Nonaffine Models of Yield Term Structure

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Abstract. The equation of term structure for the price of a zero-coupon bond is considered, the solution of which in analytical form is known, basically, for the simplest models and has an affine structure with respect to the short-term rate. The paper constructs solutions of this equation for a family of term structure models that are based on short-term rate processes in which the square of volatility is proportional to the third power of the short-term rate in stochastic differential equations. The solution of the equation is sought in the form of a definite functional series and, as a result, is reduced to a confluent hypergeometric function. Three versions of the underlying stochastic differential equations for short-term rate processes are considered: with zero drift, linear drift, and quadratic drift. Numerical examples are given for the yield curve and the forward rate curve for these versions. Some conditions for the existence of nontrivial solutions of the equation of time structure in the family of processes under consideration are formulated.

Keywords: The equation of the yield term structure
The price of zero-coupon bond · The CIR(1980) model
The Ahn – Gao model · The yield curve · The forward curve

1 Introduction

Suppose that the state of the financial market is described by the interest rate $r(t)$, which follows a Markov process homogeneous in time, generated by the stochastic differential equation

$$dr(t) = \mu(r(t))\,dt + \sigma(r(t))\,dw(t)$$

with the drift function $\mu(x)$, the volatility function $\sigma(x)$, and the standard Wiener process $w(t)$. For convenience of reasoning, we denote the drift function $m(r) = \mu(r) - \lambda(r)\sigma(r)$ and the diffusion function $s(r) = 0.5\sigma^2(r)$. Here $\lambda(r)$ is the so-called market risk price. Previously [1], the problem of determining the time structure of the yield of a zero-coupon bond, when the functions $m(r)$ and $s(r)$ are polynomials was considered. It turned out, in this case, whether the yield curves can be polynomials or power series in the variable r. It turned

A. Dudin et al. (Eds.): ITMM 2018/WRQ 2018, CCIS 912, pp. 16–26, 2018.
https://doi.org/10.1007/978-3-319-97595-5_2

out that this happens if only $m(r)$ and $s(r)$ are polynomials of not more than first degree. In this case, the models of the yield term structure are affine.

In this paper, we consider a similar problem, but the term structure of the price of a zero-coupon bond is sought in the form of a functional series that differs from the power series. It is found out that for some cases such solutions exist. The resulting term structure turns out to be non-affine and is described by confluent hypergeometric functions. This family includes such known models of interest rates as the model CIR(1980) [2] and the model Ahn – Gao [3].

2 The General Equation for the Price of a Bond and Its Components

Consider the equation of term structure for the price of the zero-coupon bond $P(r, \tau)$ [4]

$$-\frac{\partial P(r,\tau)}{\partial \tau} + m(r)\frac{\partial P(r,\tau)}{\partial r} + s(r)\frac{\partial^2 P(r,\tau)}{\partial r^2} - rP(r,\tau) = 0, \quad P(r,0) = 1. \quad (1)$$

Here $m(r)$ is the function of the short-term interest rate drift, and $s(r)$ is the square of its volatility. We seek a solution of this equation in the form

$$P(r,\tau) = \sum_{n=0}^{\infty} \left(\frac{a(\tau)}{r}\right)^{\alpha+n} c_n, \quad (2)$$

where $a(\tau)$, α and c_n, $n = 0, 1, 2, \ldots$, are the function and coefficients to be determined.

The corresponding derivatives used in Eq. (1) have the form

$$\begin{aligned}
\frac{\partial P(r,\tau)}{\partial \tau} &= \frac{a'(\tau)}{a(\tau)} \sum_{n=0}^{\infty} (\alpha+n) \left(\frac{a(\tau)}{r}\right)^{\alpha+n} c_n, \\
\frac{\partial P(r,\tau)}{\partial r} &= \frac{1}{a(\tau)} \sum_{n=0}^{\infty} (\alpha+n) \left(\frac{a(\tau)}{r}\right)^{\alpha+n+1} c_n, \\
\frac{\partial^2 P(r,\tau)}{\partial r^2} &= \frac{1}{a(\tau)^2} \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n+1) \left(\frac{a(\tau)}{r}\right)^{\alpha+n+2} c_n. \quad (3)
\end{aligned}$$

Suppose that the drift and volatility of the short-term interest rate are such that the functions $m(r)$ and $s(r)$ are polynomials of order p and q, respectively:

$$m(r) = \sum_{k=0}^{p} m_k r^k, \quad s(r) = \sum_{k=0}^{q} s_k r^k. \quad (4)$$

Now substituting expressions (2) – (4) in Eq. (1), we obtain

$$\sum_j \big(-I(j\,|\,0)\,(\alpha+j)\;a'(\tau)\;a(\tau)^{\alpha+j-1}\,c_j - I(j\,|-1)\;a(\tau)^{\alpha+j+1}\,c_{j+1}$$

$$-I(j\,|\,1-p) \sum_{k=Max\{0,\,1-j\}}^{p} (\alpha+j+k-1)\;m_k\,a(\tau)^{\alpha+j+k-1} c_{j+k-1}$$

$$+I\,(j\,|\,2-q) \sum_{k=Max\{0,2-j\}}^{q} [(\alpha+j+k-2t)\,(\alpha+j+k-1)$$

$$\times\; s_k\,a\,(\tau)^{\alpha+j+k-2}\,c_{j+k-2}]\big)\;\left(\frac{1}{r}\right)^{\alpha+j} = 0. \tag{5}$$

A certain complexity in the expression (5) is caused by the fact that the summation over the index j for each term starts in different ways: for the first term $j \geq 0$, for the second term $j \geq -1$, for the third term $j \geq 1-p$, for the fourth summand $j \geq 2-q$. Therefore, in expressions of the terms, the factors $I(j \mid k)$ appeared, representing indicator functions equal to one if $j \geq k$, and zero otherwise.

Equality (5) must be satisfied uniformly with respect to the variable r. In this case, since the functions $r^{-j}(j = 0, \pm 1, \pm 2, ...)$ are linearly independent, the coefficients in front of these functions in expression (5) must be zero. This leads to a system of equations for the unknown parameters α, $a(\tau)$ and c_n, $n = 0$, 1, 2, ..., in the representation (2) of the solution of Eq. (1), if it exists in in this form. Note that each term in each element of the sum (5) has a nonzero factor $a(\tau)^{\alpha}$, therefore, for simplicity, it can be reduced in all elements of the sum.

3 The CIR(1980) Model

Among the models of short-term rate $r(t)$ processes with zero drift, the CIR(1980) model [2] is widely known, in which the rate is generated in the general case by the diffusion process

$$dr = \sigma r^{\gamma} dw. \tag{6}$$

Despite the fact that the model is known for a long time, the time structure of its zero-coupon yield has not been described so far. It turns out that the proposed method for finding the time structure allows this. In Eq. (6) we take $\gamma = 1.5$ and $s \equiv 0.5\sigma^2$. Equation (1) for the price of the zero-coupon bond $P(r,\tau)$ takes the form

$$-\frac{\partial P(r,\tau)}{\partial \tau} + sr^3 \frac{\partial^2 P(r,\tau)}{\partial r^2} - rP(r,\tau) = 0, \quad P(r,0) = 1. \tag{7}$$

We seek a solution of this equation in the form (2). The corresponding derivatives have the form (3). After substituting these expressions into Eq. (7), we obtain the following equality

$$-\frac{a'(\tau)}{a(\tau)}\sum_{n=0}^{\infty}(\alpha+n)\left(\frac{a(\tau)}{r}\right)^{\alpha+n}c_n - \ a(\tau)\sum_{n=0}^{\infty}\left(\frac{a(\tau)}{r}\right)^{\alpha+n-1}c_n$$
$$+s\,a(\tau)\sum_{n=0}^{\infty}(\alpha+n)(\alpha+n+1)\left(\frac{a(\tau)}{r}\right)^{\alpha+n-1}c_n = 0.$$

This equality can be rewritten in a more convenient form:

$$\sum_{n=0}^{\infty}\left[(\alpha+n)\,\frac{a'(\tau)}{a(\tau)}c_n + a(\tau)\,c_{n+1} - s\,a(\tau)\,(\alpha+n+1)(\alpha+n+2)c_{n+1}\right]$$
$$\times\left(\frac{a(\tau)}{r}\right)^{\alpha+n} + \ a(\tau)\,(1-s\,\alpha\,(\alpha+1))\ c_0\left(\frac{a(\tau)}{r}\right)^{\alpha-1} = 0.$$

Since the expressions $(a(\tau)/r)^k$ as functions of the variable r for different values of k are linearly independent, and the equality must be satisfied uniformly with respect to r, then the coefficients before these expressions for different k must be zero. And we get a system of equations for the unknowns α, $a(\tau)$ and c_n, $n = 0, 1, 2, ...$

$$s\alpha(\alpha+1) = 1, \tag{8}$$

$$(\alpha+n)\frac{a'(\tau)}{a(\tau)^2}c_n + c_{n+1} - s(\alpha+n+1)(\alpha+n+2)c_{n+1} = 0, \quad n = 0, 1, 2, ... \tag{9}$$

From the Eq. (8) the parameter α is determined by

$$\alpha = \frac{1}{2}\left(\sqrt{1+\frac{4}{s}} - 1\right) \equiv \frac{1}{2}\left(\frac{\sqrt{8+\sigma^2}}{\sigma} - 1\right) > 0. \tag{10}$$

Generally speaking, Eq. (8) has two roots: positive and negative. However, with a negative solution, as will be shown below, the price function $P(r, \tau)$ acquires properties that the price of the zero-coupon bond does not possess. Therefore, we take the root (10). Consider the Eq. (9) for $n = 0$.

$$a'(\tau)\alpha c_0 + a(\tau)^2 c_1[1 - s(\alpha+1)(\alpha+2)] = 0.$$

Taking into account equality (8), it can be rewritten as

$$a'(\tau) = a(\tau)^2\frac{2\omega}{\alpha^2}, \tag{11}$$

where for brevity we denote $\omega = c_1 \ / \ c_0$. Equation (11) is a differential equation with respect to the function $a(\tau)$. The solution of the equation has the form

$$a(\tau) = -\frac{\alpha^2}{2\omega\tau + \eta}, \tag{12}$$

up to a constant η, which, if necessary, is determined from the properties of the bond price. We note that it follows from (11) that

$$\frac{a'(\tau)}{a(\tau)^2} = \frac{2\omega}{\alpha^2}.$$

Now consider Eq. (9) for an arbitrary $n \geq 1$. It can be written as a recurrence relation that determines the coefficient c_{n+1} in terms of the coefficient c_n:

$$c_{n+1} = \frac{2(\alpha+n)\omega}{[s(\alpha+n+1)(\alpha+n+2)-1]\,\alpha^2}\, c_n, \tag{13}$$

Note that

$$\frac{2(\alpha+n)}{[s(\alpha+n+1)(\alpha+n+2)-1]\,\alpha^2} = \frac{\theta}{n+1}\left(\frac{n+\alpha}{n+\xi}\right),$$

where for brevity is denoted $\xi = 2(\alpha+1), \theta = \xi/\alpha$. Thus, the sequence of coefficients $\{c_n,\ n = 0,\ 1,\ 2,\ ...\}$ is as follows

$$c_0, \quad c_1 = c_0\omega = c_0\,\omega\,\theta\,\frac{\alpha}{\xi}\,, \quad c_2 = c_0\frac{\omega\,\theta}{2}\,\frac{(1+\alpha)}{(1+\xi)},$$

$$c_3 = c_0\frac{(\omega\,\theta)^2}{1\times2\times3}\,\frac{(1+\alpha)}{(1+\xi)}\frac{(2+\alpha)}{(2+\xi)}, \quad ..., \quad c_n = c_0\frac{(\omega\,\theta)^{n-1}}{n!}\prod_{k=1}^{n-1}\frac{(k+\alpha)}{(k+\xi)}, \quad ...$$

Then the solution (2) of Eq. (7) can be represented in the form

$$P(r,\tau) = c_0\left(\frac{a(\tau)}{r}\right)^\alpha\left[1+\left(\frac{\omega\theta a(\tau)}{r}\right)\frac{\alpha}{\xi}+\sum_{n=2}^{\infty}\left(\frac{\omega\theta a(\tau)}{r}\right)^n\frac{1}{n!}\prod_{k=0}^{n-1}\frac{(k+\alpha)}{(k+\xi)}\right].$$

We note that among the special functions there is a so-called confluent hypergeometric function (Kummer function) ${}_1F_1(x,\ y,\ z)$ (in the notation of the Wolfram Mathematica system), which is defined by

$${}_1F_1(x,y,z) = 1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\prod_{k=1}^{n}\frac{(x+k-1)}{(y+k-1)} = 1+\frac{\Gamma(y)}{\Gamma(x)}\sum_{n=1}^{\infty}\frac{z^n}{n!}\,\frac{\Gamma(x+n)}{\Gamma(y+n)}.$$

Using these notations, the price $P(r,\ \tau)$ can be written in the form

$$\begin{aligned} P(r,\tau) &= c_0\left(\frac{a(\tau)}{r}\right)^\alpha\left(1+\frac{\Gamma(\xi)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\omega\theta\,\frac{a(\tau)}{r}\right)^n\frac{\Gamma(\alpha+n)}{\Gamma(\xi+n)}\right) \\ &= c_0\left(\frac{a(\tau)}{r}\right)^\alpha {}_1F_1\left(\alpha,\ \xi,\ \omega\theta\,\frac{a(\tau)}{r}\right). \end{aligned}$$

In terms of its economic properties, the bond price as a function of the maturity term τ is a continuous monotonically decreasing function that for any $r > 0$ has limits [9]

$$\lim_{\tau\to0} P(r,\tau) = 1, \quad \lim_{\tau\to\infty} P(r,\tau) = 0.$$

These requirements can be satisfied by determining the so far undetermined constants c_0 and η by appropriate way. The final expression for the price of the zero-coupon bond becomes

$$P(r,\tau) = \frac{(1+\alpha)\sqrt{\pi}}{2^{1+2\alpha}\,\Gamma(\alpha+1.5)}\,\left(\frac{1}{s\,r\,\tau}\right)^{\alpha}\,{}_1F_1\left(\alpha,\ 2(1+\alpha),\ -\frac{1}{s\,r\,\tau}\right), \tag{14}$$

where $s \equiv 0.5\sigma^2$, $\alpha = 0.5\left(\sqrt{1+4/s}-1\right) > 0$, and $\Gamma(x)$ is gamma function. Here it is assumed that $\alpha > 0$. When $\alpha < 0$, the gamma function $\Gamma(x)$ used in formula (14) can have undesirable properties. For example, for integer negative values of an argument, it has unbounded discontinuities, on intervals $(2\kappa, 2\kappa+1)$, $\kappa = 0,\,1,\,2,\,...$, it is negative, etc., which is not corresponds to the properties of the bond price. Therefore, negative values of the parameter α are undesirable.

Typically, the term structure is in practice not represented through the bond price, but through yield. By definition, the yield to maturity of the zero-coupon bond (yield curve) $y(r,\tau)$ and the yield of the forward rates (forward curve) $f(r,\tau)$ are determined by the expressions [5]:

$$y(r,\tau) = -\frac{\ln P(r,\tau)}{\tau}, \quad f(r,\tau) = -\frac{\partial \ln P(r,\tau)}{\partial \tau} \tag{15}$$

and, unfortunately, are not presented in a compact analytical form and can only be investigated numerically.

4 The Ahn – Gao Model

Now let the polynomials $m(r)$ and $s(r)$ be such that $p = 2, q = 3$, that is $1 - p = 2 - q = -1$. Then the components of the sum (9) differ from zero only for $j \geq -1$, where the first term differs from zero only for $j \geq 0$. In this case we obtain the following system of equations:
for $j = -1$

$$-\,c_0 - \alpha\, m_2\, c_0 + \alpha\,(\alpha+1)\, s_3\, c_0 \;= 0\,; \tag{16}$$

for $j = 0$

$$-\alpha a'(\tau)a(\tau)^{-1}c_0 - a(\tau)c_1 - \sum_{k=1}^{2}(\alpha+k-1)m_k a(\tau)^{k-1}c_{k-1} + \sum_{k=2}^{3}(\alpha+k-2)(\alpha+k-1)s_k a(\tau)^{k-2}c_{k-2} = 0; \tag{17}$$

for $j = 1$

$$-(\alpha+1)a'(\tau)c_1 - a(\tau)^2c_2 - \sum_{k=0}^{2}(\alpha+k)m_k a(\tau)^k c_k + \sum_{k=1}^{3}(\alpha+k-1)(\alpha+k)s_k a(\tau)^{k-1}c_{k-1} = 0; \tag{18}$$

for $j > 1$

$$-(\alpha+j)a'(\tau)a(\tau)^{j-1}c_j - a(\tau)^{j+1}c_{j+1} - \sum_{k=0}^{2}(\alpha+j+k-1)m_k a(\tau)^{j+k-1}c_{j+k-1}$$
$$+\sum_{k=0}^{3}(\alpha+j+k-2)(\alpha+j+k-1)s_k a(\tau)^{j+k-2}c_{j+k-2} = 0. \quad (19)$$

From the Eq. (16), which under the assumption that $c_0 \neq 0$ has the form $\alpha(\alpha+1)s_3 = \alpha m_2 + 1$, the parameter α is determined.

$$\alpha_1 = \frac{1}{2s_3}\left(m_2 - s_3 - \sqrt{4s_3 + (m_2 - s_3)^2}\right),$$
$$\alpha_2 = \frac{1}{2s_3}\left(m_2 - s_3 + \sqrt{4s_3 + (m_2 - s_3)^2}\right). \quad (20)$$

Since Eq. (16) is quadratic, it has two roots, which means that the solution of Eq. (1) can have two components of the form (2), a compromise between them, and also the initial condition $P(r, 0) = 1$ can affect on the choice of the coefficient c_0.

Equation (17) is an ordinary differential equation with respect to the function $a(\tau)$. Its solution has the form

$$a(\tau) = \frac{\lambda}{\mu + exp[(\tau + \alpha\xi c_0)(m_1 - (\alpha+1)s_2)]}, \quad (21)$$

where for compactness we denote by $\lambda = \alpha c_0((1+\alpha)s_2\alpha - m_1)$, $\mu = c_1(1 + (\alpha+1)m_2 - (\alpha+1)(\alpha+2)s_3)$, and ξ is a constant integration of the differential equation, whose choice is made depending on the properties of the solution of Eq. (1).

Equation (18) determines the coefficient c_2, and Eq. (19) can be considered as the basis for constructing a recurrence formula for calculating the coefficients c_{n+1} in terms of the previous coefficients c_j, $j \leq n$. Consider first the Eq. (19). It allows us to express the coefficient c_{j+1} in terms of the previous coefficients c_j, c_{j-1}, c_{j-2} by the formula

$$c_{j+1} = \frac{1}{a(\tau)^3[(1+\alpha+j)(2+\alpha+j)s_3 - (1+\alpha+j)m_2 - 1]}$$
$$\times[a(\tau)(\alpha+j)(a'(\tau) + a(\tau)(m_1 - (1+\alpha+j)s_2))c_j$$
$$-[a(\tau)(-m_0 + (\alpha+j)s_1))c_{j-1} + (\alpha+j-2)s_0c_{j-2}](\alpha+j-1)]. \quad (22)$$

However, by the definition of the coefficients c_n in the expression (2), they must be constant coefficients independent of the variable τ. This means that in the formula (22) the right-hand side of the equality must not depend on τ. This is only if $m_0 = 0$, $s_0 = 0$, $s_1 = 0$, $s_2 = 0$. This requirement is a necessary condition for the existence of a non-trivial solution (2), which says that a non-trivial solution does not hold for any polynomials $m(r)$ and $s(r)$ are of order 2 and 3, respectively, but only for

$$m(r) = m_1 r + m_2 r^2, \quad s(r) = s_3 r^3. \tag{23}$$

Substitution of the required necessary conditions into the formula (22) for the coefficient c_{n+1} leads to the recurrence relation

$$c_{n+1} = \frac{\alpha + n}{(1+\alpha+n)(2+\alpha+n)s_3 - (1+\alpha+n)m_2 - 1} \frac{a'(\tau) + a(\tau)m_1}{a(\tau)^2} c_n. \tag{24}$$

We note that the denominator of the first factor of the right-hand side of (24) can be represented in the form

$$(1+\alpha+n)(2+\alpha+n)s_3 - (1+\alpha+n)m_2 - 1 = s_3(1+n)(\beta+n),$$

where

$$\beta = \begin{cases} \beta_1 \equiv \frac{1}{s_3}\left(s_3 - \sqrt{4s_3 + (m_2 - s_3)^2}\right) & for \quad \alpha = \alpha_1, \\ \beta_2 \equiv \frac{1}{s_3}\left(s_3 + \sqrt{4s_3 + (m_2 - s_3)^2}\right) & for \quad \alpha = \alpha_2. \end{cases} \tag{25}$$

When the necessary conditions are fulfilled, the function a(τ), determined by the formula (21), is somewhat simplified

$$a(\tau) = \frac{\lambda}{\mu + exp[(\tau + \alpha\xi c_0)m_1]}, \tag{26}$$

where $\lambda = -\alpha c_0 m_1$, $\mu = (1 + (\alpha+1)m_2 - (\alpha+1)(\alpha+2)s_3)c_1$. Substituting into the right-hand side of (24) an explicit expression for the function $a(\tau)$, determined by formula (15), we obtain

$$\frac{a'(\tau) + a(\tau)m_1}{a(\tau)^2} = \frac{\mu m_1}{\lambda} = \frac{s_3\beta\omega}{\alpha},$$

where $\omega \equiv c_1/c_0$. In this case, the dependence on the variable τ on the right-hand side of formula (24) vanishes. Thus, the recurrence formula (24) for the coefficient c_{n+1} is transformed to the final form

$$c_{n+1} = \frac{\beta(\alpha+n)\omega c_n}{\alpha(1+n)(\beta+n)}. \tag{27}$$

Now we turn to the solution of the last Eq. (18), from which it is necessary to determine c_2. Since there are $s_0 = 0$ among the necessary conditions, Eq. (18) will coincide with Eq. (15) for $n = 1$ and therefore the coefficient c_2 is calculated from formula (26) for $n = 1$.

It turns out that if the polynomials $m(r)$ and $s(r)$ of the order 2 and 3, respectively, are determined by the expressions (23), the solution of Eq.,(1) can be represented as the sum of two series of the type (2), each of which has the following structure

$$\left(\frac{a(\tau)}{r}\right)^{\alpha} c_0 \left(1+\sum_{n=1}^{\infty}\left(\frac{a(\tau)}{r}\,\frac{\omega\beta}{\alpha}\right)^{n}\frac{1}{n!}\prod_{k=0}^{n-1}\frac{\alpha+k}{\beta+k}\right).$$

Using again the confluent hypergeometric function, the result can be compactly written in the analytical form

$$c_0\left(\frac{a(\tau)}{r}\right)^{\alpha} {}_1F_1\left(\alpha,\ \beta,\ \frac{a(\tau)}{r}\,\frac{\omega\beta}{\alpha}\right). \tag{28}$$

As already mentioned, since Eq. (16) has two solutions (20), the solution of Eq. (1) can consist of two components of the form (2) with different sets of parameters $(\alpha,\ \beta)$, whose values are determined by formulas (20) and (25):

$$\begin{aligned}P(r,\tau) &= c_{01}\left(\frac{a(\tau)}{r}\right)^{\alpha_1} {}_1F_1\left(\alpha_1,\ \beta_1,\ \frac{a(\tau)}{r}\,\frac{\omega\beta_1}{\alpha_1}\right)\\ &+c_{02}\left(\frac{a(\tau)}{r}\right)^{\alpha_2} {}_1F_1\left(\alpha_2,\ \beta_2,\ \frac{a(\tau)}{r}\,\frac{\omega\beta_2}{\alpha_2}\right).\end{aligned} \tag{29}$$

Before concretizing the solution, we will make some preliminary analysis. First, we consider the properties of the diffusion process, given by drift and volatility, determined by the functions (1) and (23). According to the assumptions made, the process of short-term interest rate $r(t)$, corresponding to these functions, is described by equation

$$dr(t) = (m_1 r(t) + m_2 r(t)^2)dt + \sqrt{2s_3}\, r(t)^{3/2} dw.$$

The marginal probability density of this process has the form

$$f(r) = \frac{\delta^{2-\gamma} e^{-\delta/r}}{r^{3-\gamma}\Gamma(2-\gamma)}, \quad \delta = \frac{m_1}{s_3} > 0, \quad \gamma = \frac{m_2}{s_3} < 2, \quad s_3 > 0, \quad r \geq 0,$$

where $\Gamma(x)$ is the gamma function. Taking into account these inequalities, we note that the parameters of expression (28), according to formulas (20) and (25), take the values $\alpha_1 < 0$, $\alpha_2 > 0$, $\beta_2 > 0$, and β_1 can take positive values only when the volatility parameter is $s_3 > 4$, which practically does not occur in real cases.

As is well known, the bond price for $r > 0$ is a monotonically decreasing function with respect to the variable $\tau \in (0,\ \infty)$ from $P(r,0) = 1$ to $P(r,\infty) = 0$. Therefore, expression (28) must have the same properties. The function ${}_1F_1(x,y,z)$ has suitable properties only for $x > 0$, $y > 0$, $z \in (-\infty, 0)$. Therefore, the first term in the representation (29) must be absent. In addition, for the argument z to ${}_1F_1$ to take values in the interval $(-\infty, 0)$ as τ changes in the interval $(0,\infty)$, it is necessary to define the integration constant ξ in expression (15) by the equality $\xi = \ln(\beta\omega s_3)/\alpha c_0 m_1$. Then

$$a(\tau) = \frac{\lambda}{\mu + exp[(\tau + \alpha\xi c_0)m_1]} = \frac{-m_1}{(e^{m_1\tau}-1)s_3}.$$

Finally, in order for the requirement $P(r,0)=1$, to be satisfied, it is necessary that the uncertain so far parameter c_0 be defined by the equality $c_0 = \Gamma(\beta-\alpha)/\Gamma(\beta)$. Thus, the final form of the solution (2) of Eq. (1) in the case under consideration has a final form

$$P\left(r,\tau\right)=\frac{\Gamma\left(\beta-\alpha\right)}{\Gamma\left(\beta\right)}\left(\frac{m_1}{r\,s_3\left(e^{m_1\tau}-1\right)}\right)^{\alpha}{}_1F_1\left(\alpha,\ \beta,\ \frac{-m_1}{r\,s_3\left(e^{m_1\tau}-1\right)}\right),$$

where the parameters α and β are determined by means of formulas (20) and (25):

$$\alpha=\frac{1}{2s_3}\left(m_2-s_3+\sqrt{4s_3+\left(m_2-s_3\right)^2}\right)>0,$$
$$\beta=\frac{1}{s_3}\left(s_3+\sqrt{4s_3+\left(m_2-s_3\right)^2}\right)>0.$$

We note that this solution completely coincides with the solution obtained in another way by Ahn and Gao [3], where in the notation of these authors $m_1 = \kappa\theta-\lambda_1>0$, $m_2=-\kappa-\lambda_2<0$, $s_3=\sigma^2/2$. In principle, using expression (18), we can find, by formulas (15), analytical expressions for the yield curve $y(r,\tau)$ and the forward curve $f(r,\tau)$. However, these expressions are very cumbersome and more practical to use numerical methods for expressing these functions for the necessary numerical parameters.

The functions $y(r,\tau)$ and $f(r,\tau)$, defined by formulas (15) in terms of the representation $P(r,\tau)$, can be investigated only by numerical methods. True, the limiting values of these functions can be found in an analytical form:

$$\lim_{\tau\to 0} y(r,\tau)=\lim_{\tau\to 0} f(r,\tau)=r,\quad \lim_{\tau\to\infty} y(r,\tau)=\lim_{\tau\to\infty} f(r,\tau)=\alpha m.$$

As you can see, the left limit is determined only by the state of the market and does not depend on the model parameters, and the right limit is determined only by the structure of the model and does not depend on the state of the market at a certain moment in time.

5 Conclusion

The article presents models for which yield curves of zero-coupon bonds and corresponding forward curves can be found that are not related to the class of affine models. Unfortunately, models that admit such solutions are few and, in particular, include some well-known models: the CIR(1980) model [2] and the Ahn-Gao model [3]. Let us formulate the requirements for the structure of the short-term interest rate model, which would allow obtaining the term structure of the bond price in the form (2).

The parameters of the series (2) are determined by the Eq. (9), in fact, from which we obtain a system of equations with respect to the unknowns $\alpha, a(\tau)$ and $c_n, n=0,1,2,...$

1. To obtain a non-trivial solution (that is, for the presence of $c_n \neq 0$), it is necessary that the degrees p and q of the polynomials $m(r)$ and $s(r)$, determining the drift and volatility of the short-term interest rate, satisfy one of the following conditions: $\{p \leq 2,\ q = 3\}$, $\{p = 2,\ q \leq 3\}$, $\{p > 2,\ q = p + 1\}$. In these cases, the equations are found from which the positive parameter is determined.
2. Another necessary condition is related to the existence of $a(\tau)$, which does not depend on the summation index of the series (2).
3. In addition, it is necessary that the coefficients $\{c_n\}$ do not depend on the variable τ.

Simultaneous fulfillment of these necessary conditions significantly narrows the family of models for which the solution of the term structure equation (1) has the form (2).

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