pmatrix}$$

- E_0 is the matrix representation of the rate of arrival of type B customers and it depends neither on the number of customers currently undergoing service at the main station nor the number of customers waiting in the orbit of passive customers.
- E_1 is the matrix representation of the rates corresponding to the transitions when there are i customers in the main station and j customers in the offer zone.
- E_2^j is the matrix representation of the rates at which customers leave the offer-zone after completing their service in the offer zone when there are j customers in the offer zone.

E_1 is an $(M+1)\times(M+1)$ matrix with sub-blocks given by

$$E_1 = \begin{pmatrix} F_1 & & & & & \\ F_2 & F_1 & & & & \\ & F_2 & F_1 & & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & \\ & & & & F_2 & F_1 \end{pmatrix}$$

- F_1 is the matrix representation of the transition rates corresponding to the environmental changes, phase changes of the environmental process and the phase changes of the arrival process when there are i customers in the main station and j customers in the offer zone and k customers in the orbit where $j = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, M$
- F_2 is the matrix representation of the rates at which passive customers leave the orbit without retrying for service at the main station

F_1 is given by

$$F_1 = \begin{pmatrix} C_1 & S_1^0 \otimes p_2\beta_2 \otimes I_m & S_1^0 \otimes p_3\beta_3 \otimes I_m & \ldots & S_1^0 \otimes p_n\beta_n \otimes I_m \\ S_2^0 \otimes p_1\beta_1 \otimes I_m & C_2 & S_2^0 \otimes p_3\beta_3 \otimes I_m & \ldots & S_2^0 \otimes p_n\beta_n \otimes I_m \\ S_3^0 \otimes p_1\beta_1 \otimes I_m & S_3^0 \otimes p_2\beta_2 \otimes I_m & C_3 & \ldots & S_3^0 \otimes p_n\beta_n \otimes I_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_n^0 \otimes p_1\beta_1 \otimes I_m & S_n^0 \otimes p_2\beta_2 \otimes I_m & \ldots & \ldots & C_n \end{pmatrix}$$

For $j = 1, 2, ...N-1$, if $i \leq (L-1)$ then

$$C_l = [S_l - (i\mu + j\mu_l + k\nu + \nu^* + \zeta)] \oplus D_0$$

and if $i \geq L$ then,

$$C_l = [S_l - (i\mu + j\mu_l + k\nu + \zeta)] \oplus D_0$$

For $j = N$, if $i \leq (L-1)$ then

$$C_l = [S_l - (i\mu + j\mu_l + k\nu + \nu^* + \zeta)] \oplus [D_0 + (1-\gamma)D_2]$$

and for $j = N$, if $i \geq L$ then

$$C_l = [S_l - (i\mu + j\mu_l + k\nu + \zeta)] \oplus [D_0 + (1-\gamma)D_2]$$

Let $M^* = \sum_{i=1}^{n} M_i$ and $M^{**} = (M+1)\sum_{i=1}^{n} M_i$.

$$F_2 = \zeta I_{mM^*}$$

$$E_0 = I_{M^{**}} \otimes D_2$$

For a fixed value of j, E_2^j is a block-diagonal matrix of order $(M+1)\times(M+1)$ given by

$$E_2^j = \begin{pmatrix} O & G & & & & \\ & O & G & & & \\ & & O & G & & \\ & & & \ddots & \ddots & \\ & & & & O & G \\ & & & & O & G \end{pmatrix}$$

where

$$G = diag(I_{M_1} \otimes (1-\eta)j\mu_1 I, I_{M_2} \otimes (1-\eta)j\mu_2 I, \ldots, I_{M_n} \otimes (1-\eta)j\mu_n I).$$

Here diag(a, b, c, .., .) represents a diagonal matrix whose diagonal entries are listed and

$$I = I_{m(M+1)(N+1)}$$

The matrix G represents the rate at which the customers enter the orbit of passive customers and the entry is restricted to a maximum number$M$ of the passive customers in the orbit.

The matrix A_0^i corresponding to the arrival of a customer to the main station can be written as

$$A_0^i = \begin{pmatrix} U_1 & & & & \\ U_2^1 & U_1 & & & \\ & U_2^2 & U_1 & & \\ & & \ddots & \ddots & \\ & & & U_2^N & U_1^N \end{pmatrix}$$

U_1 represents the transitions from $i \rightarrow i+1$ without making any changes in the number of customers in the offer zone

$$U_1 = \begin{pmatrix} V_1 & & & & & \\ V_2^1 & V_1 & & & & \\ & V_2^2 & V_1 & & & \\ & & \cdots\cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & V_2^{(M-1)} & V_1 & \\ & & & & V_2^M & V_1 \end{pmatrix}$$

where for $j = 1, 2, \ldots, (N-1)$

$$V_1 = I_{M^{**}} \otimes D_1$$

and for $j = N$

$$V_1 = I_{M^{**}} \otimes [D_1 + \gamma D_2]$$

For $k = 1, 2, \ldots, M$, the matrices V_2^k represents the rate at which customers from the orbit of passive customers enter the main station.

In this case there are two possibilities depending on i, the number of customers in the main station. Whenever the number of customers is greater than or equal to L, the virtual search mechanism is in off condition and only retrials from the orbit increases the number of customers in the main station and whenever this i drops down to $(L-1)$, the search mechanism starts search for customers from the orbit.

For a fixed i and j, if $i \geq L$ then

$$V_2^k = k\nu I_{mM^*}$$

and if $i \leq (L-1)$ then

$$V_2^k = [k\nu + \nu^*] I_{mM^*}$$

For $j = 1, 2, \ldots N$, the matrix U_2^j gives the rates at which customers from the offer zone proceeds to the main station without discontinuing their service

$$U_2^j = diag(I_{M_1} \otimes \eta j \mu_1 I, I_{M_2} \otimes \eta j \mu_2 I, \ldots, I_{M_n} \otimes \eta j \mu_n I)$$

where diag(a, b, c, .., .) represents a diagonal matrix whose diagonal entries are listed and

$$I = I_{m(M+1)(N+1)}$$

The matrices A_2^i, representing the rates at which service completion occurs from the main station are given by

$$A_2^i = i\mu I_{(N+1)mM^{**}}$$

3.1 Stability Condition

The present model is a level dependent QBD and we apply Neuts-Rao truncation for the analysis of the model. We assume that when the number of customers in the main station exceeds a certain limit, say K, service occurs at constant rates $K\mu$. In that situation the matrices A_2^i becomes A_2^K for $i \geq K$. We also assume that the truncation level K is greater than the number L at which the search must be switched off. The infinitesimal generator Q^1 of the modified model becomes

$$Q^1 = \begin{pmatrix} A_1^0 & A_0^0 & & & & & \\ A_2^1 & A_1^1 & A_0^1 & & & & \\ & A_2^2 & A_1^2 & A_0^2 & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & A_2 & A_1 & A_0 & \\ & & & & A_2 & A_1 & A_0 \\ & & & & & \cdots & \cdots \\ & & & & & & \cdots \end{pmatrix}$$

where $A_1 = A_1^K, A_2 = A_2^K$ and $A_0 = A_0^K$.

Let the matrix A be defined as $A = A_0 + A_1 + A_2$. We can see that A is an irreducible infinitesimal generator matrix of the underlying process and so there exists the stationary $1 \times (N+1)(M+1)mM^*$ vector $\boldsymbol{\pi}$ of A such that

$$\boldsymbol{\pi} A = 0$$

and

$$\boldsymbol{\pi}\mathbf{e} = 1.$$

where $M^* = \sum_{r=1}^{n} M_i$.

The vector $\boldsymbol{\pi}$ can be partitioned as

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_N)$$

For $i = 1, 2, \ldots, N$ the vectors $\boldsymbol{\pi}_i$ can be partitioned as

$$\boldsymbol{\pi}_i = \big(\boldsymbol{\pi}(i,1), \boldsymbol{\pi}(i,2), \ldots, \boldsymbol{\pi}(i,M)\big)$$

whereas

$$\boldsymbol{\pi}(i,j) = \big(\boldsymbol{\pi}(i,j,1,1), \boldsymbol{\pi}(i,j,1,2), \ldots, \boldsymbol{\pi}(i,j,1,M_1), \ldots, \boldsymbol{\pi}(i,j,n,1), \ldots, \boldsymbol{\pi}(i,j,n,M_n)\big)$$

Each vector $\boldsymbol{\pi}(i,j,k,l)$ is a $1 \times m$ vector denoted as

$$\pi(i,j,k,l) = (\pi(i,j,k,l,1), \pi(i,j,k,l,2), \ldots, \pi(i,j,k,l,m))$$

where the state $\pi(i,j,k,l,m)$ is the probability of being in state (i,j,k,l,m) where i is the number of customers at the offer zone, j the number of passive customers in the orbit, k the environment of the offer zone, l the phase of the environment and r the phase of the underlying MMAP arrival process.

Let the matrix A be of the form

$$A = \begin{pmatrix} W_1^0 & W_0 & & & & \\ W_2^1 & W_1^1 & W_0 & & & \\ & W_2^2 & W_1^2 & W_0 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \\ & & & W_2^{(N-1)} & W_1^{(N-1)} & W_0 \\ & & & & W_2^N & W_1^N \end{pmatrix}$$

where

$$W_0 = E_0$$

for $j = 1, 2, ...(N-1)$

$$W_1^j = E_1 + +U_1 + K\mu I_{mM^{**}}$$

$$W_2^j = E_2^j + U_2^j$$
$$W_1^N = E_1^N + +U_1^N + K\mu I_{mM^{**}}$$

The Markov chain with generator Q^1 is positive recurrent if and only if

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}$$

3.2 Steady State Distribution

The stationary distribution of the Markov process under consideration is obtained by solving the set of equations

$$\mathbf{x}\mathbf{Q}^1 = 0, \mathbf{x}\mathbf{e} = \mathbf{1}.$$

Let $\mathbf{x}$ be the steady-state probability vector of Q^1.

Partition this vector in conformity with Q^1 as follows:

$$\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots,)$$

where

$$\mathbf{x}_i = (\mathbf{x}_{i0}, \mathbf{x}_{i1}, \ldots, \mathbf{x}_{iN}), i \geq 0$$

For $j = 0, 1, \ldots, N$ and $k = 1, 2, \ldots, M$ the vectors

$$\mathbf{x}_{ij} = (\mathbf{x}_{ij1}, \mathbf{x}_{ij2}, \mathbf{x}_{ij3}, \ldots \mathbf{x}_{ijM})$$

$$\mathbf{x}_{ijk} = (\mathbf{x}_{ijk1}, \mathbf{x}_{ijk2}, \ldots, \ldots, \mathbf{x}_{ijkn})$$

For $r = 0, 1, \ldots, n$

$$\mathbf{x}_{ijkr} = (\mathbf{x}_{ijkr1}, \mathbf{x}_{ijkr2}, \ldots, \mathbf{x}_{ijkrM_r})$$

$$\mathbf{x}_{ijkrs} = (\mathbf{x}_{ijkrs1}, \mathbf{x}_{ijkrs2}, \ldots, \mathbf{x}_{ijkrsm})$$

$\mathbf{x}_{ijkrst}$ is the probability of being in state (i, j, k, r, s, t) where

$$i \geq 0; j = 0, 1, \ldots, N; k = 0, 1, 2, \ldots, M;$$

$$r = 1, 2, \ldots, n; s = 1, 2, \ldots, M_r; t = 1, 2, \ldots, m.$$

Under the stability condition the steady-state probability vector is obtained as

$$\mathbf{x}_{(K-1)+i} = \mathbf{x}_{(K-1)} R^i, i \geq 0$$

where R is the minimal non negative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0$$

and the vectors $\mathbf{x}_0, \ldots, \mathbf{x}_{(K-1)}$ are obtained by solving

$$\mathbf{x}_0 A_1^0 + \mathbf{x}_1 A_2^1 = 0$$

$$\mathbf{x}_{(i-1)} A_0^{(i-1)} + \mathbf{x}_i A_1^i + \mathbf{x}_{(i+1)} A_2^{(i+1)} = 0; 1 \leq i \leq (K-2)$$

$$\mathbf{x}_{(K-2)} A_0 + \mathbf{x}_{(K-1)} \left[A_1^{(K-1)} + A_2 R \right] = 0$$

subject to the normalizing condition

$$\sum_{i=0}^{(K-2)} x_i + x_{(K-1)} (I - R)^{-1} \mathbf{e} = 1.$$

4 Some Performance Measures of the System

Some measures of performance, which helps the operators of the system to make decisions concerning the optimal values of maximum capacities N and M respectively of the offer zone and the orbit of passive customers and of the cut-off point L are evaluated. Loss of type B customers can happen mainly in two ways: The first type of loss namely, type I loss is due to the lack of space in the offer zone and this happens even before getting a service at the offer zone. The other type of loss namely, type II loss happens when the orbit is full. There is one more type of loss from the orbit of passive customers and the effect of this loss on the system can be can be minimized by means of orbital search if the number of customers in the main station is less than L. We can also identify the environment of the offer zone from which the maximum expected number of type B customers join the main station which in turn help us to redefine the offers. Following are some performance measures which helps us to make a detailed study about the problem under consideration.

1. Expected Number of customers in the main station

$$E[MS] = \sum_{i=0}^{\infty} i x_i \mathbf{e}$$

   where $\mathbf{e}$ is a column vector of appropriate order consisting of all ones.
2. Expected Number of customers in the offer zone

$$E[OZ] = \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{r=1}^{n}\sum_{s=1}^{M_r}\sum_{t=1}^{m} j x_{ijkrst}$$

3. Expected Number of customers in the offer zone

$$E[OPC] = \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{r=1}^{n}\sum_{s=1}^{M_r}\sum_{t=1}^{m} k x_{ijkrst}$$

4. Expected number of customers enter the main station as a result of search

$$E[S] = \sum_{i=0}^{(L-1)}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{r=1}^{n}\sum_{s=1}^{M_r}\sum_{t=1}^{m} \nu^* x_{ijkrst}$$

5. Probability that a type B customer is lost from the system when the offer zone is full

$$P[L_{T_1}] = \sum_{i=0}^{\infty}\sum_{k=0}^{M}\sum_{r=1}^{n}\sum_{s=1}^{M_r}\sum_{t=1}^{m} (1-\gamma) x_{iNkrst}$$

6. Probability that a type B customer is lost after service completion at the offer zone

$$P[L_{T_2}] = \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{r=1}^{n}\sum_{s=1}^{M_r}\sum_{t=1}^{m} x_{ijMrst}$$

7. Expected number of non-persistent customers lost from the orbit without joining the main station

$$E[L_{OPC}] = \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{r=1}^{n}\sum_{s=1}^{M_r}\sum_{t=1}^{m} k\zeta x_{ijkrst}$$

8. Expected number of type B customers who enter the main station after service completion from environment r of the offer zone

$$E[E(r)] = \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{s=1}^{M_r}\sum_{t=1}^{m} j\mu_r\eta x_{ijkrst}$$

for $r = 1, 2, ...n$
9. Expected number of type B customers lost when the offer zone is full

$$E[L_{T_1}] = \lambda_B \times P[L_{T_1}]$$

where λ_B is the fundamental rate of arrival of customers to the offer zone
10. Expected number of type B customers lost when the orbit is full

$$E[L_{T_2}] = \lambda_B \times P[L_{T_2}]$$

11. Expected number of type B customers who enter the main station after the service completion at the offer zone

$$E[OZ \to MS] = \sum_{r=0}^{n} E[E(r)]$$

12. Expected number of type B customer lost due to the capacity restrictions of the offer zone and the orbit

$$E[L] = E[L_{T_1}] + E[L_{T_2}]$$

13. Fraction of time the offer zone is in the r^{th} environment

$$F[r] = \sum_{i=0}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{s=1}^{M_r}\sum_{t=1}^{m} x_{ijkrst}$$

where $r = 1, 2, \ldots, n$

5 An Optimization Problem

For the economic interpretation of any queueing model, cost analysis plays an important role. In this section, we propose an optimization problem which determines the level L of the main station at which the search mechanism is to be switched off. In this case we assume that all other parameters are kept fixed.

To construct an objective function we assume that the customers undergoing service in the main station provide more revenue to the system when compared to the customers undergoing service in the offer zone. An additional revenue is provided by each customer who enter the main station. Operating cost associated with the functioning of the various environments or offers in the offer zone and holding cost associated with the working of the orbit are expenditures to the system. There is a search cost associated with each customer entering the main station by means of orbital search. The search cost is also an expenditure encountered by the system. Thus we introduce the revenue and expenditure per customer as follows:

- revenue r_1 monetary units per customer undergoing service in the main station
- revenue r_2 monetary units per customer undergoing service in the offer zone where $r_2 < r_1$
- operating cost c_1 monetary units per customer for providing various offers
- holding cost c_2 monetary units per customer waiting in the orbit
- search cost c_3 monetary units per customer entering the main station as a result of orbital search

The Expected Total Profit (**ETP**) is given by

$$(\mathbf{ETP}) = r_1 E[MS] + r_2 E[OZ] - c_1 E[OZ] - c_2 E[OPC] - c_3 E[S]$$

So the objective of the service providers or the operators of the system is to determine an optimal value of 'L' for which the total expected cost (**ETP**) is maximum.

6 Conclusion

The results in this paper may be extended to tandem queueing networks consisting of more than two service stations and also to the case where the service time distributions are of so general say Phase Type distributions. Even though such a generalization essentially increases the dimensions of the state space of the Markov chain under consideration which in turn makes the computational implementations more complex and time consuming, we hope that reducing the number of environments and also the dimension of the MMAP under consideration will make it more tractable. We plan to investigate such a problem in future.

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