Optimal Control Problem for Discrete-Time Markov Jump Systems with Indefinite Weight Costs

Hongdan Li[1], Chunyan Han[2], and Huanshui Zhang[1(✉)]

[1] School of Control Science and Engineering, Shandong University, Jinan 250061, Shandong, China
lhd200908@163.com, hszhang@sdu.edu.cn

[2] School of Electrical Engineering, University of Jinan, Jinan 250022, Shandong, China
cyhan823@hotmail.com

Abstract. In this article, the optimal control problem with indefinite state and control weighting matrices in the cost function for discrete-time systems involving Markov jump and multiplicative noise is discussed. Necessary and sufficient conditions of the solvability of indefinite optimal control problem in finite-horizon are obtained by solving the forward-backward stochastic difference equations with Markov jump (FBSDEs-MJ) derived from the maximum principle, whose method is different from most previous works [12], etc.

Keywords: Optimal control · FBSDEs-MJ · Indefinite Markov jump system

1 Introduction

There are many factors to give rise to abrupt changes such as abrupt environmental disturbances, component failures or repairs and these changes often occur in many control systems, for instance, economic systems and aircraft control systems. This phenomenon can be modeled as Markov jump linear systems (MJLS). Owing to its widely application in practice, in recent years, the subject of MJLS is by now huge and is growing rapidly, see [1–7], and reference therein. Seeing that the importance of the linear quadratic (LQ) control problem in the study of control system, there are also many results about these problems with Markov jump. [8] considered the optimal control problems for discrete-time linear systems subject to Markov jump with two cases that the one without noise and the other with an additive noise in model. In [9], they illustrated the equivalence between the stability of the optimal control and positiveness of the coupled algebraic Riccati equation via the concept of weak detectability.

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It is noteworthy that all the above results are obtained under the common assumption that the weighting matrices of state and control in the quadratic performance index are required to be positive semi-definite even positive definite. However, when the weighting matrices have the requirement of symmetry only, the stochastic LQ problem may be still well posed. This case is called indefinite stochastic problem which often appear in economic fields such as portfolio selection problem. As regard to the problem, [10] and [11] investigated an indefinite stochastic LQ control problem for continuous-time linear systems subject to Markov jump in finite and infinite time horizon, respectively. [12] derived the necessary and sufficient condition for the well posedness of the indefinite LQ problem and the optimal control law were given in terms of a set of coupled generalized Riccati difference equations interconnected with a set of coupled linear recursive equations.

Inspired by the above literatures, in this paper, we study the optimal control problem for discrete-time systems involving Markov jump and multiplicative noise in which the state and control weighting matrices in the cost function are indefinite. The main contribution of this paper is that an optimal controller is explicitly shown by a generalized difference Riccati equation with Markov jump (GDRE-MJ) which is derived from the solution to the FBSDEs-MJ, which is a new method compared with the previous works studied the linear quadratic optimal problem involving Markov jump. The rest of this article is made up of the following sections. Section 2 mainly provides some results about optimal control with finite horizon. And Sect. 3 makes a summary.

The related notations in this article are expressed as follows:

$\mathbb{R}^n$: the n-dimensional Euclidean space;
$\mathbb{R}^{m\times n}$: the norm bounded linear space of all $m \times n$ matrices;
Y' : the transposition of Y;
$Y \geq 0 (Y > 0)$: the symmetric matrix $Y \in \mathbb{R}^{n\times n}$ is positive semi-definite (positive definite);
$Y^\dagger$: the Moore-Penrose pseudo-inverse of Y;
$\mathbf{Ker}(Y)$: the kernel of a matrix Y;
$(\Omega, \mathcal{G}, \mathcal{G}_k, \mathcal{P})$: a complete probability space with the σ-field generated by $\{x(0), \theta(0), \cdots, x(k), \theta(k)\}$;
$E[\cdot|\mathcal{G}_k]$: the conditional expectation with respect to $\mathcal{G}_k$ and $\mathcal{G}_{-1}$ is understood as $\{\emptyset, \Omega\}$.

2 Preliminaries

Considering the following discrete-time Markov jump linear system with multiplicative noise:

$$x(k+1) = (A_{\theta(k)} + B_{\theta(k)}\omega(k))x(k) + (C_{\theta(k)} + D_{\theta(k)}\omega(k))u(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ denotes the state, $u(k) \in \mathbb{R}^m$ denotes control process and $\omega(k)$ is scalar valued random white noise with zero mean and variance σ^2. $\theta(k)$ is a discrete-time Markov chain with finite state space $\{1, 2, \cdots, L\}$ and transition probability $\rho_{i,j} = \mathrm{P}(\theta(k+1) = j|\theta(k) = i)(i, j = 1, 2, \cdots, L)$. We set $\pi_i(k) = \mathrm{P}(\theta(k) = i)(i = 1, 2, \cdots, L)$, while $A_i, B_i, C_i, D_i (i = 1, \cdots, L)$ are matrices of appropriate dimensions. The initial value x_0 is known. We assume that $\theta(k)$ is independent of x_0.

The cost function with finite horizon is as the following description.

$$J_N = \mathrm{E}\bigg\{ \sum_{k=0}^{N} \big[x(k)' Q_{\theta(k)} x(k) + u(k)' R_{\theta(k)} u(k)\big] + x(N+1)' P_{\theta(N+1)} x(N+1) \bigg\}, \tag{2}$$

where $N > 0$ is an integer, $x(N+1)$ is the terminal state, $P_{\theta(N+1)}$ reflects the penalty on the terminal state, the matrix functions $R_{\theta(k)}$ and $Q_{\theta(k)}$ are symmetric matrices.

Problem 1. Find a $\mathcal{G}_k$-measurable controller $u(k)$ to minimize (2) subject to (1).

On the ground of the indefiniteness of weighting matrices, the above problem may be ill-posed. Hence, we should introduce next definitions and lemmas.

Definition 1: Problem 1 is called well posed if $\inf\limits_{u_0, \cdots, u_N} J_N > -\infty$ for any random variables x_0.

Definition 2: Problem 1 is called solvable if there exists an admissible control $(u_0^*, \cdots, u_N^*)$ such that (2) is minimized for any x_0.

Remark 1: From Theorem 4.3 in [15], the equivalence between the well-posedness and the solvability of Problem* can be obtained.

Due to the dependence of $\theta(k)$ on its past values, an extended version of the stochastic maximum principle which is suitable for the MJLS (1) is established in the sequel.

Lemma 1 (Maximum Principle involving Markov Jump). According to the linear system (1) and the performance index (2). If the linear quadratic problem min J_N is solvable, then the optimal $\mathcal{G}_k$-measurable control $u(k)$ satisfies the following equilibrium condition

$$0 = \mathrm{E}[(C_{\theta(k)} + D_{\theta(k)}\omega(k))'\lambda_k + R_{\theta(k)}u(k)|\mathcal{G}_k], k = 0, \cdots, N, \tag{3}$$

where the costate λ_k satisfies the following equation

$$\lambda_N = \mathrm{E}[P_{\theta(N+1)}x(N+1)|\mathcal{G}_N], \tag{4}$$

$$\lambda_{k-1} = \mathrm{E}[(A_{\theta(k)} + B_{\theta(k)}\omega(k))'\lambda_k + Q_{\theta(k)}x(k)|\mathcal{G}_{k-1}], k = 0, \cdots, N, \tag{5}$$

together the costate Eqs. (4)–(5) with state Eq. (1), the FBSDEs-MJ is established, which play a vital role in this paper.

Proof. Similar to the derivation for Maximum Principle (MP) as in [13,17], the MP (3)–(5) follows directly, the aforementioned conclusion can be derived using an analogous step, so its proof is omitted.

Now we will show the following theorem which is expressed the result of Problem 1.

Theorem 1. *Problem 1 is solvable if and only if the following generalized difference Riccati equations with Markov jump*

$$\begin{cases} P_i(k) = A_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))A_i + \sigma^2 B_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))B_i + Q_i \\ -M_i(k)'\Upsilon_i(k)^{\dagger}M_i(k), \\ \Upsilon_i(k)\Upsilon_i(k)^{\dagger}M_i(k) - M_i(k) = 0, \\ \Upsilon_i(k) \geq 0, \end{cases} \tag{6}$$

in which

$$\Upsilon_i(k) = C_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))C_i + \sigma^2 D_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))D_i + R_i, \tag{7}$$

$$M_i(k) = C_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))A_i + \sigma^2 D_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))B_i, \tag{8}$$

has a solution. If this condition is satisfied, the analytical solution to the optimal control can be given as

$$u^*(k) = -\Upsilon_i(k)^{\dagger}M_i(k)x(k), i = 1, \cdots, L, \tag{9}$$

for $k = N, \cdots, 0$. The corresponding optimal performance index is given by

$$J_N^* = E[x(0)'P_{\theta(0)}(0)x(0)]. \tag{10}$$

The relationship of the costate λ_{k-1} and the state $x(k)$ is given as

$$\lambda_{k-1} = (\sum_{j=1}^{L}\rho_{i,j}P_j(k))x(k), i = 1, \cdots, L. \tag{11}$$

Proof (Necessity). Assume that Problem 1 is solvable, we will investigate that there exist symmetric matrices $P_i(0), \cdots, P_i(N)$, $i = 1, \cdots, L$ satisfying the GDRE-MJ (6) by induction. To this end, we first set the following formula as

$$\begin{aligned} \underline{J}(k) = \inf_{u_k,\cdots,u_N} \mathrm{E}\Big[&\sum_{i=k}^{N}(x(i)'Q_{\theta(i)}x(i) + u(i)'R_{\theta(i)}u(i)) \\ &+ x(N+1)'P_{\theta(N+1)}x(N+1)|\mathcal{G}_{k-1}\Big]. \end{aligned} \tag{12}$$

It is obvious to know that for any $k_1 < k_2$, when $\underline{J}(k_1)$ is finite then $\underline{J}(k_2)$ is also finite by the stochastic optimality principle. Since Problem* is supposed to be solvable, we can see that $\underline{J}(k)$ is finite for any $0 \leq k \leq N$.

Firstly, we let $k = N$, from system (1), we know that

$$\underline{J}(N) = \inf_{u_N} \mathrm{E}\Bigg\{ x(N)'[Q_i + A_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))A_i + \sigma^2 B_i' \cdot (\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))B_i]x(N) + 2x(N)'[A_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))C_i + \sigma^2 B_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))D_i]u(N) + u(N)'[R_i + C_i' \cdot (\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))C_i + \sigma^2 D_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))D_i]u(N)|\mathcal{G}_{N-1} \Bigg\}$$

By Lemma 4.3 in [15] and the finiteness of $\underline{J}(N)$, it yields that there indeed exist symmetric matrix $P_i(N)$ satisfying

$$\underline{J}(N) = \mathrm{E}[x(N)'P_i(N)x(N)],$$

and furthermore,

$$P_i(N) = A_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))A_i + \sigma^2 B_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))B_i + Q_i - M_i(N)'\Upsilon_i(N)^\dagger M_i(N), \tag{13}$$

$$\Upsilon_i(N)\Upsilon_i(N)^\dagger M_i(N) - M_i(N) = 0, \tag{14}$$

$$\Upsilon_i(N) \geq 0, \tag{15}$$

in which

$$\Upsilon_i(N) = C_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))C_i + \sigma^2 D_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))D_i + R_i, \tag{16}$$

$$M_i(N) = C_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))A_i + \sigma^2 D_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))B_i. \tag{17}$$

The optimal controller $u(N)$ will be calculated from (1), (3) and (4).

$$\begin{aligned} 0 &= E[(C_{\theta(N)} + D_{\theta(N)}\omega(N))'\lambda(N) + R_{\theta(N)}u(N)|\mathcal{G}_N] \\ &= \Big[C_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))A_i + \sigma^2 D_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))B_i\Big]x(N) \\ &\quad + \Big[C_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))C_i + \sigma^2 D_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1))D_i + R_i\Big]u(N). \end{aligned} \tag{18}$$

So, from (16) and (17), we have that

$$u(N) = -\Upsilon_i(N)^{\dagger} M_i(N) x(N), \tag{19}$$

which is as (9) in the case of $k = N$.

As to λ_{N-1}, from (1), (4), (5) and (19), it yields that

$$\begin{aligned}
\lambda_{N-1} &= \mathrm{E}[(A_{\theta(N)} + B_{\theta(N)}\omega(N))' E[P_{\theta(N+1)} x(N+1)|\mathcal{G}_N] + Q_{\theta(N)} x(N)|\mathcal{G}_{N-1}] \\
&= \mathrm{E}\Big[A_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1)) A_i + B_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(N+1)) B_i + Q_i \\
&\quad - M_i(N)'\Upsilon_i(N)^{\dagger} M_i(N)|\mathcal{G}_{N-1}\Big] x(N) \\
&= (\sum_{i=1}^{L} \rho_{s,i} P_i(N)) x(N), s = 1, \cdots, L,
\end{aligned} \tag{20}$$

which is satisfied (11) with $k = N$.

Now we assume that GDRE-MJ (6) has a solution $P_i(m)$, $k+1 \leq m \leq N$ and satisfying $\underline{J}(m) = \mathrm{E}[x(m)' P_i(m) x(m)]$ and $u(m)$, $\lambda(m-1)$ are as (9), (11), respectively, thus for k, we have

$$\begin{aligned}
\underline{J}(k) &= \inf_{u_k} \mathrm{E}\Big\{x(k)' Q_{\theta(k)} x(k) + u(k)' R_{\theta(k)} u(k) + \mathrm{E}[x(k+1)' P_i(k+1) \\
&\quad \cdot x(k+1)]|\mathcal{G}_{k-1}\Big\} \\
&= \inf_{u_k} \mathrm{E}\Big\{x(k)'[Q_i + A_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(k+1)) A_i + \sigma^2 B_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(k+1)) B_i] \\
&\quad \cdot x(k) + 2x(k)'[A_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(k+1)) C_i + \sigma^2 B_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(k+1)) D_i] u(k) \\
&\quad + u(k)'[R_i + C_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(k+1)) C_i + \sigma^2 D_i'(\sum_{j=1}^{L} \rho_{i,j} P_j(k+1)) D_i] \\
&\quad \cdot u(k)|\mathcal{G}_{k-1}\Big\}.
\end{aligned}$$

Similarly, from Lemma 4.3 in [15] and the finiteness of $\underline{J}(k)$, we can obtain that there exist $P_i(k)$ satisfying GDRE-MJ (6). Furthermore, $\underline{J}(k) = \mathrm{E}[x(k)' P_i(k) x(k)]$. From now on by mathematical induction we obtain that GDRE-MJ (6) exists a solution.

In the case that GDRE-MJ (6) exists a solution and the inductive hypothesis, the optimal controller $u(k)$ can be obtained from (1) and (3).

$$\begin{aligned}
0 &= E[(C_{\theta(k)} + D_{\theta(k)}\omega(k))'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))x(k+1) + R_{\theta(k)}u(k)|\mathcal{G}_k] \\
&= \Big[C_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))A_i + \sigma^2 D_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))B_i\Big]x(k) \\
&\quad + \Big[C_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))C_i + \sigma^2 D_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))D_i + R_i\Big]u(k), \qquad (21)
\end{aligned}$$

i.e.,

$$u(k) = -\Upsilon_i(k)^{\dagger}M_i(k)x(k). \qquad (22)$$

From (1), (5) and (22), λ_{k-1} can be derived as that

$$\begin{aligned}
\lambda_{k-1} &= \mathrm{E}[(A_{\theta(k)} + B_{\theta(k)}\omega(k))'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))x(k+1)] + Q_{\theta(k)}x(k)|\mathcal{G}_{k-1}] \\
&= \mathrm{E}\Big[A_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))A_i + B_i'(\sum_{j=1}^{L}\rho_{i,j}P_j(k+1))B_i + Q_i \\
&\quad - M_i(k)'\Upsilon_i(k)^{\dagger}M_i(k)|\mathcal{G}_{k-1}\Big]x(k) \\
&= (\sum_{i=1}^{L}\rho_{s,i}P_i(k))x(k), s = 1, \cdots, L. \qquad (23)
\end{aligned}$$

The proof about necessity is end.

(Sufficiency): When the GDRE-MJ (6) has a solution, we will show that Problem 1 is solvable.

Denote $V_N(k, x(k)) \triangleq \mathrm{E}[x(k)'P_{\theta(k)}(k)x(k)]$. From (1) we deduce that

$$\begin{aligned}
&V_N(k, x(k)) - V_N(k+1, x(k+1)) \\
&= \mathrm{E}[x(k)'P_{\theta(k)}(k)x(k) - x(k+1)'P_{\theta(k+1)}(k+1)x(k+1)] \\
&= \mathrm{E}\Big\{x(k)'[Q_i - M_i(k)'\Upsilon_i(k)^{\dagger}M_i(k)]x(k) - x(k)'M_i'(k)u(k) \\
&\quad - u(k)'M_i(k)x(k) - u(k)'\Upsilon_i(k)u(k) + u(k)'R_iu(k)\Big\} \\
&= \mathrm{E}\Big\{x(k)'Q_ix(k) + u(k)'R_iu(k) - [u(k) + \Upsilon_i(k)^{\dagger}M_i(k)x(k)]'\Upsilon_i(k)[u(k) \\
&\quad + \Upsilon_i(k)^{\dagger}M_i(k)x(k)]\Big\}
\end{aligned}$$

Adding from $k = 0$ to $k = N$ on both sides of the above equation, we have that

$$\begin{aligned} &V_N(0, x(0)) - V_N(N+1, x(N+1)) \\ = &\mathrm{E} \sum_{k=0}^{N} \Big\{ x(k)' Q_i x(k) + u(k)' R_i u(k) \\ &- [u(k) + \Upsilon_i(k)^{\dagger} M_i(k) x(k)]' \Upsilon_i(k) [u(k) + \Upsilon_i(k)^{\dagger} M_i(k) x(k)] \Big\}. \end{aligned} \tag{24}$$

The above mentioned equation implies that

$$J_N = \mathrm{E}[x_0' P_{\theta(0)} x_0] + \sum_{k=0}^{N} [u(k) + \Upsilon_i(k)^{\dagger} M_i(k) x(k)]' \Upsilon_i(k) [u(k) + \Upsilon_i(k)^{\dagger} M_i(k) x(k)].$$

Considering $\Upsilon_i(k) \geq 0$, we have $J_N \geq \mathrm{E}[x_0' P_{\theta(0)} x_0]$. Therefore, the optimal controller can be given by $u(k) = -\Upsilon_i(k)^{\dagger} M_i(k) x(k)$ and the optimal cost is given by $J_N = \mathrm{E}[x_0' P_{\theta(0)} x_0]$.

This completes the proof.

3 Conclusions

This article mainly study the linear quadratic optimal control problem for discrete-time systems involving Markov jump and multiplicative noise. The state and control weighting matrices in the cost function are allowed to be indefinite. By solving the FBSDEs-MJ derived from the extended maximum principle, we conclude that the indefinite optimal control problem in finite-horizon is solvable if and only if the corresponding GDRE-MJ has a solution, which is an easy verifiable conclusion compared with operator type results.

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