Some Necessary and Sufficient Conditions for Consensus of Fractional-Order Multi-agent Systems with Input Delay and Sampled Data

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Abstract. In this paper, the consensus of fractional-order multi-agent systems subject to input delay is investigated by sampled data method on directed graph. By applying the Laplace transform and the technique of inequality, some necessary and sufficient conditions for achieving consensus of the delay systems are obtained. It is shown that the consensus of the delay systems has relationships with the order of the derivative, the sampling period, delay, coupling strength, and communication topology. Lastly, a numerical simulation is given to illustrate the theoretical results.

Keywords: Consensus · Fractional-order · Multi-agent systems Delay · Sampled data

1 Introduction

Consensus as one of the most essential collective behaviors in multi-agent systems has been widely studied due to its extensive applications [1–3], such as biology systems, multi-robots systems, sensor networks, to name just a few. The consensus problem is to make a group of autonomous agents converge to an agreement by only using some local information.

However, most of the researches about the consensus of multi-agent systems are focused on the integer-order dynamics, such as single-integral systems, double-integral systems. In practice, some systems cannot be exactly described by integer-order dynamics [4–6], such as porous media and electromagnetic waves, while some characteristics of these systems can be explained naturally by fractional-order dynamics. On the other hand, integer-order systems can be deemed to a special case of fractional-order systems.

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Cao *et al.* discussed the consensus of fractional-order multi-agent systems firstly [6,7], and they obtained that the stability of the fractional-order multi-agent systems had something to do with the fractional order and the networked topology. Then, many works about the consensus of fractional-order multi-agent systems were done [9–14]. Nevertheless, because of the complex circumstances and the finite reaction times, the input delay always occurs in real systems. In [11], the heterogeneous input delays and communication delay were studied by the frequency-domain method and the generalized Nyquist criterion, respectively, and the consensus condition made a big difference for the different fractional order interval. In [12], a necessary and sufficient condition for the consensus of fractional-order multi-agent systems subject to input delay was obtained and a necessity condition was given for the systems with diverse input delays. In [14], consensus of fractional-order multi-agent systems with general linear and nonlinear dynamics and input delay was researched.

In practice, due to the finite transmission bandwidth and the limited resource, continuous information transmission is unreliable. Therefore, it is more practical to use the sampled data method which means that a continuous system adopts a discrete control protocol. On the other hand, it can reduce the energy and cut down the cost. As far as we know, there is only one paper which has studied the consensus of the fractional-order multi-agent systems via sampled data method [15]. In [15], the leaderless and leader consensus of fractional-order multi-agent systems with sampled data method were investigated, and some necessary and sufficient conditions were obtained.

In this paper, based on the sampled data scheme, the delay fractional-order multi-agent system was considered. Then main contributions of this paper are presented as follows. First, a sampled data method is used for the fractional-order multi-agent systems with input delay on directed network. Compared with the existing works, the system is more general and the method reduces the energy. Second, some necessary and sufficient conditions are presented, which give the relationship between the consensus of the systems and the order of the derivative, the sampling period, delay, coupling strength, and communication topology.

Section 2 introduces some preliminaries and states the problem. Section 3 presents the main results. Section 4 gives a numerical example and some conclusions are presented in Sect. 5.

2 Preliminaries and Problem Statement

2.1 Graph Theory

Let a networked graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ be the communication topology among N agents, where $\mathcal{V} = \{0, 1, \ldots, N\}$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the set of nodes and the set of edges, respectively. $\mathcal{W} = [w_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix, where $w_{ij} > 0$ if $(j, i) \in \mathcal{E}$, else $w_{ij} = 0$, and $w_{ii} = 0$. If the communication topology is an undirected graph, $w_{ij} = w_{ji}$, which means the j-th agent can receive the information from the i-th agent, and vice verse. If the communication topology is a directed graph, $w_{ij} > 0$ means the j-th agent can receive the information from the i-th agent, but not the opposite. $N_i = \{v_i \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$ denotes the neighbors set for the i-th agent. The Laplacian matrix L associated with the communication topology $\mathcal{G}$ is defined as $L = [l_{ij}] \in \mathbb{R}^{N\times N}$, in which $l_{ii} = \sum_{j\neq i}^{N} w_{ij}$ and $l_{ij} = -w_{ij}, i \neq j$.

Lemma 1. [16] *For a directed graph, a Laplacian matrix has a simple zero eigenvalue and all of the other eigenvalues have positive real parts if and only if the directed graph has a directed spanning tree. For an undirected graph, a Laplacian matrix has a simple zero eigenvalue and all of the other eigenvalues are positive if and only if the undirected graph is connected.*

2.2 Caputo Fractional Operator

Definition 1. [17] *The definition of Caputo fractional-order integral for a function $f(t)$ can be written as*

$$ {}_{t_0}^{C}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, $$

where $0 < \alpha \leq 1$ denotes the order of integral, and $\Gamma(\cdot)$ is the Gamma function which is defined as

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt. $$

Definition 2. [17] *The definition of Caputo fractional-order derivative for a function $f(t)$ can be expressed as*

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau. $$

where $0 < \alpha \leq 1$ denotes the order of derivative.

Because just the Caputo fractional operator is studied throughout this paper, we use ${}_{0}^{C}D_t^{-\alpha} f(t)$ to replace $D^{-\alpha} f(t)$ and ${}_{0}^{C}D_t^{\alpha} f(t)$ to replace $D^{\alpha} f(t)$ for convenience.

Lemma 2. [17] *For a constant c, and a function $f(t) \in \mathbb{C}^n[a, b]$,*

(1) $D^{\alpha} c = 0$;
(2) $D^{-\alpha} D^{\alpha} f(t) = f(t) - f(a)$, $0 < \alpha \leq 1$.

In the next, the Laplace transform of the Caputo derivative is introduction.

Let $\mathcal{L}\{\cdot\}$ denote the Laplace transform of a function. The formal definition of the Laplace transform $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$ is written as

$$ \mathcal{L}\{D^{\alpha} f(t)\} = s^{\alpha} F(s) - s^{\alpha-1} f(0), \ \ \alpha \in (0, 1]. $$

2.3 System Description

Consider the fractional-order multi-agent system consisting of N agents, the dynamics of the i-th agent is presented as follows:

$$D^{\alpha}x_i(t) = u_i(t), \quad i = 1, 2, \dots, N, \tag{1}$$

with $0 < \alpha \le 1$. $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ represent the state and the control input for the i-th agent, respectively.

In practice, the input delay always cannot be negligible. Therefore, the input delay is considered in this paper. By utilizing the period sampled data method, the delay control input for the i-th agent is given by

$$u_i(t) = -\mu \sum_{j \in N_i} a_{ij}\big(x_i(t_k - \tau) - x_j(t_k - \tau)\big),\ t \in [t_k, t_{k+1}),\ i = 1, 2, \dots, N, \tag{2}$$

where μ is the coupling strength. t_k $(k = 0, 1, 2, \dots)$, denotes the sampling time instants such that $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$ and $t_{k+1} - t_k = T$, in which $T > 0$ denotes the sampling period, and $0 < \tau < T$ is the input time delay.

Definition 3. *The fractional-order multi-agent system (1) under the control protocol (2) is said to reach consensus, if for any initial conditions,*

$$\lim_{t \to \infty} \parallel x_i(t) - x_j(t) \parallel = 0, \quad \forall i = 1, 2, \dots, N.$$

Lemma 3. [18] *For the following polynomial with $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$, i=1,2,*

$$f(s) = s^2 + (a_1 + \boldsymbol{i}b_1)s + (a_2 + \boldsymbol{i}b_2).$$

$f(s)$ is stable if and only if $a_1 > 0$ and $a_1b_1b_2 + a_1^2a_2 - b_2^2 > 0$.

3 Main Results

Combining (1) and (2), the fractional-order multi-agent system can be written as

$$D^{\alpha}x_i(t) = -\mu \sum_{j \in N_i} l_{ij}x_j(t_k - \tau), \quad t \in [t_k, t_{k+1}), \quad i = 1, 2, \dots, N, \tag{3}$$

described the systems in matrix form,

$$D^{\alpha}x(t) = -\mu(L \otimes I_n)x(t_k - \tau), \quad t \in [t_k, t_{k+1}), \tag{4}$$

where $x(t) = (x_1^T(t), x_2^T(t), \dots, x_N^T(t))^T$.

For the Laplacian matrix L, there exists a nonsingular matrix U satisfying $U^{-1}LU = J$, where J is the Jordan form associated with L, and $J = diag(\Lambda_1, \Lambda_2, \dots, \Lambda_r)$. For directed graph, the eigenvalues of L may be complex,

$$\Lambda_l = \begin{bmatrix} \lambda_l & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_l \end{bmatrix}_{N_l \times N_l} \tag{5}$$

where λ_i is the eigenvalue of L with algebraic multiplicity N_l, $l = 1, 2, \ldots, r$ and $N_1 + N_2 + \cdots + N_r = N$.

For undirected graph, the Laplacian matrix L is symmetric, so the Jordon form J is a diagonal matrix with real eigenvalues of L.

Let $y(t) = (P^{-1} \otimes I_n)x(t)$, we obtain,

$$D^{\alpha} y(t) = -\mu(J \otimes I_n)x(t_k - \tau),\ t \in [t_k, t_{k+1}), \tag{6}$$

According to Lemma 1, if the networked graph has a directed spanning tree, 0 is a simple eigenvalue of L, thus,

$$D^{\alpha} y_1(t) = \mathbf{0}_n, \quad t \in [t_k, t_{k+1}), \tag{7}$$

where $\mathbf{0}_n$ denotes a n-dimensional column vector with all entries being zero.

Lemma 4. *Assume that the network is directed and has a directed spanning tree, the consensus of the system (4) can be reached if and only if, in (6)*

$$\lim_{t\to\infty} \|y_i(t)\| = 0,\ \ i = 2, 3, \ldots, N.$$

Proof (Sufficiency): Based on Lemma 2, integrating both sides of (7) from t_k to t, we have $y_1(t) - y_1(t_k) = 0$, $t \in [t_k, t_{k+1})$. Therefore, $y_1(t) = y_1(t_k) = y_1(t_{k-1}) = \cdots = y_1(0)$. Because $\lim_{t\to\infty} \|y_i(t)\| = 0$, $i = 2, 3, \ldots, N$, so we can get $\lim_{t\to\infty} y(t) = [y_1(0)^T, \mathbf{0}^T, \ldots, \mathbf{0}^T]^T$. We have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(P \otimes I_n)y(t) = (P \otimes I_n)My(0) = (P \otimes I_n)M(P^{-1} \otimes I_n)x(0)$, where the matrix $M = [m_{ij}] \in \mathbb{R}^{Nn\times Nn}$ satisfies $m_{ij} = 0$, $i = j = 1, 2, \ldots, n$, else $m_{ij} = 0$. Since the network has a directed spanning tree, so we can choose $P = [\mathbf{1}_N, p_2, \ldots, p_N]$ and $P^{-1} = [q^T, q_2^T, \ldots, q_N^T]^T$, therefore, $\mathbf{1}_N$ and q^T are the right and left eigenvector of the Laplacian matrix L associated with $\lambda_1 = 0$ and $q\mathbf{1}_N = 1$. So, $\lim_{t\to\infty} x(t) = (P \otimes I_n)M(P^{-1} \otimes I_n)x(0) = (\mathbf{1}_N q^T \otimes I_n)x(0)$, that is, $\lim_{t\to\infty} x_i(t) = (q^T \otimes I_n)x(0)$. The consensus is reached.

(Necessity): If the consensus can be reached, then there must exist a vector $x^*(t)$ such that $\lim_{t\to\infty} x_i(t) = x^*(t)$ and $\lim_{t\to\infty} x(t) = \mathbf{1}_N \otimes x^*(t)$. Because $\mathbf{0}_N = P^{-1}L\mathbf{1}_N = JP^{-1}\mathbf{1}_N = J[q\mathbf{1}_N, q_2\mathbf{1}_N, \ldots, q_N\mathbf{1}_N]^T$. Based on Lemma 1, if the network has a directed spanning tree, the eigenvalues of L satisfy that $0 = \lambda_1 < Re(\lambda_2) \le Re(\lambda_3) \le \cdots \le Re(\lambda_N)$. Combining with the Jordan form J, we have $q_i\mathbf{1}_N = 0$, $i = 2, 3, \ldots, N$. Hence, $\lim_{t\to\infty} \|y_i(t)\| = \lim_{t\to\infty} \|(P^{-1} \otimes I_n)x_i(t)\| = \lim_{t\to\infty} \|(q_i \otimes I_n)x(t)\| = \lim_{t\to\infty} \|(q_i\mathbf{1}_N) \otimes x^*(t)\| = 0$, $i = 2, 3, \ldots, N$.

Corollary 1. *Assume that the network is directed and has a directed spanning tree, the system (4) can reach consensus if and only if the following $r-1$ systems are asymptotically stable*

$$D^{\alpha} z_i(t) = -\mu\lambda_i z_i(t_k - \tau), \quad t \in [t_k, t_{k+1}), \quad i = 2, 3, \ldots, r, \tag{8}$$

where $0 < \alpha \le 1$.

Proof. The proof is similar to Corollary 1 in [15].

Theorem 1. *If the network has a directed spanning tree and* $0 < \tau < T$*, then, the fractional-order multi-agent system (1) under the control protocol (2) can reach consensus if and only if*

$$T^{\alpha} - (T-\tau)^{\alpha} < a, \tag{9}$$

and

$$(3T^{\alpha} - 2(T-\tau)^{\alpha} - 2a)b^2 - (T^{\alpha} - (T-\tau)^{\alpha} - a)^2(T^{\alpha} - 2(T-\tau)^{\alpha} + 2a) < 0, \tag{10}$$

where $a = \frac{\Gamma(1+\alpha)}{\mu\|\lambda_i\|^2} Re(\lambda_i)$, $b = \frac{\Gamma(1+\alpha)}{\mu\|\lambda_i\|^2} Im(\lambda_i)$. α *is the order of derivative.* λ_i *is the nonzero eigenvalues of the Laplacian matrix* L, $Re(\lambda_i)$ *and* $Im(\lambda_i)$ *denote the real part and imaginary part of* λ_i*, respectively.*

Proof. For $t \in [t_k, t_{k+1})$, taking the Laplace transform of (8), we have

$$s^{\alpha} z_i(s) - s^{\alpha-1} z_i(t_k) = -\frac{\mu\lambda_i}{s} z_i(t_k - \tau), \tag{11}$$

so,

$$z_i(s) = \frac{1}{s} z_i(t_k) - \frac{\mu\lambda_i}{s^{\alpha+1}} z_i(t_k - \tau). \tag{12}$$

Taking the inverse Laplace transform of (12), we can obtain,

$$z_i(t) = z_i(t_k) - \frac{\mu\lambda_i}{\Gamma(1+\alpha)} (t - t_k)^{\alpha} z_i(t_k - \tau), \quad t \in [t_k, t_{k+1}), \tag{13}$$

and

$$z_i(t-\tau) = z_i(t_k) - \frac{\mu\lambda_i}{\Gamma(1+\alpha)} (t - t_k - \tau)^{\alpha} z_i(t_k - \tau). \tag{14}$$

Since the sampling time instants $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $t_{k+1} - t_k = T$ and $\tau < T$, we can get

$$z_i(t_k) = z_i(t_{k-1}) - \frac{\mu\lambda_i}{\Gamma(1+\alpha)} T^{\alpha} z_i(t_{k-1} - \tau), \tag{15}$$

and

$$z_i(t_k - \tau) = z_i(t_{k-1}) - \frac{\mu\lambda_i}{\Gamma(1+\alpha)} (T-\tau)^{\alpha} z_i(t_{k-1} - \tau), \tag{16}$$

Let $w_i(t) = [z_i(t), z_i(t-\tau)]^T$, we have

$$w_i(t_k) = E(T)w_i(t_{k-1}) = E^k(T)w_i(t_0), w_i(t) = E(t-t_k)w_i(t_k) = E(t-t_k)E^k(T)w_i(t_0),$$

where $E(t) = \begin{bmatrix} 1 & -\frac{\mu\lambda_i}{\Gamma(1+\alpha)} t^{\alpha} \\ 1 & -\frac{\mu\lambda_i}{\Gamma(1+\alpha)} (t-\tau)^{\alpha} \end{bmatrix}$.

Therefore, $\lim t \to \infty z_i(t) \to 0$ is equivalent to $w_i(t) \to 0$ as $t \to \infty$. Because it is obvious that $E(t-t_k)$ is bounded on $t \in [t_k, t_{k+1})$, so $\lim t \to \infty z_i(t) \to 0$ is equivalent to all the eigenvalues γ of the matrix $E(T)$ satisfy $\|\gamma\| < 1$.

Let $|\gamma I_2 - E(T)| = 0$, we can get

$$\gamma^2 + \big(\frac{\mu\lambda_i}{\Gamma(1+\alpha)}(T-\tau)^\alpha - 1\big)\gamma + \frac{\mu\lambda_i}{\Gamma(1+\alpha)}\big(T^\alpha - (T-\tau)^\alpha\big) = 0. \tag{17}$$

Let $\gamma = \frac{s+1}{s-1}$, hence, $\|\gamma\| < 1$ if and only if $Re(s) < 0$. (17) is equivalent to

$$s^2 + 2(a_1 + \mathbf{i}b_1)s + (a_2 + \mathbf{i}b_2) = 0, \tag{18}$$

where $a_1 = -1 + (1-\frac{\tau}{T})^\alpha + \frac{\Gamma(1+\alpha)}{\mu T^\alpha \|\lambda_i\|^2} Re(\lambda_i)$, $a_2 = 1 - 2(1-\frac{\tau}{T})^\alpha + 2\frac{\Gamma(1+\alpha)}{\mu T^\alpha \|\lambda_i\|^2} Re(\lambda_i)$, $b_1 = b_2 = -2\frac{\Gamma(1+\alpha)}{\mu T^\alpha \|\lambda_i\|^2} Im(\lambda_i)$. Based on Lemma 3, (19) is stable if and only if $a_1 > 0$ and $a_1 b_1 b_2 + a_1^2 a_2 - b_2^2 > 0$. By solving the above inequation, we can get the conditions (9) and(10). Therefore, $\lim_{t\to\infty} z_i(t) = 0$ if and only if (9) and(10) are satisfied.

Remark 1. In Theorem 1, a necessary and sufficient condition for the fractional-order multi-agent systems subject to input delay is proposed. For a fixed directed network, we can see that the consensus of the fractional-order multi-agent systems depends on the sampling period, delay, coupling strength, communication topology, and depends on the order of the derivative.

Remark 2. When $\tau = 0$, the system (1) degrades as the system without input delay, and the conditions (9) and(10) simplify as

$$T^\alpha < 2a = \frac{2\Gamma(1+\alpha)}{\mu\|\lambda_i\|^2} Re(\lambda_i),$$

which is the same as the Theorem 1 in [15].

Corollary 2. *If the network is undirected and connected, and $0 < \tau < T$. Then the fractional-order multi-agent system (1) under the control protocol (2), can reach consensus if and only if*

$$T^\alpha - (T-\tau)^\alpha - a_1 < 0, \tag{19}$$

and

$$T^\alpha - 2(T-\tau)^\alpha + 2a_1 > 0. \tag{20}$$

where $a_1 = \frac{\Gamma(1+\alpha)}{\mu\lambda_N}$, α is the order of derivative, μ is the coupling strength, and λ_N is the largest eigenvalue of L.

Proof. If the network is undirected and connect, we have all the eigenvalues of the Laplacian matrix L are positive real number, that is $Re(\lambda_i) = \lambda_i > 0$ and $Im(\lambda_i) = 0$. Then, $a = \frac{\Gamma(1+\alpha)}{\mu\lambda_i}$ and $b = 0$. Then, the conditions in Theorem 1 degrade as (19) and (20).

4 Simulation Example

Consider that the fractional-order multi-agent system (4) contains four agents, and the elements of weighted adjacency matrix are $w_{12} = 0.5$, $w_{14} = 0.7$, $w_{24} = 1$, $w_{31} = 0.3$, $w_{43} = 0.8$, and $w_{ij} = 0$, otherwise. We can get that the eigenvalues of Laplacian matrix are $\lambda_1 = 0$, $\lambda_2 = 0.95 + 0.5454\mathbf{i}$, $\lambda_3 = 0.95 - 0.5454\mathbf{i}$, and $\lambda_4 = 1.4$. Choosing the order of derivative $\alpha = 0.9$, the coupling strength $\mu = 0.2$, the sampling period $T = 1s$, and the input delay $\tau = 0.5s$, which can make sure that the conditions in Theorem 1 are satisfied. The initial condition are $x_1 = -1$, $x_2 = -4$, $x_3 = -8$, and $x_4 = 2$. The state of x_i $(i = 1, 2, 3, 4)$ is presented in Fig. 1, which illustrates that the consensus of the fractional-order multi-agent system (4) is achieved.

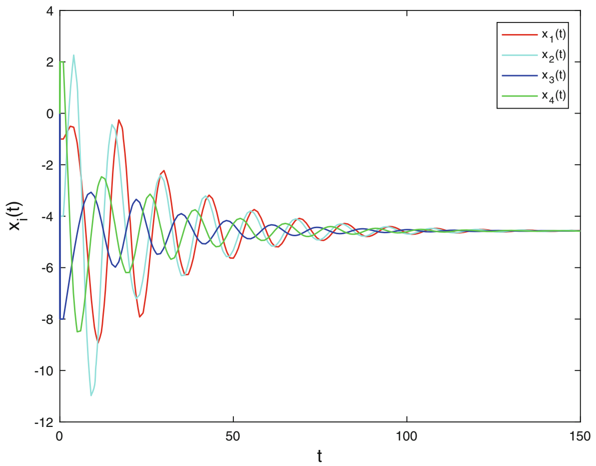

Fig. 1. The state of $x_i(t), i = 1, 2, 3, 4$.

5 Conclusion

In this paper, under the period sampled data method, the consensus of fractional-order multi-agent systems with the order $0 < \alpha < 1$ subject to input delay over directed communication topology is studied. Some necessary and sufficient conditions are established which give the relationship of the achieving consensus and the systems parameters (sampling period, delay, coupling strength, communication topology and the order of the derivative). Research of the aperiodic sampled data and diverse delays for the fractional-order multi-agent systems will be done in the future work.

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